# Tensor Products and Statistics

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ABSTRACT

A dictionary between operator-based and matrix-based languages in multivariate statistical analysis is proposed. Then this formulary is applied to asymptotic factorial analyses, especially for giving asymptotic covariance matrices and operators in an explicit form. Finally, we present the mathematical foundations on which are based the functional tools, i.e. tensor products of linear spaces, of vectors, and of operators.

## 1. INTRODUCTION

In the multivariate statistics literature, we may find two distinct approaches. The first one uses essentially the matrix calculus and language, and the second one tends to exploit at best the possibilities of the functional tools.

In numerous papers, the matrix-based approach is preferred for historical reasons and also for computing convenience. Nevertheless, this point of view has several disadvantages. First of all, the notions considered are not coordinate-free; secondly, some of these notions are obscured by the language form; finally, as infinite dimension is a necessary assumption in the formalization and resolution of some problems, it is difficult and sometimes impossible to extend the results obtained in this form.

The functional or operator-based language is less used by statisticians. However, it gives an intrinsic form of the study to which it applies (that is, without basis choice constraint); it clarifies some notions that seem more complex in matrix form




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(think, for example, of the commutation transformation), and finally, for the study of not necessarily finite dimensional cases it gives a natural mathematical background.

Contrarily to a widespread idea, using linear maps (or operators) instead of matrix language does not present any difficulty in finite dimensions: in fact, the definition spaces are better specified (think, for example, of the adjoint and the transpose maps, which have the same matrix representations). Moreover, a computation may be more efficient than in matrix form, and the eventual extension to infinite dimension needs, at least at the first stage, minimal knowledge and care. For these reasons, we present our functional results in finite dimensional real pre-Hilbert spaces, that is, in Euclidean spaces.

The fundamental argument for the choice of the matrix calculus may be its convenience in a computational environment. The fast evolution of programming languages and of computers suggests that this reason will soon be less important and will perhaps disappear. Nevertheless, the statistical software packages extensively use the matrix calculus, and even in a functional context we need a translation from operator into matrix language. This translation must be in both directions and will give supplementary and new results for each approach.

Thus, starting from a functional point of view, the main goal of this work is to give a matrix-operator dictionary in finite dimension: that is the scope of Section 3. Before that, in Section 2, we present functional tools which are useful for the matrix-operator translation: whenever multiindices are used (as, for example, for the formulation of a matrix of matrices), we show that classical matrix calculus implicitly needs a predetermined convention for the basis enumeration.

Although applications of the dictionary are not restricted to factorial analyses (see Section 6 for this topic), we give in Section 4 two examples chosen in this area, namely principal component analysis and the canonical analysis. Finally, in the Section 5, we present the mathematical background on which the functional tools are based, such as tensor products of linear spaces, of vectors, and of operators.

## 2. TENSOR PRODUCTS AND MATRIX EXPRESSIONS

For each Euclidean space (that is, a finite dimensional pre-Hilbert space), the inner product is denoted by $\langle\cdot,\cdot\rangle$; for simplicity of notation, with one exception, no subscript will be added to the inner product, since it operates on a space which is easily identified from its variables. Let $X$ and $Y$ be two Euclidean spaces, and $\mathcal{L}(X, Y)$ denote the space of the linear maps from $X$ into $Y$. The Hilbert-Schmidt inner product on $\mathcal{L}(X, Y)$ is defined as follows:

$$(a, b) \in [\mathcal{L}(X, Y)]^2 \mapsto \langle a, b\rangle_2 = \operatorname{tr} ab^* = \operatorname{tr} ba^*,$$

where tr stands for the linear trace map and $a^*$ for the adjoint of $a$ [as usual, $a^*$ is the unique operator of $\mathcal{L}(Y, X)$ defined by $\langle a(x), y\rangle = \langle x, a^*(y)\rangle$ for each $(x, y)$ of $X \times Y$]. Let $\sigma_2(X, Y)$ [$\sigma_2(X)$] denote the Euclidean space obtained when $\mathcal{L}(X, Y)$ [$\mathcal{L}(X, X)$] is equipped with the Hilbert-Schmidt inner product. This product is systematically indexed by 2, the context allowing one to identify without ambiguity the space on which it operates.

In this whole section, $X$, $Y$, $Z$, and $T$ are Euclidean spaces, and in the whole paper, $\mathcal{X}$ and $\mathcal{Y}$ will denote $\sigma_2(X, Y)$ and $\sigma_2(Z, T)$ respectively.

### 2.1. *Functional Tools and Products*

Definition 2.1. For each $(x, y)$ in $X \times Y$, the tensor product of $x$ and $y$ is the rank one operator in $\mathcal{X}$ defined by

$$x \otimes y : u \in X \mapsto \langle u, x\rangle\, y \in Y.$$

This product clearly distributes over addition. It is also easily verified that the adjoint of $x \otimes y$ is

$$(x \otimes y)^* = y \otimes x$$

and, using the elementary properties

$$\begin{aligned} \operatorname{tr}(x \otimes y) &= \langle y, x\rangle, \qquad (x, y) \in X^2, \\ e(x \otimes y) &= x \otimes e(y), \qquad x \in X, \quad y \in Y, \quad e \in \sigma_2(Y, Z), \end{aligned} \tag{2.1a}$$

that we have

$$(u \otimes z)(x \otimes y) = \langle y, u\rangle x \otimes z, \qquad x \in X, \quad (y, u) \in Y^2, \quad z \in Z, \tag{2.1b}$$

$$(x \otimes y)d^* = d(x) \otimes y, \qquad x \in X, \quad y \in Y, \quad d \in \sigma_2(X, Z), \tag{2.2}$$

and then

$$\langle a, x \otimes y\rangle_2 = \big\langle a(x), y\big\rangle\, [= \operatorname{tr} y \otimes a(x)], \qquad a \in \mathcal{X}, \quad x \in X, \quad y \in Y.$$

So we have the equality

$$\langle u \otimes v, x \otimes y\rangle_2 = \langle x, u\rangle\langle y, v\rangle, \qquad (x, u) \in X^2, \quad (y, v) \in Y^2, \tag{2.3}$$

which can also be written

$$\operatorname{tr}[(u \otimes v)(y \otimes x)] = \operatorname{tr}(u \otimes x)\operatorname{tr}(v \otimes y).$$

In the same way, we may consider the tensor product of an element of $\mathcal{X}$ and an element of $\mathcal{Y}$. In order to distinguish it from the previous one, it is denoted by $\widetilde{\otimes}$.

Thus, this product is the element of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ defined by

$$a \widetilde{\otimes} b : d \in \mathcal{X} \mapsto \langle d, a\rangle_2 b = [\operatorname{tr}(ad^*)]b \in \mathcal{Y}, \qquad a \in \mathcal{X}, \quad b \in \mathcal{Y}.$$

*2.1.1. Tensor Product of Linear Maps*

DEFINITION 2.2. For $a \in \mathcal{X}$ and $b \in \mathcal{Y}$, the tensor product of the linear maps $a$ and $b$ is the element $a \overset{\ell}{\otimes} b$ of $\mathcal{L}(\sigma_2(X, Z), \sigma_2(Y, T))$ defined by

$$a \overset{\ell}{\otimes} b : d \in \sigma_2(X, Z) \mapsto bda^* \in \sigma_2(Y, T).$$

For $d \in \sigma_2(X, Z)$ and $e \in \sigma_2(Y, T)$, we have

$$\begin{aligned}\left\langle \left(a \overset{\ell}{\otimes} b\right)(d), e\right\rangle_2 &= \operatorname{tr}(bda^*e^*) = \operatorname{tr}(da^*e^*b) \\ &= \operatorname{tr}(d(b^*ea)^*) = \left\langle d, a^* \overset{\ell}{\otimes} b^*(e)\right\rangle_2\end{aligned}$$

and so

$$\left(a \overset{\ell}{\otimes} b\right)^* = a^* \overset{\ell}{\otimes} b^* \tag{2.4}$$

The map $\overset{\ell}{\otimes}$ is called the tensor product of linear maps, with reference to the corresponding notion in the general theory of tensor products (see Section 5.3). In fact, for any $(x, z) \in X \times Z$, using (2.1a) and (2.2) we have

$$\left(a \overset{\ell}{\otimes} b\right)(x \otimes z) = a(x) \otimes b(z), \quad a \in \mathcal{X}, \quad b \in \mathcal{Y}. \tag{2.5}$$

This definition is directly connected with the general notion of tensor product, and in this respect it seems more appropriate than the slightly different one used by Eaton [8, p. 34].

Finally, note that we have the equality

$$(x \otimes y) \overset{\ell}{\otimes} (z \otimes t) = (x \otimes z) \widetilde{\otimes} (y \otimes t), \qquad (x, y, z, t) \in X \times Y \times Z \times T. \tag{2.6}$$

Indeed, for each $d \in \sigma_2(X, Z)$,

$$\begin{aligned}(x \otimes y) \overset{\ell}{\otimes} (z \otimes t)(d) &= (z \otimes t)d(y \otimes x) = y \otimes [z \otimes t(dx)] \\ &= y \otimes (\langle z, d(x)\rangle t) = \langle d, x \otimes z\rangle_2 y \otimes t \\ &= (x \otimes z) \widetilde{\otimes} (y \otimes t)(d).\end{aligned}$$

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*2.1.2. Permutation Operator.* The linear map defined by

$$c_{Y,X} : a \in \mathcal{X} \to a^* \in \sigma_2(Y, X)$$

is called the permutation (or commutation) operator on $\mathcal{X}$.

The permutation operator on $\sigma_2(X)$ is denoted by $c_X$ (instead of $c_{X,X}$). It is immediately verified that $c_{X,Y} c_{Y,X}$ is the identity on $\mathcal{X}$ and that $c_{X,Y}$ is the adjoint of $c_{Y,X}$, that is,

$$c_{X,Y}\, c_{Y,X} = i_{\mathcal{X}} \quad \text{and} \quad c^*_{Y,X} = c_{X,Y}. \tag{2.7}$$

In particular, $c_X$ is involutive and self-adjoint:

$$c_X^2 = i_{\sigma_2(X)} \quad \text{and} \quad c_X^* = c_X.$$

The following final property justifies the name of the $c$-type operators:

$$c_{T,Y}\Big[\Big(a \overset{\ell}{\otimes} b\Big)(d)\Big] = (bda^*)^* = ad^*b^* = \Big(b \overset{\ell}{\otimes} a\Big) c_{Z,X}(d), \qquad d \in \sigma_2(X, Z),$$

that is, for all $a \in \mathcal{X}$ and $b \in \mathcal{Y}$,

$$c_{T,Y}\Big(a \overset{\ell}{\otimes} b\Big) = \Big(b \overset{\ell}{\otimes} a\Big) c_{Z,X}. \tag{2.8}$$

*2.1.3. Vectorization Operator.* Consider the Euclidean space $E$. The Riesz isomorphism, denoted by $\chi_E$, leads us to identify $E$ with its dual space $E^* = \mathcal{L}(E, \mathbb{R})$; it is the application

$$\chi_E : x \in E \mapsto \chi_E(x) = \langle x, \cdot\rangle \in E^*.$$

The map $\chi_E$ is an isomorphism from $E$ onto $E^*$, so the inner product $\langle\cdot,\cdot\rangle$, defined on $E^*$ by

$$\langle m, n\rangle = \big\langle \chi_E^{-1}(m),\ \chi_E^{-1}(n)\big\rangle = \langle x, y\rangle \qquad \text{for} \quad m = \langle x, \cdot\rangle \text{ and } n = \langle y, \cdot\rangle \in E^*,$$

may be considered.

The adjoint of $\chi_E$ is then $\chi_E^{-1}$. For all linear form $m = \langle x, \cdot \rangle$ on $E$, all $y$ in $E$, and all real $\alpha$, we have

$$\langle m(y), \alpha \rangle = \alpha \langle x, y \rangle = \langle y, \alpha x \rangle = \langle y, m^*(\alpha) \rangle;$$

so $m^*$ is the map $\alpha \in \mathbb{R} \mapsto \alpha x \in E$.

According to the previous notation, $nm^*$ is the map

$$\alpha \in \mathbb{R} \mapsto n\left(m^*(\alpha)\right) = \langle y, \alpha x \rangle = \alpha \langle y, x \rangle.$$

So we have $\operatorname{tr} nm^* = \langle y, x \rangle$, which shows that the induced inner product is the Hilbert-Schmidt inner product on the space $E^*$.

For the sake of simplicity, let $F$ denote the space $\sigma_2(E)$, and $\chi$ the isomorphism $\chi_F$. We have

$$\chi(a) = \langle a, \cdot \rangle_2 = \operatorname{tr}(\cdot\, a^*), \qquad a \in F.$$

Following the definition, we may note that

$$a \,\widetilde{\otimes}\, b(\cdot) = \chi(a)(\cdot) b, \qquad (a, b) \in F^2,$$

and, from what precedes,

$$[\chi(a)]^* : \alpha \in \mathbb{R} \mapsto \alpha a \in F.$$

Denoting by $c$ the permutation operator $c_E$, we have, for $(a, b) \in F^2$,

$$\chi(a)(b) = \langle a, b \rangle_2 = \operatorname{tr}[a\, c(b)] = \operatorname{tr}\left[c(b)(a^*)^*\right] = \langle a^*, c(b) \rangle_2,$$

that is,

$$\chi(a) = \chi(a^*) c. \tag{2.9}$$

Furthermore, for all $a, b, d,$ and $e \in F$, we obtain

$$\begin{aligned}\chi(a\, b\, d)(e) &= \operatorname{tr}(a\, b\, d\, e^*) = \operatorname{tr}\left[b\,(a^*\, e\, d^*)^*\right] \\ &= \langle b,\, a^*\, e\, d^* \rangle_2 = \chi(b)\left[\left(d \overset{\ell}{\otimes} a^*\right)(e)\right],\end{aligned}$$

and then

$$\chi(a\, b\, d) = \chi(b)\left(d \overset{\ell}{\otimes} a^*\right), \qquad (a,\ b,\ d) \in F^3. \tag{2.10}$$

In a more general way, let $\chi_{X,Y}$ be the Riesz isomorphism defined on $\mathcal{X}$. For

$a$ and $d$ in $\mathcal{X}$ and $b$ in $\mathcal{Y}$, we have

$$\left(a \tilde{\otimes} b\right)(d) = \langle a, d\rangle_2 b = [\chi_{Z,T}(b)]^*(\langle a, d\rangle_2) = [\chi_{Z,T}(b)]^* \chi_{X,Y}(a)(d).$$

Then it follows that

$$a \tilde{\otimes} b = [\chi_{Z,T}(b)]^* \chi_{X,Y}(a). \tag{2.11a}$$

For $d \in \sigma_2(\mathbb{R}, \mathcal{Y})$,

$$\begin{aligned}[\chi_{Z,T}(b)]^* \tilde{\otimes} \chi_{X,Y}(a)(d) &= \langle d, [\chi_{Z,T}(b)]^*\rangle_2 \chi_{X,Y}(a) \\ &= (\mathrm{tr}[\chi_{Z,T}(b)]^* d^*) \chi_{X,Y}(a) \\ &= (\mathrm{tr}[\chi_{Z,T}(b)] d) \chi_{X,Y}(a) \\ &= \chi_{Z,T}(b) d \chi_{X,Y}(a)\end{aligned}$$

so it can be deduced that

$$[\chi_{Z,T}(b)]^* \tilde{\otimes} \chi_{X,Y}(a) = [\chi_{X,Y}(a)]^* \overset{\ell}{\otimes} \chi_{Z,T}(b). \tag{2.11b}$$

### 2.2. *Matrix Representation*

*2.2.1. Usual Notation.* The $\mathbb{R}$-linear space of the $n \times p$ matrices ($n$ rows and $p$ columns) is denoted by $\mathcal{M}_{n,p}$. For each element $A$ of this space, ${}^tA$ stands for its transpose matrix.

Let $(x_i)_i$ $[(y_j)_j, (z_k)_k, (t_\ell)_\ell]$ be an orthonormal basis of the $m$ $[n, p, q]$ dimensional Euclidean space $X$ $[Y, Z, T]$. Using (2.3), we easily verify that $(x_i \otimes y_j)_{i,j}$ is an orthonormal basis of $\mathcal{X}$ and that $((x_i \otimes y_j) \tilde{\otimes} (z_k \otimes t_l))_{i,j,k,l}$ (or equivalently $((x_i \otimes z_k) \overset{\ell}{\otimes} (y_j \otimes t_l))_{i,j,k,l}$) is an orthonormal basis of $\sigma_2(\mathcal{X}, \mathcal{Y})$.

In order to establish the link with classical matrix calculus notions, the following conventions will be adopted:

(1) If possible, a linear map is denoted by lowercase letter; the associated matrix, with reference to the chosen bases, is then denoted by the corresponding capital letter. When that is not done, $M(a)$ denotes the matrix of the linear map $a$; conversely, each matrix $A$ can be associated with a convenient linear map denoted by $m(A)$.

(2) A linear map $a$ from $X$ into $Y$ is denoted by

$$a = \sum_{i,j} a_{ji} x_i \otimes y_j.$$

Its matrix $A$ is then $(a_{ij})$, where, as usual, the first index stands for the row number.

(3) The enumeration of the basis $(x_i \otimes y_j)_{i,j}$ will use the lexicographic order:

$$x_1 \otimes y_1, \ldots, x_1 \otimes y_j, \ldots, \; x_1 \otimes y_n, x_2 \otimes y_1, \ldots, x_2 \otimes y_n, \ldots, x_i \otimes y_1, \ldots,$$
$$x_i \otimes y_j, \ldots, x_i \otimes y_n, \ldots, x_m \otimes y_1, \ldots, x_m \otimes y_n;$$

that is, the index $i$ first equals 1, $j$ varying from 1 to $n$; then $i$ equals 2, $j$ varying from 1 to $n$, and so on until $i$ equals $m$.

2.2.2. *Tensor Products of Vectors and Linear Maps.* The bases $(x_i)_{i=1,\ldots,m}$ and $(y_j)_{j=1,\ldots,n}$ are respectively defined on the Euclidean spaces $X$ and $Y$. By an easy verification we prove that, for $u \in X$ and $v \in Y$,

$$M(u \otimes v) = V\,{}^tU. \tag{2.12}$$

where $U$ represents the column matrix of the $u$ components, for all $u$ of a linear space.

From the definition of $\chi_{X,Y}$, for all $(i, j)$ in $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and all $a = \sum_{k,\ell} a_{\ell k} x_k \otimes y_\ell$, we have

$$\chi_{X,Y}(a)(x_i \otimes y_j) = \langle a, x_i \otimes y_j \rangle_2 = \langle y_j, a(x_i) \rangle = a_{ji}.$$

Taking into account the enumeration order for the basis $(x_i \otimes y_j)_{i,j}$ of $\mathcal{X}$, we have

$$M(\chi_{X,Y}(a)) = (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots,$$
$$a_{1i}, \ldots, a_{ji}, \ldots, a_{ni}, \ldots, a_{1m}, \ldots, a_{nm}).$$

The classical matrix operation vec (see, for example, Henderson and Searle [11]) stacks the columns of a matrix one underneath the other to form a single vector. More generally, we can prove that

PROPOSITION 2.1. *For all $a$ in $\mathcal{X}$, we have*

$$M(\chi_{X,Y}(a)) = {}^t\mathrm{vec}\, A, \qquad M([\chi_{X,Y}(a)]^*) = \mathrm{vec}\, A.$$

Of course, a specific vec application corresponds to each space $\mathcal{M}_{n,m}$, so it can be denoted by $\mathrm{vec}_{n,m}$, for more precision.

Let $a$ and $b$ be two elements of $\mathcal{X}$ and $\mathcal{Y}$ respectively. It follows from (2.11a) that

$$M(a \,\tilde{\otimes}\, b) = M([\chi_{Z,T}(b)]^*) M(\chi_{X,Y}(a));$$

hence

$$M(a \,\tilde{\otimes}\, b) = \mathrm{vec}\, B\, {}^t(\mathrm{vec}\, A), \qquad a \in \mathcal{X}, \quad b \in \mathcal{Y}. \tag{2.13}$$

For $a = \sum_{j,i} a_{ji} x_i \otimes y_j$ and $b = \sum_{\ell,k} b_{\ell k} z_k \otimes t_\ell$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively, it follows from (2.5) that

$$\left(a \overset{\ell}{\otimes} b\right)(x_i \otimes z_k) = a(x_i) \otimes b(z_k) = \sum_{j,\ell} a_{ji} b_{\ell k} y_j \otimes t_\ell;$$

this result shows that, with the previous enumeration convention for the bases, we have

$$M\left(a \overset{\ell}{\otimes} b\right) = (a_{ji} B) = A \overset{K}{\otimes} B, \qquad a \in \mathcal{X}, \quad b \in \mathcal{Y}, \tag{2.14}$$

where $\overset{K}{\otimes}$ is the usual Kronecker product of two matrices (see also Henderson and Searle [11]).

Considering the same elements $a$ and $b$ and denoting by $d$ an element of $\sigma_2(X, Z)$, we can deduce from $\left(a \overset{\ell}{\otimes} b\right)(d) = b\, d\, a^*$ that

$$M\left[\left(a \overset{\ell}{\otimes} b\right)(d)\right] = BD^t A.$$

Consequently, a tensor product on a matrix space may be considered as follows: for $A$ in $\mathcal{M}_{n,m}$ and $B$ in $\mathcal{M}_{q,p}$, $A \overset{L}{\otimes} B$ denotes the map from $\mathcal{M}_{p,m}$ into $\mathcal{M}_{q,n}$ defined by

$$A \overset{L}{\otimes} B : D \in \mathcal{M}_{p,m} \mapsto BD^t A \in \mathcal{M}_{q,n}.$$

So we can write, for $a \in \mathcal{X}$, $b \in \mathcal{Y}$, and $d \in \sigma_2(X, Z)$,

$$M\left[\left(a \overset{\ell}{\otimes} b\right)(d)\right] = \left(A \overset{L}{\otimes} B\right)(D), \tag{2.15}$$

and, using (2.1a), we have

$$a \overset{\ell}{\otimes} b(d) = b\left[d\left(\sum_{j,i} a_{ji} y_j \otimes x_i\right)\right] = b\left[\sum_{j,i} a_{ji} y_j \otimes d(x_i)\right].$$

By the linearity of the tensor product, (2.6) implies

$$a \overset{\ell}{\otimes} b = \sum_{i,j,k,\ell} a_{ji} b_{\ell k} (x_i \otimes z_k) \tilde{\otimes} (y_j \otimes t_\ell). \tag{2.16}$$

We may also note that, for the same maps $a$ and $b$, we have

$$\begin{aligned} a \,\widetilde{\otimes}\, b &= \left(\sum_{j,i} a_{ji} x_i \otimes y_j\right) \widetilde{\otimes} \left(\sum_{\ell,k} b_{\ell k} z_k \otimes t_\ell\right) \\ &= \sum_{i,j,k,\ell} a_{ji} b_{\ell k} (x_i \otimes y_j) \,\widetilde{\otimes}\, (z_k \otimes t_\ell). \end{aligned}$$

By (2.11b), (2.13), and (2.14), and recalling that $M([\chi_{X,Y}(a)]^*) = \operatorname{vec} A$, we have

$$\begin{aligned} \operatorname{vec} A^t \operatorname{vec} B &= M([\chi_{Z,T}(b)]^* \,\widetilde{\otimes}\, \chi_{X,Y}(a)) \\ &= M\left([\chi_{X,Y}(a)]^* \overset{\ell}{\otimes} \chi_{Z,T}(b)\right) = \operatorname{vec} A \overset{K}{\otimes}{}^t \operatorname{vec} B, \end{aligned}$$

that is,

$$\operatorname{vec} A \overset{K}{\otimes}{}^t \operatorname{vec} B = \operatorname{vec} A^t \operatorname{vec} B, \qquad A \in \mathcal{M}_{m,p}, \quad B \in \mathcal{M}_{n,q}. \tag{2.17}$$

*2.2.3. Permutation Matrix.* For every $a = \sum_{i,j} a_{ji} x_i \otimes y_j$, of $\mathcal{X}$, we have

$$c_{Y,X}(a) = \sum_{i,j} a_{ji} y_j \otimes x_i = \sum_{i,j} \langle a(x_i), y_j \rangle y_j \otimes x_i.$$

It can be deduced from (2.1a) and (2.1b) that

$$\begin{aligned} c_{Y,X}(a) &= \sum_{i,j} (y_j \otimes x_i)[y_j \otimes a(x_i)] = \sum_{i,j} (y_j \otimes x_i) a (y_j \otimes x_i) \\ &= \sum_{i,j} (x_i \otimes y_j) \overset{\ell}{\otimes} (y_j \otimes x_i)(a), \end{aligned}$$

and then

$$c_{Y,X} = \sum_{i,j} (x_i \otimes y_j) \overset{\ell}{\otimes} (x_i \otimes y_j)^*. \tag{2.18a}$$

Let $E_{ji} = (e_{k\ell})$ denote the matrix of $x_i \otimes y_j$ whose single nonnull element is $e_{ji} = 1$. From (2.14) and (2.18a), the matrix of $c_{Y,X}$ is

$$C_{m,n} = \sum_{i,j} E_{ji} \overset{K}{\otimes}{}^t E_{ji}. \tag{2.18b}$$

Therefore, the matrix of the permutation operator $c_{Y,X}$ is the classical permutation (or commutation) matrix (Magnus and Neudecker [12]). From a matrix point of view, the relations (2.7) and (2.8) give

$$C_{n,m}C_{m,n} = I_{mn}, \qquad {}^tC_{m,n} = C_{n,m}, \qquad C_{q,n}(A \overset{K}{\otimes} B) = (B \overset{K}{\otimes} A)C_{m,p},$$

where $A$ and $B$ are matrices of elements of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

## 3. MATRIX OPERATOR TRANSLATION FORMULARY

### 3.1. *Usual Framework in Statistics*

Most of the applications in statistics consider the case $X = Y = Z = T = E$, where $E$ is a $q$-dimensional Euclidean space; let $F$ denote the associated $q^2$-dimensional space $\sigma_2(E)$. The orthonormal bases $\mathcal{E} = (e_i)_{i=1,\dots,q}$ and $\mathcal{F} = (e_{ij})_{i,j} = (e_i \otimes e_j)_{i,j}$ are defined on the spaces $E$ and $F$ respectively. The elements of $\mathcal{F}$ are enumerated according to the convention given in Section 2.2.1. For each $m$ in $\{1, 2, \dots, q^2\}$, the $m^{\text{th}}$ element $\varepsilon_m$ of the basis $\mathcal{F}$ is then the element $e_i \otimes e_j$, where $m = (i-1)q + j$. Denoting by $[x]$ the integer part of a positive real number $x$, we have, for all $m$ strictly positive integer

$$i(m) = \left[\frac{m-1}{q}\right] + 1, \qquad j(m) = m - q\left[\frac{m-1}{q}\right];$$

this is equivalent to

$$\varepsilon_m = e_{i(m)} \otimes e_{j(m)} \qquad \text{for} \quad m \in \{1, 2, \dots, q^2\}.$$

Let $\mathcal{M}_q$ ($\mathcal{M}_{q^2}$) be the $\mathbb{R}$-linear space of the square order $q$ (order $q^2$) matrices. The space $\mathcal{M}_q$ is equipped with the basis $\mathcal{G} = (E_{ij})_{i,j} = (E_j\, {}^tE_i)_{i,j}$, where $E_i$ is the $q \times 1$ vector of the $e_i$ components. In order to display clearly the matrix formats used in the expressions, the following notation will be used:

(1) $M_q$ is the map which associates with each element $a$ of $F$ its matrix $A$ in $\mathcal{M}_q$; conversely, $m_q$ is the map which associates with each matrix of $\mathcal{M}_q$ the corresponding linear map with respect to the basis $\mathcal{E}$ defined on $E$.

(2) $M_{\bar{q}^2}$ $[M_{q^2}]$ is the map from $\sigma_2(F)$ $[\sigma_2(\mathcal{M}_q)]$ into $\mathcal{M}_{q^2}$ which associates with each element of $\sigma_2(F)$ $[\sigma_2(\mathcal{M}_q)]$ its matrix according to the basis $\mathcal{F}$ of $F$ $[\mathcal{G}$ of $\mathcal{M}_q]$; $m_{\bar{q}^2}$ $[m_{q^2}]$ is the inverse map of $M_{\bar{q}^2}$ $[M_{q^2}]$.

(3) $i_q(i_{\bar{q}^2})$ is the identity map of $E$ ($F$), and $I_q$ ($I_{q^2}$) is the unit matrix of

$\mathcal{M}_q$ ($\mathcal{M}_{q^2}$). We may write

$$i_q = \sum_{i=1}^{q} e_{ii}, \qquad i_{\tilde{q}^2} = \sum_{i,\,j=1}^{q} e_{ij} \mathbin{\widetilde{\otimes}} e_{ij} = i_q \overset{\ell}{\otimes} i_q,$$

$$m_q(I_q) = i_q, \quad M_{\tilde{q}^2}(i_{\tilde{q}^2}) = I_q \overset{K}{\otimes} I_q = I_{q^2}.$$

(4) $c$ is the permutation operator of $F$, and $C$ its associated matrix.
Then we have [see (2.18 a) and (2.18 b)]

$$m_{\tilde{q}^2}(C) = c, \qquad \text{with} \quad c = \sum_{i,\,j=1}^{q} e_{ij} \mathbin{\widetilde{\otimes}} e_{ji} \text{ and } C = \sum_{i,\,j=1}^{q} E_{ij} \overset{K}{\otimes} {}^tE_{ij}.$$

By (2.6), we also have $c = \sum_{i,\,j=1}^{q} e_{ij} \overset{\ell}{\otimes} e_{ji}$.

(5) The Riesz isomorphism on $F$ is denoted by $\chi$.

(6) Let $\varphi$ be the map defined by $a \in F \mapsto \varphi(a) = a + a^* \in F$. Then

$$\varphi(a) = (i_{\tilde{q}^2} + c)(a). \tag{3.1}$$

Following this notation, the equalities (2.12), (2.13) and (2.17), (2.14) may be written

$$M_q(v \otimes u) = U\,{}^tV, \quad (u, v) \in E^2,$$

$$\begin{aligned} M_{\tilde{q}^2}(a \mathbin{\widetilde{\otimes}} b) &= \operatorname{vec} M_q(b) \overset{K}{\otimes} {}^t[\operatorname{vec} M_q(a)] \\ &= \operatorname{vec} M_q(b)\,{}^t[\operatorname{vec} M_q(a)], \qquad (a, b) \in F^2, \end{aligned}$$

$$M_{\tilde{q}^2}(a \overset{\ell}{\otimes} b) = M_q(a) \overset{K}{\otimes} M_q(b), \qquad (a, b) \in F^2.$$

It is also to be noted that

$$M_{q^2}(A \overset{L}{\otimes} B) = A \overset{K}{\otimes} B, \qquad (A, B) \in (\mathcal{M}_q)^2.$$

Recall that for each $a$ of $F$, the matrix of $\chi(a)$ is ${}^t\operatorname{vec} A$ [with $A = M_q(a)$]. Since $c$ is self-adjoint, the relation (2.9) may also be written

$$[\chi(a)]^* = c[\chi(a^*)]^*,$$

that is, from a matrix-based approach (see Proposition 2.1),

$$\operatorname{vec} A = C \operatorname{vec} {}^tA. \tag{3.2}$$

For $a$ in $F$, we have

$$[\chi(\varphi(a))]^* = [\chi(a)]^* + [\chi(a^*)]^*;$$

this can be rewritten [see (3.2)]

$$\mathrm{vec}\, M_q(\varphi(a)) = \mathrm{vec}\, A + \mathrm{vec}\, {}^t A = \mathrm{vec}\, A + C(\mathrm{vec}\, A) = (I_{q^2} + C)\ \mathrm{vec}\, A.$$

Then we get

$$\mathrm{vec}\, M_q(\varphi(a)) = (I_{q^2} + C)\ \mathrm{vec}\, A. \tag{3.3}$$

By (2.4), Equation (2.10) can be expressed as

$$[\chi(abd)]^* = (d \overset{\ell}{\otimes} a^*)^*[\chi(b)]^* = (d^* \overset{\ell}{\otimes} a)[\chi(b)]^*, \qquad (a, b, d) \in F^3,$$

which gives in matrix form

$$\mathrm{vec}\, ABD = ({}^t D \overset{K}{\otimes} A)\, \mathrm{vec}\, B, \qquad (A, B, D) \in (\mathcal{M}_q)^3. \tag{3.4}$$

Let $a$ and $b$ be two elements of $F$, and $v$ an element of $\sigma_2(F)$. From the equalities

$$\chi(v(a))(b) = \langle v(a), b\rangle_2 = \langle a, v^*(b)\rangle_2 = \chi(a)[v^*(b)],$$

we obtain

$$\chi(v(a)) = \chi(a)v^*.$$

In another form,

$$[\chi(v(a))]^* = v[\chi(a)]^*,$$

and the matrix transcription is

$$\mathrm{vec}\, M_q(v(a)) = M_{\bar{q}^2}(v)\, \mathrm{vec}\, M_q(a).$$

The equality (3.3) may of course be derived from this result. Clearly, preceding matrix and functional equalities characterize $v = m_{\bar{q}^2}(V)$ or $V = M_{\bar{q}^2}(v)$. As $F$ and $\mathcal{M}_q$ play analogous roles, we have

PROPOSITION 3.1.

(1) Let $v$ be an element of $\sigma_2(F)$. Then

$$\begin{aligned} v = m_{\bar{q}^2}(V) \quad &\Leftrightarrow \quad V = M_{\bar{q}^2}(v) \\ &\Leftrightarrow \quad \mathrm{vec}\ \ M_q(v(a)) = V\, \mathrm{vec} M_q(a), \quad a \in F. \end{aligned}$$

(2) Let $v$ be an element of $\sigma_2(\mathcal{M}_q)$. Then

$$v = m_{q^2}(V) \quad \Leftrightarrow \quad V = M_{q^2}(v) \quad \Leftrightarrow \quad \operatorname{vec} v(A) = V \operatorname{vec} A, \quad A \in \mathcal{M}_q.$$

Let $D$ be the subspace of the "diagonal" operators (always with reference to $\mathcal{F}$) in $F$, and let $i_\delta$ be the orthogonal projector from $F$ into $D$. Then we have

$$D = \operatorname{span}\{e_{ii};\ i = 1, \ldots, q\} \quad \text{and} \quad i_\delta = \sum_{i=1}^{q} e_{ii} \mathbin{\tilde{\otimes}} e_{ii}.$$

The map from $\mathcal{M}_q$ into $\mathcal{M}_q$ which associates with each matrix $A = (a_{ij})$ its diagonal matrix $\operatorname{diag}(a_{11}, \ldots, a_{ii}, \ldots, a_{qq})$ is denoted by diag. From Proposition 3.1, we have

$$\operatorname{vec} M_q(i_\delta(a)) = M_{\tilde{q}^2}(i_\delta) \operatorname{vec} A, \qquad a \in F,$$

that is,

$$\operatorname{vec} \operatorname{diag} A = M_{\tilde{q}^2}(i_\delta) \operatorname{vec} A, \qquad A \in \mathcal{M}_q.$$

$M_{\tilde{q}}(i_\delta)$ is then the matrix $I_\delta$ that transforms vec$A$ into vec diag $A$, so

COROLLARY 3.1. *The matrix $I_\delta$ of the map $i_\delta = \sum_{i=1}^{q} e_{ii} \mathbin{\tilde{\otimes}} e_{ii}$ is the matrix that transforms the matrix* vec $A$ *into* vec diag $A$ *by premultiplication.*

### *3.2. Basic Dictionary*

The matrix-operator (or operator-matrix) dictionary may now be established. The most useful results are presented in Table 1.

The following diagram illustrates the direct links between the various products:

$$\begin{tikzcd}
\otimes, \tilde{\otimes} \arrow[rr, shift right, "P_3"'] \arrow[dr, shift left] & & \overset{L}{\otimes}, \overset{\ell}{\otimes} \arrow[ll, shift right] \arrow[dl, shift right] \\
& \overset{K}{\otimes} \arrow[ul, shift left, "P_5"] \arrow[ur, shift right, "P_2"'] &
\end{tikzcd}$$

The property $P_1$ is a particular case of (2.15), $P_6$ and $P_6'$ rewrite (3.3) and (3.2) respectively, and $P_3$ is an immediate consequence of (2.16). The relation $P_2$ is

TABLE 1

| | |
|---|---|
| $P_1$ | For $(A, B, D) \in (\mathcal{M}_q)^3$,<br>$(A \overset{L}{\otimes} B)(D) = M_q[(a \overset{\ell}{\otimes} b)(d)]$. |
| $P_2$ | For $(A, B, D) \in (\mathcal{M}_q)^3$,<br>$\text{vec}\,[(A \overset{L}{\otimes} B)(D)] = (A \overset{K}{\otimes} B)\,\text{vec}\,D$. |
| $P_3$ | For $a = \sum_i \alpha_i a_i \otimes a_i \in F,\ b = \sum_j \beta_j b_j \otimes b_j \in F$,<br>$(a \overset{\ell}{\otimes} b) = \sum_{ij} \alpha_i \beta_j (a_i \otimes b_j) \widetilde{\otimes} (a_i \otimes b_j)$. |
| $P_4$ | For $(x, y, z, t) \in E^4,\ d \in F$,<br>$\text{vec}\,M_q[(x \otimes y) \widetilde{\otimes} (z \otimes t)(d)] = C\,\text{vec}\,M_q[(x \otimes y) \widetilde{\otimes} (t \otimes z)(d)]$. |
| $P_5$ | For $(a, b, d) \in F^3$,<br>$\text{vec}\,M_q[(a \widetilde{\otimes} b)(d)] = [(\text{vec}\,B) \overset{K}{\otimes}{}^t (\text{vec}\,A)]\,\text{vec}\,D$<br>$= [\text{vec}\,B^t \text{vec}\,A)]\,\text{vec}\,D$. |
| $P_6$ | For $a \in F$,<br>$\text{vec}\,M_q(\varphi(a)) = (I_{q^2} + C)\,\text{vec}\,A$. |
| $P_6$' | For $A \in \mathcal{M}_q$,<br>$\text{vec}\,A = C\,\text{vec}\,{}^t A$. |

another form of (3.4). Apply (3.2) to the matrix of $(x \otimes y) \widetilde{\otimes} (z \otimes t)(d)$ and remark that

$$[(x \otimes y) \widetilde{\otimes} (z \otimes t)(d)]^* = [\langle d, x \otimes y \rangle_2 z \otimes t]^* = \langle d, x \otimes y \rangle_2 t \otimes z;$$

then $P_4$ follows. Finally, setting $v = a \widetilde{\otimes} b$ in Proposition 3.1, $P_5$ is obtained by remarking [see (2.13) and (2.17)] that $M_{\tilde{q}^2}(a \widetilde{\otimes} b) = \text{vec}\,B^t(\text{vec}\,A)$.

## 4. APPLICATION TO THE ASYMPTOTIC THEORY OF FACTORIAL ANALYSES

### *4.1. Covariance Operators and Matrices*

The first given result concerns the covariance operators. In the sequel, $(\Omega, \mathcal{A}, P)$ is a probability space; a Borel $\sigma$-field is associated with each Euclidean space. As $E$ is a Euclidean space, each random variable (r.v.) $u$ from $(\Omega, \mathcal{A}, P)$ into $F = \sigma_2(E)$ is called a random operator.

Let $X : (\Omega, \mathcal{A}, P) \to E$ be a centered r.v. with a $P$-integrable square norm. The random operator $X \otimes X : \omega \in \Omega \mapsto X(\omega) \otimes X(\omega) \in F$ is $P$-integrable; its expectation $\mathbb{E}(X \otimes X)$ is the covariance operator of $X$ and is denoted by $\operatorname{cov} X$. With respect to the basis $\mathcal{E}$, the matrix COV $X$ of this operator is the covariance matrix of the random vector (also denoted by $X$) of the components of $X$ with respect to the basis $\mathcal{E}$, and we have $\mathrm{COV}\, X = \mathbb{E}(X^t X)$. For a random operator $u : \Omega \mapsto F$, $\operatorname{cov} u$ denotes the covariance operator when it exists.

PROPOSITION 4.1. *Let $u$ be an $F$-valued random operator, let $U$ be its matrix, and suppose that it admits a covariance operator. Let $a$ be an element of $F$, with matrix $A$. Then we have*

$$P_7 \quad [\mathrm{COV}(\operatorname{vec} U)]\operatorname{vec} A = \operatorname{vec} M_q[(\operatorname{cov} u)(a)],$$

*which also means $M_{\tilde{q}^2}(\operatorname{cov} u) = \mathrm{COV}\,(\operatorname{vec} U)$.*

*Proof.*

$$\begin{aligned}\operatorname{vec} M_q[(\operatorname{cov} u)(a)] &= \operatorname{vec} M_q[\mathbb{E}(u \mathbin{\widetilde{\otimes}} u)(a)] = \mathbb{E}(\operatorname{vec} M_q[(u \mathbin{\widetilde{\otimes}} u)(a)]) \\ &= \mathbb{E}[(\operatorname{vec} U)^t(\operatorname{vec} U)\operatorname{vec} A] \qquad (\text{see} P_5) \\ &= [\mathrm{COV}\,(\operatorname{vec} U)]\operatorname{vec} A. \qquad \blacksquare\end{aligned}$$

*4.2. Principal Component Analysis*

The asymptotic theory of factorial analyses was first developed under a normality assumption (see Anderson [1] for a basic paper), and then with only hypotheses of the existence of moments. In the latter case, the asymptotic covariances of the sample elements can be given in an explicit form only under elliptical distribution assumptions. Depending on the authors' approaches, the results are obtained and presented in either a matrix or an operator language form. The above dictionary permits comparison of the two points of view. It will be applied here to two classical multivariate methods: principal component analysis (PCA) and canonical analysis (CA). Here, the results obtained for the operator-based approach are exploited only in finite dimension.

Let $X$ be a r.v. with values in the $q$-dimensional Euclidean space $E$; $X$ is supposed to be centered, and its norm admits a fourth order moment. Consider an independent identically distributed (i.i.d.) sample $(X_i)_{i=1,\ldots,n}$ of $X$.

In the operator-based context (see Dauxois and Pousse [5], Dauxois, Pousse, and Romain [6]), the population PCA [respectively the sample PCA] of $X$ is obtained by the diagonalization of the covariance operator $v = \mathbb{E}(X \otimes X)\left[v_n = (1/n)\sum_{i=1}^n X_i \otimes X_i\right]$. The centered asymptotic normality of $\sqrt{n}(v_n - v)$ in $F$ follows from the application of the central limit theorem to the sequence $(X_i \otimes X_i)_{i=1,\ldots,n}$.

When $X$ admits an elliptical distribution, the results can be given in an explicit form (see Arconte [2]; Pousse [15]). Recall that the distribution of $X$ is elliptical with mean zero and covariance $\gamma$ when its characteristic function is of the form

$$\Phi_X : t \in E \mapsto h(\langle t, \gamma(t) \rangle),$$

where $h : \mathbb{R} \to \mathbb{R}$ is a specific function with at least two derivatives at the origin. The real coefficient $\kappa = 4h''(0) - 1$ is called the kurtosis of the elliptical distribution. The centered elliptical family includes the centered normal distributions (when $\kappa$ is null) and spherical distributions ($\gamma = i_q$).

So under the elliptical assumption, we have

$$\sqrt{n}(v_n - v) \underset{n \to +\infty}{\xrightarrow{\mathcal{L}}} N(0, (1+\kappa)\eta + \kappa\zeta), \tag{4.1}$$

where $\eta$ is the asymptotic covariance operator of $\sqrt{n}(v_n - v)$ under the normal assumption and $\zeta$ is the operator $v \tilde{\otimes} v$.

Let $\sum_{i=1}^{q} \lambda_i e_{ii}$ be a Schmidt decomposition of $v$ in $F$ (i.e. the functional version of the singular value decomposition of $V$). Then $\eta$ may be written

$$\sum_{i<j} \lambda_i \lambda_j \varphi(e_{ij}) \tilde{\otimes} \varphi(e_{ij}) + 2 \sum_i \lambda_i^2 e_{ii} \tilde{\otimes} e_{ii}$$

(where $\varphi$ is defined as in Section 3.1), or more simply,

$$\sum_{i,j} \lambda_i \lambda_j e_{ij} \tilde{\otimes} \varphi(e_{ij}).$$

In a matrix-based approach, the population PCA of $X$ (the sample PCA of $X$) is derived by diagonalization of the covariance matrix $V$ ($V_n$). Still under the centered elliptical assumption, Tyler [17] has shown that

$$\sqrt{n}\,\mathrm{vec}(V_n - V) \underset{n \to +\infty}{\xrightarrow{\mathcal{L}}} N\Big(0, (1+\kappa)(I_{q^2} + C)\Big(V \overset{K}{\otimes} V\Big) + \kappa(\mathrm{vec}\,V)^t(\mathrm{vec}\,V)\Big).$$

With this last result, and denoting by $U$ the limit of $\sqrt{n}(V_n - V)$ as $n$ increases infinitely, we have

$$\mathrm{COV}(\mathrm{vec}\,U) = (1+\kappa)(I_{q^2} + C)\Big(V \overset{K}{\otimes} V\Big) + \kappa(\mathrm{vec}\,V)^t(\mathrm{vec}\,V).$$

For all $A$ in $\mathcal{M}_q$, we can write

$$\begin{aligned}[\mathrm{COV}(\mathrm{vec}\,U)]\,\mathrm{vec}\,A = {} & (1+\kappa)(I_{q^2} + C)\Big(V \overset{K}{\otimes} V\Big)\mathrm{vec}\,A \\ & + \kappa[(\mathrm{vec}\,V)^t(\mathrm{vec}\,V)]\mathrm{vec}\,A.\end{aligned}$$

Applying $P_2$, $P_1$, and $P_3$, we have

$$\left(V \overset{K}{\otimes} V\right)\mathrm{vec}\, A = \mathrm{vec}\left[\left(V \overset{L}{\otimes} V\right)(A)\right] = \mathrm{vec}\, M_q\left[\left(v \overset{\ell}{\otimes} v\right)(a)\right]$$

$$= \mathrm{vec}\, M_q\left[\sum_{i,j} \lambda_i \lambda_j e_{ij} \,\widetilde{\otimes}\, e_{ij}(a)\right],$$

and using $P_6$ and $P_4$,

$$(I_{q^2} + C)\left(V \overset{K}{\otimes} V\right)\mathrm{vec}\, A = \mathrm{vec}\, M_q\left[\varphi\left(\sum_{i,j} \lambda_i \lambda_j e_{ij} \,\widetilde{\otimes}\, e_{ij}(a)\right)\right]$$

$$= \mathrm{vec}\, M_q\left[\sum_{i,j} \lambda_i \lambda_j e_{ij} \,\widetilde{\otimes}\, \varphi(e_{ij})(a)\right].$$

Furthermore, we know from $P_5$ that

$$[(\mathrm{vec}\, V)^t(\mathrm{vec}\, V)]\,\mathrm{vec}\, A = \mathrm{vec}\, M_q[(v \,\widetilde{\otimes}\, v)(a)].$$

So

$$[\mathrm{COV}\,(\mathrm{vec}\, U)]\,\mathrm{vec}\, A = \mathrm{vec}\, M_q\left[\left((1+\kappa)\sum_{i,j} \lambda_i \lambda_j e_{ij} \,\widetilde{\otimes}\, \varphi(e_{ij}) + \kappa v \,\widetilde{\otimes}\, v\right)(a)\right],$$

which is, with $P_7$, the same result as (4.1).

### 4.3. *Canonical Analysis*

This example is more informative. In fact, the asymptotic studies that can be found in the literature, either with a functional or with a matrix method, have not the same background (i.e., they are based on the diagonalization of different matrices or operators). The operator-based approach is given in Arconte [2] and Pousse [15], and the matrix-based approach may be found in Tyler [17]. For a general presentation of canonical analysis in Hilbert spaces, see Dauxois and Pousse [5].

*4.3.1. Asymptotic Study of CA.* Let $X = (X_1, X_2)$ be a r.v. with values in the $q = q_1 + q_2$ dimensional Euclidean space $E = E_1 \times E_2$ (where $q_1 < q_2$, for instance) and whose norm admits a fourth order moment. The CA of $X_1$ and $X_2$ is given by the spectral analysis of $\Pi_2\,\Pi_1\,\Pi_2$, for example, where $\Pi_k$, $k = 1, 2$, is the orthogonal projector on the subspace generated by the components of $X_k$.

As we know that the CA is invariant under one-to-one bimeasurable mappings, the covariance matrix of $X$ may be written

$$V = \begin{pmatrix} I_1 & V_{12} \\ V_{21} & I_2 \end{pmatrix},$$

where $I_k$ is the identity matrix on $E_k$, and $V_{k\ell}$ $(k \neq \ell)$ is the matrix of the cross covariance operator $v_{k\ell} = \mathbb{E}(X_\ell \otimes X_k)$, $1 \leq k, \ \ell \leq 2$. The CA is obtained by the spectral analysis of $r = v_{12}\, v_{21}$ or of equivalent operators.

A self-adjoint estimator of $r$ based on a $n$-i.i.d. sample $X^i = (X_1^i, X_2^i)$, $i = 1, 2, \ldots, n$, of $X$ is the operator defined by

$$r_n = (v_1^n)^{-1/2}\, v_{12}^n (v_2^n)^{-1}\, v_{21}^n\, (v_1^n)^{-1/2},$$

where $v_1^n$, $v_2^n$, $v_{12}^n$, and $v_{21}^n$ are the operators whose respective matrices constitute the sample covariance matrix

$$V^n = \begin{pmatrix} V_1^n & V_{12}^n \\ V_{21}^n & V_2^n \end{pmatrix}$$

of the operator $(1/n) \sum_{i=1}^n \mathbb{E}(X^i \otimes X^i)$.

Under the Gaussian (the elliptical) assumption, the asymptotic covariance operator of $r_n$ may be found in Arconte [2] (in Pousse [15]). Let $\rho_i^2$ be the $i$th canonical coefficient, and $e_i$ be an associated canonical factor (i.e. an eigenvector of $r$ which is associated with the possibly nonsimple eigenvalue $\rho_i^2$).

If $\kappa$ is the $X$ distribution kurtosis, the asymptotic distribution of $s_n = \sqrt{n}(r_n - r)$ is Gaussian with mean zero and covariance operator given by

$$\operatorname{cov} s = (1 + \kappa) \left[ \sum_{i<j}^{q_1} \mu_{ij} \varphi(e_{ij}) \,\widetilde{\otimes}\, \varphi(e_{ij}) + \sum_{i=1}^{q_1} \mu_i e_{ii} \,\widetilde{\otimes}\, e_{ii} \right],$$

with

$$\mu_{ij} = -\tfrac{3}{4}\left(\rho_i^2 + \rho_j^2\right)^2 + \left(\rho_i^2 + \rho_j^2\right)\left(\rho_i^2 \rho_j^2 + 1\right) - \rho_i^2 \rho_j^2$$

for each $(i, j) \in \{1, \ldots, q_1\}^2$, $i < j$,

$$\mu_i = 4\rho_i^2\left(1 - \rho_i^2\right)^2,$$

and

$$e_{ij} = e_i \otimes e_j,$$

as defined in Section 3.1. This expression may be simplified by remarking that, for each $i$, $2\mu_{ii} = \mu_i$; thus

$$\operatorname{cov} s = (1+\kappa) \sum_{i,j=1}^{q_1} \mu_{ij} e_{ij} \widetilde{\otimes} \varphi(e_{ij}) = (1+\kappa)\left(i_{\tilde{q}_1^2} + c\right) \sum_{i,j=1}^{q_1} \mu_{ij} e_{ij} \widetilde{\otimes} e_{ij}. \tag{4.2}$$

Under the same hypotheses on $X$, but without assuming the $X_k$ are standardized, Tyler [17] presents an asymptotic study of the CA, giving the asymptotic behavior of the nonsymmetric matrix $T_n$ of the operator $t_n = (v_1^n)^{-1} v_{12}^n (v_2^n)^{-1} v_{21}^n$. Setting $U_n = \sqrt{n}(T_n - T)$, where $T$ is the matrix of $v_1^{-1}\, v_{12}\, v_2^{-1}\, v_{21}$, the author shows that the asymptotic covariance matrix of $\operatorname{vec} U_n$ is

$$\begin{aligned} \operatorname{COV}(\operatorname{vec} U) = (1+\kappa) \Big[ & V_1 Z \overset{K}{\otimes} V_1^{-1}\, {}^tT + V_1 T Z \overset{K}{\otimes} V_1^{-1} ({}^tZ - {}^tT) \\ & + C\Big( Z \overset{K}{\otimes} {}^t(TZ) + TZ \overset{K}{\otimes} {}^tZ \Big)\Big], \end{aligned} \tag{4.3}$$

where $Z$ is defined by $Z = I_1 - T$.

So the two given results are not directly comparable, but using the dictionary, the corresponding version of each case can be given. This is the object of the two following sub-subsections.

*4.3.2. Canonical Analysis Based on Self-Adjoint Operators.* This part is devoted to the corresponding matrix-based version of (4.2). We have

$$i_{q_1} = i_1 = \sum_{i=1}^{q_1} e_{ii}, \qquad r = \sum_{i=1}^{q_1} \rho_i^2 e_{ii}, \quad \text{and} \quad r^2 = \sum_{i=1}^{q_1} \rho_i^4 e_{ii}$$

and, for all $(i, j)$ in $\{1, \ldots, q_1\}^2$,

$$\mu_{ij} = -\tfrac{5}{2}\rho_i^2\rho_j^2 - \tfrac{3}{4}\rho_i^4 - \tfrac{3}{4}\rho_j^4 + \rho_i^2 + \rho_j^2 + \rho_i^4\rho_j^2 + \rho_i^2\rho_j^4.$$

So

$$\begin{aligned} \sum_{i,j} \mu_{ij} e_{ij} \widetilde{\otimes} (e_{ij}) = & -\frac{5}{2} \sum_{i,j} \rho_i^2 \rho_j^2 e_{ij} \widetilde{\otimes} e_{ij} - \frac{3}{4} \sum_{i,j} \rho_i^4 e_{ij} \widetilde{\otimes} e_{ij} \\ & - \frac{3}{4} \sum_{i,j} \rho_j^4 e_{ij} \widetilde{\otimes} e_{ij} + \sum_{i,j} \rho_i^2 e_{ij} \widetilde{\otimes} e_{ij} + \sum_{i,j} \rho_j^2 e_{ij} \widetilde{\otimes} e_{ij} \\ & + \sum_{i,j} \rho_i^4 \rho_j^2 e_{ij} \widetilde{\otimes} e_{ij} + \sum_{i,j} \rho_i^2 \rho_j^4 e_{ij} \widetilde{\otimes} e_{ij}. \end{aligned}$$

Using $P_3$, it follows that

$$\sum_{i,j} \mu_{ij} e_{ij} \widetilde{\otimes} e_{ij} = -\tfrac{5}{2} r \overset{\ell}{\otimes} r - \tfrac{3}{4} r^2 \overset{\ell}{\otimes} i_1 - \tfrac{3}{4} i_1 \overset{\ell}{\otimes} r^2 + r \overset{\ell}{\otimes} i_1$$
$$+ i_1 \overset{\ell}{\otimes} r + r^2 \overset{\ell}{\otimes} r + r \overset{\ell}{\otimes} r^2.$$

So, from (2.4), (3.1), and then (2.8), we may write

$$\sum_{i,j} \mu_{ij} e_{ij} \widetilde{\otimes} e_{ij} = (i_{\bar{q}_1^2} + c)\Big( - \tfrac{5}{4} r \overset{\ell}{\otimes} r - \tfrac{3}{4} r^2 \overset{\ell}{\otimes} i_1 + r \overset{\ell}{\otimes} i_1 + r \overset{\ell}{\otimes} r^2 \Big)$$
$$= \Big( - \tfrac{5}{4} r \overset{\ell}{\otimes} r - \tfrac{3}{4} r^2 \overset{\ell}{\otimes} i_1 + r \overset{\ell}{\otimes} i_1 + r \overset{\ell}{\otimes} r^2 \Big)(i_{\bar{q}_1^2} + c).$$

Then we get

$$\operatorname{cov} s = (1+\kappa)\Big[ \tfrac{3}{4}(r \overset{\ell}{\otimes} r - r^2 \overset{\ell}{\otimes} i_1) + r \overset{\ell}{\otimes} (r - i_1)^2 \Big](i_{\bar{q}_1^2} + c).$$

Under the previous assumptions for $X = (X_1, X_2)$ and with the same notation, the CA of $X_1$ and $X_2$ may be obtained from (the diagonalization of) the matrix $R = V_{12} V_{21}$. From a $n$-i.i.d. sample of $X$ denoted by $(X^i) = (X_1^i, X_2^i)_{i=1,\dots,n}$, an estimator of $R$ is

$$R_n = (V_{11}^n)^{-1/2} V_{12}^n (V_{22}^n)^{-1} V_{21}^n (V_{11}^n)^{-1/2},$$

where, for all $k, \ell$, we have $V_{kl}^n = (1/n) \sum_{i=1}^n \mathbb{E}(X_k^i \, {}^t X_l^i)$. Consequently we obtain

PROPOSITION 4.2. *Let $R_n$ be the sample symmetric matrix involving the CA of $X_1$ and $X_2$. When $X_1$ and $X_2$ are standardized and admit a joint elliptical distribution, the asymptotic covariance matrix of the Gaussian limit distribution of $S_n = \sqrt{n}(R_n - R)$ is given by*

$$\operatorname{COV}(\operatorname{vec} S) = (1+\kappa)\Big[ \tfrac{3}{4}\Big(R \overset{K}{\otimes} R - R^2 \overset{K}{\otimes} I_1\Big) + R \overset{K}{\otimes} (R - I_1)^2 \Big](I_{q_1^2} + C).$$

*4.3.3. Canonical Analysis Based on Nonsymmetric Matrices.* From Tyler's result given in (4.3), we may obtain the corresponding operator-based version. As we have seen before, one can replace $V_{kk} = V_k$, $k = 1, 2$, by $I_k$ without loss of generality. The limit of the nonsymmetric matrix is then the symmetric matrix $T = V_{12} V_{21}$, and we are led to

$$\operatorname{COV}(\operatorname{vec} U) = (1+\kappa) \Big[ Z \overset{K}{\otimes} T + (TZ) \overset{K}{\otimes} (Z - T)$$

$$+ C\Big(Z \overset{K}{\otimes} (TZ) + (TZ) \overset{K}{\otimes} Z\Big)\Big].$$

For all $a$ in $F$, with matrix $A$, and by $P_7$, we have

$$\begin{aligned}
&\operatorname{vec} M_q[(\operatorname{cov} u)(a)] \\
&\quad = [\mathrm{COV}(\operatorname{vec} U)] \operatorname{vec} A \\
&\quad = (1+\kappa)\Big[\Big(Z \overset{K}{\otimes} T\Big) \operatorname{vec} A \\
&\qquad + \Big[(TZ) \overset{K}{\otimes} (Z - T)\Big] \operatorname{vec} A + C\Big(Z \overset{K}{\otimes} (TZ) + (TZ) \overset{K}{\otimes} Z\Big) \operatorname{vec} A\Big] \\
&\quad = (1+\kappa) \operatorname{vec}\Big[\Big(Z \overset{L}{\otimes} T) + (TZ) \overset{L}{\otimes} (Z - T)\Big)(A)\Big] \\
&\qquad + (1+\kappa) C \operatorname{vec}\Big[\Big(Z \overset{L}{\otimes} (TZ) + (TZ) \overset{L}{\otimes} Z\Big)(A)\Big],
\end{aligned}$$

where the last equality comes from $P_2$. Furthermore, using $P_6'$,

$$\begin{aligned}
C \operatorname{vec}[(TZ)AZ + ZA(ZT)] &= \operatorname{vec}(Z\,{}^t\!A\,ZT + TZ\,{}^t\!AZ) \\
&= \operatorname{vec}\Big[\Big((TZ) \overset{L}{\otimes} Z + Z \overset{L}{\otimes} (TZ)\Big)({}^t A)\Big].
\end{aligned}$$

So applying $P_1$ leads to

$$\begin{aligned}
\operatorname{vec} M_q[(\operatorname{cov} u)(a)] = \operatorname{vec} M_q\Big((1+\kappa)\Big[z \overset{\ell}{\otimes} t + (t\,z) \overset{\ell}{\otimes} (z - t) \\
+ \Big((t\,z) \overset{\ell}{\otimes} z + z \overset{\ell}{\otimes} (t\,z)\Big) c\Big] a\Big);
\end{aligned}$$

hence

$$\operatorname{cov} u = (1+\kappa)\Big[z \overset{\ell}{\otimes} t + (t\,z) \overset{\ell}{\otimes} (z - t) + \Big((t\,z) \overset{\ell}{\otimes} z + z \overset{\ell}{\otimes} (t\,z)\Big) c\Big],$$

and finally, using (2.8),

$$\operatorname{cov} u = (1+\kappa)\Big[z \overset{\ell}{\otimes} t + (t\,z) \overset{\ell}{\otimes} (z - t) + c\Big(z \overset{\ell}{\otimes} (t\,z) + (t\,z) \overset{\ell}{\otimes} z\Big)\Big],$$

and we have established

PROPOSITION 4.3. *The CA of two standardized r.v.'s $X_1$ and $X_2$ with elliptical joint distribution may be approximated by the spectral analysis of the non-self-adjoint sample operator $t_n = (v_1^n)^{-1} v_{12}^n (v_2^n)^{-1} v_{21}^n$. The asymptotic covariance* *operator* cov $u$ *of the Gaussian limit distribution of* $u_n = \sqrt{n}(t_n - t)$ *is given by*

$$\operatorname{cov} u = (1+\kappa)\Big[z \overset{\ell}{\otimes} t + (t\,z) \overset{\ell}{\otimes} (z-t) + c\left(z \overset{\ell}{\otimes} (t\,z) + (t\,z) \overset{\ell}{\otimes} z\right)\Big].$$

## 5. MATHEMATICAL BACKGROUND

### 5.1. *Tensor Product of Linear Spaces*

The notion of the tensor product of linear spaces is a fundamental tool in linear algebra with many applications throughout mathematics. One of its interests is to replace the study of a bilinear map by the study of a linear map. The different products previously introduced (such as $\otimes$, $\widetilde{\otimes}$, $\overset{L}{\otimes}$, $\overset{\ell}{\otimes}$, $\overset{K}{\otimes}$) are connected with this notion, as we will see in this part, and we start by giving some elementary results (see Bourbaki [3] or Guichardet [10]).

The proposed definition is limited here to the finite dimensional case, and is a direct consequence of the following property:

PROPOSITION 5.1. *Let $X$ and $Y$ be two finite dimensional linear spaces. There exists a pair $(X \otimes Y, T)$, uniquely determined up to an isomorphism, of a linear space and a bilinear map $T$ from $X \times Y$ into $X \otimes Y$ satisfying*:

(i) *$T(X \times Y)$ generates $X \otimes Y$ linearly,*

(ii) *for each basis $(x_i)_i$ of $X$ and each basis $(y_j)_j$ of $Y$, $(T(x_i, y_j))_{i,j}$, is a basis of $X \otimes Y$.*

The space $X \otimes Y$ is called *tensor product* of $X$ and $Y$; its elements are called *tensors*. Each tensor of the form $T(x, y)$ is said to be *decomposable* and denoted by

$$x \underset{X,Y}{\otimes} y.$$

The uniqueness of the previous result is expressed in this way: if $(\Lambda, \Gamma)$ is a pair consisting of a linear space and a bilinear map satisfying conditions (i) and (ii), there exists a unique isomorphism $J$ from $X \otimes Y$ into $\Lambda$ such that $\Gamma = JT$. The dimension of $X \otimes Y$ is clearly the product of the dimensions of $X$ and $Y$. It may be noted that each tensor is, of course, sum of decomposable tensors, but this decomposition is not unique (for instance, the null element of $X \otimes Y$ can be $0 \underset{X,Y}{\otimes} y$ for all $y$ in $Y$, or $x \underset{X,Y}{\otimes} 0$ for all $x$ in $X$).

The links between the bilinear and linear maps are then given by

PROPOSITION 5.2. *Let $B$ be a bilinear map from $X \times Y$ into the linear space $Z$. Then there exists a unique linear map $L$ from $X \otimes Y$ to $Z$ satisfying*

$$L\Big(x \underset{X,Y}{\otimes} y\Big) = B(x, y), \qquad (x, y) \in X \times Y.$$

For Euclidean spaces, $X \otimes Y$ is also Euclidean with the following property:

PROPOSITION 5.3. *When $X$ and $Y$ are Euclidean spaces, there exists on $X \otimes Y$ a unique inner product satisfying*

$$\left\langle x \underset{X,Y}{\otimes} y, u \underset{X,Y}{\otimes} v \right\rangle = \langle x, u\rangle\langle y, v\rangle, \qquad (x, u) \in X^2, \quad (y, v) \in Y^2.$$

Although many results are valid in more general contexts, the presentation from now on is limited to Euclidean linear spaces.

*5.2. Tensor Product of Vectors*

*5.2.1.* In Section 2, we considered the product $\otimes$ defined by

$$x \otimes y : u \in X \mapsto \langle u, x\rangle y, \qquad (x, y) \in X \times Y.$$

As the map

$$B_1 : (x, y) \in X \times Y \mapsto x \otimes y \in \mathcal{X}$$

is bilinear, then there exists a linear map $L_1$ (obviously injective) from $X \otimes Y$ into $\mathcal{X}$ defined by

$$L_1\left(x \underset{X,Y}{\otimes} y\right) = x \otimes y.$$

As $X \otimes Y$ and $\mathcal{X}$ have the same finite dimension, $L_1$ is an isomorphism from $X \otimes Y$ onto $\mathcal{X}$. In what follows, $L_1$ is also denoted by $L_{1,X,Y}$, or $L_{1,X}$ when $X$ and $Y$ are the same space. By (2.3) and Proposition 5.3, for all $(x, u)$ in $X^2$ and all $(y, v)$ in $Y^2$, we have

$$\begin{aligned}\left\langle L_1\left(x \underset{X,Y}{\otimes} y\right), L_1\left(u \underset{X,Y}{\otimes} v\right)\right\rangle_2 &= \langle x \otimes y, u \otimes v\rangle_2 = \langle x, u\rangle\langle y, v\rangle \\ &= \left\langle x \underset{X,Y}{\otimes} y, u \underset{X,Y}{\otimes} v\right\rangle.\end{aligned}$$

This equality shows that $L_1$ is an isometric isomorphism from $X \otimes Y$ onto $\mathcal{X}$. With the same notation and hypotheses as in Proposition 5.2, the pair $(\mathcal{X}, LL_1^{-1})$ can be identified with $(X \otimes Y, L)$, up to an isomorphism. The formulation has the advantage that $B_1$, also denoted by $\otimes$, is completely explicit, and the following diagram summarizes these facts:

$$
\begin{tikzcd}
X \times Y \arrow[rr, "\otimes"] \arrow[dr, "\underset{X,Y}{\otimes}"] \arrow[ddr, "B"'] & & \mathcal{X} \arrow[ddl, "L L_1^{-1}"] \\
& X \otimes Y \arrow[ur, "L_1"] \arrow[d, "L"] & \\
& Z &
\end{tikzcd}
$$

*5.2.2.* In an analogous way, consider $\mathcal{X}$ instead of $X$, $\mathcal{Y}$ instead of $Y$, and the bilinear map

$$B_2 : (a, b) \in \mathcal{X} \times \mathcal{Y} \mapsto a \widetilde{\otimes} b \in \sigma_2(\mathcal{X}, \mathcal{Y}),$$

where

$$a \widetilde{\otimes} b : d \in \mathcal{X} \mapsto \operatorname{tr}(d\, a^*) b \in \mathcal{Y}.$$

Then, an isomorphism $L_2$ from $\mathcal{X} \otimes \mathcal{Y}$ onto $\sigma_2(\mathcal{X}, \mathcal{Y})$ is defined by

$$L_2(a \underset{\mathcal{X},\mathcal{Y}}{\otimes} b) = a \widetilde{\otimes} b, \qquad (a, b) \in \mathcal{X} \times \mathcal{Y}.$$

With the notation of Section 5.2.1 we have $L_2 = L_{1,\mathcal{X},\mathcal{Y}}$. The pair $(\sigma_2(\mathcal{X}, \mathcal{Y}), LL_2^{-1})$ may also be identified with the pair $(\mathcal{X} \otimes \mathcal{Y}, L)$ (where $L$ is the linear map associated with $B$).

*5.2.3.* The map

$$B_3 : (a, b) \in \mathcal{X} \times \mathcal{Y} \mapsto a \overset{\ell}{\otimes} b \in \sigma_2\big(\sigma_2(X, Z), \sigma_2(Y, T)\big),$$

where $a \overset{\ell}{\otimes} b$ is defined by

$$d \in \sigma_2(X, Z) \mapsto bda^* \in \sigma_2(Y, T),$$

is bilinear. Thus, it corresponds to the isomorphism $L_3$ from $\mathcal{X} \otimes \mathcal{Y}$ onto $\sigma_2(\sigma_2(X, Z), \sigma_2(Y, T))$ defined by

$$L_3\Big(a \underset{\mathcal{X},\mathcal{Y}}{\otimes} b\Big) = a \overset{\ell}{\otimes} b.$$

With the notation of Proposition 5.2, the pair $(\mathcal{X} \otimes \mathcal{Y}, L)$ may be identified with $(\sigma_2(\mathcal{X}, \mathcal{Y}), LL_3^{-1})$.

*5.2.4.* The Bilinear Map

$$B_4\colon (A, B) \in \mathcal{M}_{n,m} \times \mathcal{M}_{q,p} \mapsto A \overset{L}{\otimes} B \in \sigma_2(\mathcal{M}_{p,m}, \mathcal{M}_{q,n})$$

leads to the isomorphism $L_4$ from $\mathcal{M}_{n,m} \otimes \mathcal{M}_{q,p}$ onto $\sigma_2(\mathcal{M}_{p,m}, \mathcal{M}_{q,n})$ defined by

$$L_4\Big(A \underset{\mathcal{M}_{n,m},\mathcal{M}_{q,p}}{\otimes} B\Big) = A \overset{L}{\otimes} B.$$

*5.3. Tensor Product of Operators*

Consider two linear maps $a$ and $b$, respectively elements of $\mathcal{L}(X, Y)$ and $\mathcal{L}(Z, T)$, where $X$, $Y$, $Z$, and $T$ are finite dimensional spaces. The map

$$B_5 \,:\, (x, z) \in X \times Z \mapsto a(x) \underset{Y,T}{\otimes} b(z) \in Y \otimes T$$

is bilinear. From Proposition 5.2, there exists a unique linear map $L_5$ from $X \otimes Z$ into $Y \otimes T$ satisfying

$$L_5\Big(x \underset{X,Z}{\otimes} z\Big) = a(x) \underset{Y,T}{\otimes} b(z), \qquad (x, z) \in X \times Z.$$

This map is denoted by $a \underset{\ell}{\otimes} b$ and is called the *tensor product* of the *linear maps a and b* (see Bourbaki [3, p. 13]).

If $(x_i)_i$, $(y_j)_j$, $(z_k)_k$, and $(t_\ell)_\ell$ denote respective bases of $X$, $Y$, $Z$, and $T$, and if

$$a(x_i) = \sum_j a_{ji} y_j \quad \text{and} \quad b(z_k) = \sum_\ell b_{\ell k} t_\ell,$$

it follows that

$$\Big(a \underset{\ell}{\otimes} b\Big)\Big(x_i \underset{X,Z}{\otimes} z_k\Big) = \sum_{j,\ell} a_{ji} b_{\ell k} y_j \underset{Y,T}{\otimes} t_\ell;$$

this shows that the matrix of $a \underset{\ell}{\otimes} b$ is the Kronecker product of the matrix $A$ of $a$ and the matrix $B$ of $b$ for the considered bases and for the assumed index order convention.

The product $\underset{\ell}{\otimes}$ leads us to consider the bilinear mapping

$$B_6 \,:\, (a, b) \in \mathcal{L}(X, Y) \times \mathcal{L}(Z, T) \mapsto a \underset{\ell}{\otimes} b \in \mathcal{L}(X \otimes Z, Y \otimes T).$$

Then there exists a unique isomorphism $L_6$ from $\mathcal{L}(X, Y) \otimes \mathcal{L}(Z, T)$ onto $\mathcal{L}(X \otimes$

$Z$, $Y \otimes T$) satisfying

$$L_6\Big(a \underset{\mathcal{L}(X,Y),\,\mathcal{L}(Z,T)}{\otimes} b\Big) = a \underset{\ell}{\otimes} b.$$

Let $\varphi$ be an isomorphism from $X$ onto $Y$, $\psi$ an isomorphism from $Z$ onto $T$. Then we may observe that $\varphi \underset{\ell}{\otimes} \psi$ is an isomorphism from $X \otimes Z$ onto $Y \otimes T$ with inverse isomorphism $\varphi^{-1} \underset{\ell}{\otimes} \psi^{-1}$. When $X$, $Y$, $Z$, and $T$ are Euclidean spaces, the following diagram obtains:

$$\begin{tikzcd}
\mathcal{X} \times \mathcal{Y} \arrow[rr, "\underset{\mathcal{X},\mathcal{Y}}{\otimes}"] \arrow[dr, "\underset{\ell}{\otimes}"] \arrow[ddr, "\overset{\ell}{\otimes}"'] & & \mathcal{X} \otimes \mathcal{Y} \arrow[dl, "L_6"'] \arrow[ddl, "L_3"] \\
& \sigma_2(X \otimes Z, Y \otimes T) \arrow[d, "L_3\, L_6^{-1}"] & \\
& \sigma_2(\sigma_2(X, Z), \sigma_2(Y, T)) &
\end{tikzcd}$$

and we have

$$\overset{\ell}{\otimes} = (L_3\, L_6^{-1}) \underset{\ell}{\otimes},$$

which shows that $\overset{\ell}{\otimes}$ and $\underset{\ell}{\otimes}$ are defined up to an isometry and justifies the notation $\overset{\ell}{\otimes}$.

For each $a$ of $\mathcal{X}$ and each $b$ of $\mathcal{Y}$, an isomorphism between $a \overset{\ell}{\otimes} b$ and $a \underset{\ell}{\otimes} b$ may also be built. Indeed, for all $(x, z)$ in $X \times Z$, we have

$$\begin{aligned}
\Big(a \overset{\ell}{\otimes} b\Big)(x \otimes z) = a(x) \otimes b(z) &= L_{1,Y,T}\Big(a(x) \underset{Y,T}{\otimes} b(z)\Big) \\
&= L_{1,Y,T}\Big(\Big(a \underset{\ell}{\otimes} b\Big)\Big(x \underset{X,Z}{\otimes} z\Big)\Big) \\
&= L_{1,Y,T}\Big(a \underset{\ell}{\otimes} b\Big) L_{1,X,Z}^{-1}(x \otimes z),
\end{aligned}$$

so

$$a \overset{\ell}{\otimes} b = L_{1,Z,T}\Big(a \underset{\ell}{\otimes} b\Big) L_{1,X,Y}^{-1}, \qquad a \in \mathcal{X}, \quad b \in \mathcal{Y}.$$

*5.4. Summarizing Diagrams*

Consider the special case where $X$, $Y$, $Z$, and $T$ are the same Euclidean space $E$ (this is the usual framework in statistics). The spaces $E \otimes E$ and $F = \sigma_2(E)$ are isomorphic by $L_{1,E}$. Then $(E \otimes E) \times (E \otimes E)$ $[(E \otimes E) \otimes (E \otimes E)]$ and $F \times F\,[F \otimes F]$ are in correspondence by the isomorphism $(L_{1,E}, L_{1,E})$ $[L_{1,E} \underset{\ell}{\otimes} L_{1,E}]$.

For each $(a, b)$ in $F^2$, the following diagram may be established:

$$\begin{tikzcd}
E \times E \arrow[rr, "\otimes"] \arrow[dr, "\underset{E}{\otimes}"'] & & F = \sigma_2(E) \arrow[dd, "a \overset{\ell}{\otimes} b"] \\
& E \otimes E \arrow[ur, "L_{1,E}"] \arrow[dd, "a \underset{\ell}{\otimes} b"] & \\
& & F \\
& E \otimes E \arrow[ur, "L_{1,E}"'] &
\end{tikzcd}$$

This last diagram gives finally the links between the different considered tensor products, as shown in Figure 1.

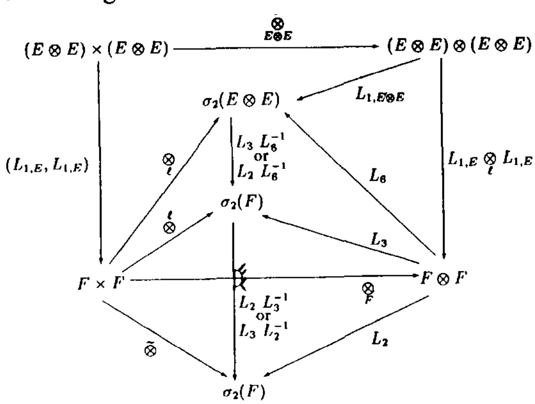


Fig. 1

## 6. CONCLUSIONS

We conclude this paper with some extensions and remarks.

Two examples of application of the dictionary have been given in Section 4. In the same area, we might have presented the *PCA* for a correlation matrix or operator (see Neudecker and Wesselman [13] for the matrix form, and Fine and Romain [9] for the functional form). Another example is the comparison of eigenspaces for two symmetric matrices or operators related to different populations (see Chen and Robinson [4] and Dauxois, Romain, and Viguier [7] for the respective cases). In fact for all factorial analyses, such as discriminant or correspondence analysis, complex *PCA*, and functional models, the asymptotic results presented in a functional context are available (the interested reader may obtain these references directly from the authors).

The applications of the formulary are multipurpose and may be used in an environment that is not necessarily asymptotic or that is not necessarily Gaussian or elliptical. Moreover, note that other authors use the same tools in a very different context (see, for example the recent paper of Wong and Wang [18]).

Furthermore, other Kronecker-product-related tools may be considered in a functional form. We may particularly think of the Hadamard product, which has interesting properties in multivariate statistics (see Styan [16] on this topic). This will give further developments.

Finally, we have chosen not to speak about the canonical isomorphism between $X^* \otimes Y$ and $\mathcal{L}(X, Y)$. So we have avoided the use of the duality bracket $\langle\ ,\ \rangle_{X^*,Y}$ even if the image of an element $x \underset{X^*,Y}{\otimes} y$ is by definition the element of $\mathcal{L}(X, Y)$ which maps $z$ of $X$ into $\langle x, z\rangle_{X^*,X}\ y$ of $Y$, and so the identification of $x \underset{X^*,Y}{\otimes} y$, $x \underset{X,Y}{\otimes} y$, and then $x \otimes y$ is immediate (see Bourbaki [3] and Pollock [14] for further developments on that subject).

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