Positive Solutions of Some Second-Order Nonlinear Singular Differential Equations

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(Received September 2001; revised and accepted April 2002)

Abstract—With new schemes, the existence of the positive solutions is proved to some second-order nonlinear singular boundary value problems and initial value problems, which have higher-order singularities. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Singular problems, Higher-order singularities, Positive solution, Diagonal sequence.

1. INTRODUCTION

It is well known that singular problems arise in the fields of applied mathematics and applied physics and then are always noticed by the scholars at home and abroad (see, e.g., [1-11] and their references).

Singular problems, which are not related to first-order derivative $x'$, were studied satisfactorily. Agarwal and O'Regan [1,2] studied superlinear, delay singular boundary value problems and singular Dirichlet problems, Lan and Webb [3] gave the existence results of semilinear singular boundary value problems, and Cheng and Zhang [4] obtained positive solutions for a class of singular boundary value problems. Yang [5] established minimal and maximal solutions by fixed-point theorems of increasing operators in ordered space without any assumptions of continuity, compactness, concavity, or convexity.

If singular problems are related to first-order derivative $x'$, without Green's function, the study of singular problems is much more difficult than that mentioned above. Agarwal and O'Regan [6,7] (applying some known theorems), Kelevedjiev [8] (applying the topological transversality theorem), and Lepin and Ponomarev [9] (using lower and upper function) offered some valuable results. Yang [10,11] studied singular boundary value problems and initial value problems with higher-order singularities at $x' = 0$. As far as we know, the study of boundary value problems with higher-order singularities at $x = 0$ is very rare and very few results [7,11] are available for singular initial value problems.

I am grateful to Prof. and Editor-in-Chief Ervin Y. Rodin for his help and the referee for his careful reading and valuable suggestions.
In this paper, we study positive solutions to the following boundary value problem with higher-order singularities at \( x = 0 \) and \( x' = 0 \):

\[
x'' + f(t, x, x') = 0, \quad 0 < t < 1,
\]

\[
x(0) = x'(1) = 0,
\]

and the initial value problem with higher-order singularities at \( x' = 0 \),

\[
x'' = f(t, x, x'), \quad 0 < t < 1,
\]

\[
x(0) = x'(0) = 0,
\]

\[(1.1)\]

\( f(t, x, y) \) has the following singularities at \( x = 0 \) and \( y = 0 \):

\[
\lim_{x \to 0^+} f(., x, .) = f_{\infty}, \quad \lim_{y \to 0^+} f(., ., y) = f_{\infty}.
\]

\[(1.3)\]

To obtain the approximate solutions of \((1.1)\) and \((1.2)\), the assumption of \( f(t, x, y) \leq k(t) \)

\( F(x)G(y) \) is usually needed (see, e.g., [7-9,11]). In Section 2, we make some preliminaries. We study the boundary value problem \((1.1)\) and the initial value problem \((1.2)\) in Section 3. Also, the examples are given.

## 2. SOME PRELIMINARIES

Throughout this paper, suppose that \( f(t, x, y) \in C((0, 1) \times (0, \infty)^2, \{0, \infty\}) \), \( k(t) \in C((0, 1), \{0, \infty\}) \), \( F(x), G(y) \in C((0, \infty), (0, \infty)) \), \( f(t, x, y) \leq k(t)F(x)G(y) \), and the following conditions are satisfied.

(H1) \( F(x) \) is a decreasing function, i.e., \( x_1 \leq x_2 \) implies \( F(x_1) \geq F(x_2) \).

(H2) \( \lim_{y \to +\infty} G(y) < +\infty \).

Remark 2.1. For any fixed \( z \in (0, +\infty) \), \( \sup_{y \in [z, +\infty)} G(y) = \sup \{ G(y) : z \leq y < +\infty \} < +\infty \) by (H2).

Definition. A function \( x(t) \) is called to be the positive solution of \((1.1)\) (or \((1.2)\)) if and only if

(i) \( x(t) > 0 \), for \( t \in (0, 1) \) and \( x(t) \) belongs to \( C[0, 1] \cap C^2(0, 1) \);

(ii) \( x(t) \) satisfies equation \((1.1)\) (or \((1.2)\)) with boundary conditions.

For \( y(t) \in C[0, 1] \cap L[0, 1] \) (or \( y(t) \in C[0, 1] \cap L[0, 1] \)), define \( Ay(t) = \int_0^t y(s) ds \). It is easy to verify directly that if \( y(t) \in C[0, 1] \cap L[0, 1] \), \( y(t) > 0 \), \( t \in (0, 1) \), and \( y(t) \) satisfies equation \((2.1),(2.2)\),

\[
y(t) = \int_t^1 f(s, Ay(s), y(s)) ds, \quad t \in (0, 1).
\]

\[(2.1)\]

\[
y(t) = \int_0^t f(s, Ay(s), y(s)) ds, \quad t \in [0, 1).
\]

\[(2.2)\]

Then \( x(t) = \int_0^t y(s) ds \) is a positive solution of \((1.1),(1.2)\).

Lemma 2.1.

(i) If \( tk(t) \in L[0, 1] \), then there exists \( y_n(t) \in C[0, 1] \), \( y_n(t) \geq 0 \) such that

\[
y_n(t) = \begin{cases} 
\frac{1}{n} + \int_t^1 f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) ds, & t \in \left[ \frac{1}{n}, 1 \right], \\
\frac{1}{n} + \int_0^1 f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) = y_n \left( \frac{1}{n} \right), & t \in \left[ 0, \frac{1}{n} \right].
\end{cases}
\]

\[(2.3)\]
(ii) If \((1 - t)k(t) \in L[0, 1]\), then there exists \(y_n(t) \in C[0, 1]\), \(y_n(t) \geq 0\) such that

\[
y_n(t) = \begin{cases}
\frac{1}{n} + \int_0^t f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) ds, & t \in \left[ 0, 1 - \frac{1}{n} \right], \\
\frac{1}{n} + \int_0^{1/n} f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) ds = y_n \left( 1 - \frac{1}{n} \right), & t \in \left( 1 - \frac{1}{n}, 1 \right).
\end{cases}
\]

Proof.

(i) Put \(a_n = \int_1^{1/n} k(s) ds, a_n \leq \int_1^{1/n} \frac{1}{s/(1/n)} k(s) ds \leq n \int_0^1 k(s) ds < +\infty\). By Remark 2.1, for any fixed natural number \(n\), we may choose a sufficiently large \(R_n > 0\) to fit \(1 + a_n F(1/n)\). 

Put \(C_n = \{y(t) : y(t) \in C[0, 1], 1/n \leq y(t) \leq R_n, y(t)\) is decreasing\}. For \(y(t) \in C_n\), define an operator \(T_n\) as follows:

\[
T_n y(t) = \begin{cases}
\frac{1}{n} + \int_t^1 f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds, & t \in \left[ \frac{1}{n}, 1 \right], \\
\frac{1}{n} + \int_1^{1/n} f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds - T_n y \left( \frac{1}{n} \right), & t \in \left[ 0, \frac{1}{n} \right).
\end{cases}
\]

By \(T_n y(t) \leq 1 + \int_1^{1/n} f \left( s, Ay(s) + 1/n, y(s) \right) ds \leq 1 + \int_1^{1/n} k(s) F(Ay(s) + 1/n) ds \leq 1 + \int_1^{1/n} \frac{1}{s/(1/n)} k(s) ds F(1/n) \sup G([1/n, +\infty)) \leq R_n\), we know that \(T_n\) is an operator from \(C_n\) into \(C_n\). To obtain the desired result, we prove that \(T_n\) is continuous and compact.

Let \(y_k(t), y(t) \in C_n, \|y_k - y\| \to 0\), then \(1/n \leq y_k(t) \leq R_n, 1/n \leq y(t) \leq R_n, t \in [0, 1]\).

\[
\|T_n y_k(t) - T_n y(t)\| = \left\| \int_t^1 f \left( s, Ay_k(s) + \frac{1}{n}, y_k(s) \right) - f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds \right\|, \quad t \in \left[ \frac{1}{n}, 1 \right],
\]

\[
\|T_n y_k(t) - T_n y(t)\| = \left\| \int_1^{1/n} f \left( s, Ay_k(s) + \frac{1}{n}, y_k(s) \right) - f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds \right\|, \quad t \in \left[ 0, \frac{1}{n} \right).
\]

Then \(\|T_n y_k(t) - T_n y(t)\| \leq \int_1^{1/n} \|f \left( s, Ay_k(s) + 1/n, y_k(s) \right) - f \left( s, Ay(s) + 1/n, y(s) \right)\| ds, \quad t \in [0, 1]\).

Clearly, \(\{f \left( s, Ay_k(s) + 1/n, y_k(s) \right)\}\) converges to \(f \left( s, Ay(s) + 1/n, y(s) \right)\), for any \(s \in (0, 1)\). By 

the dominated convergence theorem (the dominated function \(F_0(s) = F(1/n) \sup G([1/n, +\infty))\)) 

\(k(s) \in L[1/n, 1], \|T_n y_k(t) - T_n y(t)\| \to 0\), i.e., \(T_n\) is a continuous operator.

Let \(t_1, t_2 \in [0, 1], t_1 < t_2, y(t) \in C_n\),

\[
\|T_n y(t_2) - T_n y(t_1)\| = \begin{cases}
\int_{t_1}^{t_2} f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds, & t_1, t_2 \in \left[ \frac{1}{n}, 1 \right], \\
\int_{1/n}^{t_2} f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) ds, & t_1 \in \left[ 0, \frac{1}{n} \right), t_2 \in \left[ \frac{1}{n}, 1 \right], \\
0, & t_1, t_2 \in \left[ 0, \frac{1}{n} \right),
\end{cases}
\]

\[
\leq \begin{cases}
\int_{t_1}^{t_2} k(s) ds F \left( \frac{1}{n} \right) \sup G \left( \left[ \frac{1}{n} + \infty \right) \right), & t_1, t_2 \in \left[ \frac{1}{n}, 1 \right], \\
\int_{1/n}^{t_2} k(s) ds F \left( \frac{1}{n} \right) \sup G \left( \left[ \frac{1}{n} + \infty \right) \right), & t_1 \in \left[ 0, \frac{1}{n} \right), t_2 \in \left[ \frac{1}{n}, 1 \right], \\
0, & t_1, t_2 \in \left[ 0, \frac{1}{n} \right).
\end{cases}
\]
According to the absolute continuity of Lebesgue integral, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \), when \( t', t'' \in [1/n, 1] \), \( 0 < t'' - t' < \delta \), \( \int_{t'}^{t''} k(s) \, ds < \varepsilon \). From this, we know that \( \{T_n y(t) : y(t) \in C_1 \} \) is equicontinuous. Schauder fixed theorem tells us that equation \( T_n y(t) = y(t) \) has a positive solution \( y_n(t) \in C_1 \), i.e., (i) holds.

(ii) Put \( c_n = \int_0^{1-1/n} k(s) \, ds \), then \( c_n \leq \int_0^{1-1/n} (1 - s/(1 - (1 - 1/n))) k(s) \, ds \leq n \int_0^1 (1 - s) k(s) \, ds < +\infty \). For any fixed natural number \( n \), we choose a sufficiently large \( R_n > 0 \) to fit \( 1 + c_n F'(1/n) \sup G([1/n, +\infty)) \leq R_n \).

Put \( D_n = \{y(t) : y(t) \in C_1 \}, \) \( 1/n < y(t) \leq R_n, y(t) \) is increasing \}. For \( y(t) \in D_n \), define an operator \( S_n \) as follows:

\[
S_n y(t) = \begin{cases} 
\frac{1}{n} + \int_0^t f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) \, ds, & t \in \left[0, 1 - \frac{1}{n} \right], \\
\frac{1}{n} + \int_{1-1/n}^t f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) \, ds = S_n y \left( \frac{1 - t}{n} \right), & t \in \left(1 - \frac{1}{n}, 1 \right).
\end{cases}
\]

By \( S_n y(t) \leq 1 + \int_0^{1-1/n} f(s, Ay(s) + 1/n, y(s)) \, ds \leq 1 + \int_0^{1-1/n} k(s) F(Ay(s) + 1/n) G(y(s)) \, ds \leq 1 + c_n F'(1/n) \sup G([1/n, +\infty)) \leq R_n \), \( t \in [0, 1] \), we know that \( S_n \) is an operator from \( D_n \) into \( D_n \).

Let \( y_k(t), y(t) \in D_n \), \( \|y_k - y\| \to 0 \), then \( 1/n \leq y_k(t) \leq R_n, 1/n \leq y(t) \leq R_n, t \in [0, 1] \).

\[
|S_n y_k(t) - S_n y(t)| = \begin{cases} 
\int_0^t \left| f(s, Ay_k(s) + \frac{1}{n}, y_k(s)) - f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) \right| \, ds, & t \in \left[0, 1 - \frac{1}{n} \right], \\
\int_{1-1/n}^t \left| f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) - f \left( s, Ay_k(s) + \frac{1}{n}, y_k(s) \right) \right| \, ds, & t \in \left(1 - \frac{1}{n}, 1 \right).
\end{cases}
\]

Then \( |S_n y_k(t) - S_n y(t)| \leq \int_0^{1-1/n} |f(s, Ay_k(s) + 1/n, y_k(s)) - f(s, Ay(s) + 1/n, y(s))| \, ds, t \in [0, 1] \).

Clearly, \( \{f(s, Ay_k(s) + 1/n, y_k(s))\} \) converges to \( f(s, Ay(s) + 1/n, y(s)) \), for any \( s \in (0, 1) \). By the dominated convergence theorem (the dominated function \( F_0(s) = F(1/n) \sup G([1/n, +\infty)) k(s) \in L[0, 1/n] \)), \( \|S_n y_k(t) - S_n y(t)\| \to 0 \), i.e., \( S_n \) is continuous operator.

Let \( t_1, t_2 \in [0, 1] \), \( t_1 < t_2, y(t) \in D_n \),

\[
|S_n y(t_2) - S_n y(t_1)| = \begin{cases} 
\int_{t_1}^{t_2} f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) \, ds, & t_1, t_2 \in \left[0, 1 - \frac{1}{n} \right], \\
\int_{1-1/n}^{t_1} f \left( s, Ay(s) + \frac{1}{n}, y(s) \right) \, ds, & t_1 \in \left[0, 1 - \frac{1}{n} \right], \\
0, & t_1, t_2 \in \left(1 - \frac{1}{n}, 1 \right).
\end{cases}
\]

According to the absolute continuity of Lebesgue integral \( k(s) \in L[0, 1 - 1/n] \), we know that \( \{S_n y(t) : y(t) \in D_n \} \) is equicontinuous. This shows equation \( S_n y(t) = y(t) \) has a positive solution \( y_n(t) \in D_n \), i.e., (ii) holds.
3. MAIN RESULTS

**THEOREM 3.1.** Let \((H_1)\) and \((H_2)\) hold and \(tk(t) \in L[0, 1]\), then the boundary value problem (1.1) has a positive solution at least.

**PROOF.** Proof is divided into two parts. In this section, \(y_n(t)\) satisfies (2.3) in Lemma 2.1.

(1) \(\{y_n(t)\}\) is equicontinuous on any \([a, b] \subseteq (0, 1)\).

Put \(w(t) = \inf\{y_n(t)\}, W(t) = \sup\{y_n(t)\}, t \in (0, 1)\), then

\[
0 < w(t), W(t) < +\infty, \quad t \in (0, 1). \tag{3.1}
\]

In fact, if there exists \(t_0 \in (0, 1)\) such that \(w(t_0) = 0\). Then there exists \(\{y_{n_k}(t_0)\} \subseteq \{y_n(t_0)\}\), \(\{y_{n_k}(t_0)\}\) converges to 0. The decrease in \(y_{n_k}(t)\) implies that \(\{y_{n_k}(t)\}\) converges uniformly to 0, \(t \in [t_0, 1]\). By (1.3), there exists \(\delta > 0\), when \(0 < y < \delta, f(t, y) \geq 1\). Choose a natural number \(N\), when \(n_k \geq N, y_{n_k}(t) \leq \delta, t \in [t_0, 1]\). From (2.3), we obtain \(\{n_k > \max\{N, 1/t_0\}\}\).

\(y_{n_k}(t_0) \geq \int_{t_0}^1 1 \, ds = 1 - t_0 > 0\). This is a contradiction. Put \(\eta(t) = \inf\{\int_t^1 y(s) \, ds\}\), then \(\eta(t) \geq \int_t^1 w(s) \, ds > 0\) by \(w(t) > 0, t \in (0, 1)\). Clearly, \(w(t)\) is decreasing in \((0, 1]\) and \(\eta(t)\) is increasing in \((0, 1]\).

Fixed \(t \in (0, 1]\), we have from (2.3),

\[
-y''(s) = f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) \leq k(s)F \left( Ay_n(s) + \frac{1}{n} \right) G(y_n(s)), s > t, \quad n > \frac{1}{t}. \tag{3.2}
\]

Notice \(Ay_n(s) + 1/n \geq \eta(t), s \in [t, 1]\), then by (3.2),

\[
-y''(s) \leq k(s)F \left( Ay_n(s) + \frac{1}{n} \right) \leq k(s)F[\eta(t)], s \in [t, 1], \quad \left( n \geq \frac{1}{t} \right). \tag{3.3}
\]

Integrating (3.3) from \(t\) to 1, we have

\[
\int_{1/n}^{y_n(t)} \frac{dy}{G(y)} \leq \int_t^1 k(s) \, ds F[\eta(t)] < +\infty. \tag{3.4}
\]

Equation (3.4) and \((H_3)\) imply that \(\{y_n(t)\}\) is bounded, i.e., \(W(t) < +\infty\).

For fixed \([a, b] \subseteq (0, 1)\), we obtain by (3.1) and (3.2) \((n \geq 1/a)\):

\[
-y''(t) \leq k(t)F(\eta(a)) \sup G([w(b), +\infty)) \in L[a, b]. \tag{3.5}
\]

Equation (3.5) and the absolute continuity of Lebesgue integral show that \(\{y_n(t)\}\) is equicontinuous on \([a, b]\). Arzela-Ascoli theorem guarantees that there exists a subsequence of \(\{y_n(t)\}\), which converges uniformly on \([a, b]\). Put \(a = 1/3k, b = 1 - 1/3k, k\) is a natural number. When \(k = 1\), there exists a subsequence \(\{y_{(1)}(t)\}\) of \(\{y_n(t)\}\), which converges uniformly on \([1/3, 2/3]\). When \(k = 2\), there exists a subsequence \(\{y_{(2)}(t)\}\) of \(\{y_{(1)}(t)\}\), which converges uniformly on \([1/6, 5/6]\). In general, there exists a subsequence \(\{y_{(k+1)}(t)\}\) of \(\{y_{(k)}(t)\}\), which converges uniformly on \([1/3(k+1), 1 - (1/3(k+1))]\). Then the diagonal sequence \(\{y_{(k)}(t)\}\) converges everywhere in \((0, 1)\) and it is easy to verify that \(\{y_{(k)}(t)\}\) converges uniformly on any interval \([c, d] \subseteq (0, 1)\). Without loss of generality, let \(\{y_{(k)}(t)\}\) be itself of \(\{y_n(t)\}\) in the rest. Put \(y(t) = \lim_{n \to +\infty} y_n(t), t \in (0, 1]\), then \(y(t)\) is continuous and decreasing in \((0, 1]\).

(2) \(y(t)\) is a positive solution of (2.1). First, we prove

\[
\lim_{h \to 0^+} \sup \left\{ \int_0^h y_n(s) \, ds \right\} = 0. \tag{3.6}
\]
From (2.3), we have: 
\[ y_n(t) - y_n(t/2) \leq \int_{t/2}^{1/2} f(s, A_{y_n}(s) + 1/n, y_n(s)) \, ds \leq \int_{t/2}^{1/2} k(s) F(A_{y_n}(s) + 1/n) G(y_n(s)) \, ds, \quad t \in [0, 1/2] \quad (n > 2). \]
Notice \( t \leq s \), \( A_{y_n}(s) = \int_0^s y_n(\mu) \, d\mu \geq \int_0^t y_n(\mu) \, d\mu = A_{y_n}(t) \), and the decrease in \( F(x) \), we have 
\[ y_n(t) \leq \int_{t/2}^{1/2} k(s) F(A_{y_n}(t)) \sup G([y(1/2), +\infty)) + \sup \{y_n(t)\}, \quad t \in (0, 1/2]. \]
Hence, 
\[ \frac{y_n(t)}{F(A_{y_n}(t)) + 1} \leq \int_{t/2}^{1/2} k(s) ds \sup G \left( \left\lfloor \frac{1}{2}, +\infty \right\rfloor \right) + W \left( \frac{1}{2} \right) \frac{1}{F(A_{y_n}(t)) + 1}. \] (3.7)

Put \( A_{y_n}(t) = \mu \), we have from (3.7) 
\[ \int_0^{A_{y_n}(t)} \frac{d\mu}{F(\mu) + 1} \leq \int_0^t \int_{t/2}^{1/2} k(s) ds \sup G \left( \left\lfloor \frac{1}{2}, +\infty \right\rfloor \right) + W \left( \frac{1}{2} \right) \frac{1}{h}. \] (3.8)
Since \( \int_0^1 k(s) ds \, dt = \int_0^1 sk(s) ds \), i.e., \( \int_1^s k(s) ds \in L[0, 1] \), (3.8) and (H1) lead that \( \{A_{y_n}(1/2)\} \) is bounded (consequently, \( A_{y_n}(1) \) is bounded from the decrease in \( y_n(t) \)).

Put \( C(h) = \sup \{\int_0^1 y_n(s) \, ds\} \), then \( C(h) \) is increasing in \((0, 1/2]\). If (3.6) is false, i.e., there exists \( c > 0 \) such that \( C(0^+ \geq c). \) For \( h = 1/k \), \( k \) is a natural number, there exists \( n_k \) such that \( \int_0^1 k(s) ds \in L[0, 1] \) and \( \{A_{y_n}(1/2)\} \) is bounded by \( c/2 \). Putting \( h = 1/k \), \( n = n_k \) in (3.8) and letting \( k \rightarrow +\infty \), we have 
\[ \int_0^{A_{y_n}(1/2)} \frac{d\mu}{F(\mu) + 1} = 0. \] This leads to a contradiction.

Since \( y_n(t) \) converges uniformly on \([a, b] \in (0, 1)\), (3.6) lead that \( \{A_{y_n}(s)\} \) converges to \( A_y(s) \), for any \( s \in (0, 1) \) and \( y(s) \in L[0, 1] \). Fixed \( t \in (0, 1) \), choose \( \gamma (1/2, 1) \) such that \( t < \gamma \). \( \{f(s, A_{y_n}(s) + 1/n, y_n(s))\} \) converges to \( f(s, A_y(s), y(s)) \), \( s \in [t, \gamma] \). By (2.3), we have \( (n \geq 1/t) \)
\[ y_n(t) - y_n(\gamma) = \int_t^\gamma f \left( s, A_{y_n}(s), \frac{1}{n}, y_n(s) \right) \, ds, \quad s \in [t, \gamma]. \] (3.9)
\( w(\gamma) \leq A_{y_n}(s), w(\gamma) \leq y_n(s), s \in [t, \gamma], \{A_{y_n}(s)\}, \) and \( \{y_n(s)\} \) are bounded on \([t, \gamma]\), the equicontinuity of \( \{y_n(s)\}, \{A_{y_n}(s)\} \) on \([t-\gamma] \) lead 
\[ y(t) - y(\gamma) = \int_t^\gamma f(s, A_y(s), y(s)) \, ds. \] (3.10)
Putting \( t = \gamma \) in (3.4) and letting \( n \rightarrow +\infty \) in (3.4), we have 
\[ \int_0^{y(\gamma)} \frac{dy}{G(y)} \leq \int_0^1 \int_t^\gamma k(s) ds \, dt \sup \left[ \eta \left( \frac{1}{2} \right) \right]. \] (3.11)
Equation (3.11) leads \( y(1^-) = \lim_{n \rightarrow -1} y(\gamma) = 0 \). Letting \( \gamma \rightarrow 1^- \) in (3.10), we have 
\[ y(t) = \int_t^1 f(s, A_y(s), y(s)) \, ds, \quad t \in (0, 1]. \]
Hence, \( x(t) = \int_t^1 y(s) \, ds \) is a positive solution of (1.1).

Our proof is completed.

**Example 3.1.** For the following boundary value problem:
\[ x'' + t^{-\alpha} (1 - t)^{-\beta} \left[ x^{-\gamma} + \cos x^{-\delta} \left( (x')^{-\sigma} + (x')^{-\rho} \right) \right] = 0, \quad 0 < t < 1, \]
\[ x(0) = x(1) = 0. \] (3.12)
Where \( 1 < \alpha < 2, 0 < \beta < 1, \gamma > 0, \delta > 0, \sigma > 0, \mu > 0. \)

Put \( f(t, x, y) = t^{-\alpha} (1 - t)^{-\beta} \left[ x^{-\gamma} + |\cos x^{-\delta} (x')^{-\sigma} + (x')^{-\rho}| \right], k(t) = t^{-\alpha} (1 - t)^{-\beta}, F(x) = 1 + x^{-\gamma}, G(y) = 1 + y^{-\sigma} + y^{-\rho}, \) then \( f(t, x, y) \leq k(t) F(x) G(y), \) \( tk(t) = t^{-\alpha+1} (1 - t)^{-\beta} \in L[0, 1]. \)
By Theorem 3.1, (3.12) has a positive solution at least.

**For initial value problems (1.2), we suppose**
\( (H_3) \lim_{t \rightarrow 0^+} k(t) < +\infty \) and \( \int_0^1 F(x) \, dx < +\infty. \)
THEOREM 3.2. Let \((H_1)-(H_3)\) hold and \((1-t)k(t) \in L[0,1]\), then the initial value problem (1.2) has a positive solution at least.

PROOF. In this section, \(y_n(t)\) satisfies (2.4) in Lemma 2.1.

\(1\) \(\{y_n(t)\}\) is equicontinuous on any \([a,b] \subseteq (0,1)\).

Put \(w(t) = \inf\{y_n(t)\}\), \(W(t) = \inf\{y_n(t)\}\), \(\eta(t) = \inf\{\int_0^t y_n(s) \, ds\}, t \in [0,1]\). First, we prove

\[0 < w(t), \quad 0 < \eta(t), \quad W(t) < +\infty, \quad t \in (0,1).\] (3.13)

In fact, if there exists \(t_0 \in (0,1)\) such that \(w(t_0) = 0\), then there exists \(\{y_{n_k}(t_0)\} \subseteq \{y_n(t_0)\}\), \(\{y_{n_k}(t_0)\}\) converges to 0. The increase in \(y_n(t)\) implies that \(\{y_{n_k}(t)\}\) converges uniformly to 0, \(t \in [0,t_0]\). By (1.3) there exists \(\delta > 0\), when \(0 < y \leq \delta, \, f(.,., y) \geq 1\). Choose a natural number \(N\), when \(n \geq N, \, y_{n_k}(t) \leq \delta, \, t \in [0,t_0]\). From (2.4), we obtain \(\{y_{n_k}(t_0)\} : \, y_{n_k}(t_0) \geq \int_0^1 \delta \, ds = 1 - t_0 > 0\). This is a contradiction. \(\eta(t) > 0, \, t \in (0,1)\) by \(w(t) > 0, \, t \in (0,1)\). Clearly, \(w(t)\) is an increasing function. So is \(\eta(t)\).

If there exists \(t_0 \in (0,1)\) such that \(W(t_0) = +\infty\), then there exists \(\{y_{n_k}(t_0)\} \subseteq \{y_n(t_0)\}\) such that \(y_{n_k}(t_0) \to +\infty\). Without loss of generality, let \(\{y_{n_k}(t_0)\}\) be itself of \(\{y_n(t_0)\}\). Put \(d_0 = \sup\{k(t) : 0 < t \leq t_0\}\), then \(d_0 < +\infty\) by \((H_3)\). By (2.4), we have

\[y'_n(t) = f \left( t, Ay_n(t) + \frac{1}{n}, y_n(t) \right) \leq k(t)F(Ay_n(t))G(y_n(t)), \quad t \leq t_0, \quad n \geq \frac{1}{1-t_0}.\] (3.14)

By (3.14), we have

\[
\frac{y'_n(t_0)y_n(t_0)}{G(y_n(t_0))} \leq d_0F(Ay_n(t_0))y_n(t_0), \quad t \leq t_0, \quad n \geq \frac{1}{1-t_0}.\] (3.15)

Integrating (3.15) from 0 to \(t_0\), we have

\[
\int_{y_{n_0}(t_0)}^{y_{n_0}(t_0)} y \, dy \leq d_0 \int_0^{A_{t_0}(t_0)} F(x) \, dx \leq d_0 \left[ \int_0^{1+A_{t_0}(t_0)} F(x) \, dx \right].\] (3.16)

By (3.16), we obtain (when \(y_{n_0}(t_0) \geq 1\))

\[
\frac{y_{n_0}^2(t_0) - 1}{2 \sup G([1, +\infty))} \leq d_0 \left[ \int_0^{1} F(x) \, dx + \int_1^{1+A_{t_0}(t_0)} F(x) \, dx \right].\] (3.17)

Notice \(Ay_n(t_0) \leq y_{n_0}(t_0)\), (3.17) implies

\[
\frac{y_{n_0}^2(t_0) - 1}{2 \sup G([1, +\infty))} \leq d_0 \left[ \int_0^{1} F(x) \, dx + F(1)Ay_n(t_0) \right] \leq d_0 \left[ \int_0^{1} F(x) \, dx + F(1)y_{n_0}(t_0) \right].\] (3.18)

Letting \(n \to +\infty\) in (3.18), we have \(1/2 \sup G([1, +\infty)) = 0\). It leads to a contradiction.

For fixed \([a,b] \subseteq (0,1), t \in [a,b]\), by (2.4) and (3.13), we have \((n \geq 1/(1-b))\),

\[y'_n(t) \leq k(t)F \left( Ay_n(t) + \frac{1}{n}, G(y_n(t)) \right) \leq k(t)F(\eta(a)) sup G([w(a), +\infty)).\] (3.19)

Equation (3.19) and the absolute continuity of Lebesgue integral \((k(t) \in L[a,b])\) show that \(\{y_n(t)\}\) is equiconstant on \([a,b]\). Arzela-Ascoli theorem guarantees that there exists a subsequence of \(\{y_n(t)\}\), which converges uniformly on \([a,b]\). Put \(a = 1/3k, \, b = 1 - 1/3k\), \(k\) is a natural number. Similarly to the proof of Theorem 3.1, there exists the subsequence (the diagonal sequence) of \(\{y_n(t)\}\), which converges everywhere in \((0,1)\) and converges uniformly on any interval \([c,d] \subseteq (0,1)\). Without loss of generality, let the subsequence (the diagonal sequence) be itself of
\{y_n(t)\} in the rest. Put \(y(t) = \lim_{n \to +\infty} y_n(t), \ t \in (0,1)\), then \(y(t)\) is continuous and increasing in \((0,1)\).

(2) \(y(t)\) is a positive solution of (2.2).

Fixed \(t \in (0,1)\), choose \(\gamma \in (0,1)\) such that \(\gamma < t\). Clearly, \(\{f(s, Ay_n(s) + 1/n, y_n(s))\}\) converges to \(f(s, Ay(s), y(s))\), \(s \in [\gamma, t]\). By (2.3), we have \((n \geq 1/(1-t))\)

\[
y_n(t) - y_n(\gamma) = \int_{\gamma}^{t} f \left( s, Ay_n(s) + \frac{1}{n}, y_n(s) \right) \, ds,
\]

\((3.20)\)

\(w(\gamma) \leq y_n(s), \ \eta(\gamma) \leq Ay_n(s), \ s \in [\gamma, t]\), the equicontinuity of \(\{y_n(s)\}\) on \([\gamma, t]\), \(Ay_n(s)\) converges to \(Ay(s) (0 \leq s < 1)\) and (3.20) lead

\[
y(t) - y(\gamma) = \int_{\gamma}^{t} f(s, Ay(s), y(s)) \, ds.
\]

\((3.21)\)

We prove

\[
\int_{0}^{\gamma + \epsilon} \frac{d\mu}{F[Ay(y^{-1}(\mu))]G(\mu)} \leq \int_{0}^{\gamma + \epsilon} k(t) \, dt, \quad z \in \left(0, \frac{1}{4}\right).
\]

\((3.22)\)

which leads \(y(0^+)=\lim_{n \to +\infty} y(z) = 0\).

From (2.4), \(y_n(t)\) is strictly increasing in \([0,1/2]\), \(y(t)\) is continuous and strictly increasing in \((0,1/2]\) from (3.21). Let \(y_n^{-1}(w)\) be an inverse function of \(y_n(t) (t \in (0,1/2])\). It is easy to verify that \(\{y_n^{-1}(w)\}\) converges to \(y^{-1}(w)\) in \((y(0^+), y(1/2)]\). Where \(y^{-1}(w)\) is an inverse function of \(y(t) (t \in (0,1/2])\). (In fact, if there exists \(w \in (y(0^+), y(1/2)\), \(c > 0\) and an infinite sequence \(\{n_k\}\) such that \(|y_n^{-1}(w) - y^{-1}(w)| \geq \epsilon\), i.e., \(y_n^{-1}(w) \leq y^{-1}(w) - \epsilon\) or \(y_n^{-1}(w) \geq y^{-1}(w) + \epsilon\). Then \(w \leq y_n(y_n^{-1}(w) - \epsilon)\) or \(w \geq y_n(y_n^{-1}(w) + \epsilon)\). Letting \(n_k \to +\infty\), we obtain \(w \leq y(y^{-1}(w) - \epsilon) < y(y^{-1}(w)) = w\) or \(w \geq y(y^{-1}(w) + \epsilon) > y(y^{-1}(w)) = w\), which lead to contradictions.)

By (2.4) and the assumptions, we have \((n > 2)\): \(y_n(t)/F(Ay_n(t) + (1/n))G(y_n(t)) \leq k(t), t \in [0,1/2]\), then \(\int_{0}^{z} \frac{dy(t) \, dt}{F(Ay(t) + (1/n))G(y(t))} \leq \int_{0}^{z} k(t) \, dt, \ z \in (0,1/4]\). Put \(y_n(t) - 1/n = \mu\), we obtain

\[
\int_{0}^{\gamma + \epsilon} \frac{d\mu}{F[Ay_n(y_n^{-1}(\mu) + 1/n)]G(\mu + 1/n)} \leq \int_{0}^{\gamma + \epsilon} k(t) \, dt.
\]

\((3.23)\)

For fixed \(z \in (0,1/4]\), since \(\{y_n(z)\}\) converges to \(y(z)\), there exists \(N_1 > 0\) such that \((1/2) y(z) \leq y_n(z) \leq (3/2) y(z), n > N_1\). When \(y(z) \leq \mu < y_n(z) - 1/n\) or \(y_n(z) - 1/n \leq \mu \leq y(z), (1/2) y(z) \leq \mu + 1/n \leq (3/2) y(z) + 1\) holds, \(n \geq N_1\).

On the other hand, \(\{y_n^{-1}(y(z))\}\) converges to \(y^{-1}(y(z)) = z\), there exists \(N_2 > 0\) such that \((1/2) z \leq y_n^{-1}(y(z)) \leq (3/2) z, n \geq N_2\). When \(y(z) \leq \mu \leq y_n(z) - 1/n, z/2 \leq y_n^{-1}(\mu + 1/n) \leq z\) holds, \(n \geq N_2\). When \(y_n(z) - 1/n \leq \mu \leq y(z), y(z) - y_n^{-1}((\mu + 1/n) \leq y_n(z) + 1/n)\) holds, \(n \geq N_2\). Since \(z < 1/3\), we choose \(N > 0\) such that \(y(z) + 1/n \leq y(1/3), n \geq N\). \(\{y_n^{-1}(y(1/3))\}\) converges to \(1/3\), there must be \(N_3 > N\) and constant \(0 < c < 1/2\) such that \(z/2 \leq y_n^{-1}(\mu + 1/n) \leq c, n \geq N_3\geq N_2\).

Notice \(Ay_n(y_n^{-1}(\mu + 1/n)) = \int_{0}^{y_n^{-1}(\mu + 1/n)} y_n(s) \, ds \geq Aw(y_n^{-1}(\mu + 1/n))\), then when \(y(z) \leq \mu \leq y_n(z) - 1/n\) or \(y_n(z) - 1/n \leq \mu \leq y(z), n \geq N_3\),

\[
Aw \left(\frac{z}{2}\right) \leq Ay_n \left(y_n^{-1} \left(\mu + \frac{1}{n}\right)\right) \leq Ay_n(c) \leq W(c) < +\infty.
\]

\((3.24)\)

From (3.24), we know

\[
(n \geq \max\{N_1, N_3\}): \int_{0}^{y_n(z) - 1/n} \frac{dy}{F[Ay_n(y_n^{-1}(\mu + 1/n)) + 1/n]G(\mu + 1/n)} \leq \frac{|y_n(z) + 1/n - y(z)|}{\min \{FW((1/2) y(z), W(c) + 1)\} min \{G((1/2) y(z), (3/2) y(z) + 1)\}} \quad (\to 0, n \to +\infty).
\]
For fixed and sufficiently small \(\eta > 0\), when \(n > 1/\eta\), \(0 \leq y_n^{-1}(\mu + 1/n) - y_n^{-1}(\mu) \leq y_n^{-1}(\mu + \eta) - y_n^{-1}(\mu)\), we have \(\lim(y_n^{-1}(\mu + 1/n) - y_n^{-1}(\mu)) \leq y^{-1}(\mu + \eta) - y^{-1}(\mu)\). Letting \(\eta \to 0\), \(\lim(y_n^{-1}(\mu + 1/n) - y_n^{-1}(\mu)) = 0\). Notice

\[
\lim A_{y_n}(y_n^{-1}(\mu + \frac{1}{n})) = \lim \left( \int_0^{y_n^{-1}(\mu + 1/n)} y_n(s) \, ds \right)
= \lim \left( \int_0^{y_n^{-1}(\mu)} y_n(s) \, ds + \int_{y_n^{-1}(\mu)}^{y_n^{-1}(\mu + 1/n)} y_n(s) \, ds \right)
= \lim \int_0^{y_n^{-1}(\mu)} y_n(s) \, ds,
\mu \in \left( y(0^+), y \left( \frac{1}{2} \right) \right).
\]

By (3.23) and the Fatou's theorem of Lebesgue integral, we have

\[
\int_0^z k(t) \, dt \geq \lim \int_0^{y_n(z)-1/n} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)}
= \lim \left( \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)}
+ \int_{y_n^{-1}(\mu)}^{y_n(z)-1/n} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)} \right)
\geq \lim \left( \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)}
+ \int_{y_n^{-1}(\mu)}^{y_n(z)-1/n} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)} \right)
= \lim \left( \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)}
- \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)} \right)
\geq \lim \left( \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)}
- \int_0^{y_n^{-1}(\mu)} \frac{d\mu}{F \left[ A_{y_n} \left( y_n^{-1}(\mu + 1/n) \right) + 1/n \right] G(\mu + 1/n)} \right).
\]

Letting \(\gamma \to 0^+\) in (3.21), we have

\[
y(t) - \int_0^t f(\varepsilon, A_{y}(s), y(s)) \, ds, \quad t \in [0, 1].
\]

From (3.25), we have

\[
y(t) - y \left( \frac{1}{2} \right) = \int_{1/2}^t f(\varepsilon, A_{y}(s), y(s)) \, ds, \quad t \in \left[ \frac{1}{2}, 1 \right].
\]

For \(t \in [1/2, 1]\), \(A_{y}(t) \geq \eta(1/2)\), we obtain from (3.26),

\[
\int_{1/2}^1 y(t) \, dt = \int_{1/2}^t \int_{1/2}^t k(s) f(A_{y}(s), y(s)) \, ds \, dt + \frac{1}{2} y \left( \frac{1}{2} \right)
\leq \int_{1/2}^t \int_{1/2}^t k(s) \, ds \, dt F \left( \eta \left( \frac{1}{2} \right) \sup C \left( y \left( \frac{1}{2} \right), 1, \infty \right) \right)
< +\infty \left( \int_{1/2}^1 k(s) \, ds = \int_{1/2}^1 k(s) \, dt \, ds = \int_{1/2}^1 (1 - s)k(s) \, ds \right).
The increase in \( y(t) \) leads \( y(t) \in L[0, 1] \). Hence, \( x(t) = \int_0^t y(s) \, ds \) is a positive solution.

**Example 3.2.** For the following initial value problem:

\[
x'' = (1 - t)^{-\beta} \left[ x^{-\gamma} + \left| \cos x^{-\delta} \right| (x')^{-\sigma} + (x')^{-\mu} \right], \quad 0 < t < 1,
\]

\[
x(0) = x'(0) = 0.
\]

Where \( 1 < \beta < 2, 0 < \gamma < 1, \delta > 0, \sigma > 0, \mu > 0 \).

Put \( f(t, x, y) = (1 - t)^{-\beta} \left[ x^{-\gamma} + \left| \cos x^{-\delta} \right| y^{-\sigma} + y^{-\mu} \right], k(t) = (1 - t)^{-\beta}, F(x) = 1 + x^{-\gamma}, G(y) = 1 + y^{-\sigma} + y^{-\mu} \), then \( f(t, x, y) \leq k(t)F(x)G(y), F(x) \in L[0, 1], (1 - t)k(t) = (1 - t)^{-\beta+1} \in L[0, 1] \).

By Theorem 3.2, (3.27) has a positive solution at least.

**Remark 3.1.** In Theorems 3.1 and 3.2, the approximate solutions \( \{y_n(t)\} \) may not be unbounded. Neither the Arzela-Ascoli theorem nor the Helly selection principle (see, e.g., [10]) can be directly applied to the singular problem. Hence, the theorems (see, e.g., [7,11]) cannot be applied to Examples 3.1 and 3.2. The examples [7] show that the singularities at \( x = 0 \) and \( x' = 0 \) is not higher order, the singularities [11] at \( x = 0 \) is restricted to be bounded.

**References**

10. G.C. Yang et al., On the solvability of \( z'' + p(t)f(x) + q(t)g(x') = 0 \) of singular type, *Journal of Sichuan University* 38, 630–634, (2001).