



# E-theory for $C^*$ -algebras over topological spaces <sup>☆</sup>

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## Abstract

We define E-theory for separable  $C^*$ -algebras over second countable topological spaces and establish its basic properties. This includes an approximation theorem that relates the E-theory over a general space to the E-theories over finite approximations to this space. We obtain effective criteria for determining the invertibility of E-theory elements over possibly infinite-dimensional spaces. Furthermore, we prove a Universal Multicoefficient Theorem for  $C^*$ -algebras over totally disconnected metrisable compact spaces. © 2012 Elsevier Inc. All rights reserved.

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## 1. Introduction

Eberhard Kirchberg [17] proved a far-reaching classification theorem for non-simple, strongly purely infinite, stable, nuclear, separable  $C^*$ -algebras. Roughly speaking, two such  $C^*$ -algebras are isomorphic once they have homeomorphic primitive ideal spaces – call this space  $X$  – and are  $KK(X)$ -equivalent in a suitable bivariant K-theory for  $C^*$ -algebras over  $X$ . To apply this classification theorem, we need tools to compute this bivariant K-theory. Following Mikael Rørdam [28] and Alexander Bonkat [3], who dealt with the simplest non-trivial case, the non-Hausdorff space

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with two points, Universal Coefficient Theorems for  $KK(X)$  have now been established over several finite spaces  $X$  in [14,22,26,27]. Here we concentrate on the special issues for infinite  $X$ .

Recall that Kasparov theory only satisfies excision for  $C^*$ -algebra extensions with a completely positive section. Similar technical restrictions appear for all variants of Kasparov theory, including Kirchberg’s. This is a severe limitation. For instance, excision does not hold in general for extensions of the form  $A(U) \hookrightarrow A \rightarrow A/A(U)$  for an open subset  $U$ , where  $A(U)$  denotes the restriction of  $A$  to  $U$ , extended by 0 to a  $C^*$ -algebra over the original space, even if  $A$  is nuclear. In the non-equivariant case, such technical problems are resolved by passing to E-theory, which satisfies excision for all  $C^*$ -algebra extensions (see [5]). Here we define an analogue of E-theory for separable  $C^*$ -algebras over a second countable topological space  $X$ . We establish that our new theory has the expected properties, including a universal property and exactness for all extensions of  $C^*$ -algebras over  $X$ . If  $X$  is a locally compact Hausdorff space, then our definitions agree with previous ones by Efton Park and Jody Trout in [24] and by Radu Popescu in [25]. We also formulate sufficient criteria for the natural map  $E_*(X; A, B) \rightarrow KK_*(X; A, B)$  to be invertible. For instance, this works if  $X$  is locally compact and Hausdorff and  $A$  is a continuous field of nuclear  $C^*$ -algebras over  $X$ .

Our definition of  $E_*(X; A, B)$  is based on asymptotic homomorphisms satisfying an approximate equivariance condition. An asymptotic homomorphism  $\varphi_t : A \rightarrow B, t \in [0, \infty)$ , is called *approximately  $X$ -equivariant* if for each open subset  $U \subseteq X$ , we have

$$\lim_{t \rightarrow \infty} \|\varphi_t(a)\|_{X \setminus U} = 0 \quad \text{for all } a \in A(U),$$

where  $\|\varphi_t(a)\|_{X \setminus U}$  denotes the norm of  $\varphi_t(a)$  in the quotient  $B(X \setminus U) := B/B(U)$  of  $B$ .

Let  $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$  be a countable basis for the topology of  $X$ . For each  $n \in \mathbb{N}$ , the open subsets  $U_1, \dots, U_n$  generate a finite topology  $\tau_n$  on  $X$ . Let  $X_n$  be the  $T_0$ -quotient of  $(X, \tau_n)$ , this is a finite  $T_0$ -space. The quotient map  $X \rightarrow X_n$  allows us to view  $C^*$ -algebras over  $X$  as  $C^*$ -algebras over  $X_n$  for all  $n \in \mathbb{N}$ . Our first main result is a short exact sequence

$$\varinjlim_{n \in \mathbb{N}} E_{*+1}(X_n; A, B) \hookrightarrow E_*(X; A, B) \rightarrow \varprojlim_{n \in \mathbb{N}} E_*(X_n; A, B) \tag{1.1}$$

for all separable  $C^*$ -algebras  $A$  and  $B$  over  $X$ . This is made plausible by the observation that an asymptotic homomorphism  $A \rightarrow B$  is approximately  $X$ -equivariant if and only if it is approximately  $X_n$ -equivariant for all  $n \in \mathbb{N}$ . Hence the space of approximately  $X$ -equivariant asymptotic homomorphisms is the intersection of the spaces of approximately  $X_n$ -equivariant asymptotic homomorphisms for  $n \in \mathbb{N}$ . Since there are, in general, technical problems with computing homotopy groups of intersections, we use a mapping telescope to establish the long exact sequence (1.1).

As an important application of (1.1), we give an effective criterion for invertibility of E-theory elements: an element in  $E_*(X; A, B)$  is invertible if and only if its image in  $E_*(A(U), B(U))$  is invertible for all  $U \in \mathcal{O}(X)$ . As a consequence, if all two-sided closed ideals of a separable nuclear  $C^*$ -algebra  $A$  with Hausdorff primitive spectrum  $X$  are  $KK$ -contractible, then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

This result solves the problem of characterising the trivial continuous fields with fibre  $\mathcal{O}_2 \otimes \mathbb{K}$  within the class of strongly purely infinite, stable, continuous fields of  $C^*$ -algebras. It is worth noting that in general the  $KK$ -contractibility of ideals does not follow from the  $KK$ -contractibility of the fibres. Indeed, there are examples of separable nuclear continuous fields  $A$  over the Hilbert cube with all fibres isomorphic to  $\mathcal{O}_2$  and yet such that  $K_0(A) \neq 0$ , see [8].

While (1.1), in principle, reduces the computation of  $E_*(X; A, B)$  for infinite spaces  $X$  to the corresponding problem for the finite approximations  $X_n$ , this does not yet lead to a Universal Coefficient Theorem. If  $E_*(X_n; A, B)$  is computable by Universal Coefficient Theorems for all  $n \in \mathbb{N}$ , the latter will usually involve short exact sequences. Thus we have to combine two short exact sequences, as in the computation of the  $K$ -theory for crossed products by  $\mathbb{Z}^2$  using the Pimsner–Voiculescu exact sequence twice. This can only be carried through if we have some extra information. In terms of the general homological machinery developed in [21], we find that the homological dimension of  $E$ -theory over an infinite space  $X$  may be one larger than the homological dimensions of the finite approximations  $X_n$ . Thus it is usually 2, which does not suffice for classification theorems.

In fact, it is well known that filtrated  $K$ -theory cannot be a complete invariant for  $C^*$ -algebras over the one-point compactification of  $\mathbb{N}$ . Here we observe that the counterexample in [10] may be transported easily to any compact Hausdorff space.

The good excision properties of  $E$ -theory are particularly useful to study the  $E$ -theoretic analogue of the bootstrap class. For a finite space  $X$ , the bootstrap class for  $KK(X)$  is studied in [20]. When we replace  $KK(X)$  by  $E(X)$ , the technical assumptions in [20] about completely positive sections disappear, so that a  $C^*$ -algebra  $A$  over a finite space  $X$  belongs to the  $E$ -theoretic bootstrap class over  $X$  if and only if all the distinguished ideals  $A(U)$  for open subsets  $U \subseteq X$  belong to the usual non-equivariant  $E$ -theoretic bootstrap class. As we shall see, the latter criterion becomes a useful *definition* of the bootstrap class over an infinite space  $X$ . In  $KK(X)$ , this condition would not yet be sufficient for a reasonable definition of the bootstrap class.

If  $X$  is the Cantor set or, more generally, a totally disconnected metrisable compact space, then we may resolve the counterexamples mentioned above by taking into account coefficients. Our second main result is a Universal *Multicoefficient* Theorem for  $E_*(X; A, B)$  for two  $C^*$ -algebras  $A$  and  $B$  over  $X$ . It assumes that  $A(U)$  belongs to the  $E$ -theoretic bootstrap class for all open subsets  $U \subseteq X$  and yields a natural exact sequence

$$\text{Ext}_{C(X, \Lambda)}(\underline{K}(A)[1], \underline{K}(B)) \rightarrow E(X; A, B) \rightarrow \text{Hom}_{C(X, \Lambda)}(\underline{K}(A), \underline{K}(B)),$$

where  $\underline{K}$  denotes the  $K$ -theory of  $A$  with coefficients, viewed as a countable module over the  $\mathbb{Z}/2$ -graded ring  $C(X, \Lambda)$  of locally constant functions from  $X$  to the  $\mathbb{Z}/2$ -graded ring  $\Lambda$  of Bökstein operations (see [11]). As a consequence, two  $C^*$ -algebras  $A$  and  $B$  in the  $E$ -theoretic bootstrap class over  $X$  are  $E(X)$ -equivalent if and only if  $\underline{K}(A)$  and  $\underline{K}(B)$  are isomorphic as  $C(X, \Lambda)$ -modules.

## 2. $E$ -theory for $C^*$ -algebras over non-Hausdorff spaces

We recall some definitions from [20] regarding  $C^*$ -algebras over possibly non-Hausdorff topological spaces and then introduce equivariant  $E$ -theory for them. Following the approach of Alain Connes and Nigel Higson in [5], we first describe  $E$ -theory concretely using asymptotic morphisms, then abstractly using a universal property. For a locally compact Hausdorff space  $X$ ,

our definition is equivalent to previous ones for  $C_0(X)$ -algebras by Efton Park and Jody Trout in [24] and by Radu Popescu in [25].

2.1.  $C^*$ -algebras over non-Hausdorff spaces

Here we recall some basic definitions from [20].

For a  $C^*$ -algebra  $A$ , let  $\text{Prim}(A)$  denote its primitive ideal space, equipped with the hull-kernel topology, and let  $\mathbb{I}(A)$  be the set of ideals in  $A$ , partially ordered by inclusion. For a topological space  $X$ , let  $\mathbb{O}(X)$  be the set of open subsets of  $X$ , partially ordered by inclusion. Both  $\mathbb{I}(A)$  and  $\mathbb{O}(X)$  are complete lattices, that is, any subset has both an infimum and a supremum. It is shown in [13, §3.2] that there is a canonical lattice isomorphism

$$\mathbb{O}(\text{Prim}(A)) \cong \mathbb{I}(A), \quad U \mapsto \bigcap \{ \mathfrak{p} : \mathfrak{p} \in \text{Prim}(A) \setminus U \}. \tag{2.1}$$

**Definition 2.2.** Let  $X$  be a topological space.

A  $C^*$ -algebra over  $X$  is a  $C^*$ -algebra  $A$  with a continuous map  $\psi$  from  $\text{Prim}(A)$  to  $X$ .

For an open subset  $U$  of  $X$ , we let  $A(U) \in \mathbb{I}(A)$  be the ideal that corresponds to  $\psi^{-1}(U) \in \mathbb{O}(\text{Prim} A)$  under the isomorphism (2.1).

For a closed subset  $S$  of  $X$ , we let  $A(S) := A/A(X \setminus S)$ . For  $a \in A$ , we write  $\|a\|_S$  for the norm of the image of  $a$  in the quotient  $C^*$ -algebra  $A(S)$ .

More generally, if  $S \subseteq X$  is locally closed, that is,  $S = U \setminus V$  with open subsets  $V \subseteq U \subseteq X$ , then we let  $A(S) := A(U)/A(V)$ . This quotient is independent of the choice of the open sets  $U$  and  $V$  with  $S = U \setminus V$ .

Let  $A$  and  $B$  be  $C^*$ -algebras over  $X$ . A  $*$ -homomorphism  $f : A \rightarrow B$  is called  $X$ -equivariant or a  $*$ -homomorphism over  $X$  if  $f$  maps  $A(U)$  into  $B(U)$  for all open subsets  $U$  of  $X$ .

Let  $\mathcal{C}^*\text{alg}(X)$  be the category whose objects are the  $C^*$ -algebras over  $X$  and whose morphisms are the  $*$ -homomorphisms over  $X$ . Let  $\mathcal{C}^*\text{sep}(X)$  be the full subcategory of separable  $C^*$ -algebras over  $X$  with  $*$ -homomorphisms over  $X$  as morphisms.

We usually drop the map  $\text{Prim}(A) \rightarrow X$  from our notation and simply call  $A$  a  $C^*$ -algebra over  $X$ .

Although the above definition involves  $X$ , all that really matters is the lattice  $\mathbb{O}(X)$ . It is explained in [20] that it is essentially no loss of generality to assume  $X$  to be *sober*. In that case, we may recover  $X$  from the lattice  $\mathbb{O}(X)$  and the map  $\text{Prim}(A) \rightarrow X$  from the map  $\mathbb{O}(X) \rightarrow \mathbb{I}(A)$ ,  $U \mapsto A(U)$  (see [20, Lemma 2.25]), which may be any map that commutes with finite infima and arbitrary suprema. Thus if  $X$  is a second countable, sober space, a  $C^*$ -algebra over  $X$  is a  $C^*$ -algebra  $A$  endowed with an order preserving map  $\mathbb{O}(X) \rightarrow \mathbb{I}(A)$ ,  $U \mapsto A(U)$ , which satisfies the following conditions:

- (1)  $A(\emptyset) = 0, A(X) = A,$
- (2)  $A(U_1 \cap U_2) = \overline{A(U_1) \cdot A(U_2)},$
- (3)  $A(\bigcup_{n=1}^\infty U_n) = \sum_{n=1}^\infty A(U_n).$

If a  $C^*$ -algebra  $A$  satisfies the conditions (1) and (2) and

$$(3') \quad A(U_1 \cup U_2) = A(U_1) + A(U_2),$$

then we say that  $A$  is a *quasi*  $C^*$ -algebra over  $X$ . If  $B$  is a  $C^*$ -algebra over  $X$  then  $C_b(T, B)$  and  $C_b(T, B)/C_0(T, B)$  for  $T := [0, \infty)$  become quasi  $C^*$ -algebras over  $X$ , via the maps  $U \mapsto C_b(T, B(U))$  and

$$U \mapsto C_b(T, B(U)) + C_0(T, B)/C_0(T, B).$$

However, they do not satisfy the condition (3) above.

Let  $X$  be a locally compact Hausdorff space and let  $A$  be a  $C^*$ -algebra over  $X$ . The continuous map  $\text{Prim}(A) \rightarrow X$  induces a  $*$ -homomorphism

$$C_b(X) \rightarrow C_b(\text{Prim}(A)) \cong Z\mathcal{M}(A),$$

where  $Z\mathcal{M}(A)$  denotes the centre of the multiplier algebra of  $A$ . One verifies that  $C_0(X)A$  is dense in  $A$ , so that  $A$  becomes a  $C_0(X)$ - $C^*$ -algebra. This yields an isomorphism of categories between  $C^*\text{-alg}(X)$  and the category of  $C_0(X)$ - $C^*$ -algebras with  $C_0(X)$ -linear  $*$ -homomorphisms as morphisms by [20, Proposition 2.11].

### 2.2. Approximately equivariant asymptotic morphisms

Recall:

**Definition 2.3.** An *asymptotic morphism* between two  $C^*$ -algebras  $A$  and  $B$  is a map  $\varphi : A \rightarrow C_b(T, B)$  for  $T := [0, \infty)$  that induces a  $*$ -homomorphism

$$\dot{\varphi} : A \rightarrow B_\infty := C_b(T, B)/C_0(T, B).$$

The map  $\varphi$  is equivalent to a family of maps  $\varphi_t : A \rightarrow B$  for  $t \in T$  such that  $t \mapsto \varphi_t(a)$  is a bounded continuous function from  $T$  to  $B$  for each  $a \in A$ . Such a family is an asymptotic morphism if and only if

$$\varphi_t(a^* + \lambda b) - \varphi_t(a)^* - \lambda\varphi_t(b) \quad \text{and} \quad \varphi_t(a \cdot b) - \varphi_t(a) \cdot \varphi_t(b)$$

converge to 0 in the norm topology for  $t \rightarrow \infty$  for all  $a, b \in A, \lambda \in \mathbb{C}$ .

Two asymptotic morphisms  $\varphi$  and  $\varphi'$  are called *equivalent* if  $\dot{\varphi} = \dot{\varphi}'$ , that is,  $\varphi_t(a) - \varphi'_t(a)$  converges to 0 for  $t \rightarrow \infty$  for all  $a \in A$ .

**Definition 2.4.** An asymptotic morphism  $\varphi_t : A \rightarrow B$  from  $A$  to  $B$  is called *approximately  $X$ -equivariant* if, for any open subset  $U \subseteq X$ ,

$$\lim_{t \rightarrow \infty} \|\varphi_t(a)\|_{X \setminus U} = 0 \quad \text{for all } a \in A(U). \tag{2.5}$$

Let  $\text{Asymp}(A, B)_X$  be the set of approximately  $X$ -equivariant asymptotic morphisms  $A \rightarrow B$ .

Our definition of  $\text{Asymp}(A, B)_X$  requires  $X$ -equivariance only in the limit, the individual maps  $\varphi_t$  need not be  $X$ -equivariant.

**Remark 2.6.** If  $\varphi$  is equivalent to an approximately  $X$ -equivariant asymptotic morphism, then  $\varphi$  itself is approximately  $X$ -equivariant.

**Lemma 2.7.** An asymptotic morphism  $\varphi$  is approximately  $X$ -equivariant if and only if, for all closed subsets  $S$  of  $X$ ,

$$\limsup_{t \rightarrow \infty} \|\varphi_t(a)\|_S \leq \|a\|_S \quad \text{for all } a \in A.$$

**Proof.** Let  $U := X \setminus S$ . The lim sup-criterion specialises to the definition of  $X$ -equivariance for  $a \in A(U)$ . Conversely, for any  $\varepsilon > 0$  we may split  $a \in A$  as  $a = a_1 + a_2$  with  $a_1 \in A(U)$  and  $\|a_2\| < \|a\|_S + \varepsilon$  and estimate

$$\limsup \|\varphi_t(a)\|_S \leq \limsup \|\varphi_t(a_1)\|_S + \limsup \|\varphi_t(a_2)\|.$$

The  $X$ -equivariance of  $\varphi$  and  $a_1 \in A(U)$  imply  $\lim \|\varphi_t(a_1)\|_S = 0$ , and

$$\limsup \|\varphi_t(a_2)\| = \|\dot{\varphi}(a_2)\| \leq \|a_2\| < \|a\|_S + \varepsilon.$$

Thus  $\limsup \|\varphi_t(a)\|_S < \|a\|_S + \varepsilon$  for all  $\varepsilon > 0$ .  $\square$

Let  $U \in \mathbb{O}(X)$  and  $S := X \setminus U$ . The quotient map  $\pi_S : B \rightarrow B(S)$  induces a map  $\tilde{\pi}_S : C_b(T, B) \rightarrow C_b(T, B(S))$  whose kernel is  $C_b(T, B(U))$ . Condition (2.5) is equivalent to

$$\tilde{\pi}_S \circ \varphi(A(U)) \subseteq C_0(T, B(S)). \tag{2.8}$$

**Lemma 2.9.** An asymptotic morphism  $\varphi$  is approximately  $X$ -equivariant if and only if, for all open subsets  $U$  of  $X$ ,

$$\varphi(A(U)) \subseteq C_b(T, B(U)) + C_0(T, B). \tag{2.10}$$

**Proof.** It is clear that (2.10) implies (2.8). To verify the converse, it suffices to prove

$$(\tilde{\pi}_S)^{-1}(C_0(T, B(S))) = C_b(T, B(U)) + C_0(T, B).$$

The Bartle–Graves Theorem provides a continuous section  $\gamma : B(S) \rightarrow B$  of  $\pi_S$ . Any  $f \in C_b(T, B)$  decomposes as  $f = g + h$  with  $g := f - \gamma \circ \tilde{\pi}_S(f)$  and  $h := \gamma \circ \tilde{\pi}_S(f)$ . We have  $g \in C_b(T, B(U))$  and  $h \in C_0(T, B)$  whenever  $\tilde{\pi}_S(f) \in C_0(T, B(S))$  because  $\gamma$  is continuous.  $\square$

For Hausdorff spaces  $X$ , Park and Trout [24] and Popescu [25] defined an E-theory  $\mathcal{RE}_*(X; A, B)$  for  $C_0(X)$ -algebras based on asymptotic morphisms  $\varphi$  that are *asymptotically  $C_0(X)$ -equivariant* in the sense that  $\varphi(fa) - f\varphi(a) \in C_0(T, B)$  for all  $a \in A$  and  $f \in C_0(X)$ ; equivalently,  $\dot{\varphi} : A \rightarrow B_\infty$  is  $C_0(X)$ -linear.

**Proposition 2.11.** Let  $X$  be a second countable locally compact Hausdorff space and let  $A$  and  $B$  be  $C_0(X)$ -algebras. Then an asymptotic morphism from  $A$  to  $B$  is asymptotically  $C_0(X)$ -equivariant if and only if it is approximately  $X$ -equivariant.

**Proof.** Clearly, an asymptotically  $C_0(X)$ -equivariant asymptotic morphism satisfies (2.10) since  $A(U) = C_0(U)A$  and  $C_0(U)C_b(T, B) \subseteq C_b(T, B(U))$ . Conversely, let  $\varphi$  be approximately  $X$ -equivariant. Let  $B_\infty^X := C_0(X) \cdot B_\infty \subseteq B_\infty$ , this is a  $C_0(X)$ -algebra. We are going to show that  $\hat{\varphi}(C_0(U)A)$  is contained in  $C_0(U) \cdot B_\infty^X = C_0(U) \cdot B_\infty$  for all  $U \in \mathbb{O}(X)$ . This is equivalent to the  $C_0(X)$ -linearity of  $\hat{\varphi} : A \rightarrow B_\infty^X$  by [20, Proposition 2.11].

For any  $f \in C_0(U)$  and any  $\varepsilon > 0$ , there are a relatively compact open subset  $U_\varepsilon \subseteq \bar{U}_\varepsilon \subseteq U$  and  $f_\varepsilon \in C_0(U_\varepsilon)$  with  $\|f - f_\varepsilon\| < \varepsilon$ . Since  $A$  is a  $C_0(X)$ - $C^*$ -algebra, the same approximation applies to all  $a \in A(U) = C_0(U) \cdot A$ . Therefore, it suffices to prove  $\hat{\varphi}(A(U')) \subseteq C_0(U) \cdot B_\infty$  for all relatively compact open subsets  $U'$  of  $U$  with  $\bar{U}' \subseteq U$ .

Since there is a function  $w$  in  $C_0(U)$  with  $w(x) = 1$  for all  $x \in U'$ , we have

$$C_b(T, B(U')) \subseteq w \cdot C_b(T, B) \subseteq C_0(U) \cdot C_b(T, B)$$

for all  $n \in \mathbb{N}$ . Since  $\varphi$  maps  $A(U')$  into  $C_b(T, B(U')) + C_0(T, B)$  by (2.10),  $\hat{\varphi}$  maps  $A(U')$  into  $C_0(U) \cdot B_\infty$  for all  $n \in \mathbb{N}$ .  $\square$

### 2.3. Homotopy of asymptotic morphisms

**Definition 2.12.** A *homotopy* of asymptotic morphisms from  $A$  to  $B$  is an asymptotic morphism from  $A$  to  $C([0, 1], B)$ . Let  $\llbracket A, B \rrbracket_X$  denote the set of homotopy classes of approximately  $X$ -equivariant asymptotic morphisms from  $A$  to  $B$ .

Equivalent asymptotic morphisms are homotopic.

We do not know whether there is a natural topology on  $\text{Asymp}(A, B)_X$  such that  $\llbracket A, B \rrbracket_X = \pi_0(\text{Asymp}(A, B)_X)$ . It is easy to avoid this question by using quasi-topological spaces in the sense of Edwin H. Spanier (see [30]).

**Definition 2.13.** A *quasi-topological space* is a set  $W$  together with distinguished sets of maps  $C(Y, W)$  from  $Y$  to  $W$  for each compact Hausdorff space  $Y$ , called *quasi-continuous maps*  $Y \rightarrow W$ . These quasi-continuous maps are required to satisfy the following conditions:

- constant maps are quasi-continuous;
- a function defined on a disjoint union  $Y_1 \sqcup Y_2$  is quasi-continuous if and only if its restrictions to  $Y_1$  and  $Y_2$  are quasi-continuous;
- if  $f : Y_1 \rightarrow Y_2$  is a quasi-continuous map and  $h : Y_2 \rightarrow W$  is quasi-continuous, so is  $h \circ f$ ; and, conversely,
- if  $f$  is surjective and continuous (so that  $f$  is an open surjection), then  $h$  is quasi-continuous provided  $h \circ f$  is quasi-continuous.

Since  $W$  is the set of quasi-continuous functions from the one-point space to  $W$ , we may also view a quasi-topological space as a contravariant functor from the category of compact Hausdorff spaces to the category of sets with some additional properties.

We define a quasi-topology on  $\text{Asymp}(A, B)_X$  by letting

$$C(Y, \text{Asymp}(A, B)_X) := \text{Asymp}(A, C(Y, B))_X$$

for each compact Hausdorff space  $Y$ .

Furthermore,  $\text{Asymp}(A, B)_X$  has a canonical base point, the zero map. Thus  $\text{Asymp}(A, B)_X$  becomes a pointed quasi-topological space.

Homotopy groups for pointed quasi-topological spaces may be defined as for ordinary topological spaces, using quasi-continuous maps instead of continuous maps. By definition,  $\llbracket A, B \rrbracket_X = \pi_0(\text{Asymp}(A, B)_X)$ .

2.4. E-theory: Definition and universal property

The original approach of Alain Connes and Nigel Higson in [5] only works well for separable  $C^*$ -algebras. The same restriction applies to our equivariant generalisation. Hence we (tacitly) assume all  $C^*$ -algebras to be separable from now on. For similar reasons, we assume the underlying space  $X$  to be second countable, that is, its topology must have a countable basis.

**Definition 2.14.** Let  $X$  be a second countable topological space and let  $A$  and  $B$  be separable  $C^*$ -algebras over  $X$ . Following [5], we define

$$E_0(X; A, B) := \llbracket C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K} \rrbracket_X.$$

The orthogonal direct sum turns  $E_0(X; A, B)$  into an abelian group. This also holds for  $E_1(X; A, B) := E_0(X; C_0(\mathbb{R}, A), B)$ .

**Proposition 2.15.** The composition of asymptotic morphisms induces a product

$$\llbracket A, B \rrbracket_X \times \llbracket B, C \rrbracket_X \rightarrow \llbracket A, C \rrbracket_X.$$

The proof is similar to the non-equivariant case outlined in [4]. In addition to the arguments from [4, Appendix B of Chapter II], we need the following lemma to take care of approximate  $X$ -equivariance.

Recall that an asymptotic morphism  $\varphi$  is called *uniformly continuous* if the map  $\varphi : A \rightarrow C_b(T, B)$  is continuous. By the Bartle–Graves Theorem, every asymptotic morphism is equivalent to a uniformly continuous one.

**Lemma 2.16.** Let  $X$  be a second countable topological space, let  $A, B$  and  $C$  be separable  $C^*$ -algebras, and let  $\varphi : A \rightarrow C_b(T, B)$  and  $\psi : B \rightarrow C_b(T, C)$  be uniformly continuous, approximately  $X$ -equivariant asymptotic morphisms. Let  $A_0$  be a  $\sigma$ -compact dense  $*$ -subalgebra of  $A$ . There is an increasing, continuous map  $r_0 : T \rightarrow T$  such that for any other increasing, continuous map  $r : T \rightarrow T$  with  $r(t) \geq r_0(t)$  for all  $t \in T$ , there is an approximately  $X$ -equivariant asymptotic morphism  $\theta : A \rightarrow C_b(T, C)$  such that  $\lim_{t \rightarrow \infty} \|\theta_t(a) - \psi_{r(t)} \circ \varphi_t(a)\| = 0$  for all  $a \in A_0$ .

**Proof.** Let  $(U_i)_{i=1}^\infty$  be a basis of open sets for the topology of  $X$ . Choose a dense sequence  $(a_{ij})_{j=1}^\infty$  in  $A(U_i)$  for each  $i \geq 1$ . We will find a map  $r_0$  such that, for all  $r \geq r_0$ ,

- (i)  $(\psi_{r(t)}\varphi_t)$  is a bounded asymptotic morphism from  $A_0$  to  $C$ , and
- (ii)  $\lim_{t \rightarrow \infty} \|\psi_{r(t)} \circ \varphi_t(a_{ij})\|_{X \setminus U_i} = 0$  for all  $i, j$ .



Then  $\psi_{r(t)} \circ \varphi_t$  defines a bounded  $*$ -homomorphism  $A_0 \rightarrow C_\infty$  by (i). It extends to a  $*$ -homomorphism  $\hat{\theta}$  on  $A$ . Let  $\theta : A \rightarrow C_b(T, C)$  be a lifting of  $\hat{\theta}$ . Then  $\theta$  is approximately  $X$ -equivariant by (ii).

It remains to construct  $r_0$ . By the usual non-equivariant case, there is a continuous map  $r_{00}$  such that (i) holds for all  $r \geq r_{00}$ . Since  $\varphi(A(U_i)) \subseteq C_b(T, B(U_i)) + C_0(T, B)$ , there are  $f_{ij} \in C_b(T, B(U_i))$  and  $g_{ij} \in C_0(T, B)$  such that  $\varphi(a_{ij}) = f_{ij} + g_{ij}$  for all  $i, j \geq 1$ . Consider the following countable families of compact sets:

$$K_n := \bigcup_{i,j=1}^n f_{ij}[1, n+1] \cup g_{ij}[1, n+1] \subseteq B,$$

$$L_{i,n} := \bigcup_{j=1}^n f_{ij}[1, n+1] \subseteq B(U_i).$$

Since  $\psi$  is a uniformly continuous asymptotic morphism, we can inductively construct an increasing sequence  $(s_n)_n$  such that for any  $s \geq s_n$

$$\|\psi_s(x+y) - \psi_s(x) - \psi_s(y)\| < 1/n \quad \text{for all } x, y \in K_n, \tag{2.17}$$

$$\|\psi_s(x)\| < \|x\| + 1/n \quad \text{for all } x \in K_n. \tag{2.18}$$

Since  $\psi$  is approximately  $X$ -equivariant and  $L_{i,n} \subseteq B(U_i)$ , for each  $i$  there is an increasing sequence  $(r_{i,n})_n$  such that

$$\|\psi_s(x)\|_{X \setminus U_i} < 1/n \quad \text{for all } x \in L_{i,n} \text{ and all } s \geq r_{i,n}. \tag{2.19}$$

Choose an increasing continuous map  $r_0 : T \rightarrow T$  with  $r_0(t) \geq r_{00}(t)$  and  $r_0(n) \geq \max\{s_n, r_{1,n}, r_{2,n}, \dots, r_{n,n}\}$  for all  $n \geq 1$ . We claim that any increasing, continuous function  $r \geq r_0$  satisfies (ii). This will finish the proof.

Fix  $i, j$  and  $\varepsilon > 0$ . Choose  $n$  such that  $n \geq i, n \geq j$  and  $1/n < \varepsilon/3$ . We shall show that for any  $t \geq n$ ,

$$\|\psi_{r(t)} \circ \varphi_t(a_{ij})\|_{X \setminus U_i} < \varepsilon + \|g_{ij}(t)\|.$$

This will conclude the proof since  $\lim_{t \rightarrow \infty} g_{ij}(t) = 0$  by construction. If  $t \geq n$ , then there is an integer  $m \geq n$  such that  $m \leq t < m + 1$ . Therefore  $f_{ij}(t)$  and  $g_{ij}(t)$  are in  $K_m$  and  $r(t) \geq r(m) \geq s_m$ . Eq. (2.17) yields

$$\|\psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| < 1/m < \varepsilon/3. \tag{2.20}$$

Since  $i, j \leq n \leq m$  and  $t < m + 1$ , we have  $f_{ij}(t) \in L_{i,m}$  and  $r(t) \geq r(m) \geq r_{i,m}$ . Inequality (2.19) yields

$$\|\psi_{r(t)}(f_{ij}(t))\|_{X \setminus U_i} < 1/m < \varepsilon/3. \tag{2.21}$$

Similarly, (2.18) yields

$$\|\psi_{r(t)}(g_{ij}(t))\| \leq \|g_{ij}(t)\| + 1/m < \|g_{ij}(t)\| + \varepsilon/3. \tag{2.22}$$

Putting together (2.20), (2.21) and (2.22), we get

$$\begin{aligned} \|\psi_{r(t)}\varphi_t(a_{ij})\|_{X \setminus U_i} &\leq \|\psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| \\ &\quad + \|\psi_{r(t)}(f_{ij}(t))\|_{X \setminus U_i} + \|\psi_{r(t)}(g_{ij}(t))\| \\ &< \varepsilon + \|g_{ij}(t)\|. \quad \square \end{aligned}$$

For any extension of separable  $C^*$ -algebras  $I \hookrightarrow A \xrightarrow{p} B$ , there is a canonical asymptotic morphism from  $C_0((0, 1), B)$  to  $I$ . If  $A$  is a  $C^*$ -algebra over  $X$ , then  $I$  and  $B$  become  $C^*$ -algebras over  $X$  in a unique natural way, such that the given extension is an extension of  $C^*$ -algebras over  $X$ . Specifically,  $I(U) = I \cap A(U)$  and  $B(U) = p(A(U))$  for all  $U$  open in  $X$ .

**Proposition 2.23.** *Let  $I \hookrightarrow A \twoheadrightarrow B$  be an extension of  $C^*$ -algebras over  $X$ . Then the associated asymptotic morphism from  $C_0((0, 1), B)$  to  $I$  is approximately  $X$ -equivariant.*

**Proof.** Having an extension of  $C^*$ -algebras over  $X$  means that we have  $C^*$ -algebra extensions

$$I(U) \hookrightarrow A(U) \twoheadrightarrow B(U)$$

for all open subsets  $U$  of  $X$ . Since the map  $B(U) \rightarrow B$  is injective, this implies  $I(U) = I \cap A(U) = I \cdot A(U)$ .

We fix a positive and contractive continuous approximate unit  $(u_t)_{t \in T}$  of  $I$  which is quasi-central in  $A$ . The canonical asymptotic morphism

$$\gamma : SB := C_0((0, 1), B) \rightarrow C_b(T, I)$$

is defined in two steps. First, we define a homomorphism

$$\gamma' : SA \rightarrow C_b(T, I)/C_0(T, I), \quad \gamma'_t(f \otimes a) := f(u_t) \cdot a.$$

Secondly, since the restriction of  $\gamma'$  to  $SI$  is equivalent to the null asymptotic morphism,  $\gamma'$  induces an asymptotic morphism from  $SB$  to  $I$ . Clearly,  $\gamma'$  is approximately  $X$ -equivariant because  $I \cdot A(U) \subseteq I(U)$ . This is inherited by  $\gamma$  because  $\dot{\gamma} \circ p = \dot{\gamma}'$ , where  $p : A \rightarrow B$  is the quotient map.  $\square$

Let  $I \hookrightarrow B \xrightarrow{p} C$  be an extension of  $C^*$ -algebras over  $X$ . Let  $A$  be a  $C^*$ -algebra over  $X$  and let  $\varphi : A \rightarrow C$  be an  $X$ -equivariant  $*$ -homomorphism. Let  $E$  be the  $C^*$ -algebra defined by the pullback diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \varphi \\
 0 & \longrightarrow & I & \longrightarrow & B & \xrightarrow{p} & C \longrightarrow 0,
 \end{array}$$

that is,  $E = \{(a, b) \in A \oplus B : \varphi(a) = p(b)\}$ . For  $U \in \mathbb{O}(X)$ , set  $E(U) := E \cap (A(U) \oplus B(U))$ .

**Lemma 2.24.** *E is a C\*-algebra over X and  $I \twoheadrightarrow E \twoheadrightarrow A$  is an extension of C\*-algebras over X. The same conclusions hold if B and C are only quasi C\*-algebras over X.*

**Proof.** Recall that for a quasi C\*-algebra B over X, the map  $U \mapsto B(U)$  preserves only finite suprema in general. The map  $U \mapsto E(U)$  is obviously order-preserving. The conditions  $E(\emptyset) = 0$ ,  $E(X) = E$  and  $E(U_1 \cap U_2) = E(U_1) \cap E(U_2)$  are easily verified. Let us show that  $E(U_1 \cup U_2) \subset E(U_1) + E(U_2)$ , the reverse inclusion being obvious. Let  $(a, b) \in E(U_1 \cup U_2)$ . Then  $a \in A(U_1 \cup U_2) = A(U_1) + A(U_2)$  and hence there are  $a_i \in A(U_i)$ ,  $i = 1, 2$  such that  $a = a_1 + a_2$ . Since  $\varphi$  is X-equivariant,  $\varphi(a_i) \in C(U_i)$  and hence there are  $b_i \in B(U_i)$  such that  $p(b_i) = \varphi(a_i)$ ,  $i = 1, 2$ . It follows that  $b_1 + b_2 - b \in B(U_1 \cup U_2)$  and  $p(b_1 + b_2 - b) = \varphi(a_1) + \varphi(a_2) - \varphi(a) = 0$ . Therefore,  $b_1 + b_2 - b \in I \cap B(U_1 \cup U_2) = I(U_1 \cup U_2) = I(U_1) + I(U_2)$ . This shows that there are  $x_i \in I(U_i)$ ,  $i = 1, 2$ , such that  $b_1 + b_2 - b = x_1 + x_2$ . It follows that  $(a_i, b_i - x_i) \in E(U_i)$  and  $(a, b) = (a_1, b_1 - x_1) + (a_2, b_2 - x_2)$ .

It remains to show that  $E(\bigcup U_n)$  is the closure of  $\bigcup E(U_n)$  for any increasing sequence  $(U_n)$  in  $\mathbb{O}(X)$ . The sequence of C\*-algebras

$$I(U) \twoheadrightarrow E(U) \twoheadrightarrow A(U)$$

is exact for each open set U. Since A and I are C\*-algebras over X,

$$\begin{aligned}
 A(U) &= \overline{\bigcup A(U_n)} = \varinjlim A(U_n), \\
 I(U) &= \overline{\bigcup I(U_n)} = \varinjlim I(U_n).
 \end{aligned}$$

Since the C\*-algebra inductive limit functor is exact, we get another extension of C\*-algebras

$$I(U) \twoheadrightarrow \overline{\bigcup E(U_n)} \twoheadrightarrow A(U)$$

because  $\varinjlim E(U_n) = \overline{\bigcup E(U_n)}$ . This implies that  $E(U)$  is the supremum of  $\{E(U_n)\}$ , so that E is a C\*-algebra over X. □

**Theorem 2.25.** *The equivariant E-theory defined above carries a composition product and hence yields a category  $\mathfrak{E}(X)$ . The canonical functor from the category  $\mathfrak{C}^*\mathfrak{sep}(X)$  of separable C\*-algebras over X to  $\mathfrak{E}(X)$  is the universal half-exact, C\*-stable homotopy functor.*

**Proof.** The composition product is described in Proposition 2.15. The same argument as in the non-equivariant case shows that it is associative. The functor  $\mathfrak{C}^*\mathfrak{sep}(X) \rightarrow \mathfrak{E}(X)$  is a C\*-stable homotopy functor by definition. Next we check its exactness.

Let  $I \hookrightarrow E \xrightarrow{p} Q$  be an extension of  $C^*$ -algebras over  $X$ . The cone

$$C_p := \{(f, a) \in C_0((0, 1], Q) \oplus E \mid f(1) = p(a)\},$$

$$C_p(U) := (C_0((0, 1], Q(U)) \oplus E(U)) \cap C_p \quad \text{for } U \in \mathbb{O}(X),$$

is a  $C^*$ -algebra over  $X$  by Lemma 2.24. The asymptotic morphism  $\gamma_t : SC_p \rightarrow SI$  induced by the extension  $SI \hookrightarrow CE \rightarrow C_p$  is approximately  $X$ -equivariant. There is a natural  $X$ -equivariant inclusion  $i : I \rightarrow C_p, i(a) = (0, a)$ . The proof of [7, Theorem 13] with no essential change yields that  $\gamma$  is a homotopy inverse of  $Si$ , that is,  $\llbracket \gamma \circ Si \rrbracket_X = \llbracket id_{SI} \rrbracket_X$  and  $\llbracket Si \circ \gamma \rrbracket_X = \llbracket id_{SC_p} \rrbracket_X$ . As in the non-equivariant case, this excision result and Proposition 2.15 show that  $E_0(X; A, B) := \llbracket SA \otimes \mathbb{K}, SB \otimes \mathbb{K} \rrbracket_X$  is a periodic exact functor in both variables  $A$  and  $B$ , that is, if  $I \hookrightarrow E \rightarrow Q$  is an extension in  $\mathcal{C}^*sep(X)$  and  $B$  is a separable  $C^*$ -algebra over  $X$ , then there are six-term exact sequences

$$\begin{array}{ccccc} E_0(X; Q, B) & \longrightarrow & E_0(X; E, B) & \longrightarrow & E_0(X; I, B) \\ \uparrow \partial & & & & \downarrow \partial \\ E_1(X; I, B) & \longleftarrow & E_1(X; E, B) & \longleftarrow & E_1(X; Q, B) \end{array}$$

and

$$\begin{array}{ccccc} E_0(X; B, I) & \longrightarrow & E_0(X; B, E) & \longrightarrow & E_0(X; B, Q) \\ \uparrow \partial & & & & \downarrow \partial \\ E_1(X; B, Q) & \longleftarrow & E_1(X; B, E) & \longleftarrow & E_1(X; B, I). \end{array}$$

The horizontal maps in both exact sequences are induced by the given maps  $I \rightarrow E \rightarrow Q$ , and the vertical maps are, up to signs, products with the class of the approximately  $X$ -equivariant asymptotic morphism associated to the extension as in Proposition 2.23.

It remains to verify universality. Again this is similar to the proof of the non-equivariant case in [2, Theorem 25.6.1], using Lemma 2.26 below as a substitute for [2, Proposition 25.6.2].  $\square$

**Lemma 2.26.** Any element of  $E_0(X; A, B)$  may be written as  $[\rho] \circ [\pi]^{-1}$  for  $X$ -equivariant  $*$ -homomorphisms  $\rho$  and  $\pi$ .

**Proof.** Let  $\varphi : A \rightarrow C_b(T, B)$  be an approximately  $X$ -equivariant asymptotic morphism. We shall use Lemma 2.24 to show that the  $C^*$ -algebra

$$E := \{(a, b) \in A \oplus C_b(T, B) : \varphi(a) - b \in C_0(T, B)\},$$

becomes a  $C^*$ -algebra over  $X$  by

$$E(U) := E \cap (A(U) \oplus C_b(T, B(U))).$$

As a consequence of the Bartle–Graves Theorem, for any two closed two-sided ideals  $J_1$  and  $J_2$  in a  $C^*$ -algebra  $D$ ,  $C_b(T, J_1 + J_2) = C_b(T, J_1) + C_b(T, J_2)$ . From this we see that

$$C_0(T, B) \twoheadrightarrow C_b(T, B) \twoheadrightarrow C_b(T, B)/C_0(T, B) = B_\infty$$

is an extension of quasi  $C^*$ -algebras over  $X$ . By Lemma 2.24, its pullback under the  $X$ -equivariant  $*$ -homomorphism  $\dot{\varphi} : A \rightarrow B_\infty$  is an extension of  $C^*$ -algebras over  $X$ :

$$C_0(T, B) \twoheadrightarrow E \xrightarrow{\pi} A$$

with  $\pi(a, b) := \varphi(a)$ . The map  $\pi$  becomes an isomorphism in  $\mathfrak{E}(X)$  because  $C_0(T, B)$  is contractible over  $X$ . Let  $\rho' : E \rightarrow C_b(T, B)$  be the  $*$ -homomorphism  $\rho'(a, b) = b$ . When regarded as an asymptotic morphism from  $E$  to  $B$ ,  $\rho'$  is homotopic to the constant asymptotic morphism  $\rho(a, b) = b(0)$ . We have  $[\varphi] \circ [\pi] = [\rho']$  because  $\varphi(\pi(a, b)) - \rho'(a, b) \in C_0(T, B)$  for all  $(a, b) \in E$ . Hence  $[\varphi] = [\rho] \circ [\pi]^{-1}$ .  $\square$

2.5. Further properties of E-theory

Like the category  $\mathfrak{K}\mathfrak{K}(X)$ , the category  $\mathfrak{E}(X)$  carries the additional structure of a triangulated category (see [19,23]). As in KK-theory, the translation automorphism is the suspension functor  $A \mapsto SA := C_0((0, 1), A)$ , and a triangle is exact if it is isomorphic to the mapping cone triangle of some  $X$ -equivariant  $*$ -homomorphism.

**Theorem 2.27.** *The category  $\mathfrak{E}(X)$  is triangulated.*

**Proof.** The argument is essentially the same as in the appendix of [19]. The only axiom that requires a different treatment is the one that requires each  $\varphi \in E_0(X; A, B)$  to embed in an exact triangle. Here we use the Bott periodicity in conjunction with the factorisation  $\varphi = [\rho] \circ [\pi]^{-1}$  of Lemma 2.26 with  $X$ -equivariant  $*$ -homomorphisms  $\rho : E \rightarrow B$  and  $\pi : E \rightarrow A$ . Since  $[\pi]$  is invertible in E-theory, the mapping cone triangle

$$SB \rightarrow C_\rho \rightarrow E \xrightarrow{\rho} B$$

is isomorphic to an exact triangle  $SB \rightarrow C_\rho \rightarrow A \xrightarrow{[\varphi]} B$ .  $\square$

The proof that E-theory is exact shows that any extension  $I \twoheadrightarrow E \twoheadrightarrow Q$  of  $C^*$ -algebras over  $X$  gives rise to an exact triangle  $SQ \rightarrow I \rightarrow E \rightarrow Q$ , where the map  $SQ \rightarrow I$  is the Connes–Higson construction (see Proposition 2.23) and the maps  $I \rightarrow E \rightarrow Q$  are the given ones. Such triangles are called *extension triangles*. This works for all extensions, so that we need no admissibility assumption as in  $\mathfrak{K}\mathfrak{K}(X)$ .

Since there is no admissibility hypothesis, several constructions in Kasparov theory simplify in E-theory. For instance, the colimit  $\varinjlim(A_n, \varphi_n)$  of any inductive system  $\varphi_n : A_n \rightarrow A_{n+1}$ ,  $n \in \mathbb{N}$ , in  $\mathfrak{E}^*\mathfrak{sep}(X)$  is also a homotopy colimit in  $\mathfrak{E}(X)$ , by the argument in [19, Section 2.4].

**Proposition 2.28.** *If  $A$  is the inductive limit of an inductive system  $(A_n, \varphi_n)$  in  $\mathfrak{E}^*\mathfrak{sep}(X)$ , then there is a natural short exact sequence*

$$0 \rightarrow \varinjlim^1 E_1(X; A_n, B) \rightarrow E(X; A, B) \rightarrow \varinjlim E(X; A_n, B) \rightarrow 0.$$

**Proof.** The functor  $A \mapsto E(X; A, B)$  is seen to be countably additive as in the proof of [15, Proposition 7.1]. Then we follow the standard argument based on mapping telescopes in [2, Section 21.3.2].  $\square$

For locally compact Hausdorff spaces, we may compare our definition of equivariant E-theory with previous ones in [24,25]. Since we use the original Connes–Higson model of E-theory instead of iterated asymptotic algebras, this does not yet follow directly from Proposition 2.11 and [25].

**Proposition 2.29.** *Let  $X$  be Hausdorff and locally compact and let  $A$  and  $B$  be  $C^*$ -algebras over  $X$ . Then  $E_*(X; A, B)$  is naturally isomorphic to  $\mathcal{R}E_*(X; A, B)$ .*

**Proof.** Both theories satisfy the same universal property. Alternatively, the statement follows from Proposition 2.11 and [24].  $\square$

Recall that for a compact Hausdorff space  $X$ , there is a canonical isomorphism

$$KK_*(X; C(X, A), B) \cong KK_*(A, B)$$

for any  $C^*$ -algebra  $A$  and any  $C^*$ -algebra  $B$  over  $X$ . The same isomorphism holds in E-theory as well:

**Lemma 2.30.** *Let  $X$  be a compact Hausdorff space. Then*

$$E_*(X; C(X, A), B) \cong E_*(A, B)$$

for any  $C^*$ -algebra  $A$  and any  $C^*$ -algebra  $B$  over  $X$ .

**Proof.** We may view  $C(X, A)$  as a  $C^*$ -algebra over  $X$  using the obvious map  $\text{Prim } C(X, A) \rightarrow X$ , so that  $C(X, A)(U) := C_0(U, A)$  for  $U \in \mathbb{O}(X)$ . We have to show that the functor

$$\mathfrak{E} \rightarrow \mathfrak{E}(X), \quad A \mapsto C(X, A),$$

is left adjoint to the functor

$$\mathfrak{E}(X) \rightarrow \mathfrak{E}, \quad B \mapsto B(X).$$

First of all, both maps on objects clearly induce functors on E-theory categories because of the universal properties. For the adjointness, we have to furnish the unit and counit of adjunction and verify the two zigzag equations (see [18]). The unit is the  $X$ -equivariant  $*$ -homomorphism  $C(X, B) = C(X) \otimes B(X) \rightarrow B$  that comes from viewing a  $C^*$ -algebra  $B$  over  $X$  as a  $C(X)$ - $C^*$ -algebra. The counit is the embedding  $A \rightarrow C(X, A)(X) = C(X, A)$ ,  $a \mapsto 1 \otimes a$ , as constant functions. The zigzag equations are trivial to verify and hold already on the level of  $*$ -homomorphisms.  $\square$

**Proposition 2.31.** *Let  $Y \subseteq X$  be a locally closed subset. Then there exists a natural restriction functor  $E_*(X; A, B) \rightarrow E_*(Y; r_X^Y(A), r_X^Y(B))$  for  $C^*$ -algebras  $A$  and  $B$  over  $X$ .*

**Proof.** The restriction functor  $\mathcal{C}^*\mathfrak{sep}(X) \rightarrow \mathcal{C}^*\mathfrak{sep}(Y)$  is defined in [20] by  $r_X^Y A(Z) := A(Y \cap Z)$  for  $Z \in \mathcal{O}(Y)$ . It evidently maps extensions again to extensions and commutes with stabilisation. Hence it induces a functor on E-theory by the universal property.  $\square$

### 3. Approximation by finite spaces

Let  $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$  be a countable basis for the topology of  $X$ . For each  $n \in \mathbb{N}$ , let  $\tau_n$  be the topology generated by the open subsets  $U_1, \dots, U_n$ . That is, the subsets  $U_j$  are a subbasis for  $\tau_n$ , so that the intersections

$$U_F := \bigcap_{i \in F} U_i$$

for  $F \subseteq \{1, \dots, n\}$  are a basis for  $\tau_n$ .

Since the topology  $\tau_n$  is finite, it is pulled back from a finite  $T_0$ -space  $X_n$ ; namely, we equip  $X$  with the equivalence relation

$$x \sim_n y \iff \{1 \leq j \leq n: x \in U_j\} = \{1 \leq j \leq n: y \in U_j\}$$

for  $x, y \in X$ . We may view  $\tau_n$  as a topology on the finite set  $X/\sim_n$ . A point in  $X/\sim_n$  is parametrised by the set  $\{1 \leq j \leq n: x \in U_j\}$ .

**Remark 3.1.** The minimal open neighbourhood in  $X_n$  that contains the point corresponding to  $F \subseteq \{1, \dots, n\}$  is the image in  $X_n$  of  $U_F := \bigcap_{i \in F} U_i$ .

In the following, we view  $C^*$ -algebras over  $X$  as  $C^*$ -algebras over  $(X, \tau_n)$  or, equivalently, over  $X_n := (X/\sim_n, \tau_n)$  by forgetting most of the distinguished ideals.

**Theorem 3.2.** *Let  $A$  and  $B$  be  $C^*$ -algebras over  $X$ , viewed as  $C^*$ -algebras over  $X_n := (X/\sim_n, \tau_n)$  for  $n \in \mathbb{N}$ . Then there is a natural extension of  $\mathbb{Z}/2$ -graded abelian groups*

$$\varprojlim^1 E_{*+1}(X_n; A, B) \rightarrow E_*(X; A, B) \rightarrow \varprojlim E_*(X_n; A, B).$$

**Proof.** Recall the description of  $[[A, B]]_X$  as the zeroth homotopy group of a quasi-topological space  $\text{Asymp}(A, B)_X$  in Section 2.3. This also applies to E-theory: we have  $E_0(X; A, B) \cong \pi_0(\Gamma_X)$  with

$$\Gamma_X := \text{Asymp}(C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K})_X.$$

The same definitions for  $X_n$  yield quasi-topological spaces  $\Gamma_n := \Gamma_{X_n}$  for  $n \in \mathbb{N}$  with  $E_0(X_n; A, B) \cong \pi_0(\Gamma_n)$ . The quasi-topological spaces  $\Gamma_n$  form a projective system because approximate  $X_{n+1}$ -equivariance implies approximate  $X_n$ -equivariance.

We claim that

$$\Gamma_X = \bigcap_{n \in \mathbb{N}} \Gamma_n, \quad C(Y, \Gamma_X) = \bigcap_{n \in \mathbb{N}} C(Y, \Gamma_n)$$

for each compact Hausdorff space  $Y$ , where  $C(Y, \Gamma_n)$  denotes the space of quasi-continuous maps  $Y \rightarrow \Gamma_n$ .

The inclusion  $C(Y, \Gamma_X) \subseteq \bigcap C(Y, \Gamma_n)$  is evident. The intersection of  $C(Y, \Gamma_n)$  consists of those asymptotic morphisms that satisfy (2.10) for all  $U \in \mathcal{U}$ . Since the set of open subsets for which (2.10) holds is closed under arbitrary unions and  $\mathcal{U}$  is a basis for the topology of  $X$ , this implies (2.10) for all open subsets of  $X$ , proving the claim.

The claim above shows that  $\Gamma_X$  is the inverse limit of the projective system  $\Gamma_n$ . The homotopy groups of inverse limits of ordinary topological spaces are computed by an exact sequence of the desired form if the maps  $\Gamma_{n+1} \rightarrow \Gamma_n$  have the *homotopy covering property*, see [32]. It is easy to see that this carries over to quasi-topological spaces; but in our case the maps  $\Gamma_{n+1} \rightarrow \Gamma_n$  are injective and therefore cannot have the homotopy covering property. Nevertheless, we can get the desired result by following part of the argument in [32].

First we observe that [32, Theorem C], which computes the homotopy groups of homotopy equalisers remains true for quasi-topological spaces. Let  $f, g : A \rightrightarrows B$  be two base point preserving quasi-continuous maps between pointed quasi-topological spaces. The *homotopy equaliser* of  $f, g$  is the quasi-topological space  $D(f, g)$  defined so that, for all  $Y$  compact Hausdorff,

$$C(Y, D(f, g)) = \{(a, b) \in C(Y, A) \times C(Y \times I, B) \mid f \circ a = b(\_, 0), g \circ a = b(\_, 1)\}.$$

Let  $Y$  be a compact Hausdorff space. Then there is an exact sequence of pointed sets

$$* \rightarrow T \rightarrow [Y, D(f, g)] \rightarrow K \rightarrow * \tag{3.3}$$

where  $[Y, X]$  denotes homotopy classes of quasi-continuous maps  $Y \rightarrow X$ ,  $K := \{a \in [Y, A] \mid f_*(a) = g_*(a)\}$ , and  $T$  is the orbit space for a certain canonical action of  $[Y \times \mathbb{S}^1, A]_*$  on  $[Y \times \mathbb{S}^1, B]_*$ , where  $[Y \times \mathbb{S}^1, \_ ]_*$  means that we restrict attention to quasi-continuous maps and homotopies that map  $Y \times \{1\} \subseteq Y \times \mathbb{S}^1$  to the base point.

Next we apply (3.3) to the pair of maps

$$\text{Id}, f : \prod_{n=0}^{\infty} \Gamma_n \rightrightarrows \prod_{n=0}^{\infty} \Gamma_n,$$

where  $f$  is the shift map from the definition of the projective limit. Letting  $\gamma_{n+1}^n : \Gamma_{n+1} \rightarrow \Gamma_n$  denote the maps in the projective system, we have  $f((x_n)_{n \in \mathbb{N}}) := (\gamma_{n+1}^n(x_{n+1}))_{n \in \mathbb{N}}$ . The homotopy equaliser of  $(\text{id}, f)$  is quasi-homeomorphic to the quasi-topological space  $\Gamma_\infty$  defined by

$$C(Y, \Gamma_\infty) := \left\{ (f_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([0, 1] \times Y, \Gamma_n) \mid f_n(1) = \gamma_{n+1}^n(f_{n+1}(0)) \text{ for all } n \in \mathbb{N} \right\}.$$

This is a familiar mapping telescope construction. The quasi-topological version of [32, Theorem C] shows that the homotopy groups of  $\Gamma_\infty$  are computed by an exact sequence of exactly the desired form.

To finish the proof of the theorem, it remains to show that the homotopy limit  $\Gamma_\infty$  and the limit  $\Gamma_X$  of the projective system  $(\Gamma_n)$  have isomorphic  $\pi_0$ . Lacking the homotopy covering property used in [32], we do this by hand.



Let us describe the homotopy limit  $\Gamma_\infty$  more concretely. The maps  $\gamma_{n+1}^n : \Gamma_{n+1} \rightarrow \Gamma_n$  are just the inclusion maps. It is convenient to identify  $C(Y, \Gamma_\infty)$  with

$$C(Y, \Gamma_\infty) = \left\{ (f_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([n, n+1] \times Y, \Gamma_n) \mid f_n(n+1) = f_{n+1}(n+1) \text{ for all } n \in \mathbb{N} \right\}.$$

We view each  $f_n$  as an approximately  $X_n$ -equivariant asymptotic morphism from  $A'$  to  $C([n, n+1] \times Y, B')$ , where  $A' := C_0(\mathbb{R}, A) \otimes \mathbb{K}$  and  $B' := C_0(\mathbb{R}, B) \otimes \mathbb{K}$ . We may piece together these asymptotic morphisms to a single family of maps  $\varphi_{s,t} : A' \rightarrow C(Y, B')$ ,  $s, t \in T$ , where  $\varphi_{s,t}|_{s \in [n, n+1]}$  is  $f_n$ . That is,  $\varphi_{s,t}$  is an asymptotic morphism for fixed  $s$ , uniformly for  $s \in [n, n+1]$  for all  $n$ , and hence uniformly for  $s$  in compact subsets of  $T$ ; furthermore, this asymptotic morphism is (uniformly) approximately  $X_n$ -equivariant for  $s \in [n, n+1]$  and hence for  $s$  in compact subsets of  $[n, \infty)$ .

We map  $\Gamma_X$  to  $\Gamma_\infty$  by taking a constant family of asymptotic morphisms. It remains to show that this map  $\Gamma_X \rightarrow \Gamma_\infty$  induces an isomorphism on homotopy classes.

Let  $(\varphi_{s,t}) \in \Gamma_\infty$  and let  $A_0 \subseteq A'$  be a countably generated dense subalgebra. The same considerations as in the construction of the product of asymptotic homomorphisms show that there is an increasing continuous function  $h_0 : T \rightarrow T$  such that  $\varphi_{t,h(t)} : A_0 \rightarrow B'$  extends to an  $X$ -equivariant asymptotic morphism for all continuous  $h \geq h_0$ . Here we use that an asymptotic morphism is  $X$ -equivariant once it satisfies (2.10) for all  $U \in \mathcal{U}$ . Furthermore, we may choose  $h_0$  such that the convex homotopies  $\varphi_{s,rh(t)+(1-r)t}$  from  $\varphi_{s,t}$  to  $\varphi_{s,h(t)}$  and  $\varphi_{rt+(1-r)s,h(t)}$  from  $\varphi_{s,h(t)}$  to  $\varphi_{t,h(t)}$  are homotopies in  $\Gamma_\infty$  for  $h \geq h_0$ . We discuss this in detail below. Thus  $(\varphi_{s,t})$  is homotopic to the constant family of asymptotic morphism  $(\varphi_{t,h(t)})$  in  $\Gamma_\infty$ , so that the map  $\pi_0(\Gamma_X) \rightarrow \pi_0(\Gamma_\infty)$  is surjective. A similar argument may be applied to homotopies in  $\Gamma_\infty$  and shows that two elements of  $\Gamma_X(A, B)$  that become homotopic in  $\Gamma_\infty$  are already homotopic in  $\Gamma_X$ .

Let us now show how to construct the function  $h_0$  for given  $(\varphi_{s,t}) \in \Gamma_\infty$ . The first homotopy from  $\varphi_{s,t}$  to  $\varphi_{s,h(t)}$  is a homotopy of asymptotic morphisms provided  $h(t) \geq t$ , for obvious reasons. Thus it only remains to study the second homotopy. Let  $A_0 = \{a_1, a_2, \dots\} \subseteq A'$  be a countable dense  $*$ -subalgebra. Let  $\{\lambda_1, \lambda_2, \dots\}$  be a sequence dense in  $\mathbb{C}$ . Let  $(U_i)_{i=1}^\infty$  be a basis of open sets for the topology of  $X$ . Choose a dense sequence  $(a_{ij})_{j=1}^\infty$  in  $A'(U_i)$  for each  $i \geq 1$ .

For each integer  $m \geq 1$  choose  $\alpha_m > 0$  such that for all  $1 \leq i, j, k \leq m$  and all  $t \geq \alpha_m$ ,

$$\sup_{s \in [0, m+1]} \|\varphi_{s,t}(a_i^* + \lambda_k a_j) - \varphi_{s,t}(a_i)^* - \lambda_k \varphi_{s,t}(a_j)\| < 1/m, \tag{3.4}$$

$$\sup_{s \in [0, m+1]} \|\varphi_{s,t}(a_i a_j) - \varphi_{s,t}(a_i) \varphi_{s,t}(a_j)\| < 1/m. \tag{3.5}$$

For each integer  $n \geq 1$  we construct a sequence  $(\tau_{m,n})_{m=1}^\infty$  such that

$$\sup_{s \in [n, m+1]} \|\varphi_{s,t}(a_{ij})\|_{X \setminus U_i} < 1/m, \tag{3.6}$$

for all  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  and all  $t \geq \tau_{m,n}$ . Moreover, once the sequence  $(\tau_{m,n})_{m=1}^\infty$  is constructed, we construct the next sequence  $(\tau_{m,n+1})_{m=1}^\infty$  such that  $\tau_{m,n+1} \geq \tau_{m,n}$  for all

$m \geq 1$ . Let  $h_0 : T \rightarrow T$  be a continuous increasing function with  $h_0(m) \geq \max\{\alpha_m, \tau_{m,m}\}$  and  $\lim_{t \rightarrow \infty} h_0(t) = \infty$ .

Let  $h \geq h_0$  be a continuous function. The homotopy  $\varphi_{rt+(1-r)s,h(t)}$  is defined by an element  $H$  in

$$C([0, 1] \times Y, \Gamma_\infty) = \left\{ (H_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([0, 1] \times [n, n+1] \times Y, \Gamma_n) \mid H_n(n+1) = H_{n+1}(n+1) \text{ for all } n \in \mathbb{N} \right\},$$

where for  $r \in [0, 1]$ ,  $(H_n)_r := (\varphi_{rt+(1-r)s,h(t)})_{s \in [n,n+1], t \in T}$ .

In order to verify that  $H$  is an element of  $C([0, 1] \times Y, \Gamma_\infty)$ , it is sufficient to show that for all  $i, j, k \geq 1$

$$\lim_{t \rightarrow \infty} \sup_{s \in [n,n+1], r \in [0,1]} \left\| \varphi_{rt+(1-r)s,h(t)}(a_i^* + \lambda_k a_j) - \varphi_{rt+(1-r)s,h(t)}(a_i)^* - \lambda_k \varphi_{rt+(1-r)s,h(t)}(a_j) \right\| = 0, \tag{3.7}$$

$$\lim_{t \rightarrow \infty} \sup_{s \in [n,n+1], r \in [0,1]} \left\| \varphi_{rt+(1-r)s,h(t)}(a_i a_j) - \varphi_{rt+(1-r)s,h(t)}(a_i) \varphi_{rt+(1-r)s,h(t)}(a_j) \right\| = 0, \tag{3.8}$$

and that for all  $1 \leq i \leq n, j \geq 1$

$$\lim_{t \rightarrow \infty} \sup_{s \in [n,n+1], r \in [0,1]} \left\| \varphi_{rt+(1-r)s,h(t)}(a_{ij}) \right\|_{X \setminus U_i} = 0. \tag{3.9}$$

We deal first with (3.7) and (3.8). Let  $i, j, k \geq 1$  and  $\varepsilon > 0$  be given. We claim that for any  $t \geq \max\{n, i, j, k, 1/\varepsilon\} + 1$ , the quantities whose limits are taken in (3.7) and (3.8) are smaller than  $\varepsilon$ . If  $m$  is the integer part of  $t$ , then  $\max\{n, i, j, k, 1/\varepsilon\} < m \leq t < m + 1$ . Moreover, for any  $s \in [n, n + 1]$  and  $r \in [0, 1]$ ,  $rt + (1 - r)s \in [0, m + 1]$  and  $h(t) \geq h_0(t) \geq h_0(m) \geq \alpha_m$ . Since  $1/m < \varepsilon$  our claim follows now from (3.4) and (3.5).

Let us now check (3.9). Let  $1 \leq i \leq n, j \geq 1$  and  $\varepsilon > 0$  be given and suppose that  $t \geq \max\{n, j, 1/\varepsilon\} + 1$ . Then there is an integer  $m$  such that  $\max\{n, j, 1/\varepsilon\} < m \leq t < m + 1$ . Observe that for any  $s \in [n, n + 1]$  and  $r \in [0, 1]$ ,  $rt + (1 - r)s \in [n, m + 1]$  and  $h(t) \geq h_0(t) \geq h_0(m) \geq \tau_{m,m} \geq \tau_{m,n}$ . Since  $1/m < \varepsilon$ , it follows from (3.6) that the quantity whose limit is taken in (3.9) is smaller than  $\varepsilon$  whenever  $t \geq \max\{n, j, 1/\varepsilon\} + 1$ .  $\square$

**Theorem 3.10.** *Let  $X$  be a second countable topological space. An element in  $E_*(X; A, B)$  is invertible if and only if its image in  $E_*(A(U), B(U))$  is invertible for all  $U \in \mathcal{O}(X)$ .*

**Proof.** The necessity of the condition is trivial. Next we sketch why the condition is sufficient if  $X$  is a finite space. The proof is similar to the proof of a similar statement in KK-theory in [20, Proposition 4.9]. If  $X$  is finite, any point  $x \in X$  is contained in a minimal open subset  $U_x$ . For a  $C^*$ -algebra  $A$ , let  $i_x A$  be  $A$  viewed as a  $C^*$ -algebra over  $X$  concentrated at  $x \in X$ , that is,  $i_x(A)(U) = A$  for  $x \in U$  and  $i_x(A)(U) = 0$  for  $x \notin U$ . An argument similar to the proof of [20, Proposition 3.13] yields

$$E_*(X; i_x(A), B) \cong E_*(A, B(U_x))$$

for  $x \in X$ , a  $C^*$ -algebra  $A$  and a  $C^*$ -algebra  $B$  over  $X$ . An argument similar to the proof of [20, Proposition 4.7] shows that objects of the form  $i_x(A)$  generate  $\mathfrak{E}(X)$ , that is, no proper triangulated subcategory of  $\mathfrak{E}(X)$  contains  $i_x(A)$  for all  $A$  (see also Proposition 4.5 below). Hence a map in  $E_*(X; A, B)$  is invertible if the induced map  $E_*(X; i_x(D), A) \rightarrow E_*(X; i_x(D), B)$  is invertible for all  $x \in X$  and all  $D$ . By the isomorphism above, this is equivalent to invertibility of the induced map  $E_*(D, A(U_x)) \rightarrow E_*(D, B(U_x))$ , which is equivalent to invertibility in  $E_*(A(U_x), B(U_x))$  for all  $x$ . This finishes the argument for finite  $X$ .

If  $X$  is infinite, let  $\mathcal{U}$  be a countable basis for its topology and let  $X_n$  be the resulting finite approximations to  $X$ . Theorem 3.2 shows that an arrow in  $\mathfrak{E}(X)$  is invertible if and only if its image in  $\mathfrak{E}(X_n)$  is invertible for all  $n \in \mathbb{N}$ . (The naturality of the extension in Theorem 3.2 implies that the kernel  $\varprojlim^1 \dots$  is nilpotent.) This reduces the general case to the finite case already settled.  $\square$

**Theorem 3.11.** *Let  $A$  be a separable nuclear  $C^*$ -algebra with Hausdorff primitive spectrum  $X$ . Suppose that each two-sided closed ideal of  $A$  is  $KK$ -contractible. Then*

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

**Proof.** By a result of Fell,  $A$  is a continuous  $C_0(X)$ -algebra with nonzero simple fibres. Set  $B := C_0(X) \otimes \mathcal{O}_2$ . Then  $0 \in E(X; A, B)$  is an  $E(X)$ -equivalence by Theorem 3.10. Theorem 5.4 yields  $E_*(X; C, D) \cong KK_*(X; C, D)$  for  $C, D \in \{A, B\}$  because  $A$  and  $B$  are nuclear and continuous  $C_0(X)$ -algebras. Hence  $0 \in KK(X; A, B)$  is a  $KK(X)$ -equivalence, and we may apply the main result of [17] to conclude that  $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ .  $\square$

#### 4. The E-theoretic bootstrap category

Recall that the bootstrap class  $\mathcal{B}$  in  $\mathfrak{K}\mathfrak{K}$  is the localising subcategory of the triangulated category  $\mathfrak{K}\mathfrak{K}$  that is generated by the object  $\mathbb{C}$ . Similarly, we define the E-theoretic bootstrap class  $\mathcal{B}_E \subseteq \mathfrak{E}$  as the localising subcategory of  $\mathfrak{E}$  generated by  $\mathbb{C}$ . This is the class of all separable  $C^*$ -algebras  $A$  for which  $E_*(A, B)$  fulfills the Universal Coefficient Theorem for all  $B$ .

For a finite topological space  $X$ , a bootstrap class  $\mathcal{B}(X)$  in  $\mathfrak{K}\mathfrak{K}(X)$  is defined in [20] along similar lines. Here we follow a different approach:

**Definition 4.1.** Let  $\mathcal{B}_E(X) \subseteq \mathfrak{E}(X)$  for a second countable topological space  $X$  be the class of all separable  $C^*$ -algebras  $A$  over  $X$  with  $A(U) \in \mathcal{B}_E$  for all  $U \in \mathbb{O}(X)$ .

Since the functors  $\mathfrak{E}(X) \rightarrow \mathfrak{E}, A \mapsto A(U)$ , are triangulated and commute with direct sums and  $\mathcal{B}_E$  is a localising subcategory of  $\mathfrak{E}$ ,  $\mathcal{B}_E(X)$  is a localising subcategory of  $\mathfrak{E}(X)$ . Furthermore, if  $A \in \mathcal{B}_E(X)$ , then  $A(Y) \in \mathcal{B}_E$  for all locally closed subsets  $Y \subseteq X$  because of the extension  $A(U) \twoheadrightarrow A(V) \twoheadrightarrow A(Y)$  with  $Y = V \setminus U$  and suitable open subsets  $U$  and  $V$  in  $X$ .

**Proposition 4.2.** *Let  $X$  be a finite topological space and let  $A$  be a separable  $C^*$ -algebra over  $X$ . Then  $A \in \mathcal{B}_E(X)$  if and only if  $A(\{x\}) \in \mathcal{B}_E$  for all  $x \in X$ .*

If  $A$  is tight, that is, the map  $\text{Prim}(A) \rightarrow X$  is a homeomorphism, then the  $C^*$ -algebras  $A(\overline{\{x\}})$  for  $x \in X$  are precisely the *prime quotients* of  $A$ .

**Proof.** Since  $\mathcal{B}_E$  is triangulated, the class  $\text{Good}$  of locally closed subsets  $Y$  of  $X$  with  $A(Y) \in \mathcal{B}_E$  has the following property: if  $Y \subseteq Z$  and if two of  $Y, Z, Z \setminus Y$  belong to  $\text{Good}$ , then so does the third. We are going to prove that a set  $\text{Good}$  of subsets must contain all locally closed subsets if it has this two-out-of-three property and contains all point closures  $\overline{\{x\}}$ . The proof is by induction on the length of the subspace  $\bar{Y}$ , that is, the length of the largest chain  $x_0 < x_1 < \dots < x_\ell$  in the specialisation preorder on the closure  $\bar{Y}$ . If  $\ell = 0$ , the subspace  $Y$  is a set of closed points of  $X$ , and the assertion is easy.

Let  $Y$  be a locally closed subset of  $X$  of length  $\ell$ . Then  $Y = \bar{Y} \setminus \partial Y$ , so that it suffices to prove  $\bar{Y}, \partial Y \in \text{Good}$ . Therefore, we may assume without loss of generality that  $Y$  is closed. Let  $Z \subseteq Y$  be the set of all open points of  $Y$ . The difference  $\bar{Y} \setminus Z$  has length  $\ell - 1$  and is therefore good by induction assumption. If  $x \in Z$ , then the closure  $\overline{\{x\}}$  is good by assumption, and  $\overline{\{x\}} \setminus \{x\}$  is good because its length is at most  $\ell - 1$ . Hence  $\{x\}$  is good for all  $x \in Z$ . Since  $Z$  is discrete, it follows that  $Z$  is good. Hence so is  $Y$ .  $\square$

Similarly, if  $X$  is finite, then  $A \in \mathcal{B}_E(X)$  if and only if  $A(U_x) \in \mathcal{B}_E$  for all  $x \in X$ , where  $U_x$  denotes the minimal open subset of  $X$  containing  $x$ .

Proposition 4.2 remains true for some infinite spaces  $X$  as well. For instance, let  $X$  be a finite-dimensional, compact, metrisable Hausdorff space. It is proved in [8] that a continuous, separable and nuclear  $C(X)$ -algebra  $A$  lies in the bootstrap class  $\mathcal{B}$  if all its fibres  $A(x) = A(\overline{\{x\}})$  are in  $\mathcal{B}$ . Applying this to all closed subsets of  $X$ , we get  $A \in \mathcal{B}_E(X)$  under the same assumptions.

For finite spaces  $X$ , we may also describe the bootstrap class in terms of generators. For  $x \in X$  and a  $C^*$ -algebra  $A$ , let  $i_x A$  be  $A$  viewed as a  $C^*$ -algebra over  $X$  concentrated over  $x \in X$ , that is,  $i_x(A)(U) = A$  for  $x \in U$  and  $i_x(A)(U) = 0$  for  $x \notin U$ . This  $C^*$ -algebra over  $X$  satisfies

$$\text{KK}_*(X; i_x(A), B) \cong \text{KK}_*(A, B(U_x))$$

for all  $B$  by [20, Proposition 3.13]. The same argument with E-theory instead of KK-theory yields

$$E_*(X; i_x(A), B) \cong E_*(A, B(U_x)) \tag{4.3}$$

for  $x \in X$ , a  $C^*$ -algebra  $A$  and a  $C^*$ -algebra  $B$  over  $X$ . Here  $U_x$  denotes the minimal open neighbourhood of  $x$ , which exists because  $X$  is finite. Furthermore,

$$E_*(X; A, i_x(B)) \cong E_*(A(\overline{\{x\}}), B) \tag{4.4}$$

as in [20], even for infinite  $X$ , but we will not use this in the following.

**Proposition 4.5.** *Let  $X$  be a finite topological space. Then  $\mathcal{B}_E(X)$  is the localising subcategory of  $\mathfrak{E}(X)$  that is generated by  $i_x \mathbb{C}$  for all  $x \in X$ . The whole category  $\mathfrak{E}(X)$  is generated by  $C^*$ -algebras of the form  $i_x A$  for separable  $C^*$ -algebras  $A$  and  $x \in X$ .*

**Proof.** It is clear that  $i_x \mathbb{C} \in \mathcal{B}_E(X)$  and that  $\mathcal{B}_E(X)$  is localising, so that it contains the localising subcategory generated by  $i_x \mathbb{C}$  for  $x \in X$ . The same proof as for [20, Proposition 4.7] shows that

a  $C^*$ -algebra  $A$  over  $X$  belongs to the localising subcategory of  $\mathfrak{E}(X)$  generated by  $i_x(A(x))$  for all  $x \in X$ . The admissibility assumptions in [20] are only needed for  $KK$ , they become automatic in  $E$ -theory. In particular, this shows that  $\mathfrak{E}(X)$  is generated by  $C^*$ -algebras of the form  $i_x A$ , while  $\mathcal{B}_E(X)$  is generated by  $i_x A$  with  $A \in \mathcal{B}_E$ . Since  $\mathcal{B}_E$  is generated by  $\mathbb{C}$ , we may replace the set of  $i_x A$  with  $A \in \mathcal{B}_E(X)$  by  $i_x \mathbb{C}$  here.  $\square$

**Theorem 4.6.** *Let  $X$  be a second countable topological space and let  $A$  and  $B$  belong to  $\mathcal{B}_E(X)$ . An element in  $E_*(X; A, B)$  is invertible if and only if it induces invertible maps  $K_*(A(U)) \rightarrow K_*(B(U))$  for all  $U \in \mathbb{O}(X)$ .*

**Proof.** It is well known that an element in  $KK_*(A, B)$  that induces an isomorphism on  $K$ -theory is invertible in  $KK$  provided  $A$  and  $B$  belong to the bootstrap category. The same argument applies to  $E$ -theory. Finally, apply Theorem 3.10 and the definition of  $\mathcal{B}_E(X)$ .  $\square$

### 5. Comparing $KK$ - and $E$ -theory

In the definition of  $E$ -theory, we may restrict attention to asymptotic morphisms  $\varphi$  for which the maps  $\varphi_t$  are all completely positive contractions. It is shown by Houghton-Larsen and Thomsen [16] that the resulting variant of  $E$ -theory agrees with Kasparov’s  $KK$ . A corresponding result for equivariant  $KK$ - and  $E$ -theory is established by Thomsen in [31]. It is a routine exercise to show that the same works in our situation.

**Definition 5.1.** Let  $[[A, B]]_X^{\text{cp}}$  denote the space of homotopy classes of  $X$ -equivariant, completely positive, linear, contractive asymptotic morphisms  $\varphi$  from  $A$  to  $B$ , where homotopy is defined using  $X$ -equivariant, completely positive, linear, contractive asymptotic morphisms  $A \rightarrow C_b(T, C([0, 1], B))$ .  $X$ -equivariance means  $\varphi(A(U)) \subseteq C_b(T, B(U))$  for all  $U \in \mathbb{O}(X)$ .

The map  $\varphi : A \rightarrow C_b(T, B)$  is an  $X$ -equivariant, completely positive, linear contraction if and only if all the individual maps  $\varphi_t : A \rightarrow B$  are  $X$ -equivariant, completely positive, linear contractions.

**Theorem 5.2.** *There is a natural isomorphism*

$$KK_0(X; A, B) \cong [[C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K}]]_X^{\text{cp}}.$$

**Proof.** Copy the proofs of the corresponding assertions for non-equivariant Kasparov theory and equivariant Kasparov theory for group actions in [16,31]. The main point is to go through the proof of the universal property of  $E$ -theory and to check that the variant  $[[C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K}]]_X^{\text{cp}}$  satisfies an analogous universal property, but with exactness only for extensions of  $C^*$ -algebras over  $X$  with a completely positive contractive section over  $X$ . Since  $\mathfrak{K}\mathfrak{K}(X)$  satisfies the same universal property, the two theories must be naturally isomorphic.

Our case is somewhat closer to case of non-equivariant  $KK$  in [16] because some issues like Hilbert space representations of groups and equivariance of approximate units do not occur.  $\square$

**Corollary 5.3.** *Let  $X$  be a second countable topological space and let  $A$  be a  $C^*$ -algebra over  $X$  which is  $KK(X)$ -equivalent to a  $C^*$ -algebra over  $X$ ,  $A'$  such that any extension  $I \twoheadrightarrow E \twoheadrightarrow C_0(\mathbb{R}, A') \otimes \mathbb{K}$  of  $C^*$ -algebras over  $X$  has an  $X$ -equivariant completely positive contractive*

linear section. Then the canonical map  $KK_0(X; A, B) \rightarrow E_0(X; A, B)$  is an isomorphism for any  $C^*$ -algebra  $B$  over  $X$ .

**Proof.** We may assume that  $A = A'$ . Any asymptotic morphism is equivalent to one with  $\varphi_0 = 0$  – multiply pointwise with a suitable scalar-valued function. Hence it makes no difference whether we assume this for the definition of  $[[A, B]]_X$  and  $[[A, B]]_X^{cp}$ . An asymptotic morphism from  $A$  to  $B$  with  $\varphi_0 = 0$  generates an extension  $C_0(T, B) \twoheadrightarrow E \twoheadrightarrow A$  with  $E = \varphi(A) + C_0(T, B) \subseteq C_b(T, B)$ , and two asymptotic morphisms generate the same extension if and only if they are equivalent. The asymptotic morphism itself is a section for this extension. The assumption of the corollary therefore implies  $[[C_0(\mathbb{R}, A) \otimes \mathbb{K}, D]]_X^{cp} = [[C_0(\mathbb{R}, A) \otimes \mathbb{K}, D]]_X$  for all  $D$ .  $\square$

**Theorem 5.4.** Let  $X$  be a second countable locally compact Hausdorff space, let  $A$  be a nuclear and continuous  $C^*$ -algebra over  $X$ , and let  $B$  be any separable  $C^*$ -algebra over  $X$ . Then the canonical map  $KK_0(X; A, B) \rightarrow E_0(X; A, B)$  is an isomorphism.

**Proof.** The result follows from [24, Theorem 4.7]. Alternatively, we may argue that  $A$  is  $C_0(X)$ -nuclear by [1, Theorem 7.2], so that it satisfies the assumptions of Corollary 5.3.  $\square$

**Theorem 5.5.** Let  $X$  be a finite topological space and let  $(A, \psi_A)$  and  $(B, \psi_B)$  be  $C^*$ -algebras over  $X$ . The canonical map

$$KK_*(X; A, B) \rightarrow E_*(X; A, B)$$

is an isomorphism if  $A$  belongs to the bootstrap class in  $\mathfrak{K}\mathfrak{K}(X)$  defined in [20]. In particular, this applies if the  $C^*$ -algebra  $A(X)$  is nuclear.

**Proof.** If  $A$  belongs to the bootstrap class of [20], then we may compute  $KK_*(X; A, B)$  by a spectral sequence whose first page only involves the  $K$ -theory groups of  $A(U)$  and  $B(U)$  for minimal open subsets  $U$  in  $X$ . The arguments in [20] only use the universal property of  $\mathfrak{K}\mathfrak{K}(X)$  and work equally well for  $\mathfrak{E}(X)$ , with some simplifications because we do not have to worry about equivariant completely positive sections of various extensions. Thus there is an analogous spectral sequence computing  $E_*(X; A, B)$ , and it has the same first page as the spectral sequence computing  $KK_*(X; A, B)$ . The canonical map  $\mathfrak{K}\mathfrak{K}(X) \rightarrow \mathfrak{E}(X)$  provides a morphism between these spectral sequences, which is an isomorphism on the first page and thus on all later pages. Hence the two spectral sequences are isomorphic, so that  $KK_*(X; A, B) \cong E_*(X; A, B)$ .  $\square$

**Example 5.6.** We exhibit an extension of nuclear  $C^*$ -algebras over  $[0, 1]$  which is not excisive for  $KK([0, 1]; \underline{\quad}, B)$ . Consider the extension of  $C^*$ -algebras over  $[0, 1]$

$$0 \rightarrow C_0[0, 1] \rightarrow C[0, 1] \xrightarrow{\pi} \mathbb{C} \rightarrow 0,$$

where  $\pi(f) = f(1)$ . We claim that the mapping cone  $C_\pi$  is not  $KK([0, 1])$ -equivalent to  $\ker(\pi) = C_0[0, 1]$  and that

$$KK([0, 1]; SC, C_0[0, 1]) \neq E([0, 1]; SC, C_0[0, 1]).$$

Here  $S\mathbb{C}$  is regarded as a  $C[0, 1]$ -algebra via the multiplication  $f \cdot g = f(1)g$  for  $f \in C[0, 1]$  and  $g \in S\mathbb{C}$ . Let us address first the second part of the claim. It is convenient to work with asymptotic morphisms parametrised by  $t \in [0, 1)$ . For each such  $t$  consider the map  $v_t : [0, 1] \rightarrow [0, 1]$ ,

$$v_t(s) = \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{s-t}{1-t} & \text{if } t \leq s \leq 1. \end{cases}$$

Define a continuous family of  $*$ -homomorphisms  $\varphi_t : S\mathbb{C} \rightarrow C_0[0, 1)$ ,  $t \in [0, 1)$  by  $\varphi_t(\exp(2\pi is) - 1) := \exp(2\pi i v_t(s)) - 1$ . It is easily verified that the asymptotic homomorphism  $(\varphi_t)$  is asymptotically  $[0, 1]$ -equivariant since  $\exp(2\pi i v_t(s)) - 1$  is supported on  $[t, 1)$ . Set  $A = S\mathbb{C}$  and  $B = C_0[0, 1)$ . We observe that the class of  $(\varphi_t)$  in  $E([0, 1]; A, B)$  is non-zero since its image in  $\text{Hom}(K_1(A(0, 1)), K_1(B(0, 1))) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z})$  is equal to  $\text{id}_{\mathbb{Z}}$ . On the other hand,  $\text{KK}_*([0, 1]; A, B) = \text{KK}_*(S\mathbb{C}, \bigcap_n B((1 - 1/n, 1))) = \text{KK}_*(S\mathbb{C}, \{0\}) = 0$ , by [20, Proposition 3.13].

Let us verify now the first part of the claim. The Puppe sequence for  $\text{KK}([0, 1]; \underline{\cdot}, B)$  associated to the map  $\pi$  yields  $\text{KK}([0, 1], C_\pi, B) = 0$  since  $\text{KK}_*([0, 1], C[0, 1], B) = \text{KK}_*(\mathbb{C}, B) = 0$  and  $\text{KK}_*([0, 1]; \mathbb{C}, B) = 0$  as argued above. At the same time,  $\text{KK}_*([0, 1]; B, B) \neq 0$  since the natural map  $\text{KK}_*([0, 1]; B, B) \rightarrow \text{Hom}(K_1(B(0, 1)), K_1(B(0, 1))) \cong \mathbb{Z}$  sends  $[\text{id}_B]$  to 1.

**6. A universal coefficient theorem for  $C^*$ -algebras over totally disconnected spaces**

In this section, we study  $C^*$ -algebras over a totally disconnected compact metrisable space  $X$ . Our goal is to construct a Universal Coefficient Theorem that computes  $E_*(X; A, B)$  for  $A, B \in \mathcal{B}_E(X)$ . For this purpose, we use filtrated K-theory with coefficients and obtain a Universal Coefficient exact sequence that generalises the Multicoefficient Theorem of [11]. In order to explain the key role of filtrated K-theory with coefficients, we also revisit an example from [10] showing that the spectral sequence generated by filtrated K-theory does not degenerate to an exact sequence.

In this section, all  $C^*$ -algebras are assumed separable and all groups countable.

Let  $\mathcal{P} \subseteq \mathbb{N}$  be the set consisting of 0 and all prime powers. The relevance of the set  $\mathcal{P}$  in the Universal Multicoefficient Theorem is that the groups  $\mathbb{Z}/p$  for  $p \in \mathcal{P}$  are exactly the indecomposable abelian groups.

For  $p \in \mathcal{P}$  let  $\mathbb{I}_p$  be the mapping cone of the unital  $*$ -homomorphism  $\mathbb{C} \rightarrow \mathbb{M}_p(\mathbb{C})$ . For  $p = 0$ , we let  $\mathbb{I}_0 := \mathbb{C}$ . It is convenient to denote  $\mathbb{I}_p$  by  $\mathbb{I}_p^0$  and its suspension  $S\mathbb{I}_p$  by  $\mathbb{I}_p^1$ . Then for a  $C^*$ -algebra  $A$ :

$$K_i(A; \mathbb{Z}/p) := \text{KK}_i(\mathbb{I}_p, A) \cong \text{KK}(\mathbb{I}_p^i, A), \quad i = 0, 1.$$

Let us set  $\mathbb{I} := \bigoplus_{p \in \mathcal{P}} \mathbb{I}_p$  and consider the ring  $\text{KK}_*(\mathbb{I}, \mathbb{I})$  with multiplication given by the Kasparov product. The non-unital subring

$$A = \bigoplus_{p, q \in \mathcal{P}} \text{KK}_*(\mathbb{I}_p, \mathbb{I}_q)$$

of  $\text{KK}_*(\mathbb{I}, \mathbb{I})$  is called the ring of *Böckstein operations*. It consists of matrices indexed by  $\mathcal{P} \times \mathcal{P}$  with only finitely many non-zero entries  $\lambda_{pq} \in \text{KK}_*(\mathbb{I}_p, \mathbb{I}_q)$ . The Kasparov product

$$KK_*(\mathbb{I}_p, \mathbb{I}_q) \times KK_*(\mathbb{I}_q, A) \rightarrow KK_*(\mathbb{I}_p, A)$$

induces a natural  $\Lambda$ -module structure on the  $\mathbb{Z}/2 \times \mathcal{P}$ -graded group

$$\underline{K}(A) = \bigoplus_{p \in \mathcal{P}} K_*(A; \mathbb{Z}/p).$$

The KK-class  $x_p^i$  of  $\text{id}_{\mathbb{I}_p}$  generates the group  $K_i(\mathbb{I}_p, \mathbb{Z}/p) \cong KK(\mathbb{I}_p, \mathbb{I}_p^i)$ . We shall work with  $\mathbb{Z}/2 \times \mathcal{P}$ -graded  $\Lambda$ -modules  $M = (M_p^i)$  such that for  $\lambda \in KK_j(\mathbb{I}_q, \mathbb{I}_k)$  and  $m \in M_p^i$ ,  $\lambda m \in M_q^{j+i}$  if  $k = p$  and  $\lambda m = 0$  if  $k \neq p$ . We also ask that  $x_p^i$  acts as the identity automorphism on  $M_p^i$ . In particular, this implies that  $pM_p^i = 0$ . These assumptions are modelled on the case  $M = \underline{K}(A)$  where  $M_p^i = KK_*(\mathbb{I}_p^i, A)$ .

**Definition 6.1.** A  $\Lambda$ -module isomorphic to  $\underline{K}(\mathbb{I}_p^i)$  for some  $(i, p) \in \mathbb{Z}/2 \times \mathcal{P}$  is called *basic*.

**Lemma 6.2.** For all  $(i, p) \in \mathbb{Z}/2 \times \mathcal{P}$ ,  $\underline{K}(\mathbb{I}_p^i) = \Lambda \cdot x_p^i$ . The basic  $\Lambda$ -modules are projective in the category of  $\mathbb{Z}/2 \times \mathcal{P}$ -graded  $\Lambda$ -modules.

**Proof.** The first part follows because  $KK_*(\mathbb{I}_p, \mathbb{I}_p) \cong KK_*(\mathbb{I}_p^i, \mathbb{I}_p^i)$  and  $x_p^i = [\text{id}_{\mathbb{I}_p^i}]$  is idempotent. For the second part we observe that if  $\lambda x_p^i = 0$  for some  $\lambda \in KK_*(\mathbb{I}_q, \mathbb{I}_k)$  then either  $k \neq p$  or  $\lambda = 0$ . This shows that if  $\pi : B \rightarrow C$  is a surjective morphism of  $\Lambda$ -modules, then any morphism  $\varphi : \Lambda x_p^i \rightarrow C$  lifts to a morphism  $\Phi : \Lambda x_p^i \rightarrow B$  defined by  $\Phi(\lambda x_p^i) = \lambda b_p^i$ ,  $\lambda \in \Lambda$ , where  $b_p^i$  is some lifting of  $\varphi(x_p^i)$ .  $\square$

We give a very concise proof of the following result from [11].

**Proposition 6.3.** Let  $A$  and  $B$  be separable  $C^*$ -algebras and suppose that  $A$  is in the bootstrap class  $\mathcal{B}$  with  $K_*(A)$  finitely generated. Then  $KK(A, B) \cong \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ .

**Proof.** Both sides are additive in the first variable. Thus by the UCT we may assume that  $A = \mathbb{I}_p^i$  for some  $(i, p) \in \mathbb{Z}/2 \times \mathcal{P}$ . Let us observe that any element  $h \in \text{Hom}_\Lambda(\Lambda x_p^i, \underline{K}(B))$  is completely determined by  $h(x_p^i) \in K_i(B; \mathbb{Z}/p) \cong KK(\mathbb{I}_p^i, B)$ . Moreover, the image of  $h(x_p^i)$  under the map  $KK(\mathbb{I}_p^i, B) \rightarrow \text{Hom}_\Lambda(\underline{K}(\mathbb{I}_p^i), \underline{K}(B))$  is precisely  $h$ . Indeed, the Kasparov product  $KK(\mathbb{I}_p^i, \mathbb{I}_p^i) \times KK(\mathbb{I}_p^i, B) \rightarrow KK(\mathbb{I}_p^i, B)$  gives  $[\text{id}_{\mathbb{I}_p^i}] \times \alpha = \alpha$ .  $\square$

If  $A$  is a separable  $C^*$ -algebra over a zero-dimensional space  $X$ , then  $\underline{K}(A)$  has a natural structure of module over the ring  $C(X, \Lambda)$  of locally constant functions from  $X$  to  $\Lambda$ . This is easily seen by observing that  $A \cong \bigoplus_{k=1}^n A(U_k)$  for any clopen partition  $(U_k)_{k=1}^n$  of  $X$ . A  $C^*$ -algebra over  $X$  is called *elementary* if it is isomorphic to  $\bigoplus_{k=1}^n C(U_k, A_k)$ , where  $(U_k)_{k=1}^n$  is a clopen partition of  $X$ , each  $A_k$  is a separable  $C^*$ -algebra in the bootstrap class, and  $K_*(A_k)$  is finitely generated. If  $A$  is elementary, then the  $C(X, \Lambda)$ -module  $\underline{K}(A)$  is isomorphic to  $\bigoplus_{k=1}^n C(U_k, \underline{K}(A_k))$ . Since  $K_*(A_k)$  is finitely generated, it follows from the UCT that  $A_k$  is KK-equivalent to a finite direct sum of  $\mathbb{I}_p^i$ s, so that  $\underline{K}(A_k)$  is  $\Lambda$ -projective by Lemma 6.2. It follows easily that the  $C(X, \Lambda)$ -module  $\underline{K}(A_k)$  is projective.



**Lemma 6.4.** *Suppose that  $M$  is isomorphic to the inductive limit of an inductive system  $(M_j)$  of projective  $C(X, \Lambda)$ -modules. Then for any  $C(X, \Lambda)$ -module  $N$  there is a natural isomorphism*

$$\varprojlim^1 \text{Hom}_{C(X, \Lambda)}(M_j, N) \cong \text{Ext}_{C(X, \Lambda)}(M, N).$$

**Proof.** Set  $R = C(X, \Lambda)$ . The extension

$$0 \rightarrow \bigoplus_{j \in \mathbb{N}} M_j \xrightarrow{\text{Id}-S} \bigoplus_{j \in \mathbb{N}} M_j \rightarrow M \rightarrow 0,$$

where  $S$  is the natural shift map, is a projective resolution of  $M$ . Since  $\bigoplus_{j \in \mathbb{N}} M_j$  is projective, we have an exact sequence

$$\text{Hom}_R\left(\bigoplus_{j \in \mathbb{N}} M_j, N\right) \xrightarrow{(\text{id}-S)^*} \text{Hom}_R\left(\bigoplus_{j \in \mathbb{N}} M_j, N\right) \rightarrow \text{Ext}_R(M, N) \rightarrow 0,$$

where the first map identifies with the first map of the exact sequence

$$\prod_{j \in \mathbb{N}} \text{Hom}_R(M_j, N) \rightarrow \prod_{j \in \mathbb{N}} \text{Hom}_R(M_j, N) \rightarrow \varprojlim^1 \text{Hom}_R(M_j, N) \rightarrow 0$$

that defines  $\varprojlim^1$ . Thus the two maps have isomorphic cokernels.  $\square$

**Proposition 6.5.** *Let  $A$  be a separable nuclear continuous  $C^*$ -algebra over a totally disconnected compact metrisable space  $X$ . Suppose that each fibre of  $A$  belongs to the bootstrap class  $\mathcal{B}$ . Then  $A$  is  $\text{KK}(X)$ -equivalent to the inductive limit of an inductive system of elementary  $C(X)$ -algebras.*

**Proof.** [8, Theorem 2.5] shows that  $A$  is  $\text{KK}(X)$ -equivalent to a unital continuous  $C(X)$ -algebra  $A^\sharp$  whose fibres are Kirchberg algebras. Thus we may assume that  $A = A^\sharp$ . By [12, Theorem 3.6], there is a sequence  $(A_n)_{n=1}^\infty$  of elementary unital  $C(X)$ -subalgebras of  $A$  which is exhausting  $A$  in the sense that for every finite subset  $F$  of  $A$ ,  $\lim_{n \rightarrow \infty} \text{dist}(F, A_n) = 0$ . Since  $A_n$  is locally trivial and its fibres are weakly semiprojective [9, Section 3] each inclusion map  $\gamma_n : A_n \hookrightarrow A$  can be perturbed to some  $C(X)$ -linear unital  $*$ -monomorphism  $\gamma_{n,n+k} : A_n \rightarrow A_{n+k}$  with  $\|\gamma_n(a) - \gamma_{n,n+k}(a)\| < 1/2^n$  for  $a$  in a prescribed finite subset of  $A_n$ . It follows that after passing to a subsequence of  $(A_n)$  we can represent  $A$  as the inductive limit of a system  $(A_{n_k}, \gamma_{n_k, n_{k+1}})$  of elementary  $C(X)$ -algebras.  $\square$

**Lemma 6.6.** *Let  $A$  and  $B$  be separable  $C(X)$ -algebras over a totally disconnected compact metrisable space  $X$  and suppose that  $A$  is elementary. Then  $\text{KK}(X; A, B) \cong \text{Hom}_{C(X, \Lambda)}(\underline{\text{K}}(A), \underline{\text{K}}(B))$ .*

**Proof.** Write  $A = \bigoplus_{i=1}^k C(U_i, D_i)$  where  $U_1, \dots, U_k$  is a clopen partition of  $X$  and each  $D_i$  is in the bootstrap class with  $\text{K}_*(D_i)$  finitely generated. We have  $\text{KK}(X; A, B) \cong \bigoplus_{i=1}^k \text{KK}(U_i;$

$A(U_i), B(U_i))$  and

$$\text{Hom}_{C(X, \Lambda)}(\underline{K}(A), \underline{K}(B)) \cong \bigoplus_{i=1}^k \text{Hom}_{C(U_i, \Lambda)}(\underline{K}(A(U_i)), \underline{K}(B(U_i))).$$

Thus we may assume that  $A = C(X, D)$ . In this case, the assertion follows from the commutative diagram

$$\begin{array}{ccc} \text{KK}(X; C(X, D), B) & \longrightarrow & \text{Hom}_{C(X, \Lambda)}(\underline{K}(C(X, D)), \underline{K}(B)) \\ \cong \downarrow & & \downarrow \cong \\ \text{KK}(D, B) & \xrightarrow{\cong} & \text{Hom}_{\Lambda}(\underline{K}(D), \underline{K}(B)). \end{array}$$

The bottom horizontal map of the diagram is bijective by Proposition 6.3, the left vertical map by Lemma 2.30. The right vertical map is bijective because

$$\begin{aligned} \underline{K}(C(X, D)) &\cong C(X, \underline{K}(D)) \cong C(X, \mathbb{Z}) \otimes \underline{K}(D) \\ &\cong C(X, \mathbb{Z}) \otimes \Lambda \otimes_{\Lambda} \underline{K}(D) \cong C(X, \Lambda) \otimes_{\Lambda} \underline{K}(D) \end{aligned}$$

and

$$\text{Hom}_{C(X, \Lambda)}(C(X, \Lambda) \otimes_{\Lambda} \underline{K}(D), \underline{K}(B)) \cong \text{Hom}_{\Lambda}(\underline{K}(D), \underline{K}(B)). \quad \square$$

**Lemma 6.7.** Any separable  $C(X)$ -algebra over a totally disconnected compact metrisable space  $X$  is isomorphic to the inductive limit of a sequence of locally trivial separable  $C(X)$ -algebras.

**Proof.** Let  $A$  be a separable  $C(X)$ -algebra over  $X$ . If  $\mathcal{U}$  is a finite clopen cover of  $X$  we denote by  $A_{\mathcal{U}}$  the locally trivial continuous  $C(X)$ -algebra  $\bigoplus_{U \in \mathcal{U}} C(U) \otimes A(U)$ . For each  $x \in U$  the fibre  $A_{\mathcal{U}}(x)$  is  $A(U)$ . There is a natural morphism of  $C(X)$ -algebras  $\alpha_{\mathcal{U}} : A_{\mathcal{U}} \rightarrow A$  which maps  $(f_U \otimes a_U)_{U \in \mathcal{U}}$  to  $\sum_{U \in \mathcal{U}} f_U a_U$ .

If  $V$  is a closed subset of  $U$  we have a natural restriction homomorphism  $C(U) \otimes A(U) \rightarrow C(V) \otimes A(V)$ , which maps  $f \otimes a$  to  $f|_V \otimes \pi_V(a)$ . Therefore, if  $\mathcal{V}$  is a finite clopen cover of  $X$  which refines  $\mathcal{U}$ , there is a natural morphism of  $C(X)$ -algebras  $\alpha_{\mathcal{V}}^{\mathcal{U}} : A_{\mathcal{U}} \rightarrow A_{\mathcal{V}}$  such that  $\alpha_{\mathcal{V}} \circ \alpha_{\mathcal{U}}^{\mathcal{V}} = \alpha_{\mathcal{U}}$ .

Let  $(\mathcal{U}_n)_n$  be an infinite sequence of finite clopen covers of  $X$ , with  $\mathcal{U}_{n+1}$  refining  $\mathcal{U}_n$ , and such that  $\text{diam}(\mathcal{U}_n) \rightarrow 0$  with respect to some metric inducing the topology of  $X$ . Set  $A_n = A_{\mathcal{U}_n}$ ,  $\alpha_n = \alpha_{\mathcal{U}_n}$  and  $\alpha_n^m = \alpha_{\mathcal{U}_n}^{\mathcal{U}_m}$ . We claim that the natural morphism  $\varinjlim (A_n, \alpha_n^m) \rightarrow A$  is an isomorphism. This morphism is surjective since each  $\alpha_n$  is surjective. To prove its injectivity, it suffices to show that if  $F \in A_n$  satisfies  $\alpha_n(F) = 0$ , then for any  $\varepsilon > 0$  there is  $m > n$  such that  $\|\alpha_n^m(F)\| \leq \varepsilon$ . By localising at each element of  $\mathcal{U}_n$ , we may assume that  $A_n = C(X) \otimes A(X)$  and regard  $F$  as a continuous function  $F : X \rightarrow A(X)$ . Since  $F$  is continuous, each  $x \in X$  has a neighbourhood  $V_x$  such that  $\|F(x) - F(y)\| < \varepsilon/2$  for all  $y \in V_x$ . Since  $A(X)$  is a  $C(X)$ -algebra, for each  $a \in A(X)$ , the map  $x \mapsto \|\pi_x(a)\|$  is upper semi-continuous. The assumption  $\alpha_n(F) = 0$

implies that  $\pi_x(F(x)) = 0$  for all  $x \in X$ . Thus, after shrinking each  $V_x$  if necessary, we may arrange that  $\|\pi_z(F(x))\| < \varepsilon/2$  for all  $z \in V_x$ . It follows that for any  $y, z \in V_x$ ,

$$\|\pi_z(F(y))\| \leq \|\pi_z(F(y) - F(x))\| + \|\pi_z(F(x))\| < \varepsilon.$$

Extract now a finite cover  $V_{x_1}, \dots, V_{x_r}$  of  $X$ . Since  $\text{diam}(\mathcal{U}_m) \rightarrow 0$  there is  $m > n$  such that each element of  $\mathcal{U}_m$  is contained in some  $V_{x_i}$ . It follows that  $\|\alpha_n^m(F)\| \leq \varepsilon$ .  $\square$

**Proposition 6.8.** *Any separable  $C(X)$ -algebra over a totally disconnected compact metrisable space  $X$  is  $E(X)$ -equivalent to a continuous separable  $C(X)$ -algebra.*

**Proof.** For a given  $C(X)$ -algebra  $A$ , let  $(A_n, \alpha_n^m)$  be the corresponding inductive system constructed as in the proof of the previous lemma. Let

$$T(A_n, \alpha_n^m) = \left\{ (f_n) \in \bigoplus_{n \in \mathbb{N}} C([n, n + 1], A_n) : f_{n+1}(n + 1) = \alpha_n^{n+1}(f_n(n + 1)) \right\}$$

be the associated mapping telescope. Since the mapping telescope construction is functorial, there is a natural  $C(X)$ -linear  $*$ -homomorphism

$$\alpha : T(A_n, \alpha_n^m) \rightarrow T(A, \text{id}_A) \cong SA.$$

Arguing as in the paragraphs following the proof of [19, Proposition 2.6], it follows that  $\alpha$  is an  $E(X)$ -equivalence. Indeed, let  $\tilde{T}(A_m, \alpha_m^n)$  be the variant of  $T(A_m, \alpha_m^n)$  where we require  $\lim_{t \rightarrow \infty} \alpha_m(f_m(t))$  to exist in  $A$  instead of  $\lim f_m(t) = 0$ . The algebra  $\tilde{T}(A_m, \alpha_m^n)$  is contractible over  $X$  in a natural way. There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(A_m, \alpha_m^n) & \longrightarrow & \tilde{T}(A_m, \alpha_m^n) & \longrightarrow & A \longrightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & T(A, \text{id}) & \longrightarrow & \tilde{T}(A, \text{id}) & \longrightarrow & A \longrightarrow 0, \end{array}$$

whose rows are short exact sequences. Since the algebras in the middle are contractible, it follows that  $\alpha$  induces an  $E(X)$ -equivalence. We conclude by observing that  $T(A_m, \alpha_m^n)$  is a continuous  $C(X)$ -algebra since it is a  $C(X)$ -subalgebra of a direct sum of continuous  $C(X)$ -algebras.  $\square$

**Proposition 6.9.** *A separable and nuclear  $C(X)$ -algebra  $A$  over a totally disconnected compact metrisable space  $X$  belongs to the bootstrap class  $\mathcal{B}_E(X)$  if and only if all its fibres are in the bootstrap class  $\mathcal{B}_E$ .*

**Proof.** By Propositions 6.8 and 2.29, we may assume that  $A$  is a continuous  $C(X)$ -algebra. By a result of [8] a separable nuclear continuous  $C(X)$ -algebra over a finite-dimensional compact metrisable space  $X$  belongs to  $\mathcal{B}$  if and only if all its fibres belong to  $\mathcal{B}$ . This concludes the proof, since a nuclear  $C^*$ -algebra belongs to  $\mathcal{B}$  if and only if it belongs to  $\mathcal{B}_E$ .  $\square$

**Proposition 6.10.** *Let  $A$  be a separable  $C(X)$ -algebra over a totally disconnected compact metrisable space  $X$ . If  $A(U)$  is E-equivalent to a separable nuclear  $C^*$ -algebra for each clopen set  $U \subset X$ , then  $A$  is E( $X$ )-equivalent to a separable, continuous, nuclear  $C(X)$ -algebra.*

**Proof.** The proposition applies for instance when  $A$  belongs to the bootstrap class  $\mathcal{B}_E(X)$ . It was shown in [8, Lemma 2.2] that  $A$  is  $\text{KK}(X)$ -equivalent to a  $C(X)$ -algebra  $A'$  such that  $A' \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong A'$  and that  $A'$  contains a full projection. Thus we may assume that  $A$  itself has these properties. Let  $(A_n, \alpha_n^m)$  be the inductive system constructed in the proof of Lemma 6.7, that is,  $A_n$  is of the form  $\bigoplus_{k=1}^{r(n)} C(U_k) \otimes A(U_k)$  with a partition into clopen sets  $U_k$ . It is clear that  $A(U_k) \cong A(U_k) \otimes \mathcal{O}_\infty \otimes \mathbb{K}$  and that  $A(U_k)$  contains a full projections. By assumption, each  $C^*$ -algebra  $A(U_k)$  is E-equivalent to some nuclear separable  $C^*$ -algebra and hence it is E-equivalent to some stable Kirchberg algebra  $D_k$ . For each  $k$ , Kirchberg’s Classification Theorem [29, Theorem 8.3.3] yields a  $*$ -homomorphism  $\eta_k : D_k \rightarrow A(U_k)$  which lifts the given E-equivalence. Moreover, we may arrange that  $\eta_k$  decomposes as  $\eta_k = \mu_k \oplus \theta_k$ , where  $\theta_k$  is a full  $*$ -monomorphism that factors through the stable Cuntz algebra  $\mathcal{O}_2 \otimes \mathbb{K}$ . Extending the  $\eta_k$  by  $C(X)$ -linearity and taking their direct sum, we get a  $C(X)$ -linear monomorphism  $\varphi_n : B_n \rightarrow A_n$ , where  $B_n := \bigoplus_{k=1}^{r(n)} C(U_k) \otimes D_k$ . Moreover, each  $\varphi_n$  induces an equivalence in  $\mathfrak{E}(X)$ . Another application of [29, Theorem 8.3.3] yields  $C(X)$ -linear  $*$ -monomorphisms  $\beta_n^{n+1} : B_n \rightarrow B_{n+1}$  such that for each  $n$  the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\alpha_n^{n+1}} & A_{n+1} \\ \varphi_n \uparrow & & \uparrow \varphi_{n+1} \\ B_n & \xrightarrow{\beta_n^{n+1}} & B_{n+1} \end{array}$$

commutes in  $\mathfrak{E}(X)$  and hence in the category  $\mathfrak{KR}(X)$  (since each  $D_k$  is nuclear). The uniqueness part of [29, Theorem 8.3.3] shows that we may arrange that the diagram above commutes up to unitary homotopy. By [6, Section 2] this gives a  $C(X)$ -linear  $*$ -homomorphism  $\varphi : B \rightarrow C_b(T, A)/C_0(T, A)$ , where  $B$  is the limit of the inductive system  $(B_n, \beta_n^{n+1})$ , such that the diagram

$$\begin{array}{ccc} A_n & \longrightarrow & A \\ \varphi_n \uparrow & & \uparrow \varphi \\ B_n & \longrightarrow & B \end{array}$$

commutes in  $\mathfrak{E}(X)$ . By Proposition 2.28, for any separable  $C(X)$ -algebra  $D$  there is a commutative diagram with exact rows

$$\begin{array}{ccccc} \varprojlim^1 E_1(X; A_i, D) & \longrightarrow & E(X; A, D) & \twoheadrightarrow & \varprojlim E(X; A_i, D) \\ \downarrow \varphi_n^* & & \downarrow \varphi^* & & \downarrow \varphi_n^* \\ \varprojlim^1 E_1(X; B_i, D) & \longrightarrow & E(X; B, D) & \twoheadrightarrow & \varprojlim E(X; B_i, D). \end{array}$$

Since the maps  $\varphi_n^*$  are bijective by construction, we conclude that  $A$  is  $E(X)$ -equivalent to the nuclear continuous  $C(X)$ -algebra  $B$ .  $\square$

**Theorem 6.11.** *Let  $A$  and  $B$  be separable  $C(X)$ -algebras over a totally disconnected compact metrisable space  $X$ . If  $A$  is in the bootstrap class  $\mathcal{B}_E(X)$ , then there is an exact sequence*

$$\text{Ext}_{C(X,\Lambda)}(\underline{K}(A), \underline{K}(SB)) \rightarrow E(X; A, B) \rightarrow \text{Hom}_{C(X,\Lambda)}(\underline{K}(A), \underline{K}(B)).$$

**Proof.** By Proposition 6.10 we may assume that  $A$  is a continuous nuclear  $C(X)$ -algebra with all its fibres in the bootstrap class  $\mathcal{B}$ . Then  $E(X; A, B) \cong \text{KK}(X; A, B)$  by Theorem 5.4. By Proposition 6.5 we may also assume  $A \cong \varinjlim A_n$  for an increasing sequence  $(A_n)_{n=1}^\infty$  of elementary  $C^*$ -subalgebras of  $A$ . Then we can apply the  $\varinjlim^1$ -sequence for nuclear continuous  $C(X)$ -algebras and  $\text{KK}(X; \_, \_)$  to obtain the following exact sequence:

$$\varinjlim^1 \text{KK}_1(X; A_n, B) \rightarrow \text{KK}(X; A, B) \rightarrow \varinjlim \text{KK}(X; A_n, B).$$

By Lemma 6.6

$$\begin{aligned} \varinjlim \text{KK}(X; A_n, B) &\cong \varinjlim \text{Hom}_{C(X,\Lambda)}(\underline{K}(A_n), \underline{K}(B)) \\ &\cong \text{Hom}_{C(X,\Lambda)}(\varinjlim \underline{K}(A_n), \underline{K}(B)) \cong \text{Hom}_{C(X,\Lambda)}(\underline{K}(A), \underline{K}(B)). \end{aligned}$$

Using again Lemma 6.6 and Lemma 6.4, we get

$$\begin{aligned} \varinjlim^1 \text{KK}_1(X; A_n, B) &\cong \varinjlim^1 \text{Hom}_{C(X,\Lambda)}(\underline{K}(A_n), \underline{K}(SB)) \\ &\cong \text{Ext}_{C(X,\Lambda)}(\underline{K}(A), \underline{K}(SB)). \quad \square \end{aligned}$$

**Remark 6.12.** If  $A$  is a separable nuclear continuous  $C(X)$ -algebra with all the fibres in  $\mathcal{B}$ , then  $A \in \mathcal{B}_E(X)$ , and Theorem 5.4 shows that the exact sequence from Theorem 6.11 holds with  $\text{KK}(X; A, B)$  replacing  $E(X; A, B)$ .

For abelian groups  $G$  and  $H$ ,  $\text{PExt}_{\mathbb{Z}}(G, H)$  denotes the subgroup of  $\text{Ext}_{\mathbb{Z}}(G, H)$  generated by pure extensions, that is, extensions  $H \rightarrow E \rightarrow G$  whose restrictions to all finitely generated subgroups of  $G$  split. Theorem 6.11 is a generalisation of the main result of [11], which corresponds to the case when  $X$  reduces to a point.

**Proposition 6.13.** *Let  $A$  and  $B$  be separable  $C^*$ -algebras. If  $A \in \mathcal{B}$ , there is a natural isomorphism  $\text{Ext}_{\Lambda}(\underline{K}(A), \underline{K}(B)) \cong \text{PExt}_{\mathbb{Z}}(\mathbb{K}_*(A), \mathbb{K}_*(B))$ .*

**Proof.** Consider the natural restriction map

$$\eta : \text{Ext}_{\Lambda}(\underline{K}(A), \underline{K}(B)) \rightarrow \text{Ext}_{\mathbb{Z}}(\mathbb{K}_*(A), \mathbb{K}_*(B)).$$

Let  $\underline{K}(B) \rightarrow \underline{M} \rightarrow \underline{K}(A)$  be an extension of  $\Lambda$ -modules. We claim that its  $\eta$ -image  $\mathbb{K}_*(B) \rightarrow M_* \rightarrow \mathbb{K}_*(A)$  is pure. Purity follows if any element  $x$  in  $\mathbb{K}_i(A)$  of order  $n \in \mathcal{P}_{\geq 1}$  lifts to an element in  $M_i$  of the same order. Since  $x$  has order  $n$ , there is an element  $y \in \mathbb{K}_{i+1}(A; \mathbb{Z}/n)$  such that  $\beta_n(y) = x$ , because of the exactness of the sequence

$$K_{i+1}(A; \mathbb{Z}/n) \xrightarrow{\beta_n} K_i(A) \xrightarrow{n} K_i(A),$$

where  $\beta_n \in \Lambda$ . Let  $\hat{y} \in M_n^{i+1}$  be a lifting of  $y$ . Then the image  $\hat{x} := \beta_n(\hat{y}) \in M_0^i$  of  $\hat{y}$  is a lifting of  $x$  of order  $n$ . Thus the image of  $\eta$  is contained in  $\text{PExt}_{\mathbb{Z}}(K_*(A), K_*(B))$ .

Conversely, if  $K_*(B) \twoheadrightarrow G_* \twoheadrightarrow K_*(A)$  is a pure extension of  $\mathbb{Z}/2$ -graded abelian groups, then the UCT provides a separable  $C^*$ -algebra  $E$  and an extension of  $C^*$ -algebras  $B \otimes K \twoheadrightarrow E \twoheadrightarrow A$  such that  $K_*(B) \twoheadrightarrow K_*(E) \twoheadrightarrow K_*(A)$  is isomorphic to the given extension. We claim that  $\underline{K}(B) \twoheadrightarrow \underline{K}(E) \twoheadrightarrow \underline{K}(A)$  is an extension of  $\Lambda$ -modules. Purity yields extensions

$$\text{Tor}_q(K_*(B), \mathbb{Z}/n) \twoheadrightarrow \text{Tor}_q(K_*(E), \mathbb{Z}/n) \twoheadrightarrow \text{Tor}_q(K_*(A), \mathbb{Z}/n)$$

for any  $n \in \mathcal{P}$  and for  $q = 0, 1$ . Furthermore, there is a natural extension

$$\text{Tor}_0(K_*(A), \mathbb{Z}/n) \twoheadrightarrow K_*(A; \mathbb{Z}/n) \twoheadrightarrow \text{Tor}_1(K_*(A), \mathbb{Z}/n),$$

and the same for  $E$  and  $B$ . Now a diagram chase shows that  $K_*(B; \mathbb{Z}/n) \twoheadrightarrow K_*(E; \mathbb{Z}/n) \twoheadrightarrow K_*(A; \mathbb{Z}/n)$  is an extension.

Having identified the image of  $\eta$  as  $\text{PExt}_{\mathbb{Z}}(K_*(A), K_*(B))$ , it remains to show that  $\eta$  is injective. We may assume that  $A$  is nuclear. Suppose that the extension  $K_*(B) \twoheadrightarrow K_*(E) \twoheadrightarrow K_*(A)$  splits. By the UCT, the class of the extension  $B \otimes \mathbb{K} \twoheadrightarrow E \twoheadrightarrow A$  in  $\text{KK}_1(A, B)$  is zero. It follows that the extension  $B \otimes \mathbb{K} \twoheadrightarrow E \twoheadrightarrow A$  is stably split, so that the extension  $\underline{K}(B) \twoheadrightarrow \underline{K}(E) \twoheadrightarrow \underline{K}(A)$  is trivial.  $\square$

The following example adapted from [10] shows that the map

$$E(X; A, B) \rightarrow \text{Hom}_{C(X, \mathbb{Z})}(K_*(A), K_*(B))$$

is not always surjective.

**Example 6.14.** Let  $X = \mathbb{N} \cup \{\infty\}$  be the one-point compactification of  $\mathbb{N}$ . We shall exhibit two separable continuous  $C(X)$ -algebras  $E_k$  and  $E_{k'}$  with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that  $E_k$  and  $E_{k'}$  have isomorphic filtrated  $K$ -theory but non-isomorphic filtrated  $K$ -theory with coefficients.

Let  $A$  be a Kirchberg algebra in the bootstrap category with  $K_0(A) = 0$  and  $K_1(A) = \mathbb{Z}/n$  for  $n \geq 2$ . For  $k \in \mathbb{Z}/n$  let  $\varphi_k : A \rightarrow \mathcal{O}_\infty$  be a  $*$ -homomorphism such that  $[\varphi_k] = k \in \text{KK}(A, \mathcal{O}_\infty) \cong \mathbb{Z}/n$ . Consider the  $C(X)$ -algebra

$$E_k = \{(f, a) \in C(X, \mathcal{O}_\infty) \oplus A : f(\infty) = \varphi_k(a)\}.$$

We note that  $K_*(E_k) \cong K_*(E_{k'})$  as  $C(X, \mathbb{Z})$ -modules for any  $k, k'$ , and we claim that if  $k\mathbb{Z}/n \neq k'\mathbb{Z}/n$ , then  $\underline{K}(E_k) \not\cong \underline{K}(E_{k'})$  as  $C(X, \Lambda)$ -modules. Indeed,  $K_0(E_k) = K_0(E_{k'}) = C_0(X, \mathbb{Z})$  with  $C(X, \mathbb{Z})$  acting by pointwise multiplication and  $K_1(E_k) = K_1(E_{k'}) = \mathbb{Z}/n$  with  $C(X, \mathbb{Z})$ -module structure  $fm = f(\infty)m$  for  $m \in \mathbb{Z}/n$ . On the other hand,

$$K_0(E_k; \mathbb{Z}/n) = \{(f, r) \in C(X, \mathbb{Z}/n) \oplus \mathbb{Z}/n : f(\infty) = kr\}.$$

The coefficient map  $\rho : K_0(E_k) \rightarrow K_0(E_k; \mathbb{Z}/n)$  is  $g \mapsto (\dot{g}, 0)$ . The Bökstein map  $\beta : K_0(E_k; \mathbb{Z}/n) \rightarrow K_1(E_k)$  is  $\beta(f, r) = r$ .

Suppose that  $\alpha : \underline{K}(E_k) \rightarrow \underline{K}(E_{k'})$  is an isomorphism of  $C(X, \Lambda)$ -modules. Then  $\alpha$  must act on  $K_0$  by multiplication by a function  $u : X \rightarrow \{-1, 1\}$ . Since  $\alpha$  is  $C(X, \mathbb{Z})$ -linear and commutes with  $\rho$  and  $\beta$ , there is a unit  $v \in \mathbb{Z}/n$  such that  $\alpha : K_0(E_k; \mathbb{Z}/n) \rightarrow K_0(E_{k'}; \mathbb{Z}/n)$  is given by  $\alpha(f, r) = (uf, vr)$ . Choose  $f$  such that  $(f, 1) \in K_0(E_k)$ . It follows that for all sufficiently large  $i$  we have  $u(i)f(i) = k'v$  and hence  $\pm kr = k'v$ . Thus  $k\mathbb{Z}/n = k'\mathbb{Z}/n$ .

Next we generalise the previous example, constructing a suitable continuous  $C(X)$ -algebra over any compact Hausdorff space  $X$ .

**Example 6.15.** Let  $X$  be an infinite metrisable compact space. We shall exhibit two unital separable continuous  $C(X)$ -algebras  $F$  and  $F'$  with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that  $F$  and  $F'$  have isomorphic filtered K-theory but non-isomorphic filtered K-theory with coefficients.

Using the assumption on  $X$  we find a sequence  $(x_i)_{i=1}^\infty$  of distinct elements of  $X$  which converges to some  $x_\infty \in X$ . Fix an embedding  $\mathcal{O}_\infty \subset \mathcal{O}_2$ . For each  $k \in \mathbb{Z}/n$  let  $A$  and  $\varphi_k : A \rightarrow \mathcal{O}_\infty$  be as in Example 6.14. Consider the  $C(X)$ -algebra

$$F_k := \{(f, a) \in C(X, \mathcal{O}_2) \oplus A \mid f(x_i) \in \mathcal{O}_\infty \text{ for all } i \in \mathbb{N}, f(x_\infty) = \varphi_k(a)\}.$$

Choose  $k, k' \in \mathbb{Z}/n$  such that  $k\mathbb{Z}/n \neq k'\mathbb{Z}/n$  and set  $F = F_k$  and  $F' = F_{k'}$ . Then  $F$  and  $F'$  have non-isomorphic filtered K-theory with coefficients since their restrictions to the subspace  $Y := \{x_\infty\} \cup \{x_i : i \in \mathbb{N}\}$  are isomorphic to the  $C(Y)$ -algebras  $E_k$  and  $E_{k'}$  from Example 6.14, respectively. At the same time, we have an exact sequence of  $C(X)$ -algebras  $G \twoheadrightarrow F_k \twoheadrightarrow E_k$  with  $G = C_0(X \setminus Y, \mathcal{O}_2)$ . Since  $K_*(\mathcal{O}_2) = 0$ , we see that  $K_*(G(T \setminus Y)) = 0$  for all locally closed subsets  $T$  of  $X$ . It follows that the filtered K-theory of  $F$  is isomorphic to the filtered K-theory of  $F'$  since we have seen that  $E_k$  and  $E_{k'}$  have this property.

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