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New compact forms of the trigonometric Ruijsenaars–Schneider system

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Abstract

The reduction of the quasi-Hamiltonian double of $\mathrm{SU}(n)$ that has been shown to underlie Ruijsenaars' compactified trigonometric n-body system is studied in its natural generality. The constraints contain a parameter y, restricted in previous works to $0 < y < \pi/n$ because Ruijsenaars' original compactification relies on an equivalent condition. It is found that allowing generic $0 < y < \pi/2$ results in the appearance of new self-dual compact forms of two qualitatively different types depending on the value of y. The type (i) cases are similar to the standard case in that the reduced phase space comes equipped with globally smooth action and position variables, and turns out to be symplectomorphic to $\mathbb{C}P^{n-1}$ as a Hamiltonian toric manifold. In the type (ii) cases both the position variables and the action variables develop singularities on a nowhere dense subset. A full classification is derived for the parameter y according to the type (i) versus type (ii) dichotomy. The simplest new type (i) systems, for which $\pi/n < y < \pi/(n-1)$, are described in some detail as an illustration.

1. Introduction

The integrable many-body systems discovered by Ruijsenaars and Schneider [1] are popular due to their rich mathematical structure and connections to important areas of physics. These

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systems appear in topics extending from soliton equations to gauge theories and representation theory (see e.g. [1–8]). As limiting cases they contain the non-relativistic Calogero–Moser systems that also have many applications [9–11]. Several members of this family have been realized as Hamiltonian reductions of higher dimensional "free systems" ([12–16] and references therein), which permits an understanding of their dynamics and duality properties [17,18] in group-theoretic terms. In the current work *new* variants of the Ruijsenaars–Schneider (RS) system will be derived by exploiting the reduction method.

This paper is a continuation of joint work of the first author with Klimčík [19], where the self-dual compactified trigonometric RS system of Ruijsenaars [18] was interpreted as a reduced system arising from a double of $G := \mathrm{SU}(n)$. A key point of the quasi-Hamiltonian reduction used in [19] was the fixing of the G-valued moment map to the maximally degenerate non-scalar matrix

$$\mu_0(y) := \mathrm{diag}\big(e^{2\mathrm{i}y}, \dots, e^{2\mathrm{i}y}, e^{-2(n-1)\mathrm{i}y}\big) \tag{1.1}$$

with

$$0 < |y| < \pi/n. \tag{1.2}$$

The restriction (1.2) on the angle-parameter y was adopted in [19] from the very beginning, motivated (solely) by its eventual identification with a corresponding parameter in the "$\mathrm{III_b}$-system" of Ruijsenaars [18], where it was restricted to this range based on intuitive arguments.

The observation that prompted the present work is that in the scheme of quasi-Hamiltonian reduction there is no internal reason that requires restriction of the parameter y to the above range. Our goal is to explain that for any generic[1] $y \in (-\pi/2, \pi/2)$, the reduction built on the moment map value $\mu_0(y)$ always leads to a compact version of the trigonometric RS system, which is not equivalent to the one constructed in [18] unless (1.2) holds.

Before turning to the content of this paper, we need to recall some essential points of [19]. The starting point there is the so-called internally fused double [20] of G, given by

$$G \times G = \big\{(A, B)\big\} \tag{1.3}$$

equipped with the 2-form

$$\begin{aligned}\omega^\lambda := \lambda\Big(&\big\langle A^{-1}\mathrm{d}A \overset{\wedge}{,} \mathrm{d}BB^{-1}\big\rangle + \big\langle \mathrm{d}AA^{-1} \overset{\wedge}{,} B^{-1}\mathrm{d}B\big\rangle \\ &- \big\langle (AB)^{-1}\mathrm{d}(AB) \overset{\wedge}{,} (BA)^{-1}\mathrm{d}(BA)\big\rangle\Big),\end{aligned} \tag{1.4}$$

where $\lambda \neq 0$ is an arbitrary real constant and $\langle X, Y\rangle := -\frac{1}{2}\mathrm{tr}(XY)$. The 2-form, the moment map

$$\mu\colon (A, B) \mapsto ABA^{-1}B^{-1}, \tag{1.5}$$

and the componentwise conjugation action of G on $G \times G$, whereby

$$G \times (G \times G) \ni \big(\eta, (A, B)\big) \mapsto \big(\eta A\eta^{-1}, \eta B\eta^{-1}\big) \in G \times G, \tag{1.6}$$

satisfy the axioms of a quasi-Hamiltonian space [20]. As a result, the reduced phase space

$$P(\mu_0) := \mu^{-1}(\mu_0)/G_{\mu_0} \tag{1.7}$$

becomes (whenever it is smooth) a *symplectic* manifold. By applying the smooth class functions of G to either components of the pair $(A, B) \in G \times G$, one obtains two sets of G-invariant

[1] As in the abstract, one may restrict to $0 < y < \pi/2$ without losing generality.

functions on $G \times G$ that descend to two Abelian Poisson algebras on $P(\mu_0)$. Therefore $(n-1)$ independent class functions of $G = \mathrm{SU}(n)$ may reduce to Liouville integrable Hamiltonian systems if $P(\mu_0)$ is a smooth manifold of dimension $2(n-1)$. Note that $P(\mu_0)$ is always compact and connected [20] and the choice of μ_0 matters only up to conjugation. It turns out that the dimension of $P(\mu_0)$ is $2(n-1)$ if μ_0 is conjugate to $\mu_0(y)$ of the form (1.1) with generic $y \in (-\pi/2, \pi/2)$.

Under the restriction (1.2), the reduced phase space was identified in [19] as the complex projective space $\mathbb{C}\mathrm{P}^{n-1}$ carrying a multiple of the standard Fubini–Study symplectic form. The analysis relied on the one-to-one parametrization of the conjugacy classes of $\mathrm{SU}(n)$ by the Weyl alcove

$$\mathcal{A} := \left\{\xi \in \mathbb{R}^n \;\middle|\; \xi_k \geqslant 0 \ (\forall k = 1, \dots, n),\ \xi_1 + \dots + \xi_n = \pi\right\}. \tag{1.8}$$

Concretely, $\xi \in \mathcal{A}$ labels the conjugacy class represented by the diagonal matrix

$$\delta(\xi) = \mathrm{diag}\big(\delta_1(\xi), \dots, \delta_n(\xi)\big),$$
$$\delta_1(\xi) := e^{\frac{2\mathrm{i}}{n}\sum_{j=1}^n j\xi_j}, \qquad \delta_{k+1}(\xi) := e^{2\mathrm{i}\xi_k}\delta_k(\xi). \tag{1.9}$$

In order to present the characterization of the reduced system, introduce the "Weyl alcove with thick walls"

$$\mathcal{A}_y := \left\{\xi \in \mathcal{A} \;\middle|\; \xi_k \geqslant |y| \ (\forall k = 1, \dots, n)\right\}, \quad \text{for any } 0 < |y| < \pi/n, \tag{1.10}$$

and let $\mathcal{A}_y^+$ be the interior of $\mathcal{A}_y$. Consider the torus $\mathbb{T}^{n-1}$ with elements $(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}^{n-1}$ and equip the Cartesian product $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ with the symplectic form

$$\Omega^\lambda_{\mathrm{can}} := \lambda \sum_{k=1}^{n-1} \mathrm{d}\theta_k \wedge \mathrm{d}\xi_k. \tag{1.11}$$

Finally, extend the above definitions by the convention

$$\delta_{k+n} := \delta_k, \qquad \xi_{k+n} := \xi_k, \qquad \theta_{k+n} := \theta_k \quad \text{and} \quad \theta_0 := 0. \tag{1.12}$$

In [19] a dense open submanifold of the reduced phase space $P(\mu_0(y))$ was exhibited which is symplectomorphic to $(\mathcal{A}_y^+ \times \mathbb{T}^{n-1}, \Omega^\lambda_{\mathrm{can}})$ and permits the identification of the reduction of the invariant function $\Re(\mathrm{tr}(A))$ as the local Hamiltonian

$$H_y^{\mathrm{loc}}(\xi, \theta) := \sum_{j=1}^n \cos(\theta_j - \theta_{j-1}) \prod_{k=j+1}^{j+n-1} \left|1 + 4\frac{\sin^2 y}{[(\delta_k/\delta_j)^{1/2} - (\delta_k/\delta_j)^{-1/2}]^2}\right|^{\frac{1}{2}}. \tag{1.13}$$

Here, the square roots $(\delta_k/\delta_j)^{1/2}$ are a notational convenience, and we do not actually pick a branch for the square root since the square roots formally disappear after expanding the square. This Hamiltonian can be interpreted in terms of the interaction of n "particles" on the unit circle, located at $\delta_1, \dots, \delta_n$. Using (1.12), one has

$$\delta_k = \delta_j e^{2\mathrm{i}(\xi_j + \dots + \xi_{k-1})}, \quad \forall k = j+1, \dots, j+n-1, \tag{1.14}$$

and the Hamiltonian takes the Ruijsenaars–Schneider form of $\mathrm{III_b}$ type [18]:

$$H_y^{\mathrm{loc}}(\xi, \theta) = \sum_{j=1}^n \cos(\theta_j - \theta_{j-1}) \prod_{k=j+1}^{j+n-1} \left|1 - \frac{\sin^2 y}{\sin^2(\sum_{m=j}^{k-1}\xi_m)}\right|^{\frac{1}{2}}. \tag{1.15}$$

The condition $\xi_k \geqslant |y|$ in (1.10) means that the particles have a minimal angular distance given by $2|y|$, and this ensures that all functions under the absolute values above are non-negative. Since $\xi_1 + \cdots + \xi_n = \pi$, these features can occur only for $|y| \leqslant \pi/n$. In [18] these features were deemed desirable, and hence y was restricted to the range (1.2).

It is of course superfluous to write absolute values in the formulae (1.13) and (1.15) if all the relevant functions are non-negative. Our usage anticipates that there exist *new* systems having perfectly reasonable global properties and a similar local description as above, with the difference that some factors under the absolute values in the local formula (1.15) are non-positive. In fact, we shall demonstrate that for generic parameter y from the full range $(-\pi/2, \pi/2)$ the quasi-Hamiltonian reduction built on $\mu_0(y)$ (1.1) leads to a smooth reduced phase space that contains a maximal dense open submanifold parametrized by $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$, for some open $\mathcal{A}_y^+ \subset \mathcal{A}$, on which the symplectic form is provided by $\Omega_{\mathrm{can}}^{\lambda}$ (1.11) and the principal reduced Hamiltonian $\Re(\mathrm{tr}(A))$ is given (in general up to a sign) by the formula (1.15). In the general case, the domain $\mathcal{A}_y^+ \subset \mathcal{A}$ will be identified as a certain dense open subset of the set of $\xi \in \mathcal{A}$ for which $\delta(\xi)$ represents the conjugacy class of some regular unitary matrix B entering a pair $(A, B) \in \mu^{-1}(\mu_0(y))$. One of the main issues studied in the text is the dependence of $\mathcal{A}_y^+$ on y.

We shall classify the coupling parameter y according to the criterion of whether the relation

$$\mu^{-1}\big(\mu_0(y)\big) \subset G_{\mathrm{reg}} \times G_{\mathrm{reg}} \tag{1.16}$$

is valid or not, i.e., whether it is true or not that the constraint surface contains only regular matrices. The cases verifying (1.16) will be later called *type (i)* and those that violate (1.16), *type (ii)*. The relation (1.16) is known to hold in the standard case. Its validity guarantees that the distinct eigenvalues of A and B descend to *smooth* functions on the reduced phase space and give rise to globally smooth action variables and position variables of the associated compact RS system. Said in more technical terms, if (1.16) holds, then the reduced system carries two distinguished Hamiltonian torus action.

Our main new result is that we shall find all y values verifying (1.16), and shall prove that in these cases the reduced phase space is symplectomorphic to $\mathbb{C}\mathrm{P}^{n-1}$ with a multiple of the Fubini–Study symplectic structure. In fact, in these cases $\mathcal{A}_y^+$ will turn out to be an open simplex, whose closure lies in the interior of $\mathcal{A}$ and yields the moment polytope of the corresponding torus action. As listed by Theorem 12 in Section 3, there are many new cases different from (1.2) which fall into this category. The simplest such new cases are associated with the range

$$\pi/n < |y| < \pi/(n-1), \quad n \geqslant 3, \tag{1.17}$$

for which we obtain that

$$\mathcal{A}_y^+ = \big\{\xi \in \mathcal{A} \,\big|\, \xi_k < |y| \ (\forall k = 1, \ldots, n)\big\}. \tag{1.18}$$

We shall describe these examples in some detail, and show that the compact RS systems associated with the ranges (1.2) and (1.17) represent non-equivalent many-body systems. This means that the respective many-body Hamiltonians cannot be converted into each other by a canonical transformation that maps coordinates into coordinates. The same conclusion can be reached regarding any two coupling parameters y_1 and y_2 for which $\sin^2 y_1 \neq \sin^2 y_2$. We remark in passing that if ξ belongs to the domain (1.18), then precisely two of the factors under the absolute values in (1.15) are negative for each $j = 1, \ldots, n$.

The globally smooth class functions of G descend to smooth reduced Hamiltonians in involution also in the cases for which (1.16) is not valid, and engender Liouville integrable systems.

However, the action variables and the position variables arising from the eigenvalues of A and B develop singularities at the loci of the coinciding eigenvalues, which intersect $\mu^{-1}(\mu_0(y))$ when (1.16) does not hold. The actions and the positions enjoy a duality relation in all our reduced systems, and thus their qualitative properties are the same. This duality stems from a natural $\mathrm{SL}(2,\mathbb{Z})$ symmetry between A and B in the pair $(A,B)\in G\times G$, which survives reduction for any moment map value $\mu_0\in G$ [19].

As for the content of the rest of the paper, we first note that many of our arguments will be adaptations of arguments from [19]. We do not wish to repeat those in detail, but need to state clearly what changes and what remains true if the restriction (1.2) is dropped. This is done in Section 2, where we generalize relevant results from [19]. This section contains also significant novel results, e.g., the description of the fixed points of the torus action given by Corollary 4 of Lemma 3 and the important Theorem 6. Then we present entirely new results in Section 3. Theorem 7 gives the form of $\mathcal{A}_y^+$ for any generic y. Theorems 12 and 13 describe the full set of type (i) cases, i.e., all cases satisfying (1.16). As illustration, the simplest new systems of type (i) are detailed in Section 4. An example violating (1.16) will be also exposed briefly at the end of Section 4. The results and open problems are discussed in Section 5, and certain non-trivial details are relegated to appendices.

In Sections 1 and 2 it is often assumed that $-\pi/2<y<\pi/2$, while in Section 3 it will be more convenient to speak in terms of $0<y<\pi$. This should not cause any confusion, since y enters through $\mu_0(y)$ (1.1) and thus can matter at most modulo π. It is also worth noting that componentwise complex conjugation of the pair (A,B) gives an anti-symplectic diffeomorphism between $P(\mu_0(y))$ and $P((\mu_0(y))^{-1})$. By using this, it would be possible to restrict attention to $0<y<\pi/2$ without losing generality, but we here find it advantageous not to do so.

2. Results for generic value of the coupling parameter

We are interested in those reductions for which the reduced phase space (1.7) is a smooth manifold of dimension $2(n-1)$. It is readily extracted from Section 3.1 of [19] that this holds if and only if $e^{2\mathrm{i}y}$ is not an m-th root of unity for any $m=1,2,\dots,n$. In these cases the isotropy group[2] $G_{\mu_0(y)}/\mathbb{Z}_n=\mathrm{U}(n)_{\mu_0(y)}/\mathrm{U}(1)$ acts freely on $\mu^{-1}(\mu_0(y))$. We henceforth assume that y satisfies

$$e^{2\mathrm{i}ym}\neq 1,\quad \forall m=1,2,\dots,n. \tag{2.1}$$

One of the important points explained below is that if the relation $\mu^{-1}(\mu_0(y))\subset G_{\mathrm{reg}}\times G_{\mathrm{reg}}$ (1.16) is valid, then the reduced phase space is a Hamiltonian toric manifold. This means that $P(\mu_0(y))$ carries the effective Hamiltonian action of an $(n-1)$-dimensional torus $\mathbb{T}^{n-1}$. In other words, under (1.16) we obtain a compact integrable Hamiltonian system having globally smooth action variables [21]. Independently if (1.16) holds or not, we shall prove that the reduction leads to an integrable system on $P(\mu_0(y))$, which contains a dense open submanifold where the principal Hamiltonian descending from $\Re\operatorname{tr}(A)$ with $(A,B)\in\mu^{-1}(\mu_0(y))$ takes the RS form.

2.1. Recall of the β-generated torus action

Following [19], let us define the "spectral function" $\Xi: G\to\mathcal{A}$ by the requirements

[2] Note from (1.6) that only the group $G/\mathbb{Z}_n=\mathrm{U}(n)/\mathrm{U}(1)$ acts effectively on the double. For notational convenience, we will occasionally use the non-effective $\mathrm{U}(n)$-action instead.

$$\Xi\big(\delta(\xi)\big) := \xi \quad \text{and} \quad \Xi\big(\eta g \eta^{-1}\big) = \Xi(g), \quad \forall \eta, g \in G. \tag{2.2}$$

Note that Ξ is G-invariant, its real component functions are globally continuous on G, and their restrictions to G_{reg} belong to $C^\infty(G_{\text{reg}})$. It is also important to know that Ξ is not differentiable at $G_{\text{sing}} = G \setminus G_{\text{reg}}$ consisting of matrices with multiple eigenvalues (see Appendix A). It follows that the functions

$$\alpha: (A, B) \mapsto \Xi(A) \quad \text{and} \quad \beta: (A, B) \mapsto \Xi(B) \tag{2.3}$$

engender continuous maps

$$\hat{\alpha}: P\big(\mu_0(y)\big) \to \mathcal{A} \quad \text{and} \quad \hat{\beta}: P\big(\mu_0(y)\big) \to \mathcal{A}. \tag{2.4}$$

These maps are globally smooth if (1.16) is valid, in which case they take their values in the interior of the alcove, denoted

$$\mathcal{A}^{\text{reg}} := \big\{\xi \in \mathcal{A} \,\big|\, \xi_k \neq 0 \ (\forall k = 1, \ldots, n)\big\}. \tag{2.5}$$

Throughout this section, we restrict our attention to the open submanifold

$$\hat{\beta}^{-1}\big(\mathcal{A}^{\text{reg}}\big) \subset P\big(\mu_0(y)\big), \tag{2.6}$$

where the components of $\hat{\beta}$ are C^∞ functions. This submanifold equals $P(\mu_0(y))$ if (1.16) holds, and it will be shown to be an open dense subset for any y satisfying (2.1).

The linearly independent smooth functions

$$\hat{\beta}_j^\lambda := \lambda \hat{\beta}_j, \quad j = 1, \ldots, n-1, \tag{2.7}$$

induce Hamiltonian flows on the submanifold (2.6). These are 2π-periodic and thus generate a $\mathbb{T}^{n-1}$-action [19]. To describe this torus action, let us take a representative $(A, B) \in \mu^{-1}(\mu_0(y))$ of the point $[(A, B)] \in P(\mu_0(y))$. Then diagonalize B, that is, introduce $\xi \in \mathcal{A}^{\text{reg}}$ and $g \in G$ by

$$g B g^{-1} = \delta(\xi). \tag{2.8}$$

The action of

$$\tau = (\tau_1, \ldots, \tau_{n-1}) \in \mathbb{T}^{n-1} \tag{2.9}$$

is provided by the following formula:

$$\hat{\Psi}_\tau^\beta : \big[(A, B)\big] \mapsto \big[\big(A g^{-1} \varrho(\tau) g, B\big)\big] \tag{2.10}$$

with

$$\varrho(\tau) := \text{diag}(1/\tau_1, \tau_1/\tau_2, \tau_2/\tau_3, \ldots, \tau_{n-2}/\tau_{n-1}, \tau_{n-1}). \tag{2.11}$$

It can be shown (see below) that this Hamiltonian $\mathbb{T}^{n-1}$-action on $\hat{\beta}^{-1}(\mathcal{A}^{\text{reg}})$, which we call the β-generated torus action,[3] is an effective action.

Since $P(\mu_0(y))$ is compact and connected [20], we see that $P(\mu_0(y))$ *is a Hamiltonian toric manifold under the β-generated torus action whenever (1.16) is valid*. Then we can invoke the powerful Atiyah–Guillemin–Sternberg and Delzant theorems of symplectic geometry [22,23] that determine the structure of a Hamiltonian toric manifold in terms of the moment map. In particular, under (1.16), we know that the image of the map $\hat{\beta}$ is a closed convex polytope in $\mathcal{A}^{\text{reg}}$.

[3] Of course one also has an analogously operating α-generated torus action on $\hat{\alpha}^{-1}(\mathcal{A}^{\text{reg}})$ [19].

The polytope is the convex hull of its vertices, which are the images of the fixed points of the β-generated torus action. The correspondence between the vertices and the fixed points is one-to-one. Moreover, the polytope completely characterizes the Hamiltonian toric manifold.

On account of the above, at least in the presence of (1.16), we may establish the structure of $P(\mu_0(y))$ if we can find its image under the map $\hat{\beta}$. Next, we shall present a characterization of the image $\beta(\mu^{-1}(\mu_0(y)) \cap (G \times G_{\rm reg}))$, and study the equations that determine fixed points of the β-generated torus action.

2.2. The β-regular part of the reduced phase space

The open submanifold $\hat{\beta}^{-1}(\mathcal{A}^{\rm reg})$ (2.6) will be called the β-regular part of the reduced phase space. Here, we are interested in the $\hat{\beta}$-image of this submanifold, given by

$$\mathcal{A}_y^{\rm reg} := \hat{\beta}\big(P\big(\mu_0(y)\big)\big) \cap \mathcal{A}^{\rm reg} = \beta\big(\mu^{-1}\big(\mu_0(y)\big) \cap (G \times G_{\rm reg})\big). \tag{2.12}$$

Our description of this image relies on the functions $z_\ell(\xi, y)$ defined on $\mathcal{A}^{\rm reg}$ by the formula

$$z_\ell(\xi, y) := \frac{e^{2{\rm i}y} - 1}{e^{2n{\rm i}y} - 1} \prod_{\substack{j=1\\ j\neq\ell}}^{n} \frac{\delta_j(\xi) - e^{2{\rm i}y}\delta_\ell(\xi)}{\delta_j(\xi) - \delta_\ell(\xi)}, \qquad \forall \ell = 1, \ldots, n. \tag{2.13}$$

By using formula (1.9) and the periodicity convention (1.12) we can spell out this function as

$$z_\ell(\xi, y) = \frac{\sin(y)}{\sin(ny)} \prod_{\substack{j=1\\ j\neq\ell}}^{n} \frac{e^{{\rm i}y}\delta_\ell - e^{-{\rm i}y}\delta_j}{\delta_\ell - \delta_j} = \frac{\sin(y)}{\sin(ny)} \prod_{j=\ell+1}^{\ell+n-1} \left[\frac{\sin(\sum_{m=\ell}^{j-1} \xi_m - y)}{\sin(\sum_{m=\ell}^{j-1} \xi_m)}\right]. \tag{2.14}$$

The proof of the following result can be extracted from Section 3.2 of [19]. Nevertheless we sketch it here since it is required for our later arguments.

Lemma 1. *The element $\xi \in \mathcal{A}^{\rm reg}$ belongs to the $\hat{\beta}$-image (2.12) if and only if $z_\ell(\xi, y)$ is non-negative for all $\ell = 1, \ldots, n$.*

Proof. Suppose that we have

$$ABA^{-1}B^{-1} = \mu_0(y). \tag{2.15}$$

Since B is conjugate to $\delta(\xi)$ with some $\xi \in \mathcal{A}$, (2.15) is equivalent to

$$A^g \delta(\xi) \big(A^g\big)^{-1} = \big(g\mu_0(y)g^{-1}\big)\delta(\xi), \tag{2.16}$$

where g is a unitary matrix for which

$$\delta(\xi) = gBg^{-1} \quad \text{and} \quad A^g = gAg^{-1}. \tag{2.17}$$

Denoting by $v \in \mathbb{C}^n$ the last column of the matrix g,

$$v_\ell := g_{\ell n}, \tag{2.18}$$

it is easily checked that

$$g\mu_0(y)g^{-1} = e^{2{\rm i}y}\mathbf{1}_n + \big(e^{2{\rm i}(1-n)y} - e^{2{\rm i}y}\big)vv^\dagger. \tag{2.19}$$

Eq. (2.16) implies the equality of the characteristic polynomials of the matrices on the two sides, which gives

$$\prod_{j=1}^{n}\big(\delta_j(\xi)-x\big)=\prod_{j=1}^{n}\big(\delta_j(\xi)e^{2\mathrm{i}y}-x\big)+\big(e^{2\mathrm{i}(1-n)y}-e^{2\mathrm{i}y}\big)\sum_{k=1}^{n}\Bigg(|v_k|^2\delta_k(\xi)\prod_{\substack{j=1\\ j\neq k}}^{n}\big(\delta_j(\xi)e^{2\mathrm{i}y}-x\big)\Bigg) \tag{2.20}$$

for all $x\in\mathbb{C}$. Supposing now that B is regular, evaluation of (2.20) at the n distinct values $x=\delta_\ell(\xi)e^{2\mathrm{i}y}$ leads to the equations

$$|v_\ell|^2=z_\ell(\xi,y) \tag{2.21}$$

with the functions defined in (2.13). Therefore these functions must be non-negative for all ξ in the image (2.12).

Conversely, suppose that all z_ℓ in (2.13) are non-negative at $\xi\in\mathcal{A}^{\mathrm{reg}}$. Choose $v=v(\xi,y)\in\mathbb{C}^n$ for which $|v_\ell|^2=z_\ell(\xi,y)$. Then we observe that the equality (2.20) holds at all $x\in\mathbb{C}$ since we can check that it holds at the n distinct values $\delta_\ell(\xi)e^{2\mathrm{i}y}$. Evaluating this equality at $x=0$ implies that the vector $v(\xi,y)$ has unit norm, and consequently the right-hand side of (2.19) with this vector defines a unitary matrix of unit determinant, now denoted as $\mu_{v(\xi,y)}$. Since (2.20) guarantees that the unitary matrices $\delta(\xi)$ and $\mu_{v(\xi,y)}\delta(\xi)$ have the same spectra, there exists a unitary matrix, say A_0, for which

$$A_0\delta(\xi)A_0^{-1}=\mu_{v(\xi,y)}\delta(\xi), \tag{2.22}$$

and we can normalize A_0 to have unit determinant, yielding $A_0\in G$. Then we take a unitary matrix g having $v(\xi,y)$ as its last column and conjugate both sides of (2.22) by g^{-1}. This allows to conclude that

$$A:=g^{-1}A_0g\quad\text{and}\quad B:=g^{-1}\delta(\xi)g \tag{2.23}$$

satisfy (2.15), i.e., $(A,B)\in\mu^{-1}(\mu_0(y))$ and $\beta(A,B)=\xi$ holds. □

Remark 2. The special element $\xi^*\in\mathcal{A}^{\mathrm{reg}}$ having equal components

$$\xi_k^*:=\pi/n,\quad\forall k=1,\dots,n, \tag{2.24}$$

is in the image (2.13) for all allowed values of y. Indeed, one can check that

$$z_\ell\big(\xi^*,y\big)=\frac{\sin(y)}{\sin(ny)}\prod_{k=1}^{n-1}\left[\frac{\sin(k\frac{\pi}{n}-y)}{\sin(k\frac{\pi}{n})}\right]>0,\quad\forall\ell=1,\dots,n, \tag{2.25}$$

at any admissible value of y. The point is that if $\frac{m\pi}{n}<y<\frac{(m+1)\pi}{n}$ for some $m=0,\dots,n-1$, then m factors in the above product are negative and $(n-m-1)$ factors are positive.[4] This yields exactly the right parity to cancel the possible minus sign from $\sin(ny)$.

[4] We here took y from the interval $(0,\pi)$ instead of $(-\pi/2,\pi/2)$, which is permitted since only its value modulo π appears in $\mu_0(y)$ (1.1).

As a spin-off from the above proof, we can in principle construct all elements of the β-regular part of the constraint surface $\mu^{-1}(\mu_0(y))$ by the following algorithm. First, take $\xi \in \mathcal{A}^0$ for which $z_\ell(\xi, y)$ is non-negative for all ℓ, and define

$$v_\ell(\xi, y) := \sqrt{z_\ell(\xi, y)} \tag{2.26}$$

using non-negative square roots. Choosing a unitary matrix $g := g(v)$ that has v as its last column and taking $A_0 \in G$ subject to (2.22), define (A, B) according to (2.23). Then the most general element of $\mu^{-1}(\mu_0(y))$ for which β takes the value ξ is a gauge transform of an element of the following form:

$$\big(Ag(v)^{-1}\varrho g(v), B\big) \quad \text{with } \varrho \in \mathrm{S}\mathbb{T}^n := \mathrm{SU}(n) \cap \mathbb{T}^n. \tag{2.27}$$

This holds because Eq. (2.22) determines A_0 up to right multiplication by a diagonal matrix, leading to ϱ in the formula(2.27). The result could be made more explicit by actually solving Eq. (2.22) for A_0. In fact, we shall give a fully explicit formula in the next subsection.

One sees from (2.10) that for fixed $\xi = \Xi(B) \in \mathcal{A}_y^{\mathrm{reg}}$ the set of gauge equivalence classes

$$\big\{\big[\big(Ag(v)^{-1}\varrho g(v), B\big)\big] \bigm| \varrho \in \mathrm{S}\mathbb{T}^n\big\} \tag{2.28}$$

is an orbit of the β-generated torus action. Thus the above construction implies the *transitivity* of the torus action on $\hat{\beta}^{-1}(\xi)$ for all ξ in the image (2.12).

The next lemma provides a characterization of the stability subgroups for the β-generated torus action on $\hat{\beta}^{-1}(\mathcal{A}^{\mathrm{reg}})$.

Lemma 3. *Consider ξ from the image (2.12) and define $\mathbb{T}^n[v] < \mathbb{T}^n$ to be the subgroup whose elements have $v := v(\xi, y)$ (2.26) as their eigenvector. Take an element $[(A, B)] \in P(\mu_0(y))$ that verifies $\Xi(B) = \xi$. Then $\hat{\Psi}_\tau^\beta([(A, B)]) = [(A, B)]$ holds for precisely those $\tau \in \mathbb{T}^{n-1}$ for which*

$$\varrho(\tau) = \big(gA^{-1}g^{-1}\big)\zeta\big(gAg^{-1}\big)\zeta^{-1} \quad \textit{with some } \zeta \in \mathbb{T}^n[v]. \tag{2.29}$$

Here (A, B) is a representative of $[(A, B)]$, g is any unitary matrix subject to $gBg^{-1} = \delta(\xi)$ and $\varrho(\tau)$ refers to (2.11). The mapping $\zeta \mapsto \varrho(\tau)$ defines a homomorphism from $\mathbb{T}^n[v]$ onto the stabilizer subgroup of $[(A, B)]$ with respect to the β-generated torus action, whose kernel is given by the scalar matrices in $\mathbb{T}^n$.

Proof. Suppose that $[(A, B)]$ is fixed by $\varrho := \varrho(\tau)$ (2.11). Choosing a representative (A, B), this is equivalent to the existence of some $h \in G_{\mu_0(y)}$ that satisfies

$$\big(Ag^{-1}\varrho g, B\big) = \big(hAh^{-1}, hBh^{-1}\big). \tag{2.30}$$

Allowing h to be in $\mathrm{U}(n)_{\mu_0(y)}$, the second component says that

$$h = g^{-1}\zeta g \tag{2.31}$$

for some $\zeta \in \mathbb{T}^n$. It is easily seen that h (2.31) belongs to the little group of $\mu_0(y)$ if and only if $v(\xi, y)$ is an eigenvector of the diagonal matrix ζ. We can then solve the equality

$$Ag^{-1}\varrho g = hAh^{-1} = g^{-1}\zeta gAg^{-1}\zeta^{-1}g \tag{2.32}$$

for ϱ as $\varrho = (gA^{-1}g^{-1})\zeta(gAg^{-1})\zeta^{-1}$, which is just the formula (2.29).

It remains to show that the right-hand side of (2.29) defines an element in the stabilizer of $[(A, B)]$ for any $\zeta \in \mathbb{T}^n[v]$. For this, recall that the moment map constraint is equivalent to

$$\big(gAg^{-1}\big)\delta(\xi)\big(gAg^{-1}\big)^{-1} = \mu_v\delta(\xi), \tag{2.33}$$

where v is the last column of g and μ_v is given by (2.19). (By a choice of g we may arrange that $v = v(\xi, y)$ (2.26), but this is inessential: all vectors whose components have the same absolute values are eigenvectors of the same diagonal unitary matrices.) Conjugating this equation by ζ that has v is its eigenvector, we see that

$$\big(gAg^{-1}\big)\delta(\xi)\big(gAg^{-1}\big)^{-1} = \big(\zeta\big(gAg^{-1}\big)\big)\delta(\xi)\big(\zeta\big(gAg^{-1}\big)\big)^{-1}, \tag{2.34}$$

which implies that

$$\zeta gAg^{-1} = gAg^{-1}\eta(\zeta) \tag{2.35}$$

for some $\eta(\zeta) \in \mathbb{T}^n$. Therefore $\varrho := \eta(\zeta)\zeta^{-1}$ is also diagonal, and it belongs to the stabilizer of $[(A, B)]$ since (2.35) implies (2.30) with $h := g^{-1}\zeta g \in \mathrm{U}(n)_{\mu_0(y)}$.

It is readily verified that the map $\zeta \mapsto \varrho(\tau)$ (2.29) is a homomorphism, which does not depend on the choices (of (A, B) and g) that were made in its construction. To finish the proof, suppose that ζ is in the kernel of this homomorphism. This means that

$$\zeta A^g\zeta^{-1} = A^g \quad \text{for } A^g := gAg^{-1}, \tag{2.36}$$

and since $gBg^{-1} = \delta(\xi)$ we conclude that $g^{-1}\zeta g$ fixes (A, B) by the componentwise conjugation action. Since we know [19] that $\mathrm{U}(n)_{\mu_0(y)}/\mathrm{U}(1)$ acts freely on $\mu^{-1}(\mu_0(y))$, we obtain that $g^{-1}\zeta g$ must belong to the scalar matrices $\mathrm{U}(1) < \mathrm{U}(n)$, and hence ζ has the same property. □

Those vectors $v(\xi, y)$ that have only non-vanishing components are eigenvectors of the scalar elements of $\mathbb{T}^n$ only, and therefore the β-generated torus action is *free* on the corresponding fibres $\hat{\beta}^{-1}(\xi)$ (for example on $\hat{\beta}^{-1}(\xi^*)$ with ξ^* in (2.24)). In particular, this shows that the torus action is effective. On the other hand, using the fact that the common eigenvectors of $\mathbb{T}^n$ are those vectors that have a single non-zero component, Lemma 3 implies the following useful statement.

Corollary 4. *The fixed points of the β-generated torus action are in one-to-one correspondence with the set of $\xi \in \mathcal{A}^{\mathrm{reg}}$ for which the equations*

$$z_\ell(\xi, y) = \delta_{k,\ell}, \quad \ell = 1, \dots, n, \tag{2.37}$$

hold for some arbitrarily fixed $k \in \{1, 2, \dots, n\}$.

Remark 5. The center $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ of $\mathrm{SU}(n)$ acts on $\mathcal{A}$ as given by the following action of the generator σ:

$$\sigma(\xi)_k := \xi_{k+1}. \tag{2.38}$$

One can check from (2.14) that

$$z_\ell\big(\sigma(\xi), y\big) = z_{\ell+1}(\xi, y), \quad \forall \xi \in \mathcal{A}^{\mathrm{reg}}, \tag{2.39}$$

with the convention $z_{\ell+n} := z_\ell$. It follows that the image $\mathcal{A}_y^{\mathrm{reg}}$ (2.12) as well as the set of fixed points of the β-generated torus action are invariant under this action of $\mathbb{Z}_n$. Moreover, Corollary 4 implies that the $\mathbb{Z}_n$-orbit of any chosen fixed point of the torus action consists of n different fixed points. By noting that the $\mathbb{Z}_n$-action engendered by (2.38) is inherited from the action of the center of $\mathrm{SU}(n)$ on $\mathrm{SU}(n)$ by left-multiplications, it is readily seen that the full image

$$\mathcal{A}_y := \hat{\beta}\big(P\big(\mu_0(y)\big)\big) \tag{2.40}$$

is also mapped to itself by σ.

2.3. RS system on dense open submanifold of $P(\mu_0(y))$

We show below that the reduction leads to an integrable system whose "principal Hamiltonian" takes the RS form (1.15) on a dense open submanifold of the reduced phase space. For our characterization of this system, it will be useful to decompose $\mathcal{A}_y$ (2.40) into the union of 3 disjoint subsets:

$$\mathcal{A}_y = \mathcal{A}_y^{\rm reg} \cup \mathcal{A}_y^{\rm sing} = \mathcal{A}_y^{+} \cup \mathcal{A}_y^{=} \cup \mathcal{A}_y^{\rm sing}, \tag{2.41}$$

where $\mathcal{A}_y^{\rm sing} := \mathcal{A}_y \cap \partial\mathcal{A}$ and

$$\mathcal{A}_y^{+} := \big\{\xi \in \mathcal{A}^{\rm reg} \bigm| z_\ell(\xi, y) > 0,\ \forall \ell = 1, \dots, n\big\}, \tag{2.42}$$

$$\mathcal{A}_y^{=} := \left\{\xi \in \mathcal{A}^{\rm reg} \,\middle|\, z_\ell(\xi, y) \geqslant 0,\ \forall \ell = 1, \dots, n,\ \prod_{\ell=1}^{n} z_\ell(\xi, y) = 0\right\}. \tag{2.43}$$

Their significance is that the β-generated torus action is free on $\hat{\beta}^{-1}(\mathcal{A}_y^{+})$, has non-trivial isotropy groups on $\hat{\beta}^{-1}(\mathcal{A}_y^{=})$, and is not defined at all on $\hat{\beta}^{-1}(\mathcal{A}_y^{\rm sing})$ (which is empty if (1.16) holds). It turns out that these sets depend only on the absolute value of $y \in (-\pi/2, \pi/2)$, and each of them is mapped to itself by the cyclic permutation σ (2.39) and the "partial reflection" ν that maps ξ to $\nu(x)$ according to

$$\nu(\xi)_k = \xi_{n-k} \quad \forall k = 1, \dots, n-1 \quad \text{and} \quad \nu(\xi)_n = \xi_n. \tag{2.44}$$

In order to derive the above mentioned properties, we begin by pointing out that the equalities

$$\alpha\big(\mu^{-1}(\mu_0)\big) = \beta\big(\mu^{-1}(\mu_0)\big) = \alpha\big(\mu^{-1}\big(\mu_0^{-1}\big)\big) = \beta\big(\mu^{-1}\big(\mu_0^{-1}\big)\big) \tag{2.45}$$

are valid for any moment map value $\mu_0 \in G$. To see this, first remark [19] that

$$(A, B) \in \mu^{-1}(\mu_0) \quad\Longleftrightarrow\quad S(A, B) := \big(B^{-1}, BAB^{-1}\big) \in \mu^{-1}(\mu_0). \tag{2.46}$$

On $\mu^{-1}(\mu_0)$ we thus have

$$\alpha = \beta \circ S, \tag{2.47}$$

and since S is a diffeomorphism of $\mu^{-1}(\mu_0)$ this entails that the α-image of $\mu^{-1}(\mu_0)$ is the same as its β-image. Second, by inverting the group commutator, notice that

$$(A, B) \in \mu^{-1}(\mu_0) \quad\Longleftrightarrow\quad (B, A) \in \mu^{-1}\big(\mu_0^{-1}\big), \tag{2.48}$$

which implies the second equality in (2.45).

Since $\mu_0(y)^{-1} = \mu_0(-y)$, we conclude from the above that

$$\beta\big(\mu^{-1}\big(\mu_0(y)\big)\big) = \beta\big(\mu^{-1}\big(\mu_0(-y)\big)\big). \tag{2.49}$$

We also observe from (2.13) that if ξ is such an element of $\mathcal{A}^{\rm reg}$ for which $z_\ell(\xi, y)$ is non-zero for all $\ell = 1, \dots, n$, then ξ verifies the same property for $-y$. Taking advantage of the identity $\Xi_k(\delta(\xi)^{-1}) = \nu(\xi)_k$ and componentwise complex conjugation of the pair $(A, B) \in \mu^{-1}(\mu_0(y))$, it follows that $\mathcal{A}_y = \mathcal{A}_{-y}$ (2.40) is stable under the involution $\xi \mapsto \nu(\xi)$.

We now focus on the subset of $P(\mu_0(y))$ given by the inverse image $\hat{\beta}^{-1}(\mathcal{A}_y^+)$. Note that ξ^* (2.24) always belongs to $\mathcal{A}_y^+$, which is therefore a non-empty open subset of $\mathcal{A}^{\rm reg}$. Since $\hat{\beta}$ is continuous, $\hat{\beta}^{-1}(\mathcal{A}_y^+) \subset P(\mu_0(y))$ is a non-empty open submanifold.

Define the smooth matrix function $\mathcal{L}_y^{\rm loc}$ on $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ by the formula

$$\mathcal{L}_y^{\rm loc}(\xi,\tau)_{j\ell} := \frac{\sin(ny)}{\sin(y)} \frac{e^{{\rm i}y} - e^{-{\rm i}y}}{e^{{\rm i}y}\delta_j(\xi)\delta_\ell(\xi)^{-1} - e^{-{\rm i}y}} v_j(\xi,y) v_\ell(\xi,-y) \varrho(\tau)_\ell. \tag{2.50}$$

Further, taking any vector $v \in \mathbb{R}^n$ that has unit norm and component $v_n \neq -1$, introduce the unitary matrix $g(v) \in {\rm U}(n)$ by

$$\begin{aligned} &g(v)_{jn} := -g(v)_{nj} := v_j, \quad \forall j = 1,\dots,n-1, \qquad g(v)_{nn} := v_n, \\ &g(v)_{jl} := \delta_{jl} - \frac{v_j v_l}{1+v_n}, \quad \forall j,l = 1,\dots,n-1. \end{aligned} \tag{2.51}$$

Then set

$$g_y(\xi) := g\big(v(\xi,y)\big), \quad \forall \xi \in \mathcal{A}_y^+, \tag{2.52}$$

where $v(\xi,y)$ denotes the positive vector $v_\ell(\xi,y) = \sqrt{z_\ell(\xi,y)}$.

We are ready to present the main result of this section, which generalizes Theorem 4 of [19].

Theorem 6. *For any $y \in (-\pi/2, \pi/2)$ subject to (2.1), the set of elements*

$$\big\{\big(g_y(\xi)^{-1}\mathcal{L}_y^{\rm loc}(\xi,\tau)g_y(\xi),\, g_y(\xi)^{-1}\delta(\xi)g_y(\xi)\big) \;\big|\; (\xi,\tau) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1}\big\} \subset G \times G \tag{2.53}$$

defines a cross-section of the orbits of $G_{\mu_0(y)}$ in the open submanifold $\beta^{-1}(\mathcal{A}_y^+) \cap \mu^{-1}(\mu_0(y))$ of the constraint surface. The one-to-one parametrization of this cross-section by $(\xi,\tau) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ induces Darboux coordinates on the corresponding open submanifold of the reduced phase space,

$$\hat{\beta}^{-1}\big(\mathcal{A}_y^+\big) \subset P\big(\mu_0(y)\big) = \mu^{-1}\big(\mu_0(y)\big)/G_{\mu_0(y)}, \tag{2.54}$$

since on this submanifold the symplectic form that descends from ω^λ in (1.4) can be written as

$$\omega_{\rm red}^{\rm loc} = {\rm i}\lambda \sum_{k=1}^{n-1} {\rm d}\xi_k \wedge {\rm d}\tau_k\, \tau_k^{-1} = \lambda \sum_{k=1}^{n-1} {\rm d}\theta_k \wedge {\rm d}\xi_k \quad \text{with } \tau_k = e^{{\rm i}\theta_k}. \tag{2.55}$$

The submanifold $\hat{\beta}^{-1}(\mathcal{A}_y^+)$ is a dense subset of the full reduced phase space. On this submanifold the Poisson commuting reduced Hamiltonians descending from the smooth class functions of A in $(A,B) \in G \times G$ are given by the class functions of the SU(n)*-valued "local Lax matrix" $\mathcal{L}_y^{\rm loc}(\xi,\tau)$. In particular, using $s := {\rm sign}(\frac{\sin(y)}{\sin(ny)})$ and $\theta_0 = \theta_n := 0$, the reduction of the function $\Re({\rm tr}(A))$ yields the generalized RS Hamiltonian*

$$\begin{aligned} H_y^{\rm loc}(\xi,\theta) &:= \Re\big({\rm tr}\big(\mathcal{L}_y^{\rm loc}(\xi,\tau)\big)\big) \\ &= s\sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \prod_{k=j+1}^{j+n-1} \left| 1 - \frac{\sin^2 y}{\sin^2(\sum_{m=j}^{k-1}\xi_m)} \right|^{\frac{1}{2}}. \end{aligned} \tag{2.56}$$

The first statement of the theorem requires proving that the set (2.53) lies in the "constraint surface" $\mu^{-1}(\mu_0(y))$ and its intersection with any orbit of $G_{\mu_0(y)}$ consists of at most one point. The second statement requires calculation of the pull-back of the quasi-Hamiltonian 2-form (1.4) on the set (2.53). The proof of both parts follows word-by-word the proof of the corresponding statements of Theorem 4 of [19], and hence is omitted.

The proof of the denseness statement is trivial if (1.16) holds, i.e., if $\mathcal{A}_y^{\rm sing} = \emptyset$. In such cases $P(\mu_0(y))$ is a Hamiltonian toric manifold under the β-generated torus action, and $\hat{\beta}^{-1}(\mathcal{A}_y^+)$ gives the corresponding submanifold of principal orbit type, which is known to be dense and open. Regarding the cases when $\mathcal{A}_y^{\rm sing} \neq \emptyset$, the denseness is proved in Appendix B. Finally, the formula (2.56) follows by straightforward calculation.

We finish this section with a few comments. First of all, we recall that in the case of the regime (1.2) the Hamiltonian (2.56) is just the standard RS Hamiltonian of $\mathrm{III_b}$ type [18]. The principal message of the theorem is that *the local RS Hamiltonian defined by (2.56) on the domain $\mathcal{A}_y^+ \times \mathbb{T}^{n-1}$ extends uniquely to a globally smooth Hamiltonian on the compact reduced phase space $P(\mu_0(y))$ for any parameter $y \in (-\pi/2, \pi/2)$ subject to (2.1).*

The domain $\mathcal{A}_y$ is in general different from the Weyl alcove with thick walls (1.10). We shall investigate the dependence of this domain on y in the following section. Here it is worth noting that the continuity of $\hat{\beta}: P(\mu_0(y)) \to \mathcal{A}$ and the denseness statement in Theorem 6 imply that $\mathcal{A}_y^+$ is always a dense subset of $\mathcal{A}_y$.

By the duality between the functions $\hat{\alpha}^\lambda$ and $\hat{\beta}^\lambda$, which arises from the relation (2.47), the components of $\hat{\alpha}^\lambda$ generate a free Hamiltonian torus action on the dense open submanifold $\hat{\alpha}^{-1}(\mathcal{A}_y^+) \subset P(\mu_0(y))$. This shows the Liouville integrability of the commuting set of globally smooth Hamiltonians that descend from the smooth class function of the matrix A in $(A, B) \in D$.

3. Classification of the coupling parameter

We have seen that our reduction always yields a Liouville integrable system whose leading Hamiltonian has the RS form of $\mathrm{III_b}$ type (2.56) on a dense open submanifold of the compact reduced phase space $P(\mu_0(y))$. In principle, two different types of cases can occur:

- Type (i): the constraint surface satisfies $\mu^{-1}(\mu_0(y)) \subset G_{\rm reg} \times G_{\rm reg}$.
- Type (ii): the relation $\mu^{-1}(\mu_0(y)) \subset G_{\rm reg} \times G_{\rm reg}$ does not hold.

In the type (i) cases the reduced phase space inherits globally smooth action and position variables from the double. In the type (ii) cases neither the action variables nor the position variables extend to globally smooth (differentiable) functions on the full reduced phase space $P(\mu_0(y))$. This follows from the fact that the components of the spectral function Ξ (2.2), whereby α and β (2.3) descend to action variables and position variables, develop singularities at the non-regular elements of G, and those singularities cannot disappear by the reduction. It is also worth noting that at non-regular elements the dimension of the span of the differentials of the smooth class functions of $G = \mathrm{SU}(n)$ is always smaller than $(n-1)$. These group-theoretic results are elucidated in Appendix A.

In this section we show that both type (ii) and new type (i) cases exist, and give the precise classification of the coupling parameter y according to this dichotomy. Moreover, we shall prove that in the type (i) cases the full reduced phase space is always symplectomorphic to $\mathbb{C}\mathrm{P}^{n-1}$ with a multiple of the Fubini–Study form. The final results are given by Theorems 12 and 13 below.

Using that y matters only modulo π, we here parametrize $\mu_0(y)$ by y taken from the range

$$0 < y < \pi. \tag{3.1}$$

It is proved in Appendix B that the β-image $\mathcal{A}_y$ of the constraint surface is the closure of $\mathcal{A}_y^+$ defined in (2.42). Now the domain $\mathcal{A}_y^+$ can be characterized as follows.

Theorem 7. *Take any y subject to (2.1), (3.1) and let $k \in \{0, \dots, n-1\}$ be the integer verifying*

$$k\pi/n < y < (k+1)\pi/n. \tag{3.2}$$

Then $\mathcal{A}_y^+$ consists of those elements $\xi \in \mathcal{A}^{\mathrm{reg}}$ whose components satisfy the following condition for each $\ell = 1, \dots, n$:

$$\xi_\ell > y \quad \textit{if } k = 0, \tag{3.3}$$

$$\xi_\ell + \dots + \xi_{\ell+k-1} < y \quad \textit{and} \quad \xi_\ell + \dots + \xi_{\ell+k} > y \quad \textit{if } k = 1, \dots, n-2, \tag{3.4}$$

$$\xi_\ell + \dots + \xi_{\ell+n-2} < y \quad \textit{if } k = n-1. \tag{3.5}$$

Proof. Recall from (2.42) that $\xi \in \mathcal{A}_y^+$ if and only if $z_\ell(\xi, y) > 0$ for each $\ell = 1, \dots, n$. By inspecting the formula (2.14) one sees that $z_\ell(\xi, y) > 0$ holds if and only if ξ satisfies the inequalities

$$\xi_\ell + \dots + \xi_{\ell+\kappa(\ell)-1} < y \tag{3.6}$$

and

$$\xi_\ell + \dots + \xi_{\ell+\kappa(\ell)} > y \tag{3.7}$$

for some

$$\kappa(\ell) \in \{0, 1, \dots, n-1\} \quad \text{subject to } (-1)^{\kappa(\ell)} = (-1)^k. \tag{3.8}$$

The above inequalities say that the number of ξ-dependent negative factors in the product that gives $z_\ell(\xi, y)$ (2.14) is $\kappa(\ell)$, while rest of the factors is positive. We utilized that, on account of (3.2), the sign of the "pre-factor" $\sin(y)/\sin(ny)$ in (2.14) is the same as the sign of $(-1)^k$.

The sums in (3.6) and in (3.7) contain $\kappa(\ell)$ and $(\kappa(\ell)+1)$ terms, respectively. If $\kappa(\ell) = 0$, then Eq. (3.6) is absent (automatic if the value of the empty sum is taken to be zero), and if $\kappa(\ell) = (n-1)$, then Eq. (3.7) holds automatically. In principle, $\kappa(\ell)$ could be a non-constant function of ℓ and it could also vary as ξ varies.

We now demonstrate that the inequalities (3.6) and (3.7) together with (3.8) enforce that

$$\kappa(\ell) = k. \tag{3.9}$$

To this end, we first show that the relations

$$\kappa(\ell) \leqslant \kappa(\ell+1) \quad \text{for } \ell = 1, \dots, n-1 \tag{3.10}$$

and

$$\kappa(n) \leqslant \kappa(1) \tag{3.11}$$

must hold, which entail that κ is an ℓ-independent constant. We remark that the formula of the function z_ℓ can be extended by periodicity, $z_{\ell+n} = z_\ell$, and then one must also have $\kappa(\ell+n) = \kappa(\ell)$. With this convention, (3.11) is just the $\ell = n$ instance of (3.10).

To derive (3.10), fix some $1 \leqslant \ell \leqslant (n-1)$ and note first that if $\kappa(\ell) \in \{0, 1\}$, then (because of (3.8)), there is nothing to prove. Suppose then that $\kappa(\ell) \geqslant 2$ and suppose also that (3.10) does not hold at this ℓ. Since the parity of $\kappa(\ell)$ is independent of ℓ, this means that

$$\kappa(\ell+1) \leqslant \kappa(\ell) - 2, \tag{3.12}$$

which is equivalent to

$$\ell + 1 + \kappa(\ell+1) \leqslant \ell + \kappa(\ell) - 1. \tag{3.13}$$

But then we would obtain that

$$\begin{aligned} \xi_{\ell+1} + \cdots + \xi_{(\ell+1)+\kappa(\ell+1)} &\leqslant \xi_{\ell+1} + \cdots + \xi_{\ell+\kappa(\ell)-1} \\ &\leqslant \xi_\ell + \xi_{\ell+1} + \cdots + \xi_{\ell+\kappa(\ell)-1}. \end{aligned} \tag{3.14}$$

This is a contradiction since the first sum in (3.14) is larger than y by (3.7) applied to $(\ell+1)$, while the last sum is smaller than y by (3.6) applied to ℓ.

Invoking the periodicity, $z_\ell = z_{\ell+n}$, or by direct inspection of (3.6) and (3.7) for $\ell = n$ and $\ell = 1$, we obtain equation (3.11) as well.

Let κ_0 denote the value of the constant $\kappa(\ell)$. By taking the sum of the respective inequalities in (3.6) and (3.7) for $\ell = 1, \dots, n$, we see that

$$\kappa_0 \pi < ny \quad \text{and} \quad (\kappa_0 + 1)\pi > ny. \tag{3.15}$$

Comparison with (3.2) shows that $\kappa_0 = k$, whence the proof is complete. □

The type (i) cases are precisely those for which $\mathcal{A}_y$ does not intersect the boundary $\partial\mathcal{A}$ of the alcove $\mathcal{A}$ (1.8). The subsequent analysis will lead to a complete description of the y-values when this holds. To start, introduce the affine space E by

$$E := \{\xi \in \mathbb{R}^n \mid \xi_1 + \cdots + \xi_n = \pi\}. \tag{3.16}$$

Then, for any integer $1 \leqslant p \leqslant (n-1)$ and $0 < y < \pi$ not equal to $p\pi/n$, define the closed convex polyhedron as the subset of E given by requiring the following:

- The bounding hyperplanes of $\mathcal{B}(p, y)$ are defined by the n cyclic permutations of the equation

$$\xi_1 + \cdots + \xi_p = y. \tag{3.17}$$

- The polyhedron $\mathcal{B}(p, y)$ contains the point ξ^* (2.24).

We additionally define $\mathcal{B}(0, y) = \mathcal{B}(n, y) = E$ and also let $\mathcal{B}(p, y)^\circ$ denote the interior of $\mathcal{B}(p, y)$. We remark that $\mathcal{B}(p, y)$ is not necessarily bounded.

With the above definitions, we have

$$\mathcal{A}_y = \mathcal{B}(k, y) \cap \mathcal{B}(k+1, y) \quad \text{if } k\pi/n < y < (k+1)\pi/n,\ k = 0, \dots, n-1. \tag{3.18}$$

Indeed, if (3.2) holds then $\mathcal{B}(k, y)$ and $\mathcal{B}(k+1, y)$ are respectively given by imposing

$$\xi_\ell + \cdots + \xi_{\ell+k-1} \leqslant y \quad \text{and} \quad \xi_\ell + \cdots + \xi_{\ell+k} \geqslant y, \quad \forall \ell = 1, \dots, n, \tag{3.19}$$

on $\xi \in E$. The differences of these equations imply that $\xi_{\ell+k} \geqslant 0$ for all ℓ, i.e., the intersection on the right-hand side of (3.18) lies in $\mathcal{A}$. Thus (3.18) follows from Theorem 7 and from the fact that $\mathcal{A}_y$ is the closure of $\mathcal{A}_y^+$. One should note that $\mathcal{A}_y$ is of interest only under the additional regularity condition (2.1) on y, but below it will be convenient to formulate various statements for slightly more general values of y.

Let us consider the finite ring $\mathbb{Z}/n\mathbb{Z}$. Addition and multiplication in $\mathbb{Z}/n\mathbb{Z}$ are inherited from $\mathbb{Z}$, and we choose to represent the equivalence classes by $\{1, 2, \ldots, n\}$. It is well-known that if n and $1 \leqslant p \leqslant (n-1)$ are relatively prime, $\gcd(n, p) = 1$, then multiplication by p gives a permutation of the elements of this ring. In particular, there exists a unique integer $1 \leqslant q \leqslant (n-1)$ such that $pq = 1 \bmod n$. This will be crucial in proving the following lemma, which exhibits cases when $\mathcal{B}(p, y)$ is bounded.

Lemma 8. *If the integers $1 \leqslant p \leqslant (n-1)$ and n are relatively prime and $0 < y < \pi$ satisfies $y \neq p\pi/n$, then $\mathcal{B}(p, y)$ is an $(n-1)$-dimensional simplex. Writing q for the integer $1 \leqslant q \leqslant n-1$ such that $pq = 1 \bmod n$, and defining*

$$\tilde{y} := y - \frac{p\pi}{n}, \qquad a := \frac{\pi}{n} + q\tilde{y}, \qquad b := \frac{\pi}{n} - (n-q)\tilde{y}, \tag{3.20}$$

the n vertices of $\mathcal{B}(p, y)$ are the cyclic permutations of the point $x \in E$ given by

$$\begin{aligned} &x_i = a \quad \text{for } i = jp \text{ with } j = 1, \ldots, n-q, \\ &x_i = b \quad \text{otherwise}, \end{aligned} \tag{3.21}$$

where the index i is read modulo n.

Proof. If the polyhedron $\mathcal{B}(p, y)$ is bounded, then it must be a simplex, since it is bounded by n hyperplanes in the $(n-1)$-dimensional space E and contains a neighborhood of the point ξ^*. One knows from the Minkowski–Weyl theorem [24] that $\mathcal{B}(p, y)$ is not bounded if and only if it contains a half-line, i.e., a set of elements of the form

$$\xi(\lambda) = c + \lambda d, \quad \forall \lambda \geqslant 0,\ E \ni d \neq 0. \tag{3.22}$$

We next show that such a half-line does not exist.

Let e_i $(i = 1, \ldots, n)$ be the standard basis of $\mathbb{R}^n$ and apply the convention $e_j = e_{j\pm n}$ for all $j \in \mathbb{Z}$. Define $\epsilon := e_1 + \cdots + e_n$ and

$$V_i(p) := e_i + e_{i+1} + \cdots + e_{i+p-1}, \quad \forall i \in \mathbb{Z}. \tag{3.23}$$

Supposing for definiteness that $y > p\pi/n$, $\mathcal{B}(p, y)$ consists of the elements $x \in \mathbb{R}^n$ for which

$$\epsilon \cdot x = \pi \quad \text{and} \quad V_i(p) \cdot x \leqslant y, \quad \forall i. \tag{3.24}$$

Therefore, the direction vector d of a half-line contained in $\mathcal{B}(p, y)$ must satisfy

$$d \cdot \epsilon = 0 \quad \text{and} \quad d \cdot V_i(p) \leqslant 0, \quad \forall i. \tag{3.25}$$

Since $V_1(p) + \cdots + V_n(p) = p\epsilon$, these conditions imply

$$d \cdot V_i(p) = 0, \quad \forall i. \tag{3.26}$$

Let us expand the vector d as

$$d = d_1 e_1 + \cdots + d_n e_n \tag{3.27}$$

and set $d_j := d_{j\pm n}$ for all $j \in \mathbb{Z}$. By writing

$$pq = rn + 1 \quad \text{with some } r \geqslant 0, \tag{3.28}$$

one has the identity

$$V_i(p) + V_{i+p}(p) + \cdots + V_{i+(q-1)p}(p) = r\epsilon + e_i, \quad \forall i. \tag{3.29}$$

Taking the scalar product of this identity with d, using that $d \cdot \epsilon = 0$, leads to

$$d_i = e_i \cdot d = 0, \quad \forall i. \tag{3.30}$$

Hence the polyhedron $\mathcal{B}(p, y)$ contains no half-line. A similar argument works also if $y < p\pi/n$.

Now that we know that $\mathcal{B}(p, y)$ is a simplex, we need to calculate its vertices. Since the n vertices are clearly the cyclic permutations of a single one, it is enough to find the vertex x that solves the first $n-1$ cyclic permutations of equation (3.17). Taking subsequent differences of these $n-1$ equations gives the relations

$$x_i = x_{i+p} \quad \text{for } i = 1, \ldots, n-2, \tag{3.31}$$

where the indices are understood modulo n. The assumption $\gcd(n, p) = 1$ implies that for each $i = 1, \ldots, n-2$ there exists a unique $m_i \in \{1, \ldots, n-1\} \setminus \{(n-q)\}$ such that $i = m_i p \bmod n$, where we used that $n - 1 = (n-q)p \bmod n$. It follows immediately that the relations (3.31) can be recast in the form

$$\begin{aligned} &x_p = x_{2p} = \cdots = x_{(n-q)p} := a, \\ &x_{(n-q+1)p} = x_{(n-q+2)p} = \cdots = x_{np} := b, \end{aligned} \tag{3.32}$$

with some constants a and b.

We are left with the task of calculating a and b. We have two linear equations for this task. First of all, the condition $x \in E$ is equivalent to

$$qb + (n-q)a = \pi. \tag{3.33}$$

To obtain the second equation, we sum *all* cyclic permutations of (3.17) for x. On the one hand, this sum contains each coefficient p times, so (by $x \in E$) it must be equal to πp. On the other hand, notice that for the n-th cyclic permutation that was omitted we have

$$\begin{aligned} x_n + x_1 + x_2 + \cdots + x_{p-1} &= x_n - x_p + (x_1 + \cdots + x_p) \\ &= b - a + (x_1 + \cdots + x_p) \\ &= b - a + y. \end{aligned} \tag{3.34}$$

Therefore, summing all cyclic permutations gives

$$\pi p = ny + b - a. \tag{3.35}$$

Eqs. (3.33) and (3.35) for a and b are solved uniquely by the formula (3.20). □

One sees from Lemma 8 that as y approaches $p\pi/n$ the simplex $\mathcal{B}(p, y)$ contracts onto the point ξ^*. Then as y moves away from $p\pi/n$ the simplex grows and at some value of y its vertices reach $\partial\mathcal{A}$. The range of y for which it stays inside the interior $\mathcal{A}^{\text{reg}}$ of $\mathcal{A}$ is described as follows.

Corollary 9. *For* $\gcd(n, p) = 1$, *the simplex* $\mathcal{B}(p, y)$ *is contained in* $\mathcal{A}^{\text{reg}}$ *if and only if* $y \neq \frac{p\pi}{n}$ *belongs to the open interval* $(\frac{p\pi}{n} - \frac{\pi}{nq}, \frac{p\pi}{n} + \frac{\pi}{(n-q)n})$, *where* q *is defined as in Lemma 8.*

Proof. The simplex $\mathcal{B}(p, y)$ is contained in $\mathcal{A}^{\text{reg}}$ if and only if its vertices are contained in $\mathcal{A}^{\text{reg}}$, which means that both a and b in (3.20) are positive. If $y > p\pi/n$, then $a > 0$ and the positivity of b is equivalent to $y < p\pi/n + \pi/(n(n-q))$. If $y < p\pi/n$, then $b > 0$ and the positivity of a is equivalent to $y > p\pi/n - \pi/(nq)$. □

Lemma 10. *Suppose* $\gcd(n, p) = 1$ *and take y from the interval given in Corollary 9 such that it is not an integer multiple of* π/n. *In this case the simplex* $\mathcal{B}(p, y) \subset \mathcal{A}^{\text{reg}}$ *verifies the following property.*

- *If* $\frac{p\pi}{n} < y < \frac{p\pi}{n} + \frac{\pi}{(n-q)n}$, *then* $\mathcal{B}(p, y) \subset \mathcal{B}(p+1, y)^\circ$.
- *If* $\frac{p\pi}{n} - \frac{\pi}{nq} < y < \frac{p\pi}{n}$, *then* $\mathcal{B}(p, y) \subset \mathcal{B}(p-1, y)^\circ$.

Proof. Let us pick a vertex x of the simplex $\mathcal{B}(p, y) \subset \mathcal{A}^{\text{reg}}$ and recall that it satisfies all but one of the n cyclic permutations of the equation

$$x_1 + \cdots + x_p = y. \tag{3.36}$$

In particular, it satisfies at least one of the following two equations

$$x_\ell + \cdots + x_{\ell+p-1} = y \quad \text{or} \quad x_{\ell+1} + \cdots + x_{\ell+p} = y \tag{3.37}$$

for each $\ell = 1, \ldots, n$. Suppose now that $\frac{p\pi}{n} < y < \frac{p\pi}{n} + \frac{\pi}{(n-q)n}$, which entails that the polyhedron $\mathcal{B}(p+1, y)^\circ$ is given by the inequalities

$$\xi_\ell + \cdots + \xi_{\ell+p} > y. \tag{3.38}$$

The fact that all components of x are positive implies by (3.37) that the vertex x of $\mathcal{B}(p, y)$ lies in $\mathcal{B}(p+1, y)^\circ$. The case $\frac{p\pi}{n} - \frac{\pi}{nq} < y < \frac{p\pi}{n}$ is settled quite similarly by using that in this case the defining inequalities of $\mathcal{B}(p-1, y)^\circ$ are $\xi_\ell + \cdots + \xi_{\ell+p-2} < y$ if $p > 1$ and $\mathcal{B}(0, y)^\circ = E$. □

Proposition 11. *Let* $n \geqslant 2$ *be given and pick* $1 \leqslant p \leqslant n-1$ *such that* $\gcd(n, p) = 1$. *Define q as in Lemma 8 and consider* $y \in (\frac{\pi p}{n} - \frac{\pi}{nq}, \frac{p\pi}{n} + \frac{\pi}{(n-q)n})$ *subject to (2.1). Then* $\mathcal{A}_y = \mathcal{B}(p, y)$.

Proof. This is a direct consequence of the relation (3.18), whereby $\mathcal{A}_y = \mathcal{B}(p-1, y) \cap \mathcal{B}(p, y)$ if $p\pi/n - \pi/nq < y < p\pi/n$ and $\mathcal{A}_y = \mathcal{B}(p, y) \cap \mathcal{B}(p+1, y)$ if $p\pi/n < y < p\pi/n + \pi/(n(n-q))$, and the statement of Lemma 10. □

Now we are ready to formulate the main results of the present paper.

Theorem 12. *Consider the reduction of the double* $\mathrm{SU}(n) \times \mathrm{SU}(n)$ *defined by the moment map constraint* $\mu(A, B) = \mu_0(y)$ *with* $0 < y < \pi$ *subject to (2.1). Suppose that y belongs to an open interval of the form*

$$\left(\frac{p\pi}{n} - \frac{\pi}{nq}, \frac{p\pi}{n} + \frac{\pi}{(n-q)n}\right), \tag{3.39}$$

where $\gcd(n, p) = 1$ *and* $pq = 1 \bmod n$ *with integers* $1 \leqslant p, q \leqslant (n-1)$. *Then the* β*-image* $\mathcal{A}_y$ *of the constraint surface* $\mu^{-1}(\mu_0(y))$ *is contained in* $\mathcal{A}^{\text{reg}}$. *In these cases the reduced phase space* $P(\mu_0(y))$ *is symplectomorphic to* $\mathbb{C}\mathrm{P}^{n-1}$ *with a multiple of the Fubini–Study symplectic structure.*

Proof. Proposition 11 and the preceding lemmas ensure that if y satisfies (2.1) and (3.39), then the β-image of the constraint surface is provided by the simplex $\mathcal{B}(p, y)$, which is contained in $\mathcal{A}^{\text{reg}}$. This implies that the reduced phase space is a Hamiltonian toric manifold with respect to the toric moment map $\hat{\beta}^{\lambda}$ having the image $\lambda\mathcal{B}(p, y)$, where the constant λ gives the scale of the quasi-Hamiltonian 2-form (1.4). Up to symplectomorphisms, the only toric manifold whose "Delzant polytope" is an $(n-1)$-dimensional simplex is $\mathbb{C}\mathrm{P}^{n-1}$ equipped with a multiple of the Fubini–Study symplectic form [22,23]. □

Theorem 13. *The values of y given in Theorem 12 exhausts all type (i) cases. In other words, if $0 < y < \pi$ subject to (2.1) does not belong to an open interval of the form (3.39), then $\mathcal{A}_y$ intersects the boundary $\partial\mathcal{A}$ of $\mathcal{A}$.*

The proof will follow from a few simple lemmas. First of all, for any y as in Eq. (3.2) we let $\mathcal{C}_y$ denote the set of those $\xi \in \mathcal{A}$ that satisfy the inequalities

$$\xi_\ell + \cdots + \xi_{\ell+k-1} \leqslant y \quad \text{and} \quad \xi_\ell + \cdots + \xi_{\ell+k} \geqslant y \tag{3.40}$$

for each $\ell = 1, \ldots, n$ (where the first inequality is automatic if $k = 0$). This means that $\mathcal{C}_y = \mathcal{A}_y$ if y also satisfies (2.1).

Lemma 14. *Suppose that $k\pi/n < y_1 < y_2 < (k+1)\pi/n$ and both $\mathcal{C}_{y_1} \cap \partial\mathcal{A}$ and $\mathcal{C}_{y_2} \cap \partial\mathcal{A}$ are non-empty. Then the same holds for $\mathcal{C}_y$ with any $y \in [y_1, y_2]$.*

Proof. Notice from the definition of $\mathcal{C}_y$ that if $\xi \in \mathcal{C}_{y_1}$ and $\xi' \in \mathcal{C}_{y_2}$, then

$$\big(t\xi + (1-t)\xi'\big) \in \mathcal{C}_{ty_1+(1-t)y_2} \tag{3.41}$$

holds for all $0 \leqslant t \leqslant 1$. Then apply this to such $\xi \in \mathcal{C}_{y_1}$ and $\xi' \in \mathcal{C}_{y_2}$ for which $\xi_n = \xi'_n = 0$, which exist since $\mathcal{C}_y$ is stable under cyclic permutations of the components of its elements. □

Lemma 15. *Choose $0 < y < \pi$ of the form*

$$y = \frac{\pi p}{n} - \frac{\pi}{nq} \quad \text{or} \quad y = \frac{p\pi}{n} + \frac{\pi}{(n-q)n} \tag{3.42}$$

with some p and q appearing in Theorem 12. Then $\mathcal{C}_y \cap \partial\mathcal{A} \neq \emptyset$.

Proof. Following the proof of Proposition 11, one can show that in these cases $\mathcal{C}_y$ equals the simplex $\mathcal{B}(p, y)$, whose vertices now lie in $\partial\mathcal{A}$. Incidentally, these y values do not satisfy (2.1). □

Lemma 16. *Suppose that $1 < p < (n-1)$ satisfies $\gcd(n, p) \neq 1$. Then there exists $\varepsilon > 0$ such that for any $y \in (p\pi/n, p\pi/n + \varepsilon)$ and for any $y \in (p\pi/n - \varepsilon, p\pi/n)$ one has $\mathcal{C}_y \cap \partial\mathcal{A} \neq \emptyset$.*

Proof. For definiteness, consider the case $p\pi/n < y < (p+1)\pi/n$, when

$$\mathcal{C}_y = \mathcal{B}(p, y) \cap \mathcal{B}(p+1, y). \tag{3.43}$$

Then write $\ell := \gcd(n, p)$ and define the point x by

$$x = (a_1, \ldots, a_\ell, a_1, \ldots, a_\ell, \ldots, a_1, \ldots, a_{\ell-1}, 0) \tag{3.44}$$

where

$$a_1 = \cdots = a_{\ell-1} = \frac{\frac{\ell}{p} y - a_\ell}{\ell - 1} \quad \text{and} \quad a_\ell = \frac{n}{p} y - \pi. \tag{3.45}$$

It is easily verified that $x \in E$. To see that $x \in \mathcal{A}$, we need to show that all $a_i \geqslant 0$. The fact that $a_\ell > 0$ follows directly from $p\pi/n < y$. For $a_i \geqslant 0$ for $1 \leqslant i < \ell$, we need to have $y \leqslant \frac{p\pi}{n-\ell}$, which is ensured by a suitable choice of ε.

Since $a_1 + \cdots + a_\ell = \frac{\ell}{p} y$, it is readily checked that $x \in \mathcal{B}(p, y)$. To see that $x \in \mathcal{B}(p+1, y)$, we argue as follows. A cyclic permutation of the sum

$$x_1 + \cdots + x_{p+1} \tag{3.46}$$

either contains the term x_n or it does not. In the latter case, the sum is clearly greater than y, since it contains all values $a_1, \ldots, a_\ell$ at least $\frac{p}{\ell}$ times. In the former case, its value will be equal to

$$y - a_\ell + a_i \tag{3.47}$$

for some $1 \leqslant i < \ell$. So it is sufficient if

$$a_i > a_\ell, \tag{3.48}$$

which can be ensured by possibly replacing ε by a smaller value.

Now the proof is complete for $y \in (p\pi/n, p\pi/n + \varepsilon)$. The case $y \in (p\pi/n - \varepsilon, p\pi/n)$ can be handled in an analogous manner. □

Proof of Theorem 13. Suppose that $0 < y < \pi$ subject to (2.1) does not belong to an open interval of the form (3.39). (This excludes $n = 2$ and $n = 3$.) Then, as is readily seen from Lemma 15 and Lemma 16, we can find y_1 and y_2 and integer $1 < k < (n-1)$ such that $k\pi/n < y_1 < y < y_2 < (k+1)\pi/n$ and both $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$ contain points of $\partial\mathcal{A}$. By using this and the fact that under (2.1) $\mathcal{A}_y = \mathcal{C}_y$, the required statement results from Lemma 14. □

We end this section by a few remarks and questions. We saw that the coupling parameters of the type (i) cases are the generic $0 < y < \pi$ values in the open intervals of the form

$$(a_{p,n}\pi, b_{p,n}\pi) \quad \text{with } a_{p,n} = \frac{p}{n} - \frac{1}{nq} = \frac{m_p}{q},$$
$$b_{p,n} = \frac{p}{n} + \frac{1}{n(n-q)} = \frac{p - m_p}{n - q}, \tag{3.49}$$

where $p = 1, \ldots, (n-1)$, $\gcd(n, p) = 1$ and $pq = m_p n + 1$. These intervals enjoy the relation

$$a_{n-p,n} = 1 - b_{p,n}, \qquad b_{n-p,n} = 1 - a_{p,n}. \tag{3.50}$$

It seems to be indicated by computer calculations that every $y \neq p\pi/n$ from the interval (3.49) satisfies (2.1), but we have not proved this. In the type (i) cases the reduced phase space is $\mathbb{C}\mathrm{P}^{n-1}$ carrying a multiple of the Fubini–Study structure, but the constant involved was so far calculated only when $p = 1$ or $p = (n-1)$. See Section 4.

We observe that $a_p \geqslant \frac{p-1}{n}$ having equality only for $p = 1$, and $b_p \leqslant \frac{p+1}{n}$ having equality only for $p = (n-1)$. It is also not difficult to check that if $\gcd(n, p) = 1$ and $\gcd(n, p+1) = 1$ both hold for some $1 \leqslant p \leqslant (n-2)$, then

$$b_{p,n} < a_{p+1,n} \tag{3.51}$$

except for $p = k$, $n = (2k + 1)$ when $b_{k,2k+1} = a_{k+1,2k+1} = \frac{1}{2}$. As a consequence, there exist y values associated with type (ii) reductions of $\mathrm{SU}(n) \times \mathrm{SU}(n)$ in every interval $(\frac{j}{n}\pi, \frac{j+1}{n}\pi)$ for $j = 1, \dots, n-2$, except for the interval $(\frac{k}{n}\pi, \frac{k+1}{n}\pi)$ if $n = 2k + 1$. In particular, type (ii) cases exist for every n except for $n = 2$ and $n = 3$. Taking $\mathrm{SU}(2k + 1)$ with $k \geqslant 2$,

$$b_{1,2k+1}\pi = \pi/(2k) < y < \pi/(k+1) = a_{2,2k+1}\pi \tag{3.52}$$

yields examples of type (ii) cases. For $\mathrm{SU}(2k)$ with $k \geqslant 2$, $\pi/(2k-1) < y < \pi/k$ gives examples of type (ii) cases.

We have calculated the vertices and faces of the 3-dimensional "type (ii) convex polytope" $\mathcal{A}_y$ corresponding to $n = 4$ and $\pi/3 < y < \pi/2$. The vertices turned out to be the cyclic permutations of the points

$$R(1) := (y, \pi - 2y, 3y - \pi, \pi - 2y) \quad \text{and} \quad I(1) := (y, \pi - 2y, y, 0). \tag{3.53}$$

To describe the faces, let us write $R(i)$ $(i = 1, \dots, 4)$ for the cyclic permutation $\sigma^{i-1}(R(1))$ of $R(1)$ using (2.38), and define $I(i)$ similarly. Explicit inspection shows that $\mathcal{A}_y$ possesses 4 triangular and 4 rectangular faces. One particular triangular face is incident with the vertices $R(1)$, $I(1)$ and $I(3)$, and one rectangular face is incident with the vertices $R(2)$, $R(3)$, $I(3)$ and $I(4)$. Then one can check that $I(1)$ is incident with two triangular faces and two rectangular faces. In three dimensions, this means that $I(1)$ is incident with four edges. This implies that our 3-dimensional polytope $\mathcal{A}_y$ is not a Delzant polytope, since it is known [22,23] that all vertices of the n-dimensional Delzant polytopes are incident with precisely n edges. Of course it is not a surprise that $\mathcal{A}_y$ is not a Delzant polytope, because we do not obtain a toric structure in the type (ii) cases. Interestingly, as follows from Corollary 4 in Section 2.2, the regular vertices $R(i)$ correspond to fixed points of the β-generated torus action on $\hat{\beta}^{-1}(\mathcal{A}_y^{\mathrm{reg}})$. Concerning the interpretation of the irregular vertices $I(i)$, we know from Appendix A that the position variables provided by $\hat{\beta}$ are not differentiable at the locus $\hat{\beta}^{-1}(I(i))$, and the Hamiltonian vector fields of the smooth reduced class functions depending on B from $[(A, B)] \in P(\mu_0(y))$ can span at most 2-dimensional spaces at the points of $\hat{\beta}^{-1}(I(i))$, while generically they span 3-dimensional subspaces of the tangent space. Further details of this example, and the type (ii) systems in general, will be studied elsewhere.

4. On new examples of type (i) cases

In the light of Theorem 12, the standard compact RS systems associated with the coupling parameter $0 < y < \pi/n$ represent examples of type (i) cases. We have found new type (i) cases for which the coupling parameter y belongs to the interval (3.39) for any $1 \leqslant p \leqslant (n-1)$ with $\gcd(n, p) = 1$. (The cases associated with p and $(n - p)$ are essentially the same since $P(\mu_0(y))$ and $P(\mu_0(y)^{-1})$ are related by complex conjugation on the double.) The goal of this section is to elaborate certain details of new type (i) examples and explain in what sense the corresponding compact RS systems are different from the standard ones. Specifically, we shall focus on the range of y that lies on the right-side of π/n in (3.39) for $p = q = 1$, i.e., we suppose that

$$\frac{\pi}{n} < y < \frac{\pi}{(n-1)}, \quad n \geqslant 3. \tag{4.1}$$

By Proposition 11, the β-image $\mathcal{A}_y$ of the constraint surface is then given by

$$\mathcal{A}_y = \{\xi \in \mathcal{A} \mid \xi_\ell \leqslant y, \ \forall \ell = 1, \dots, n\}. \tag{4.2}$$

The vertices of this simplex are $\xi(j)$ ($j = 1, \dots, n$) having the components

$$\xi(j)_\ell = y(1 - \delta_{\ell,j}) + \big(\pi - (n-1)y\big)\delta_{\ell,j}, \quad j, \ell = 1, \dots, n. \tag{4.3}$$

Since $\mathcal{A}_y \subset \mathcal{A}^{\text{reg}}$, the reduced phase space $P(\mu_0(y))$ is a Hamiltonian toric manifold under the $\mathbb{T}^{n-1}$-action generated by the moment map $\hat{\beta}^\lambda = \lambda\hat{\beta}$. Thus one knows from the Delzant theorem [22,23] that $(P(\mu_0(y)), \omega_{\text{red}}, \hat{\beta}^\lambda)$ is equivalent to $\mathbb{C}\mathrm{P}^{n-1}$ equipped with the toric structure possessing the same moment polytope $\lambda\mathcal{A}_y$. We next describe the equivalence explicitly. For definiteness, in what follows we assume that the overall parameter λ in (1.4) is positive.

Let us realize $\mathbb{C}\mathrm{P}^{n-1}$ as a symplectic reduction of $\mathbb{C}^n$ equipped with the symplectic form $\Omega_{\mathbb{C}^n} := \mathrm{i}\sum_{k=1}^n \mathrm{d}\bar{u}_k \wedge \mathrm{d}u_k$. This can be achieved by fixing the moment map $\chi(u) := \sum_{k=1}^n |u_k|^2$ that generates the natural U(1) action on $\mathbb{C}^n$ (whereby $u \in \mathbb{C}^n$ is mapped to $e^{\mathrm{i}\gamma}u$). Indeed, by applying the constraint

$$\chi(u) = \chi_0 := \lambda(ny - \pi) \quad (\lambda > 0), \tag{4.4}$$

the corresponding reduced phase space $\chi^{-1}(\chi_0)/\mathrm{U}(1)$ turns out to be

$$\big(\mathbb{C}\mathrm{P}^{n-1}, \chi_0\omega_{\text{FS}}\big), \tag{4.5}$$

where ω_{FS} is the standard Fubini–Study symplectic form. Realizing any point of $\mathbb{C}\mathrm{P}^{n-1}$ as an equivalence class $[u] = (u_1 : u_2 : \dots : u_n)$ of some $u \in \chi^{-1}(\chi_0)$, we introduce the smooth functions $\mathcal{J}_k$ on $\mathbb{C}\mathrm{P}^{n-1}$ by the definition

$$\mathcal{J}_k\big([u]\big) := -|u_k|^2 + \lambda y, \quad k = 1, \dots, n. \tag{4.6}$$

The definition ensures that

$$\sum_{k=1}^n \mathcal{J}_k/\lambda = \pi \quad \text{and} \quad \pi - (n-1)y \leqslant \mathcal{J}_k/\lambda \leqslant y, \quad \forall k = 1, \dots, n. \tag{4.7}$$

The linearly independent functions $\mathcal{J}_k$ ($k = 1, \dots, n-1$) define the components of the moment map of a Hamiltonian action of $\mathbb{T}^{n-1}$. This is the "rotational action" for which

$$\tau = (\tau_1, \dots, \tau_{n-1}) = \big(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{n-1}}\big) \tag{4.8}$$

operates by the map

$$\mathcal{R}_\tau : \big[(u_1, \dots, u_{n-1}, u_n)\big] \mapsto \big[(\bar{\tau}_1 u_1, \dots, \bar{\tau}_{n-1}u_{n-1}, u_n)\big], \tag{4.9}$$

i.e., by the Hamiltonian flow of $(\mathcal{J}_1, \dots, \mathcal{J}_{n-1})$ at the "time-parameters" $(\theta_1, \dots, \theta_{n-1})$.

The constants and the signs were purposefully chosen in the above definitions in such a way that the image of the above toric moment map $\mathcal{J}$, where for convenience we include in $\mathcal{J}$ the last component $\mathcal{J}_n = \lambda\pi - \sum_{k=1}^{n-1}\mathcal{J}_k$, is the same polytope $\lambda\mathcal{A}_y$ (4.2) that belongs to the β-generated $\mathbb{T}^{n-1}$-action on $P(\mu_0(y))$. The vertices of the polytope correspond to the special points of $\mathbb{C}\mathrm{P}^{n-1}$ where only one of the homogeneous coordinates $(u_1, \dots, u_n)$ is non-zero.

The Delzant theorem [22,23] guarantees the existence of a diffeomorphism

$$f_\beta : \mathbb{C}\mathrm{P}^{n-1} \to P\big(\mu_0(y)\big) \tag{4.10}$$

having the properties

$$f_\beta^*(\omega_{\text{red}}) = \chi_0\omega_{\text{FS}}, \qquad f_\beta^*\big(\hat{\beta}^\lambda\big) = \mathcal{J}. \tag{4.11}$$

Such a map, called a "Delzant symplectomorphism", is essentially unique [25], that is, it is unique up to the obvious possibility to compose it with the time-one flows of arbitrary such Hamiltonians that can be expressed as functions of the corresponding toric moment maps.

In order to construct f_β, note that in the case under inspection Theorem 6 yields a symplectomorphism between

$$\left(\mathcal{A}_y^+ \times \mathbb{T}^{n-1}, \lambda \sum_{k=1}^{n-1} \mathrm{d}\theta_k \wedge \mathrm{d}\xi_k\right), \tag{4.12}$$

where $\mathcal{A}_y^+$ is the interior of $\mathcal{A}_y$ in (4.2), and the dense open submanifold $\hat{\beta}^{-1}(\mathcal{A}_y^+) \subset P(\mu_0(y))$. Then introduce the map $\mathcal{E}$ from the same domain (4.12) onto the dense open submanifold $\mathbb{C}\mathrm{P}_0^{n-1} \subset \mathbb{C}\mathrm{P}^{n-1}$ where none of the homogeneous coordinates vanish by setting

$$\mathcal{E}(\xi, \tau) := \left[\sqrt{\lambda}\left(\bar{\tau}_1\sqrt{y-\xi_1}, \ldots, \bar{\tau}_{n-1}\sqrt{y-\xi_{n-1}}, \sqrt{y-\xi_n}\right)\right]. \tag{4.13}$$

It is easy to check that $\mathcal{E}^*(\chi_0\omega_{\mathrm{FS}}) = \lambda \sum_{k=1}^{n-1} \mathrm{d}\theta_k \wedge \mathrm{d}\xi_k$ holds.

The composition of the above parametrizations of $\mathbb{C}\mathrm{P}_0^{n-1} \subset \mathbb{C}\mathrm{P}^{n-1}$ and $\hat{\beta}^{-1}(\mathcal{A}_y^+) \subset P(\mu_0(y))$ by $\mathcal{A}_y^+ \times \mathbb{T}_{n-1}$ gives rise to a symplectomorphism between $\mathbb{C}\mathrm{P}_0^{n-1}$ and $\hat{\beta}^{-1}(\mathcal{A}_y^+)$, which admits a global extension. This is the content of the following theorem, whose proof is omitted since it is very similar to that of Theorem 5 in [19].

Theorem 17. *The symplectomorphism $f_0 : \mathbb{C}\mathrm{P}_0^{n-1} \to \hat{\beta}^{-1}(\mathcal{A}_y^+)$ defined by*

$$\begin{aligned} &f_0 : \mathcal{E}(\xi, \tau) \mapsto \left[\left(g_y(\xi)^{-1}\mathcal{L}_y^{\mathrm{loc}}(\xi, \tau) g_y(\xi), g_y(\xi)^{-1}\delta(\xi) g_y(\xi)\right)\right], \\ &\quad \forall(\xi, \tau) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1}, \end{aligned} \tag{4.14}$$

where $[(A, B)]$ denotes the gauge orbit through $(A, B) \in \mu^{-1}(\mu_0(y))$, extends to a global Delzant symplectomorphism f_β verifying the properties (4.11).

One of the key ingredients of the proof of Theorem 17 is to show that after a suitable gauge transformation the local Lax matrix $\mathcal{L}_y^{\mathrm{loc}}$ (2.50) admits a smooth extensions from $\mathbb{C}\mathrm{P}_0^{n-1}$ to $\mathbb{C}\mathrm{P}^{n-1}$. In fact, there exists a unique function $\mathcal{L}^y \in C^\infty(\mathbb{C}\mathrm{P}^{n-1}, \mathrm{SU}(n))$ that satisfies the identity

$$\left(\mathcal{L}^y \circ \mathcal{E}\right)(\xi, \tau) = \Delta(\tau)^{-1}\mathcal{L}_y^{\mathrm{loc}}(\xi, \tau)\Delta(\tau) \quad \text{with } \Delta(\tau) := \mathrm{diag}(\tau_1, \ldots, \tau_{n-1}, 1). \tag{4.15}$$

The function $\mathcal{L}^y$ is called the global Lax matrix of the associated compact RS system. Using the identification of the reduced phase space $P(\mu_0(y))$ with $\mathbb{C}\mathrm{P}^{n-1}$ by the map f_β, the compact RS system resulting from the reduction can be characterized by the following properties:

1. The global extension H_y of principal RS Hamiltonian (2.56) transferred by f_0 (4.14) to $\mathbb{C}\mathrm{P}_0^{n-1}$ is given by the real part of the trace of the global Lax matrix $\mathcal{L}^y$, whose smooth class functions generate an Abelian Poisson algebra on $(\mathbb{C}\mathrm{P}^{n-1}, \chi_0\omega_{\mathrm{FS}})$.
2. The functions $\mathcal{J}_k/\lambda = \hat{\beta}_k \circ f_\beta$ give globally smooth extension of the position variables ξ_k of the local RS system living on $\mathcal{A}_y^+ \times \mathbb{T}^{n-1} \simeq \mathbb{C}\mathrm{P}_0^{n-1}$.
3. The functions $\lambda \Xi_k \circ \mathcal{L}^y = \lambda\hat{\alpha}_k \circ f_\beta$ define globally smooth action variables for the compact RS system.

In conclusion, the outcome of the reduction in the case (4.1) is the compact RS system encoded by the triple $(\mathbb{C}P^{n-1}, \chi_0\omega_{FS}, \mathcal{L}^y)$ and the above mentioned Abelian Poisson algebras of distinguished observables.

In the rest of this section, we wish to compare the compact RS system that we just constructed using the parameter y subject to (4.1) to the original compact system of Ruijsenaars [18] having the parameter y in the range (1.2). The physical interpretation of these systems is based on the "principal local Hamiltonian" (2.56). This Hamiltonian has the same form in all cases, but different parameters y appear in it and the domain where the position variable ξ is allowed to vary also depends on y. Any two systems associated with different parameters are different in this basic sense.

We now further clarify the relation between the two systems by presenting them in terms of the same coordinate system on $\mathbb{C}P_0^{n-1}$. To elaborate this, let us denote all objects pertaining to the "old case" (1.2) by "primed" letters, and also take the parameters positive. Thus in the old case the reduced phase space is $\mathbb{C}P^{n-1}$ equipped with the symplectic form

$$\lambda'(\pi - ny')\omega_{FS} \quad \text{with } 0 < y' < \pi/n. \tag{4.16}$$

The dense open submanifold of $\mathbb{C}P^{n-1}$ where none of the homogeneous coordinates vanish is then parameterized by the domain $\mathcal{A}_{y'}^+ \times \mathbb{T}^{n-1}$, where $\mathcal{A}_{y'}^+$ is the Weyl alcove with thick walls (1.10). Concretely, the element

$$\left(\xi', e^{i\theta_1'}, \ldots, e^{i\theta_{n-1}'}\right) \in \mathcal{A}_{y'}^+ \times \mathbb{T}^{n-1} \quad \left(y' < \xi_k', \ \sum_{k=1}^n \xi_k' = \pi\right), \tag{4.17}$$

corresponds to the equivalence class

$$\left[\sqrt{\lambda'}\left(e^{i\theta_1'}\sqrt{\xi_1' - y'}, \ldots, e^{i\theta_{n-1}'}\sqrt{\xi_{n-1}' - y'}, \sqrt{\xi_n' - y'}\right)\right] \in \mathbb{C}P^{n-1}. \tag{4.18}$$

In this parametrization the symplectic form (4.16) becomes $\lambda' \sum_{k=1}^{n-1} d\theta_k' \wedge d\xi_k'$ and the principal Hamiltonian reads

$$H_{y'}^{loc}(\xi', \theta') = \sum_{j=1}^n \cos\left(\theta_j' - \theta_{j-1}'\right) \prod_{k=j+1}^{j+n-1} \left|1 - \frac{\sin^2 y'}{\sin^2(\sum_{m=j}^{k-1} \xi_m')}\right|^{\frac{1}{2}}. \tag{4.19}$$

Since otherwise the resulting systems are plainly non-equivalent, let us require that in the old and new cases the reduction equips $\mathbb{C}P^{n-1}$ with the same symplectic form, which means that the respective parameters (λ', y') and (λ, y) enjoy the relation

$$\lambda'\left(\pi - ny'\right) = \lambda(ny - \pi), \tag{4.20}$$

where y' varies according to (4.16) and $\pi/n < y < \pi/(n-1)$. The variables $\xi_k', e^{i\theta_k'}$ and $\xi_k, e^{i\theta_k}$ represent two coordinate systems on the same open dense submanifold $\mathbb{C}P_0^{n-1} \subset \mathbb{C}P^{n-1}$, and thus there is a unique relation between them. By comparing (4.18) and (4.13) under the assumption (4.20), we find that the transformation between the coordinate systems is governed by the equations

$$\theta_k = -\theta_k', \qquad \lambda(\xi_k - y) = \lambda'\left(y' - \xi_k'\right). \tag{4.21}$$

If we now express the "new Hamiltonian" H_y^{loc} in the primed variables by substituting the above formulas into (2.56), then we obtain the function

$$H_y^{\rm loc}(\xi',\theta') = -\sum_{j=1}^{n} \cos(\theta'_j - \theta'_{j-1}) \prod_{k=j+1}^{j+n-1} \left| 1 - \frac{\sin^2 y}{\sin^2(c_{j,k} + (\lambda'/\lambda)\sum_{m=j}^{k-1} \xi'_m)} \right|^{\frac{1}{2}} \tag{4.22}$$

with $c_{j,k} = \frac{\lambda'}{\lambda}(y+y')(j-k)$. It is clear that when viewed as functions of the same coordinates on $\mathbb{CP}_0^{n-1}$ the Hamiltonians $H_{y'}^{\rm loc}(\xi',\theta')$ (4.19) and $H_y^{\rm loc}(\xi',\theta')$ (4.22) are different. Since their local restrictions are different, $H_{y'}$ and H_y are different functions on the full phase space $\mathbb{CP}^{n-1}$. This holds even in those special cases for which the relations $(\pi - ny') = (ny - \pi)$ and $\lambda' = \lambda$ are satisfied. The conclusion is independent from having the overall minus sign in (4.22), which comes from s in (2.56) and could be dropped by change of conventions or by suitable shifts of the variables θ'_k.

To gain yet another perspective on the comparison, note that we can express H_y in terms its action variables $I_k := \lambda\hat{\alpha}_k$ and also express $H_{y'}$ in terms its action variables $I'_k := \lambda'\hat{\alpha}'_k$. By using that $\hat{\alpha}$ and $\hat{\beta}$ have the same images due to (2.45), the Delzant theorem guarantees the existence of a symplectomorphism that converts the respective action variables into each other according to the relation

$$\lambda(\hat{\alpha}_k - y) \iff \lambda'(y' - \hat{\alpha}'_k). \tag{4.23}$$

This is fully analogous to the second equality in (4.21), where $\lambda\xi_k$ and $\lambda'\xi'_k$ are just the values taken by the toric moment maps $\lambda\hat{\beta}$ and $\lambda'\hat{\beta}'$. The definition of the function α (2.3) implies (by Eq. (A.1) in Appendix A) that for $(A,B) \in \mu^{-1}(\mu_0(y))$ one has $A \sim \exp(-2{\rm i}\sum_{k=1}^{n-1}\hat{\alpha}_k \Lambda_k)$, where $\sim$ means conjugation and we used the $n \times n$ matrices $\Lambda_k = \sum_{j=1}^{k} E_{j,j} - \frac{k}{n}\mathbf{1}_n$. Then it is readily seen from the formulas

$$H_y = \Re\,{\rm tr}\left(\exp\left(-2{\rm i}\sum_{k=1}^{n-1}\hat{\alpha}_k\Lambda_k\right)\right) \quad \text{and} \quad H_{y'} = \Re\,{\rm tr}\left(\exp\left(-2{\rm i}\sum_{k=1}^{n-1}\hat{\alpha}'_k\Lambda_k\right)\right) \tag{4.24}$$

that H_y is *not* converted into $H_{y'}$ by the symplectomorphism that obeys (4.23). In other words, if we convert the action variables of the unprimed system into the action variables of the primed system according to (4.23), then H_y and $H_{y'}$ become different functions of the primed action variables I'_k.

The foregoing discussion can be informally summarized as follows: "The systems associated with different parameters are at the first sight obviously different, and this impression persists after closer inspection, too." It might be also possible to prove the non-existence of any symplectomorphism of $\mathbb{CP}^{n-1}$ that would convert H_y into $H_{y'}$ under the condition (4.20), but we do not have such a proof. The above arguments convinced us that no such symplectomorphism exists if one requires it to have further natural properties, i.e., that it should map either particle positions into particle positions or action variables into action variables.

5. Conclusion

In this paper we derived new compact forms of the trigonometric RS system by reducing the quasi-Hamiltonian double of $G = {\rm SU}(n)$ at the moment map value $\mu_0(y)$ (1.1) with generic angle parameter y. These systems were previously considered in [18,19] under the restriction $0 < y < \pi/n$. We have shown that the reduction always yields a Liouville integrable system whose leading Hamiltonian has the RS form (1.15) on a dense open submanifold of the compact reduced phase space. Different moment map values (with $0 < y < \pi/2$) correspond to inequivalent many-body systems in general. It turned out that two drastically different types of cases occur, which we termed type (i) and type (ii).

In the type (i) cases the reduced phase space $P(\mu_0(y))$ is a Hamiltonian toric manifold since it inherits globally smooth action and position variables from the double. Our main result (given by Theorems 12 and 13 in Section 3) is that we found all y values associated with type (i) cases, and also found that the pertinent toric moment polytope is always a simplex. This implies the existence of an equivariant symplectomorphism between the reduced phase space $P(\mu_0(y))$ and the complex projective space equipped with a multiple of its standard symplectic structure, which we detailed for the particular type (i) cases having coupling parameter $\pi/n < y < \pi/(n-1)$.

In the type (ii) cases the action and position variables lose their differentiability on a nowhere dense subset of $P(\mu_0(y))$. The existence of such cases is an unexpected new result. The properties of the corresponding compact RS systems should be further explored in the future.

We worked at the classical level, but the quantum mechanics of our systems should be also investigated. It is more or less clear how to perform such investigation in the type (i) cases, since there exist general results on the quantization of Hamiltonian toric manifolds [26] and also a detailed study [27] of the quantum mechanics of the standard compact RS systems belonging to the range $0 < y < \pi/n$. In the type (ii) cases no previous studies exist.

Finally, it is worth stressing that the compact RS systems (both type (i) and type (ii)) that we dealt with are self-dual in the sense that there exists a symplectomorphism of order 4 on their phase space exchanging the position and action variables. In the same way as explained in [19], the self-duality map descends from the natural action of the modular $\mathrm{SL}(2,\mathbb{Z})$ group on the double, which provides a finite dimensional model for describing the moduli spaces of flat $\mathrm{SU}(n)$ connections on the one-holed torus [20]. It should be possible to construct a corresponding quantum mechanical representation of the $\mathrm{SL}(2,\mathbb{Z})$ group in the compact RS systems. General arguments based on Chern–Simons field theories [28] and on Hecke algebras [29] indicate the existence of such $\mathrm{SL}(2,\mathbb{Z})$ representation, but its construction in sufficiently concrete terms was, as far as we know, not addressed before even in the standard case [27].

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Appendix A. Some properties of class functions of *G*

In this appendix we briefly survey relevant properties of the real class functions of $G := \mathrm{SU}(n)$. We first show that the derivatives of globally smooth class functions span an $(n-1)$-dimensional space at all regular points, but a smaller dimensional subspace at singular points. Then we explain that the class functions Ξ_k that we defined in (2.2) are not globally smooth. They are smooth when restricted to G_{reg} and only continuous at G_{sing}. These results are well known in Lie theory, and are described here to make our text essentially self-contained.

To begin, let us remark that at any $g \in G$ the $\mathfrak{g}$-valued derivative $\nabla h(g)$ of $h \in C^\infty(G)^G$ (which is the translate of the usual exterior derivative to the unit element) belongs to the center of the Lie algebra of the stabilizer subgroup G_g of g with respect to conjugation. This is a consequence of the equivariance property $\nabla h \in C^\infty(G, \mathfrak{g})^G$. At regular g, G_g is Abelian of dimension $(n-1)$, while at non-regular g the dimension of the center of the Lie algebra of G_g is smaller than $(n-1)$. Thus it follows that at $g \in G_{\text{sing}} := G \setminus G_{\text{reg}}$ the dimension of the span of the derivatives of the C^∞ class functions drops; it becomes zero at the center of G. Via our reduction, the smooth class function applied to A in $(A, B) \in G \times G$ descend to the globally smooth principal Hamiltonian of the compact RS systems and its commuting family. The dimension of the span of the derivatives of the functions concerned cannot increase through the reduction, which involves projections. (It can actually decrease, as is exemplified by the vertices of the Delzant polytope (4.3), where the Hamiltonian vectors fields of all reduced "smooth class functions of B" vanish.) The message is that interesting special phenomena in the behavior of the Hamiltonian flows can be expected at the points of the reduced phase space that come from gauge orbits for which A or B in $(A, B) \in \mu^{-1}(\mu_0(y))$ belongs to G_{sing}.

Next, let us focus on the "spectral functions" Ξ_k (2.2) that were crucial for our considerations. These were defined using the formula (1.9), which can be recast in the equivalent form

$$\delta(\xi) = \exp\left(-2\mathrm{i}\sum_{k=1}^{n-1} \xi_k \Lambda_k\right), \tag{A.1}$$

where the diagonal matrices $\Lambda_k = \sum_{j=1}^{k} E_{j,j} - \frac{k}{n}\mathbf{1}_n$ realize the fundamental weights of $\mathrm{su}(n)$ in the standard manner. Every conjugacy class of G admits a representative of the form $\delta(\xi)$ for a unique $\xi \in \mathcal{A}$. Thus formula (A.1) yields a one-to-one correspondence between the elements of the alcove $\mathcal{A}$ (1.8) and the conjugacy classes of G. This correspondence is known to be a *homeomorphism* [30] with respect to the topology on the set of conjugacy classes inherited from the group and the topology on the alcove $\mathcal{A}$ inherited from its embedding in $\mathbb{R}^n$ (or in the Lie algebra of the maximal torus). Hence our spectral functions Ξ_k are continuous functions on G. It is also well known that the mapping

$$\mathcal{A}^{\text{reg}} \times (G/\mathbb{T}^{n-1}) \to G_{\text{reg}} \quad \text{defined by } (\xi, \gamma\mathbb{T}^{n-1}) \mapsto \gamma\delta(\xi)\gamma^{-1}, \tag{A.2}$$

where $\mathcal{A}^{\text{reg}}$ is the interior of the alcove $\mathcal{A}$, is an analytic diffeomorphism of real analytic manifolds. In particular, the spectral functions are real analytic (and thus also smooth) functions on G_{reg}. They encode the $\mathcal{A}$-component of the analytic inverse of the above map.

The parametrization by the representatives in (A.1) is a special case of the parametrization of the conjugacy classes by a fundamental domain of the affine Weyl group, which works similarly for any connected and simply connected simple compact Lie group [30].

Finally, let us explain the non-differentiability of the spectral functions at the singular locus G_{sing}. As an illustration, consider the group SU(2) and parametrize the elements η from a small neighborhood of the identity in its maximal torus as

$$\eta(x) := \operatorname{diag}(e^{\mathrm{i}x}, e^{-\mathrm{i}x}), \quad x \in (-\epsilon, \epsilon). \tag{A.3}$$

It is not hard to see from the definition (2.2) that the first component of $\Xi := \Xi^{\text{SU}(2)}$ satisfies

$$\Xi_1^{\text{SU}(2)}(\eta(x)) = |x| \tag{A.4}$$

for small x. This function is not differentiable at $x = 0$.

In order to demonstrate that the spectral functions of $G = \mathrm{SU}(n)$ for $n > 2$ are also not differentiable at G_{sing}, suppose that the converse was true. That is, suppose that Ξ^G is smooth at $g \in G_{\mathrm{sing}}$. We show that this would imply the smoothness of $\Xi^{\mathrm{SU}(2)}$ at the identity (contradicting what we have seen). To do this, take $g \in G_{\mathrm{sing}}$ as a diagonal matrix in the normal form (A.1), and assume that $\xi_i = 0$ for some $1 \leqslant i \leqslant (n-1)$, which means that $\delta_i = \delta_{i+1}$. For simplicity, we also assume that all other components of ξ are positive. Then define the smooth map F by

$$F: \mathrm{SU}(2) \to \mathrm{SU}(n), \qquad \eta \mapsto \mathrm{diag}(\delta_1, \ldots, \delta_{i-1}, \eta\delta_i, \delta_{i+2}, \ldots, \delta_n), \tag{A.5}$$

where the instance of diag should be read as a block-diagonal matrix. It is easy to check that

$$\Xi_1^{\mathrm{SU}(2)}\big(\eta(x)\big) = \big(\Xi_i^G \circ F\big)\big(\eta(x)\big) \tag{A.6}$$

near the identity. Then, because Ξ_i^G is smooth by assumption and because F is smooth by definition, so would be $\Xi_1^{\mathrm{SU}(2)}$. This contradictions shows that our assumption is false. In other words, Ξ^G is not smooth at $\delta(\xi) \in G_{\mathrm{sing}}$. Similar arguments can be applied to demonstrate non-smoothness at arbitrary points of G_{sing}.

The local properties of the spectral functions also follow from classical results about the behavior of (ordered) eigenvalues of matrices under multi-parameter analytic perturbations [31].

Appendix B. Denseness properties

Our purpose is to show that $\hat{\beta}^{-1}(\mathcal{A}_y^+)$, where the local RS system lives according to Theorem 6, is a *dense* submanifold of the reduced phase space. If (1.16) holds, this easily follows from the fact that $\hat{\beta}^{-1}(\mathcal{A}_y^+)$ is exactly the subset of principal orbit type for the β-generated torus action on the Hamiltonian toric manifold $P(\mu_0(y))$, which is known to be dense. If (1.16) fails, however, we do not have a Hamiltonian toric manifold structure on $P(\mu_0(y))$, necessitating a separate proof.

We first demonstrate that the β-regular part of the constraint surface is dense.

Proposition B.1. *For any y in (2.1), the elements $(A, B) \in G \times G$ such that $\mu(A, B) = \mu_0(y)$ and B is regular form a dense subset of the solutions to $\mu(A, B) = \mu_0(y)$.*

Proof. Recall the definition of the discriminant of a polynomial f:

$$\Delta(f) := \prod_{i<j} (\lambda_i - \lambda_j)^2 \tag{B.1}$$

for f given by

$$f = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n). \tag{B.2}$$

It is a classical result that $\Delta(f)$ is actually a polynomial in the coefficients of f. It is clear that $\Delta(f)$ is zero exactly when f has a double zero.

We know that $\mu^{-1}(\mu_0(y))$ is a connected, regular submanifold of $G \times G$. In fact, since the moment map constraint is a set of polynomial equations, we also know that $\mu^{-1}(\mu_0(y))$ inherits an *analytic*[5] manifold structure from $G \times G$. Thus the matrix elements of A and B are analytic functions on it.

[5] Our use of 'analytic' in this appendix always means 'real analytic'.

We define the complex function ϕ on $\mu^{-1}(\mu_0(y))$ by

$$\phi : (A, B) \mapsto \Delta\big(\det(\lambda - B)\big). \tag{B.3}$$

It vanishes exactly when B has a double eigenvalue. By the above, ϕ is an analytic function on $\mu^{-1}(\mu_0(y))$. If $\phi^{-1}(\{0\})$ has non-empty interior, then ϕ must vanish identically on $\mu^{-1}(\mu_0(y))$, since it is an analytic connected manifold. This proves that either the subset of $\mu^{-1}(\mu_0(y))$ for which B is non-regular has empty interior, or it coincides with $\mu^{-1}(\mu_0(y))$.

We know, however, that there exists a solution (A, B) to the moment map constraint for which $\beta(A, B) = \xi^*$ with ξ^* defined in (2.24). Since every component of ξ^* is positive, this B is regular. This shows that ϕ does not vanish identically, and thereby the proposition is proved. □

Corollary B.2. *For any y in (2.1), $\mu^{-1}(\mu_0(y)) \cap (G_{\text{reg}} \times G_{\text{reg}})$ is a dense open submanifold of the constraint surface $\mu^{-1}(\mu_0(y))$.*

Proof. Proposition B.1 ensures that $\mu^{-1}(\mu_0(y)) \cap (G \times G_{\text{reg}})$ is a dense open subset of $\mu^{-1}(\mu_0(y))$, and $\mu^{-1}(\mu_0(y)) \cap (G_{\text{reg}} \times G)$ clearly enjoys the same property. The intersection of two dense open sets is again dense open. □

Since the image of a dense set under a continuous surjective map is dense, it follows from Proposition B.1 that the subsets given in the next line are *dense*:

$$\hat{\beta}^{-1}\big(\mathcal{A}_y^{\text{reg}}\big) \subset P\big(\mu_0(y)\big) \quad \text{and} \quad \mathcal{A}_y^{\text{reg}} \subset \mathcal{A}_y. \tag{B.4}$$

We now wish to prove that analogous statements hold also for $\mathcal{A}_y^+ \subset \mathcal{A}_y^{\text{reg}}$ defined in (2.42). Our argument will be very similar to the proof of Proposition B.1.

Proposition B.3. *The open submanifold $\beta^{-1}(\mathcal{A}_y^+) \cap \mu^{-1}(\mu_0(y))$ of the constraint surface is a dense subset of $\beta^{-1}(\mathcal{A}_y^{\text{reg}}) \cap \mu^{-1}(\mu_0(y))$.*

Proof. Using (2.14), define the real function ψ on the analytic manifold $\beta^{-1}(\mathcal{A}_y^{\text{reg}}) \cap \mu^{-1}(\mu_0(y))$ by the formula

$$\psi : (A, B) \mapsto \prod_{\ell=1}^{n} z_\ell\big(\Xi(B), y\big). \tag{B.5}$$

Since $\Xi : G_{\text{reg}} \to \mathcal{A}^{\text{reg}}$ is an analytic map, it follows that ψ is analytic.

Note that the submanifold $\beta^{-1}(\mathcal{A}_y^+) \cap \mu^{-1}(\mu_0(y))$ is exactly the subset of $\beta^{-1}(\mathcal{A}_y^{\text{reg}}) \cap \mu^{-1}(\mu_0(y))$ where ψ takes non-zero values. Suppose that it is not a dense subset. Then there exists a non-empty open subset of $\beta^{-1}(\mathcal{A}_y^{\text{reg}}) \cap \mu^{-1}(\mu_0(y))$ on which ψ vanishes identically. Because ψ is analytic, this implies that ψ vanishes identically on an entire connected component M of $\beta^{-1}(\mathcal{A}_y^{\text{reg}}) \cap \mu^{-1}(\mu_0(y))$.

Let $\hat{M}$ be the connected component of $\hat{\beta}^{-1}(\mathcal{A}_y^{\text{reg}}) \subset P(\mu_0(y))$ corresponding to M, and let $\hat{M}_0$ be the dense open subset of $\hat{M}$ containing the points of principal orbit type for the β-generated $\mathbb{T}^{n-1}$-action restricted to $\hat{M}$. Since ψ vanishes on M, it follows from (the discussion following) Lemma 3 that the $\mathbb{T}^{n-1}$-action on $\hat{M}$ has orbits of dimension strictly smaller than $n - 1$. Moreover, it follows from the theorem on principal orbit type (e.g. [30]) that $\hat{M}_0$ is a locally trivial fibre bundle. Suppose that the $\mathbb{T}^{n-1}$ orbits in $\hat{M}_0$ are of dimension r. Using that the $\mathbb{T}^{n-1}$-action is generated by the moment map $\hat{\beta}$ and is transitive on $\hat{\beta}^{-1}(x)$ for all $x \in \mathcal{A}_y^{\text{reg}}$, we then see that the restriction of the map $\hat{\beta}$ to $\hat{M}_0$ induces a smooth one-to-one map of constant rank r from the base of the bundle $\hat{M}_0$ into $\mathcal{A}^{\text{reg}}$. This would imply that the dimension of $\hat{M}_0$ equals $2r < 2(n-1)$, which contradicts $\hat{M}_0$ being an open submanifold of the reduced phase space of dimension $2(n-1)$. This contradiction shows that it is not possible for the connected component M to be fully contained in the zero set of ψ. Therefore, our assumption that the submanifold $\beta^{-1}(\mathcal{A}_y^+) \cap \mu^{-1}(\mu_0(y))$ is not dense was false, proving the proposition. □

Corollary B.4. *The following is a chain of dense open submanifolds of the reduced phase space:*

$$\hat{\beta}^{-1}\big(\mathcal{A}_y^+\big) \subset \hat{\beta}^{-1}\big(\mathcal{A}_y^{\text{reg}}\big) \subset P\big(\mu_0(y)\big), \tag{B.6}$$

and $\mathcal{A}_y^+ \subset \mathcal{A}_y$ *is a dense subset.*

Corollary B.4 shows that the local RS system of Theorem 6 always lives on an open dense submanifold of the reduced phase space, which is what we wanted to prove.

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