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The Existence of Contingent Epiderivatives for Set-Valued Maps

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Abstract—This short note deals with the issue of existence of contingent epiderivatives for set-valued maps defined from a real normed space to the real line. A theorem of Jahn-Rauh [1], given for the existence of contingent epiderivatives, is used to obtain more general existence results. The strength and the limitations of the main result are discussed by means of some examples.

Keywords—Set-valued maps, Contingent epiderivative, Contingent cone.

1. INTRODUCTION

It has been long recognized that set-valued maps provide a very convenient framework for various problems arising in diverse fields as optimization, optimal control, stochastics, variational analysis, economics, etc. For various reasons, for example, for the formulation of optimality conditions in set-valued optimization, it is of importance to be equipped with a notion of a 'derivative' for set-valued maps. Motivated by this, various notions of a 'derivative' have been proposed and analyzed in recent years (see [2–4]). One of these notions, which has attracted a great deal of attention, is the contingent derivative introduced by Aubin [2]. This concept of contingent derivatives revolves round the notion of a contingent cone and the graph of set-valued maps whose derivative is sought (cf. [2–8]). However, when the notion of contingent derivative is employed in optimization theory, it turns out that there remains a gap in necessary and sufficient optimality conditions, under standard assumptions. Motivated by these facts, a notion of contingent epiderivative for set-valued maps has been proposed in [1]. It has been shown that this notion of the contingent epiderivative is a fundamental concept for the formulation of optimality conditions in set-valued optimization. For instance, the well-known Lagrange multiplier rule can be given using contingent epiderivatives of the objective set-valued map and set-valued map defining

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the constraints (see [1,5,6,9]). At this point, it should be remarked that a notion of contingent epiderivative for functionals was introduced by Aubin [2] (see [4, Chapter VI] for details).

The aim of the present paper is to discuss the existence of the contingent epiderivative for set-valued maps defined from a real normed space to the real line.

Let X and Y be real normed spaces where the space Y is partially ordered by a nontrivial convex cone $C \subset Y$ (cf. [7]). Let $F : X \rightrightarrows Y$ be a given set-valued map. The domain and the graph of F are defined by $\mathrm{dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ and $\mathrm{graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$, respectively. Moreover, the epigraph of F is defined by

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y : x \in \mathrm{dom}(F),\ y \in F(x) + C\}.$$

If F is a single-valued map, $Y = \mathbb{R}$ and $C = \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$, then the above definition of epigraph collapses to the one known in convex analysis (see [8,10]).

Next, we recall that given a set S in a real normed space Z, the contingent cone $T(S, z^*)$ of S at a point $z^* \in \mathrm{cl}(S)$ is the set of all $z \in Z$ such that there is a sequence $(z_n)_{n\in\mathbb{N}} \subset S$ with $z_n \to z^*$ and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+\backslash\{0\}$, so that $\lambda_n(z_n - z^*) \to z$.

It is known that the contingent cone is a nonempty closed cone possessing the isotony property, that is, $T(A, x) \subseteq T(B, x)$ provided that $A \subseteq B$ and $x \in A \cap B$ (see [1–10]).

In the following, we recall the notion of contingent epiderivative.

DEFINITION 1.1. *(See [1].) Let X and Y be real normed spaces and let $C \subset Y$ be a proper convex cone. Let $F : X \rightrightarrows Y$ be a set-valued map and let $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$. A single-valued map $DF(\bar{x}, \bar{y}) : X \to Y$ is called contingent epiderivative of F at $(\bar{x}, \bar{y})$, if the following holds:*

$$\mathrm{epi}(DF(\bar{x}, \bar{y})) = T(\mathrm{epi}(F), (\bar{x}, \bar{y})). \tag{1}$$

If $(\bar{x}, \bar{y})$ belongs to the interior of $\mathrm{epi}(F)$, then $T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ coincides with the product space $X \times Y$ and in this case the contingent epiderivative $DF(\bar{x}, \bar{y})$ does not exist.

We emphasize that in the above definition the domain of $DF(\bar{x}, \bar{y})$ is X. However, it is possible that for $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$, (1) holds but the domain of $DF(\bar{x}, \bar{y})$, say Ω, is a proper subset of X. In this case, we speak of the contingent epiderivative with restricted domain. But we keep the same notion for this derivative and represent it by $DF(\bar{x}, \bar{y}) : \Omega \to Y$.

2. MAIN RESULTS AND EXAMPLES

In the sequel, the notations $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ will represent the interior and the closure of a nonempty set A, respectively. Let $\mathbb{B}_Y$ be the unit ball of some space Y.

Let X be a real normed space and let $F : X \rightrightarrows \mathbb{R}$ be a given set-valued map. To the set-valued map F, we associate another set-valued map $F_+ : X \rightrightarrows \mathbb{R}$ defined by $F_+(x) = \mathrm{cl}(F(x) + \mathbb{R}_+)$. Define a single-valued map $f_0 : X \to \mathbb{R}$ by

$$f_0(x) = \{y \in F_+(x) : y \leq z \text{ for all } z \in F_+(x)\}. \tag{2}$$

Before any advancement, we would like to stress the involvement of F_+.

EXAMPLE 2.1. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map defined by $F(x) :=]x, \infty[$. Observe that the map $f_0(x)$ given in the above assumption is not even well defined if $F_+(x)$ is replaced by $F(x)$ (or by $F(x) + \mathbb{R}_+$). Moreover, $(x, x) \in \mathrm{graph}(f_0)$, but $(x, x) \notin \mathrm{graph}(F)$, $\forall\, x \in \mathbb{R}$.

Observe that by introducing the set-valued map F_+, we circumvent the situations depicted in the above example. Furthermore, it should also be clear that the function f_0, though defined via $\mathrm{cl}(F(x) + \mathbb{R}_+)$, need not have a closed epigraph.

We begin with the following result concerning the existence of contingent epiderivatives.

PROPOSITION 2.1. *Let X be a real normed space and let $F : X \rightrightarrows \mathbb{R}$ be a set-valued map. Let the function f_0 defined via (2) be convex and lower semicontinuous. Assume that $\bar{x} \in \operatorname{int}(\operatorname{dom}(F))$ and $(\bar{x}, f_0(\bar{x})) \in \operatorname{graph}(F)$. Set $\bar{y} = f_0(\bar{x})$. Then the following holds.*

(a) *The contingent epiderivative $DF(\bar{x}, \bar{y})$ of F at $(\bar{x}, \bar{y})$ exists and is given by*

$$DF(\bar{x}, \bar{y})(x) = \lim_{\lambda \searrow 0} \frac{f_0(\bar{x} + \lambda x) - f_0(\bar{x})}{\lambda}, \qquad \forall x \in X.$$

(b) *There exists $\eta > 0$ such that*

$$\forall x \in \eta \mathbb{B}_X, \qquad f_0(\bar{x}) - f_0(\bar{x} - x) \leq DF(\bar{x}, \bar{y})(x) \leq f_0(\bar{x} + x) - f_0(\bar{x}).$$

(c) *For some $\alpha > 0$ and $x \in X$,*

$$DF(\bar{x}, \bar{y})(x) \leq \alpha \|x\|.$$

PROOF. It is evident from (2) that the epigraph of F_+ is essentially characterized by f_0 and we have $\operatorname{cl}(\operatorname{epi}(F)) = \operatorname{cl}(\operatorname{graph}(F_+)) = \operatorname{cl}(\operatorname{epi}(f_0))$. Keeping in mind the lower semicontinuity of f_0, we obtain $\operatorname{epi}(f_0) = \operatorname{cl}(\operatorname{epi}(f_0)) = \operatorname{cl}(\operatorname{epi}(F))$. By a well-known feature of the contingent cone stating that $T(A, a) = T(\operatorname{cl}(A), a)$, with $A := \operatorname{epi}(F)$ and $a := (\bar{x}, \bar{y})$, we obtain

$$T(\operatorname{epi}(f_0), (\bar{x}, \bar{y})) = T(\operatorname{cl}(\operatorname{epi}(F)), (\bar{x}, \bar{y})) = T(\operatorname{epi}(F), (\bar{x}, \bar{y})).$$

Since the function f_0 is convex and lower semicontinuous and $\bar{x} \in \operatorname{int}(\operatorname{dom}(f_0))$, it follows that the contingent epiderivative $Df_0(\bar{x}, \bar{y})(x)$ of the single-valued functional f_0 exists for all $x \in X$ (cf. [4, pp. 198–199]). This amounts to say that

$$Df_0(\bar{x}, \bar{y})(x) = \inf\{y : (x, y) \in T(\operatorname{epi}(f_0), (\bar{x}, \bar{y}))\}$$

is finite for all $x \in X$. Since the contingent cone is closed, the above identity implies

$$\operatorname{epi}(Df_0(\bar{x}, \bar{y})) = T(\operatorname{epi}(f_0), (\bar{x}, \bar{y})).$$

By virtue of the definition of contingent epiderivative $DF(\bar{x}, \bar{y})$, we have

$$\begin{aligned} \operatorname{epi}(DF(\bar{x}, \bar{y})) &= T(\operatorname{epi}(F), (\bar{x}, \bar{y})) \\ &= T(\operatorname{epi}(f_0), (\bar{x}, \bar{y})) \\ &= \operatorname{epi}(Df_0(\bar{x}, \bar{y})), \end{aligned}$$

and hence, the existence of $DF(\bar{x}, \bar{y})(x)$ follows from the existence of $Df_0(\bar{x}, \bar{y})(x)$. Finally, the characterization in (a), and Parts (b) and (c) follow from the corresponding results for the contingent epiderivative of f_0 as given in [4, pp. 198–199]. The proof is complete. ∎

Now we improve the above result by relaxing the assumption of convexity to 'local convexity'. For this purpose, the following result given will be instrumental.

THEOREM 2.1. *(See [1].) Let $F : X \rightrightarrows \mathbb{R}$ be a set-valued map and let $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$. Assume that there are functionals $f_1, f_2 : \operatorname{dom}(f_1) = \operatorname{dom}(f_2) = X \to \mathbb{R}$ such that $\operatorname{epi}(f_1) \supseteq T(\operatorname{epi}(F), (\bar{x}, \bar{y})) \supseteq \operatorname{epi}(f_2)$. Then the contingent epiderivative $DF(\bar{x}, \bar{y})$ is given by*

$$DF(\bar{x}, \bar{y})(x) = \min\{y \in \mathbb{R} \mid (x, y) \in T(\operatorname{epi}(F), (\bar{x}, \bar{y}))\}, \qquad \forall x \in X.$$

It is clear that the above theorem assures that $\operatorname{dom}(DF(\bar{x}, \bar{y})) = X$. In fact, it is a consequence of the assumptions $T(\operatorname{epi}(F), (\bar{x}, \bar{y})) \supseteq \operatorname{epi}(f_2)$ and $\operatorname{dom}(f_2) = X$. Here the domain of F is of less relevance. The following example shows that if $\mathrm{dom}(f_2)$ is a proper subset of X, then $DF(\bar{x},\bar{y})$ might not be well defined in some directions.

EXAMPLE 2.2. Let $F:\mathbb{R}\rightrightarrows\mathbb{R}$ be a set-valued map defined as follows:

$$F(x)=\begin{cases}]\sqrt{|x|},\infty[, & \text{if } x\neq 0,\\ [0,\infty[, & \text{if } x=0.\end{cases}$$

Let $(\bar{x},\bar{y})=(0,0)$. Then we have $T(\mathrm{epi}(F),(\bar{x},\bar{y}))=\{(0,y)\in\mathbb{R}^2 : y\geq 0\}$. Now let us choose a function $p_1:\mathbb{R}\to\mathbb{R}$ defined by $p_1(x)=|x|$ for all $x\in\mathbb{R}$, and another function $p_2:\{0\}\to\mathbb{R}$ defined by $p_2(0)=0$. It is easy to check that if the contingent epiderivative of p_i at $(\bar{x},\bar{y})$ is denoted by f_i, where $i=1,2$, then the required condition $\mathrm{epi}(f_1)\supseteq T(\mathrm{epi}(F),(\bar{x},\bar{y}))\supseteq\mathrm{epi}(f_2)$ holds. However, the contingent epiderivative $DF(\bar{x},\bar{y})$ exists only with the restricted domain $\{0\}\subset\mathbb{R}$ and is given by $DF(\bar{x},\bar{y})(0)=0$.

Let Z be a real normed space and let $\omega\in\Omega\subset Z$ be arbitrary. We reserve the notation $U(\Omega,\omega)$ to represent the set $\mathcal{N}(\omega)\cap\Omega$, where $\mathcal{N}(\omega)$ is an arbitrary neighborhood of ω.

Our next result for the existence of contingent epiderivative is as follows.

THEOREM 2.2. *Let X be a real normed space and let $F:X\rightrightarrows\mathbb{R}$ be a set-valued map. Assume that $\bar{x}\in\mathrm{int}(\mathrm{dom}(F))$ and $(\bar{x},f_0(\bar{x}))\in\mathrm{graph}(F)$, where the functional f_0 is defined via (2). Set $\bar{y}=f_0(\bar{x})$. Assume that there exist functionals $f_i:X\to\mathbb{R}$, $i=1,2$ satisfying*

(a) *$(\bar{x},\bar{y})\in\mathrm{graph}(f_1)\cap\mathrm{graph}(f_2)$ and $\bar{x}\in\mathrm{int}(\mathrm{dom}(f_1))\cap\mathrm{int}(\mathrm{dom}(f_2))$;*
(b) *$U(\mathrm{epi}(f_1),(\bar{x},\bar{y}))\supseteq U(\mathrm{epi}(f_0),(\bar{x},\bar{y}))\supseteq U(\mathrm{epi}(f_2),(\bar{x},\bar{y}))$;*
(c) *the functionals f_1 and f_2 are convex and lower semicontinuous.*

Then the contingent epiderivative $DF(\bar{x},\bar{y})(x)$ of F at $(\bar{x},\bar{y})$ exists for all $x\in X$.

PROOF. In view of (a) and (b), and the isotony property of the contingent cone, we have

$$T(U(\mathrm{epi}(f_1),(\bar{x},\bar{y})),(\bar{x},\bar{y}))\supseteq T(U(\mathrm{epi}(f_0),(\bar{x},\bar{y})),(\bar{x},\bar{y}))\supseteq T(U(\mathrm{epi}(f_2),(\bar{x},\bar{y})),(\bar{x},\bar{y})).$$

Moreover, from the local approximation nature of the contingent cone, it follows that

$$T(U(\mathrm{epi}(f_i),(\bar{x},\bar{y})),(\bar{x},\bar{y}))=T(\mathrm{epi}(f_i),(\bar{x},\bar{y})),\qquad i=0,1,2.$$

Combining this with the preceding set of inclusions yields

$$T(\mathrm{epi}(f_1),(\bar{x},\bar{y}))\supseteq T(\mathrm{epi}(f_0),(\bar{x},\bar{y}))\supseteq T(\mathrm{epi}(f_2),(\bar{x},\bar{y})). \tag{3}$$

Since f_i is convex and lower semicontinuous, we have the existence of function $p_i:X\to\mathbb{R}$, $i=1,2$, such that

$$\mathrm{epi}(p_i)=T(\mathrm{epi}(f_i),(\bar{x},\bar{y})),\qquad i=1,2.$$

The above identity, when combined with (3), yields

$$\mathrm{epi}(p_1)\supseteq T(\mathrm{epi}(f_0),(\bar{x},\bar{y}))\supseteq\mathrm{epi}(p_2).$$

The existence of $DF(\bar{x},\bar{y})(x)$ for all $x\in X$ now follows from Theorem 2.1. ∎

REMARK 2.1. In the above result, if we use the so-called lower semicontinuous regularization (cf. [10]) of f_i to generate p_i, then we can dispense with the assumption that f_i is lower semicontinuous. Of course, it is also possible to impose other conditions on f_i which assure the existence of p_i.

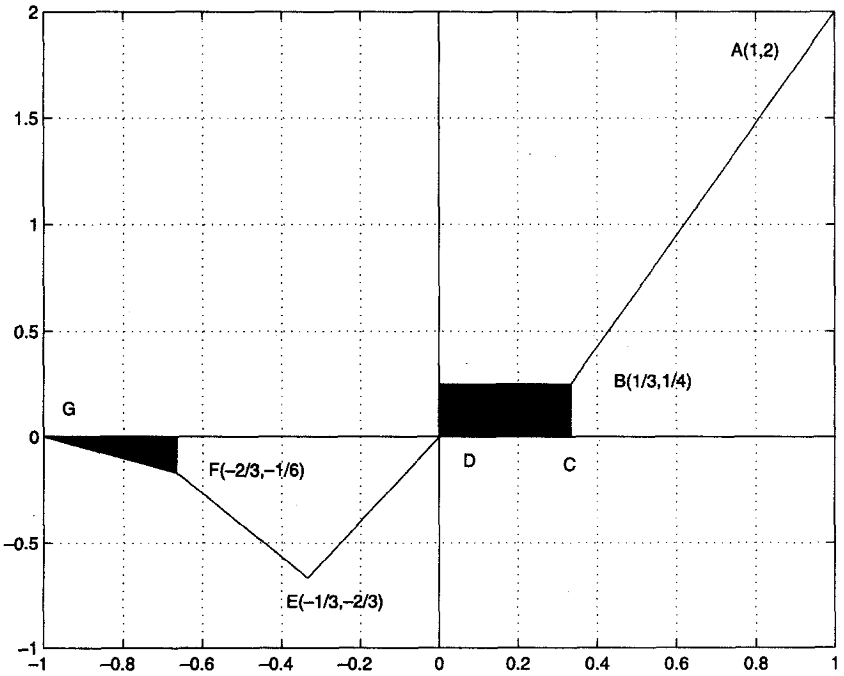
The following example highlights the use of the above result.

EXAMPLE 2.3. Consider a set-valued map $F : [-1,1] \rightrightarrows \mathbb{R}$ given by (cf. Figure 1a)

$$F(x) = \begin{cases} \left(\frac{21}{8}\right) x - \frac{5}{8}, & \text{if } \frac{1}{3} < x \leq 1, \\ \left[0, \frac{1}{4}\right], & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 2x, & \text{if } -\frac{1}{3} < x < 0, \\ \left(-\frac{3}{2}\right) x - \frac{7}{6}, & \text{if } -\frac{2}{3} < x \leq -\frac{1}{3}, \\ \left[-\frac{x}{2} - \frac{1}{2}, 0\right], & \text{if } -1 \leq x \leq -\frac{2}{3}. \end{cases}$$

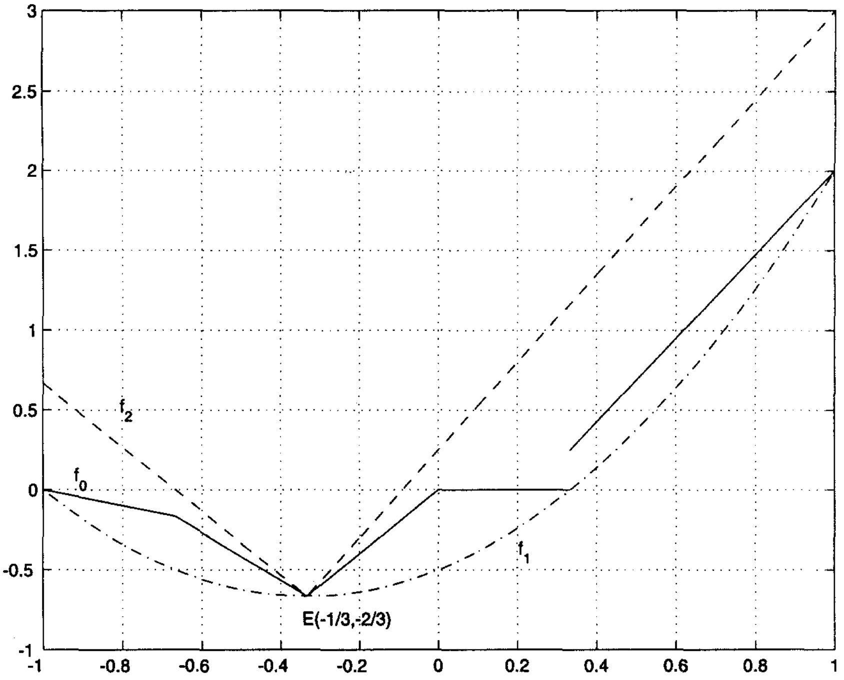
In the present case, the single-valued selection function f_0 (cf. (2)) of F is given by

$$f_0(x) = \begin{cases} \left(\frac{21}{8}\right) x - \frac{5}{8}, & \text{if } \frac{1}{3} < x \leq 1, \\ 0, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 2x, & \text{if } -\frac{1}{3} < x < 0, \\ \left(-\frac{3}{2}\right) x - \frac{7}{6}, & \text{if } -\frac{2}{3} < x \leq -\frac{1}{3}, \\ -\frac{x}{2} - \frac{1}{2}, & \text{if } -1 \leq x \leq -\frac{2}{3}. \end{cases}$$



(a) The set-valued map F.

Figure 1.



(b) The functions f_i, $i = 0, 1, 2$.

Figure 1. (cont.)

The map f_0 is discontinuous with $\operatorname{dom}(f_0) = [-1, 1]$ (cf. Figure 1b). Let us consider the point $(-1/3, -2/3) \in \operatorname{graph}(f_0)$. Define two functions $f_i : [-1, 1] \to \mathbb{R}$, $i = 1, 2$, as follows:

$$f_1(x) = \left(\frac{3}{2}\right) x^2 + x - \frac{1}{2}, \qquad \text{for all } x \in [-1, 1],$$

$$f_2(x) = \begin{cases} \left(\dfrac{11}{4}\right) x + \dfrac{1}{4}, & \text{if } -\dfrac{1}{3} < x \le 1, \\ -2x - \dfrac{4}{3}, & \text{if } -1 \le x \le -\dfrac{1}{3}. \end{cases}$$

The functions f_i, $i = 1, 2$, are convex and satisfy the following conditions (see Figure 1b).

(I) $U(\operatorname{epi}(f_1), (-1/3, -2/3)) \supset U(\operatorname{epi}(f_0), (-1/3, -2/3)) \supset U(\operatorname{epi}(f_2), (-1/3, -2/3))$.

(II) $(-1/3, -2/3) \in \operatorname{graph}(f_1) \cap \operatorname{graph}(f_2)$ and $-1/3 \in \operatorname{int}(\operatorname{dom}(f_1)) \cap \operatorname{int}(\operatorname{dom}(f_2))$.

Therefore, all the hypotheses of Theorem 2.2 are satisfied, and hence, the existence of the contingent epiderivative is assured at $(-1/3, -2/3)$. Note that the epigraph of the set-valued map f_0 is not convex, but only convex around $(-1/3, -2/3)$.

However, Theorem 2.2 does not provide any information for the existence of the contingent epiderivative at $(-2/3, -1/6)$ as there does not exist any convex map satisfying the hypotheses on the function f_1. Though the contingent epiderivative exists at $(-2/3, -1/6)$.

In order to illustrate the importance of the assumption $\bar{x} \in \operatorname{int}(\operatorname{dom}(f_1)) \cap \operatorname{int}(\operatorname{dom}(f_2))$, let us consider the point $(1/3, 0) \in \operatorname{graph}(f_0)$. Since $T(\operatorname{epi}(F), (1/3, 0)) = \{(x, y) \in \mathbb{R}^2 \mid 0 \ge x,\ y \ge 0\}$, the contingent epiderivative of F at $(1/3, 0)$ with restricted domain is given by $DF(1/3, 0)(x) = 0$, $\forall\, x \le 0$. Consequently, $\operatorname{dom}(DF(1/3, 0)) = \{x \in \mathbb{R} \mid x \le 0\}$.

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