An improvement of the Arzela–Ascoli theorem

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For the classical Arzela–Ascoli theorem and its typical modern formulation, we have improved the sufficiency part by weakening the compactness of the domain space, and the necessity part is improved by strengthening the necessity part of the classical version.

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For topological spaces Ω and X, let σΩ (resp., uΩ) be the topology on XΩ such that $f_\alpha \xrightarrow{\sigma_\Omega} f$ (resp., $f_\alpha \xrightarrow{u_\Omega} f$) if and only if $\lim_\alpha f_\alpha(\omega) = f(\omega)$ at each $\omega \in \Omega$ (resp., $\lim_\alpha f_\alpha(\omega) = f(\omega)$ uniformly for $\omega \in \Omega$), and for $S \subset X^2$, the closure of $S$ in $(X^2, \sigma_\Omega)$ (resp., $(X^2, u_\Omega)$) is denoted by $\overline{S}$ (resp., $\overline{S}$).

The Arzela–Ascoli theorem says that for compact space $\Omega$ and $S \subset C(\Omega, X)$, $S$ is relatively compact in $(C(\Omega, X), u_\Omega)$ if and only if $S$ is equicontinuous and $\{g(\omega): g \in S, \omega \in \Omega\}$ is relatively compact in $C$, i.e., $\sup_{g \in S, \omega \in \Omega} |g(\omega)| < +\infty$ [1, p. 266]. Following a recent comment on “Arzela–Ascoli theorem” (Wikipedia, the free encyclopedia, 12 March 2011), there are a few modern formulations of Arzela–Ascoli theorem [2–6], and the typical version is:

For topological spaces $\Omega$ and $X$, let $\sigma_\Omega$ (resp., $u_\Omega$) be the topology on $X^\Omega$ such that $f_\alpha \xrightarrow{\sigma_\Omega} f$ (resp., $f_\alpha \xrightarrow{u_\Omega} f$) if and only if $\lim_\alpha f_\alpha(\omega) = f(\omega)$ at each $\omega \in \Omega$ (resp., $\lim_\alpha f_\alpha(\omega) = f(\omega)$ uniformly for $\omega \in \Omega$), and for $S \subset X^2$, the closure of $S$ in $(X^2, \sigma_\Omega)$ (resp., $(X^2, u_\Omega)$) is denoted by $\overline{S}$ (resp., $\overline{S}$).

The Arzela–Ascoli theorem says that for compact space $\Omega$ and $S \subset C(\Omega, X)$, $S$ is relatively compact in $(C(\Omega, X), u_\Omega)$ if and only if $S$ is equicontinuous and $\{g(\omega): g \in S, \omega \in \Omega\}$ is relatively compact in $C$, i.e., $\sup_{g \in S, \omega \in \Omega} |g(\omega)| < +\infty$ [1, p. 266]. Following a recent comment on “Arzela–Ascoli theorem” (Wikipedia, the free encyclopedia, 12 March 2011), there are a few modern formulations of Arzela–Ascoli theorem [2–6], and the typical version is:

Let $\Omega$ be a compact Hausdorff space and $X$ a metric space. Then a subset $S \subset C(\Omega, X)$ is compact in the compact-open topology ($= u_\Omega$ by [7, Theorem 13.2.3]) if and only if it is equicontinuous, closed and pointwise relatively compact, i.e., $\{g(\omega): g \in S, \omega \in \Omega\}$ is compact for each $\omega \in \Omega$.

However, strictly speaking, these modern versions are neither proper generalizations nor proper improvement of the classical theorem because, in the modern versions, the compactness of $S$ implies that $\{g(\omega): g \in S\}$ is pointwise compact only, while the compactness of $S$ implies the compactness of $\{g(\omega): g \in S, \omega \in \Omega\}$ in the classical theorem.

We would like to improve the classical theorem and its modern formulations. First, we improve the sufficiency part by weakening the compactness of domain space $\Omega$. Then we improve the necessity part of the modern version by showing that if $S$ is relatively compact in $(C(\Omega, X), u_\Omega)$, then $\{g(\omega): g \in S, \omega \in \Omega\}$ is compact.

It is also interesting that both domain space and range space need not be Hausdorff in our improved versions.

Observe that pseudometric spaces and topological groups are uniform spaces [7, pp. 202, 242]. Let

$X^2 = \{(x, y): x, y \in X\}$.
Lemma 1. Let $X$ be a topological space. For $x \in X$ and $U \subset X^2$, let $U[x] = \{ y \in X : (x, y) \in U \}$. If $U$ is closed, then $\overline{U[x]} \subset U[x]$ for all $x \in X$.

**Proof.** For $x \in X$ and $y \in \overline{U[x]}$, there is a net $(y_\alpha)_{\alpha \in I}$ in $U[x]$ such that $y_\alpha \rightarrow y$. Then $(x, y_\alpha) \rightarrow (x, y) \in U$, i.e., $y \in U[x]$. □

Lemma 2. Let $\Omega$ be a topological space and $X$ a uniform space with the uniformity $\mathcal{U} \subset 2^{(X^2)}$ and $S \subset C(\Omega, X)$. If $S$ is equicontinuous, then $S^\sigma_\Omega$ is also equicontinuous, and in particular, $S^\alpha_\Omega \subset C(\Omega, X)$.

**Proof.** Let $U \in \mathcal{U}$ be closed and $\omega \in \Omega$. Pick a symmetric closed $V \in \mathcal{U}$ for which $V \circ V \subset U$ [7, p. 206, Problems 10]. Since $S$ is equicontinuous, there is a neighborhood $N$ of $\omega$ such that $g(N) \subset V[g(\omega)]$ for all $g \in S$.

Let $f \in S^\sigma_\Omega$. There is a net $(g_\alpha)_{\alpha \in I} \in S$ such that $g_\alpha \overset{\sigma_\Omega}{\rightarrow} f$ and so there is $\alpha_0 \in I$ such that $g_\alpha(\omega) \in V[f(\omega)]$ for all $\alpha \geq \alpha_0$. Then

$$g_\alpha(N) \subset V[g_\alpha(\omega)] \subset V[V[f(\omega)]] = (V \circ V)[f(\omega)] \subset U[f(\omega)], \forall \alpha \geq \alpha_0.$$ 

Thus, for every $\lambda \in N$, $f(\lambda) = \lim_\alpha g_\alpha(\lambda) \in U[f(\omega)]$ by Lemma 1, i.e., $f(N) \subset U[f(\omega)]$. □

A topological space $\Omega$ is called almost compact if for each open cover $A$ of $\Omega$, there is a finite number of $A_1, A_2, \ldots, A_n \in A$ such that $\Omega = \bigcup_{i=1}^n A_i$ [8, p. 239]. Following [9], if the pointwise convergence in $[0, 1]^\Omega$ coincides with the strong convergence in $[0, 1]^\Omega$, then $\Omega$ is almost compact, and if $\Omega$ is almost compact, then for every regular space $X$, the pointwise convergence in $C(\Omega, X)$ is equivalent to the strong convergence [9, Theorem 4].

Lemma 3. Let $\Omega$ be an almost compact space, $(X, \mathcal{U})$ a uniform space and $S \subset C(\Omega, X)$. If $S$ is equicontinuous, then $S^\alpha_\Omega = S^\sigma_\Omega$ (=$S$) and $(S, \sigma_\Omega) = (S, u_\Omega)$.

**Proof.** Let $(f_\alpha)_{\alpha \in (I, \leq)}$ be a net in $S$ such that $f_\alpha \overset{\sigma_\Omega}{\rightarrow} f \in X^\Omega$. Let $U \in \mathcal{U}$ be closed. Then pick a symmetric closed $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$ [7, p. 206, Problems 10].

By Lemma 2, $S^\sigma_\Omega$ is equicontinuous and so each $\omega \in \Omega$ has a neighborhood $N_\omega$ such that $g(N_\omega) \subset V[g(\omega)]$ for all $g \in S^\sigma_\Omega$. Since $\Omega$ is almost compact, there is a finite set $\{\omega_1, \omega_2, \ldots, \omega_n\} \subset \Omega$ such that $\Omega = \bigcup_{i=1}^n N_{\omega_i}$, and $f_\alpha \overset{\sigma_\Omega}{\rightarrow} f$ shows that there is $\alpha_0 \in I$ such that $f_\alpha(\omega_i) \in V[f(\omega_i)]$ for all $\alpha \geq \alpha_0$ and $i \in \{1, 2, \ldots, n\}$.

Fix $\omega \in \Omega$ and $\alpha \geq \alpha_0$. Then $\omega \in N_{\omega_i}$ for some $i \in \{1, 2, \ldots, n\}$ and so there is a net $(\omega_\beta)_{\beta \in J}$ in $N_{\omega_i}$ such that $\omega_\beta \rightarrow \omega$, and

$$f_\alpha(\omega_\beta) \in f_\alpha(N_{\omega_i}) \subset V[f_\alpha(\omega_i)] \subset V[V[f(\omega_i)]] = (V \circ V)[f(\omega_i)]$$

for all $\alpha \geq \alpha_0$ and $\beta \in I$.

Since $\omega_\beta \rightarrow \omega$, therefore $f_\alpha(\omega) = \lim_\beta f_\alpha(\omega_\beta) \in (V \circ V)f(\omega_i)$ for all $\alpha \geq \alpha_0$. Moreover, it follows from $f \in S^\sigma_\Omega \subset C(\Omega, X)$ and $V = V^{-1}$ that

$$f(\omega) = \lim_\beta f(\omega_\beta) \in f(N_{\omega_i}) \subset V[f(\omega_i)] \subset V[f(\omega_i)]$$

by Lemma 1. Then,

$$(f(\omega_i), f(\omega)) \in V = V^{-1}, \quad f(\omega_i), f(\omega_\beta) \in V \quad \text{and} \quad f(\omega) \in V[f(\omega)].$$  

By Lemma 1 again,

$$f_\alpha(\omega) \in (V \circ V)(f(\omega)) \subset (V \circ V)(V[f(\omega)]) = (V \circ V \circ V)(f(\omega)) \subset U[f(\omega)] \subset U[f(\omega)]$$

for all $\alpha \geq \alpha_0$. Observing that $\alpha_0$ is independent of the choice of $\omega \in \Omega$, this shows that $\lim_\alpha f_\alpha(\omega) = f(\omega)$ uniformly for $\omega \in \Omega$, i.e., $f_\alpha \overset{\sigma_\Omega}{\rightarrow} f \iff f_\alpha \overset{u_\Omega}{\rightarrow} f$ and so $S^\sigma_\Omega = S^\alpha_\Omega$ (=$S$).

Since $S$ is also equicontinuous, the above argument implies that for a net $(g_\alpha)_{\alpha \in I}$ in $S$ and $g \in S$, $g_\alpha \overset{\sigma_\Omega}{\rightarrow} g$ if and only if $g_\alpha \overset{u_\Omega}{\rightarrow} g$. Thus, $(S, \sigma_\Omega) = (S, u_\Omega)$. □

Theorem 4. Let $\Omega$ be an almost compact space, $X$ a uniform space and $S \subset C(\Omega, X)$. Then the following (I) and (II) hold.

(I) If $S$ is equicontinuous and $(g(\omega) ; g \in S)$ is relatively compact in $X$ for each $\omega \in \Omega$, then $S$ is relatively compact in $(C(\Omega, X), u_\Omega)$.

(II) If $S$ is equicontinuous and $(g(\omega) ; g \in S, \omega \in \Omega)$ is relatively compact in $X$, then $S$ is relatively compact in $(C(\Omega, X), u_\Omega)$.  

Proof. Suppose that $S$ is equicontinuous and $\{\bar{g}(\omega): g \in S\}$ is compact for each $\omega \in \Omega$. By Lemma 2 and 3, $\bar{S}^{\sigma \Omega} \subset C(\Omega, X)$, $\bar{S}^{\sigma \Omega} = \bar{S}^u \Omega (\bar{S} \Omega)$ and $(\bar{S}, \sigma \Omega) = (\bar{S}, u \Omega)$. Obviously, $\bar{S} = \bar{S}^{\sigma \Omega}$ is closed in $(X^{\lambda \Omega}, \sigma \Omega)$. If $f \in \bar{S}$, then there is a net $\{g_{\alpha}\}_{\alpha \in I}$ in $S$ such that

$$f(\omega) = \lim_{\alpha} g_{\alpha}(\omega) \in \{\bar{g}(\omega): g \in S\}, \quad \forall \omega \in \Omega.$$ 

This shows that

$$\{f(\omega): f \in \bar{S}\} \subseteq \{\bar{g}(\omega): g \in S\} = \{\bar{g}(\omega): g \in S\} \subseteq \{f(\omega): f \in \bar{S}\}, \quad \forall \omega \in \Omega,$$

i.e., $\{f(\omega): f \in \bar{S}\} = \{\bar{g}(\omega): g \in S\}$ is compact for each $\omega \in \Omega$. Now $(\bar{S}, \sigma \Omega)$ is compact by [5, Theorem 7.1, p. 218], but $(\bar{S}^u \Omega, \sigma \Omega) = (\bar{S}, \sigma \Omega)$, so $\bar{S}^{\sigma \Omega}$ is compact in $(C(\Omega, X), u \Omega)$, i.e., (1) holds.

Suppose that $S$ is equicontinuous and $K := \{\bar{g}(\omega): g \in S, \omega \in \Omega\}$ is relatively compact in $X$. For $\omega \in \Omega$, $S(\omega) := \{\bar{g}(\omega): g \in S\} \subset K$ and $\bar{S}(\omega) \subset \bar{K}$, i.e., $\bar{S}(\omega)$ is closed in the compact $\bar{K}$. Thus, $\{\bar{g}(\omega): g \in S\}$ is compact for each $\omega \in \Omega$ and so (II) holds by (1).

Observe that a topological space $S$ is compact if and only if every net $\{f_{\alpha}\}_{\alpha \in (I, \leq)}$ in $S$ has a convergent subnet $\{f_{\alpha'}\}_{\alpha' \in (I', \leq)}$ such that $\alpha' \leq \alpha$, whenever $\beta \leq \gamma$ [5, Chapters 2, 5].

Theorem 5. Let $\Omega$ be a compact space and $(X, d)$ a pseudometric space. For $S \subset C(\Omega, X)$, the following (1), (2) and (3) are equivalent.

1. $S$ is relatively compact in $(C(\Omega, X), u \Omega)$.
2. $S$ is equicontinuous and $K := \{\bar{g}(\omega): g \in S, \omega \in \Omega\}$ is relatively compact in $(X, d)$.
3. $S$ is equicontinuous and $\sigma \Omega := \{\bar{g}(\omega): g \in S\}$ is relatively compact in $(X, d)$ for each $\omega \in \Omega$.

Proof. (1) $\Rightarrow$ (2). First, we assume that $S = \bar{S}$, where $\bar{S}$ is the closure of $S$ in $(C(\Omega, X), u \Omega)$. For $\omega \in \Omega$, let $N_\omega$ be the family of neighborhoods of $\omega$. Suppose that $\omega \in \Omega$ but $S$ is not equicontinuous at $\omega$. Then there is $\varepsilon > 0$ for which each $N \in N_\omega$ produces $f_N \in S$ and $\omega_N \in N$ such that $d(f_N(\omega_N), f_N(\omega)) \geq \varepsilon$.

Since $N_\omega$ is directed by inclusion, both $(f_N)_{N \in N_\omega}$ and $(\omega_N)_{N \in N_\omega}$ are nets and $\lim_{N \in N_\omega} \omega_N = \omega$. Moreover, $(S, u \Omega) = (\bar{S}, u \Omega)$ is compact by (1) and so $(f_N)_{N \in N_\omega}$ has a subnet $(f_{N_{\alpha}})_{\alpha \in (I, \leq)}$ such that $f_{N_{\alpha}} \xrightarrow{u \Omega} f \in S$, i.e., $\lim_{\alpha} f_{N_{\alpha}}(\lambda) = f(\lambda)$ uniformly for $\lambda \in \Omega$, and $\lim_{\alpha} \omega_{N_{\alpha}} = \lim_{\alpha} \omega_N = \omega$.

Pick $\alpha_0 \in I$ such that $d(f(\omega_N), f(\omega)) < \varepsilon/3$ for all $\omega \in \Omega$ and $\alpha_0 \geq \alpha_0$. Since $f \in S \subset C(\Omega, X)$ and $\omega_N \rightarrow \omega$ in $\Omega$, there is $\alpha_1 \in I$ such that $d(f(\omega_N), f(\omega)) < \varepsilon/3$ for all $\alpha \geq \alpha_1$. Now pick $\beta \in I$ such that $\beta \geq \alpha_0$ and $\beta \geq \alpha_1$, then

$$d(f_{N_{\beta}}(\omega_{N_{\beta}}), f_{N_{\alpha}}(\omega)) \leq d(f_{N_{\beta}}(\omega_{N_{\beta}}), f(\omega_N)) + d(f(\omega_N), f(\omega)) + d(f(\omega), f_{N_{\alpha}}(\omega))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \varepsilon = \varepsilon.$$

This contradicts the fact that $d(f_N(\omega_N), f_N(\omega)) \geq \varepsilon$ for all $N \in N_\omega$. Hence $S$ is equicontinuous at each $\omega \in \Omega$, i.e., $S$ is equicontinuous.

As a subspace of $(X, d)$, $K = \{\bar{g}(\omega): g \in S, \omega \in \Omega\}$ is a pseudometric space. Suppose $\varphi \in C(K, \mathbb{R})$ but $\sup_{K} |\varphi(\lambda)| = +\infty$. Then $|\varphi(f_N(\omega_N))| > n$ for some sequences $\{f_n\} \subset S$ and $\{\omega_N\} \subset \Omega$. Since $(S, u \Omega) = (\bar{S}, u \Omega)$ is compact by (1), $\{f_n\}$ has a subnet $(f_{n_{\alpha}})_{\alpha \in (I, \leq)}$ such that $f_{n_{\alpha}} \xrightarrow{u \Omega} f \in S$, i.e., $\lim_{\alpha} f_{n_{\alpha}}(\lambda) = f(\lambda)$ uniformly for $\lambda \in \Omega$. By passing to a subnet if necessary, we can say that $\omega_{N_{\alpha}} \rightarrow \omega$ in the compact space $\Omega$.

Let $\varepsilon > 0$. Since $S$ is equicontinuous and $\omega_{N_{\alpha}} \rightarrow \omega$, $\lim_{\alpha} g(\omega_{N_{\alpha}}) = g(\omega)$ uniformly for $g \in S$ [1, p. 266] and so there is $\alpha_0 \in I$ such that $d(g(\omega_{N_{\alpha}}), g(\omega)) < \varepsilon/2$ for all $g \in S$ and $\alpha \geq \alpha_0$, and $d(f_{n_{\alpha}}(\omega_{N_{\alpha}}), f_{n_{\alpha}}(\omega)) < \varepsilon/2$ for all $\alpha \geq \alpha_0$. Moreover, $\lim_{\alpha} f(\omega) = f(\omega)$, so there is $\alpha_1 \in I$ such that $d(f_{n_{\alpha}}(\omega), f(\omega)) < \varepsilon/2$ for all $\alpha \geq \alpha_1$. Now pick $\alpha_2 \in I$ such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. Then

$$d(f_{n_{\alpha}}(\omega_{N_{\alpha}}), f(\omega)) \leq d(f_{n_{\alpha}}(\omega_{N_{\alpha}}), f_{n_{\alpha}}(\omega)) + d(f_{n_{\alpha}}(\omega), f(\omega))$$

$$< \frac{\varepsilon}{2} + \varepsilon = \varepsilon, \quad \forall \alpha \geq \alpha_2.$$

Therefore, $\lim_{\alpha} f_{n_{\alpha}}(\omega_{N_{\alpha}}) = f(\omega)$, $\lim_{\alpha} |\varphi(f_{n_{\alpha}}(\omega_{N_{\alpha}}))| = |\varphi(f(\omega))|$ and so there is $\alpha_3 \in I$ such that $|\varphi(f_{n_{\alpha}}(\omega_{N_{\alpha}}))| < |\varphi(f(\omega))| + 1$ for all $\alpha \geq \alpha_3$.

Pick an integer $n_0 = |\varphi(f(\omega))| + 1$ and $\alpha_4 \in I$ such that $n_{\alpha_4} \geq n_0$, and then there is $\alpha \in I$ such that $\alpha \geq \alpha_3, \alpha \geq \alpha_4$ and

$$|\varphi(f(\omega))| + 1 < n_0 \leq n_{\alpha_4} \leq n_{\alpha} < |\varphi(f_{n_{\alpha}}(\omega_{N_{\alpha}}))| < |\varphi(f(\omega))| + 1,$$

which is a contradiction. Hence $\sup_{K} |\varphi(\lambda)| < +\infty$ for all $\varphi \in C(K, \mathbb{R})$, i.e., $K$ is a pseudocompact space [7, p. 59].

Since $K$ is a pseudometric space, $K$ is normal [7, p. 62] and so the pseudocompact $K$ is countably compact [7, pp. 158, 192]. Then the pseudometric space $K$ is compact [7, p. 128].
Now consider the case of $S \neq \overline{S}$. By (1), $\overline{S}$ is compact in $(C(\Omega, X), u\Omega)$ and the above argument shows that $\overline{S}$ is equicontinuous and $(f(\omega): f \in \overline{S}, \omega \in \Omega)$ is compact in the regular space $(X, d)$ [7, pp. 61–62]. Then $\{f(\omega): f \in \overline{S}, \omega \in \Omega\}$ is also compact and so its closed subspace $\{g(\omega): g \in S, \omega \in \Omega\}$ is compact. 

(2) $\Rightarrow$ (3). Let $\omega \in \Omega$. Then $S(\omega) \subset K$ and $K$ is compact by (2). Hence $\overline{S(\omega)}$ is compact.

(3) $\Rightarrow$ (1). A special case of (1) in Theorem 4.  

**Corollary 6.** Let $\Omega$ be a compact space and $X$ a pseudometric space. Then for every equicontinuous $S \subset C(\Omega, X)$, $\{g(\omega): g \in S, \omega \in \Omega\}$ is compact if and only if $\{g(\omega): g \in \overline{S}\}$ is compact for each $\omega \in \Omega$.

**References**