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A Remark on Nets of Threshold Elements

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The necessary condition for transition functions of Simple McCulloch-Pitts Nets, given in the article: "Nets of Threshold Elements" by Kenneth Krohn and John Rhodes in Corollary 3.1. (iii) is enlarged to a sufficient criterion by addition of another necessary condition.

The symbols, designations and definitions are the same as those in Krohn and Rhodes (1965); that article contains complete algebraic characterizations of general and some special nets of threshold elements and of the transition functions associated with them. In Corollary 3.1. (iii) the authors give a criterion for transition functions of Simple McCulloch-Pitts Nets (SMPN) which is necessary but not sufficient. This paper intends to bring a sufficient criterion by addition of another necessary condition.

An example may show that equation (3.1) is not sufficient. Let $N = \{a, b, c\}$, $a \neq b$, $a \neq c$, $b \neq c$, and the map $f: \mathcal{S}(N) \to \mathcal{S}(N)$ be defined by

$$\begin{array}{ll} f(\emptyset) = \emptyset & f(\{a, b\}) = \{a\} \\ f(\{a\}) = \{a, b\} & f(\{a, c\}) = \{b, c\} \\ f(\{b\}) = \{a, c\} & f(\{b, c\}) = \{a, b, c\} \\ f(\{c\}) = \{b, c\} & f(\{a, b, c\}) = \{a, b, c\} \end{array}$$

That f satisfies equation (3.1) can easily be seen. Suppose $f = f_M$ and $M = (N, T, W) \in \text{SMPN}$. Then must be $W(b, c) = 1$ and $W(a, c) + W(b, c) < 1$, since $c \in f(\{b\})$ and $c \notin f(\{a, b\})$. Therefore follows $W(a, c) = -\omega$. But we have also $c \in f(\{c\})$ and $c \in f(\{a, c\})$ and thus we get $W(c, c) = 1$ and $W(a, c) + W(c, c) \geqq 1$, which means $W(a, c) \neq -\omega$. This is a contradiction and shows that the assumption is wrong.

In order to define uniquely the weight function W, $f: \mathcal{S}(N) \to \mathcal{S}(N)$ must satisfy equation (3.1) and the condition (3.1a), too, as we will now demonstrate.

COROLLARY 3.1.(iii). *Let N be a finite nonempty set and let $f: \mathcal{S}(N) \rightarrow \mathcal{S}(N)$. Then $f = f_M$, for M a Simple McCulloch-Pitts Net, iff for each $A \epsilon \mathcal{S}(N)$*

$$f(A) = \bigcup_{a \epsilon A} \bigcap_{b \epsilon A} f(\{a, b\}) = \bigcap_{a \epsilon A} \bigcup_{b \epsilon A} f(\{a, b\}), \tag{3.1}$$

and for each $n, n', n'' \epsilon N$

$$f(\{n'\}) \cap f(\{n'', n\}) \subseteq f(\{n', n\}). \tag{3.1a}$$

Proof. Let $f = f_M$ and $M = (N, T, W) \epsilon$SMPN. We show that there holds now (3.1a) in addition to equation (3.1). For $n = n'$ or $n = n''$ or $n' = n''$ (3.1a) is trivial. Let now $n \neq n'$, $n \neq n''$, $n' \neq n''$, and let $a \epsilon f(\{n'\}) \cap f(\{n'', n\})$, i.e. $W(n', a) = 1$ and $W(n'', a) + W(n, a) \geqq 1$. Then $W(n, a) \geqq 0$, $W(n', a) + W(n, a) \geqq 1$ and thus $n \epsilon f(\{n', n\})$, i.e. (3.1a).

Now f satisfy the conditions (3.1) and (3.1a). We show then that $f = f_M$, for M a Simple McCulloch-Pitts Net. Define $M = (N, T, W) \epsilon$SMPN as follows. For all $n \epsilon N$ let $T(n) = 1$. For $(a, n) \epsilon N \times N$ let

$$W(a, n) = \begin{cases} 1, \text{ if } n \epsilon f(\{a\}) \\ 0, \text{ if } n \epsilon f(\{n', a\}) - f(\{a\}) \text{ for some } n' \epsilon N \\ -\omega, \text{ if } n \epsilon f(\{n''\}) - f(\{n'', a\}) \text{ for some } n'' \epsilon N \\ 0 \text{ or } -\omega \text{ (either one of these two), if } n \notin f(\{n_j\}) \text{ for all } n_j \epsilon N. \end{cases}$$

Let us say W is defined for all $(a, n) \epsilon N \times N$: if $n \notin f(\{a\})$, but $n \epsilon f(\{n'\})$ for some $n' \epsilon N$, then either $n \epsilon f(\{n', a\})$ [and therefore $n \epsilon f(\{n', a\}) - f(\{a\})$], or $n \notin f(\{n', a\})$, and therefore $n \epsilon f(\{n'\}) - f(\{n', a\})$.

W is also uniquely defined: if $n \epsilon f(\{n', a\}) - f(\{a\})$ then $n \notin f(\{a\})$, but $n \epsilon f(\{n'\})$ because of (3.1). If $n \epsilon f(\{n''\}) - f(\{n'', a\})$ then $n \epsilon f(\{n''\})$, but $n \notin f(\{a\})$ because of (3.1). Suppose $n \epsilon f(\{n', a\}) - f(\{a\})$ for some $n' \epsilon N$ and $n \epsilon f(\{n''\}) - f(\{n'', a\})$ for some $n'' \epsilon N$, then $n \epsilon f(\{n''\}) \cap f(\{n', a\})$ and we get also $n \epsilon f(\{n'', a\})$ because of (3.1a). This is a contradiction.

Thus we see that $W: N \times N \rightarrow \{0, 1, -\omega\}$ is well defined.

Now we show $f = f_M$. It is clear that $f(\emptyset) = f_M(\emptyset) = \emptyset$ and $f(\{a\}) = f_M(\{a\})$ for all $a \epsilon N$. Let a and b be two elements of N and $a \neq b$. Let $n \epsilon f(\{a, b\})$. Then either $n \epsilon f(\{a\})$ or $n \epsilon f(\{b\})$. We assume $n \epsilon f(\{a\})$, i.e. $W(a, n) = 1$. If $n \epsilon f(\{b\})$, then $W(b, n) = 1$. If $n \notin f(\{b\})$, then $n \epsilon f(\{a, b\}) - f(\{b\})$ and consequently $W(b, n) = 0$. Therefore we have

$W(b, n) \geqq 0$ in all cases. It follows $W(a, n) + W(b, n) \geqq 1$, i.e. $n \epsilon f_M(\{a, b\})$ and so $f(\{a, b\}) \subseteq f_M(\{a, b\})$.

Now let $n \epsilon f_M(\{a, b\})$, which means $W(a, n) + W(b, n) \geqq 1$. If $W(a, n) = W(b, n) = 1$, then $n \epsilon f(\{a\}) \cap f(\{b\})$ and therefore $n \epsilon f(\{a, b\})$. If $W(a, n) = 1$ and $W(b, n) = 0$, then because of $n \epsilon f(\{a\})$ we get $n \epsilon f(\{n', b\}) - f(\{b\})$ for some $n' \epsilon N$ (see definition of W). As a consequence of (3.1a) we see that $n \epsilon f(\{a, b\})$ and thus $f_M(\{a, b\}) \subseteq f(\{a, b\})$. Therefore we have shown: $f(\{a, b\}) = f_M(\{a, b\})$.

Thus we get $f = f_M$, since for any $A \epsilon \mathcal{S}(N)$ we have

$$f(A) = \bigcup_{a \epsilon A} \bigcap_{b \epsilon A} f(\{a, b\}) = \bigcup_{a \epsilon A} \bigcap_{b \epsilon A} f_M(\{a, b\}) = f_M(A).$$

This proves Corollary 3.1.(iii).

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REFERENCE

KROHN, K. AND RHODES, J. (1965), Nets of threshold elements. *Inform. Control* **8**, 579–588.