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On the spectral characterization of entire operators with deficiency indices $(1, 1)$

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For entire operators and entire operators in the generalized sense, we provide characterizations based on the spectra of their selfadjoint extensions. In order to obtain these spectral characterizations, we discuss the representation of a simple, regular, closed symmetric operator with deficiency indices $(1, 1)$ as a multiplication operator in a certain de Branges space.

1. Introduction

M.G. Krein introduced the concept of entire operators in the search of a unified treatment of several classical problems in analysis [8–10,12]; a review book on this matter is [3]. In general, it may be difficult to determine whether a given operator is entire due to the lack of criteria not based on finding the entire gauge. In this work we present necessary and sufficient conditions for an operator to be entire for the case of deficiency indices $(1, 1)$. These conditions are based exclusively on the distribution of the spectra of two von Neumann selfadjoint extensions of the operator. More concretely (the precise statement is Theorem 4.5):

Let A be a simple, regular, closed symmetric operator with deficiency indices $(1, 1)$. Consider two of its selfadjoint extensions A_0 and A_γ. Then A is entire if and only if $\operatorname{Spec}(A_0)$ and $\operatorname{Spec}(A_\gamma) = \{x_n\}$ obey the following conditions:

- *The limit*

$$\lim_{r\to\infty} \sum_{0<|x_n|\leqslant r} \frac{1}{x_n}$$

exists;

- $$\lim_{n\to\infty} \frac{n}{x_n^+} = \lim_{n\to\infty} \frac{n}{x_n^-} < \infty;$$

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- *Assuming that* $\operatorname{Spec}(A_\beta) = \{b_n\}$, *define*

$$h_\beta(z) := \begin{cases} \lim_{r\to\infty} \prod_{|b_n|\leqslant r}(1 - \frac{z}{b_n}) & \text{if } 0 \notin \operatorname{Spec}(A_\beta), \\ z \lim_{r\to\infty} \prod_{0<|b_n|\leqslant r}(1 - \frac{z}{b_n}) & \text{otherwise.} \end{cases}$$

The series $\sum_{n\in\mathbb{N}} |\frac{1}{h_0(x_n)h'_\gamma(x_n)}|$ *is convergent.*

Here $\{x_n^+\}$ *and* $\{x_n^-\}$ *are, respectively, the sequences of positive and negative elements of* $\operatorname{Spec}(A_\gamma)$.

Our spectral characterization was motivated by a result, due to Woracek [19], which gives necessary and sufficient conditions for a de Branges space to have 1 as an associated function (Theorem 3.1). These conditions are formulated in terms of the spectra of two particular selfadjoint extensions of the multiplication operator in the de Branges space. On the basis of a simple result (Lemma 3.4) we reformulate the necessary and sufficient conditions in terms of two arbitrary selfadjoint extensions (Proposition 3.5). These results are then combined with spectral theory and de Branges spaces theory to obtain necessary and sufficient conditions for 1 to be in the de Branges space (Proposition 3.9).

A recent general result [20] gives, as a particular case, necessary and sufficient conditions for a de Branges space to contain the function 1. The present work presents an alternative approach to this question and, by means of Lemma 3.4, provides a way for reformulating the necessary and sufficient conditions of [20] in more general terms.

Having obtained the results mentioned above on de Branges spaces, we establish the spectral characterization of entire operators (Theorem 4.5) on the basis of the representation of any regular, simple, symmetric operator, with deficiency indices $(1,1)$, as the multiplication operator in a certain de Branges space (Section 4). The realization of this representation parallels the construction by Krein, even though the latter yields a de Branges space only when the operator is entire. We ought to mention that an alternative representation theory was developed recently in [13], although based on a quite different approach.

Besides the necessary and sufficient conditions for the spectra of two selfadjoint extensions of an entire operator, we also provide the spectral characterization for operators that are entire in the generalized sense (Theorem 5.4). The key ingredient of this characterization (Proposition 5.1) is the representation developed in Section 4.

In the process of deriving the results of this work, we touch upon the treatment of some two-spectra inverse problems involving selfadjoint extensions of the symmetric operators considered here (Corollary 3.2). This matter will be consider in a forthcoming paper. It is worth remarking that the theory of de Branges spaces has been already applied to inverse spectral problems for Schrödinger operators [14,15].

A short review on entire operators with deficiency indices $(1,1)$ is given in Section 2. As a by-product, we provide a proof of a result by Krein related to the properties of gauges (Proposition 2.2). Section 3 starts with a brief account of some facts on de Branges spaces.

2. Review on entire operators

Let $\operatorname{Sym}_R^{(1,1)}(\mathcal{H})$ denote the class of simple, regular, closed symmetric operators densely defined on a Hilbert space $\mathcal{H}$, whose deficiency indices are $(1,1)$. It is known, and easily verifiable, that because of the simplicity this class is not empty only when $\mathcal{H}$ is separable [12, Section 2]. The selfadjoint extensions of a given operator $A \in \operatorname{Sym}_R^{(1,1)}(\mathcal{H})$ shall be denoted by A_β with $\beta \in [0,\pi)$ (other parametrizations may be chosen as well). The spectra of such selfadjoint extensions are always discrete and of multiplicity one. Also, the spectra of any two selfadjoint extensions interlace and, moreover, every point of the real line belongs to the spectrum of a unique selfadjoint extension.

$\operatorname{Sym}_R^{(1,1)}(\mathcal{H})$ as a set is invariant under similarity transformations: If $V : \mathcal{H} \to \mathcal{H}'$ is one-to-one and onto, then $\operatorname{Sym}_R^{(1,1)}(\mathcal{H}') = V \operatorname{Sym}_R^{(1,1)}(\mathcal{H}) V^{-1}$ (where the domains of operators are transformed accordingly).

In what follows, the inner product will be assumed anti-linear in the first argument.

Given $A \in \operatorname{Sym}_R^{(1,1)}(\mathcal{H})$, a vector $\mu \in \mathcal{H}$ is called a *gauge* for A if

$$\mathcal{H} = \operatorname{Ran}(A - z_0 I) \dotplus \operatorname{Span}\{\mu\}, \tag{2.1}$$

for some $z_0 \in \mathbb{C}$. Once a gauge has been chosen, we look for the set of complex numbers for which (2.1) fails to hold:

$$S_\mu = \mathbb{C} \setminus \{z \in \mathbb{C}\colon \mathcal{H} = \operatorname{Ran}(A - zI) \dotplus \operatorname{Span}\{\mu\}\}.$$

That is, $w \in S_\mu$ if and only if $\mu \perp \operatorname{Ker}(A^* - \overline{w}I)$. Likewise, for every selfadjoint extension $A_\beta \supset A$ we define

$$S_\mu^{(\beta)} = \{z \in \mathbb{C} \setminus \operatorname{Spec}(A_\beta)\colon \langle \mu, \psi^{(\beta)}(\bar{z})\rangle = 0\},$$

where

$$\psi^{(\beta)}(z) := (A_\beta - z_0 I)(A_\beta - zI)^{-1}\psi_0 \tag{2.2}$$

is the generalized Cayley transform of $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$. Since $\psi^{(\beta)}(z) \in \operatorname{Ker}(A^* - zI)$ for every $z \in \mathbb{C} \setminus \operatorname{Spec}(A_\beta)$ (see [3]), we conclude that

$$S_\mu^{(\beta)} \subset S_\mu \subset S_\mu^{(\beta)} \cup \operatorname{Spec}(A_\beta).$$

It follows that S_μ is at most a countable set with no finite accumulation points. Also, it is easy to verify that

$$S_\mu = \bigcup_\beta S_\mu^{(\beta)}.$$

For symmetric operators of the class considered here, one can easily give a gauge such that the exceptional set S_μ lies entirely on the real line.

Lemma 2.1. *Let $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$. Take an eigenvector μ_0 of some selfadjoint extension A_β as a gauge for A. Then $S_{\mu_0} = \operatorname{Spec}(A_\beta) \setminus \{x_0\}$, where x_0 is the eigenvalue associated to μ_0.*

Proof. The inclusion $\operatorname{Spec}(A_\beta) \setminus \{x_0\} \subset S_{\mu_0}$ is straightforward, thus we shall only deal with the converse inclusion. Since necessarily $x_0 \notin S_{\mu_0}$, it suffices to show that S_{μ_0} is contained in $\operatorname{Spec}(A_\beta)$. Suppose that $v \in \mathbb{C} \setminus \operatorname{Spec}(A_\beta)$. For such a v, $\psi^{(\beta)}(v) = (A_\beta - z_0 I)(A_\beta - vI)^{-1}\psi_0$ is well defined, where we have chosen z_0 non-real such that $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$ is not orthogonal to μ_0. Then,

$$\langle \mu_0, \psi^{(\beta)}(v) \rangle = \langle (A_\beta - \overline{v} I)^{-1}(A_\beta - \overline{z_0} I)\mu_0, \psi_0 \rangle = \frac{x_0 - z_0}{x_0 - v} \langle \mu_0, \psi_0 \rangle \neq 0,$$

thus implying $v \notin S_{\mu_0}$. □

This result allows us to prove the following assertion first formulated by Krein without proof in [9, Theorem 8].

Theorem 2.2 *(Krein). For every $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, there exists a gauge μ such that $S_\mu \cap \mathbb{R} = \emptyset$.*

Proof. Choose μ_1 and μ_2 among the eigenstates of two different selfadjoint extensions, say, $A_\beta \mu_1 = x_1 \mu_1$ and $A_{\beta'} \mu_2 = x_2 \mu_2$ ($x_1 \neq x_2$). By Lemma 2.1 we have $S_{\mu_1} = \operatorname{Spec}(A_\beta) \setminus \{x_1\}$ and $S_{\mu_2} = \operatorname{Spec}(A_{\beta'}) \setminus \{x_2\}$, therefore

$$S_{\mu_1} \cap S_{\mu_2} = \emptyset, \tag{2.3}$$

due to the disjointness property mentioned above.

Next, we show that for every $x \in \mathbb{R} \setminus S_{\mu_2}$ there exists a unique $g(x) \in \mathbb{C}$ such that $(\mu_1 + g(x)\mu_2) \perp \operatorname{Ker}(A^* - xI)$: The uniqueness follows from the fact that $x \notin S_{\mu_2}$. As for the existence, choose some selfadjoint extension $A_{\beta''}$ such that $x \notin \operatorname{Spec}(A_{\beta''})$ and set

$$g(x) = -\frac{\langle \psi^{(\beta'')}(x), \mu_1 \rangle}{\langle \psi^{(\beta'')}(x), \mu_2 \rangle}.$$

Finally, choose some $\tilde{g} \in G := \mathbb{C} \setminus \{g = g(x)\colon\ x \in \mathbb{R} \setminus S_{\mu_2}\}$. The set G is non-empty since $g(x)$ is differentiable in any closed interval of $\mathbb{R} \setminus S_{\mu_2}$ and it cannot produce a space filling curve [17, Section 5.4]. Then $\mu = \mu_1 + \tilde{g}\mu_2$ satisfies the claimed property as a result of (2.3). □

Remark. Lemma 2.1 implies that eigenstates of two different selfadjoint extensions, of an operator in $\operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, are never orthogonal to one another. This is of course expected.

By Theorem 2.2, for operators of the class considered here it is always possible to find a gauge (indeed an uncountable number of them) such that the exceptional set S_μ lies outside the real line. A distinguished class of operators is the following one.

Definition 2.3. An operator $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ is called *entire* if there exists a gauge $\mu \in \mathcal{H}$ such that $S_\mu = \emptyset$, in which case μ is said to be an *entire gauge*.

In order to decide whether an operator is entire, the notion of universal directing functional is sometimes useful. The following statement is classical [3, Chapter 2].

Proposition 2.4. *Suppose $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ admits a universal directing functional $\Phi(\cdot, z)$ such that, for some $\mu \in \mathcal{H}$,*

$$\Phi(\mu, z) \neq 0 \quad \text{for all } z \in \mathbb{C}. \tag{2.4}$$

Then A is entire and μ is an entire gauge for A. Conversely, associated to an entire operator with entire gauge μ, there is a universal directing functional that satisfies (2.4)*.*

We note that the determination of a universal directing functional may not be easier than to find an entire gauge by brute force. One of the aims of the present work is to provide alternative criteria for determining when an operator in $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ is entire. As we already have mentioned, these criteria will rely upon the distribution of elements of the spectra of selfadjoint extensions, thus not requiring the searching of gauges with particular properties. However, we should mention that the spectral characterization discussed in the present work may not necessarily be easier to use in practice.

3. Some known and new results on de Branges spaces

Let $\mathcal{B}$ denote a non-trivial linear manifold of entire functions that is complete with respect to the norm generated by a given inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$. We say that $\mathcal{B}$ is an (*axiomatic*) *de Branges space* if, for every $f(z)$ in that space, the following conditions holds:

(A1) For every w: $\operatorname{Im} w \neq 0$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous;
(A2) for every non-real zero w of $f(z)$, the function $f(z)(z-\overline{w})(z-w)^{-1}$ belongs to $\mathcal{B}$ and has the same norm as $f(z)$;
(A3) the function $f^{\#}(z) := \overline{f(\overline{z})}$ also belongs to $\mathcal{B}$ and has the same norm as $f(z)$.

By the Riesz lemma, (A1) is equivalent to the existence of a reproducing kernel $k(z,w)$ that belongs to $\mathcal{B}$ for every non-real w and has the property $\langle k(\cdot,w), f(\cdot)\rangle_{\mathcal{B}} = f(w)$ for every $f(z) \in \mathcal{B}$. Moreover, $k(w,w) = \langle k(\cdot,w), k(\cdot,w)\rangle_{\mathcal{B}} \geqslant 0$ where, as a consequence of (A2), the positivity is strict for every non-real w unless $\mathcal{B} \cong \mathbb{C}$; see the proof of Theorem 23 in [2]. Notice that $k(z,w) = \langle k(\cdot,z), k(\cdot,w)\rangle_{\mathcal{B}}$ whenever z and w are both non-real, therefore $k(w,z) = \overline{k(z,w)}$. Finally, due to (A3) it can be shown that $\overline{k(\overline{z},w)} = k(z,\overline{w})$ for every non-real w; we refer again to the proof of Theorem 23 in [2].

Remark. By construction $k(z,w)$ is entire with respect to its first argument and, by (A3), it is anti-entire with respect to the second one (once $k(z,w)$, as a function of its second argument, has been extended to the whole complex plane [2, Problem 52]).

There is another way of defining a de Branges space. One starts by considering an entire function $e(z)$ of the Hermite–Biehler class, that is, an entire function without zeros in the upper half-plane $\mathbb{C}^+$ that satisfies the inequality $|e(z)| > |e^{\#}(z)|$ for $z \in \mathbb{C}^+$. Then, the *(canonical) de Branges space* $\mathcal{B}(e)$ associated to $e(z)$ is the linear manifold of all entire functions $f(z)$ such that both $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ belong to the Hardy space $H^2(\mathbb{C}^+)$, and equipped with the inner product

$$\langle f(\cdot), g(\cdot)\rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2}\,dx.$$

$\mathcal{B}(e)$ is indeed a Hilbert space. Both definitions of de Branges spaces are equivalent in the following sense: Every canonical de Branges space obeys (A1)–(A3); conversely, given an axiomatic de Branges space $\mathcal{B}$ there exists an Hermite–Biehler function $e(z)$ such that $\mathcal{B}$ coincides with $\mathcal{B}(e)$ as sets and the respective norms satisfies the equality $\|f(\cdot)\|_{\mathcal{B}} = \|f(\cdot)\|_{\mathcal{B}(e)}$ [2, Chapter 2]. The function $e(z)$ is not unique; a choice for it is

$$e(z) = -i\sqrt{\frac{\pi}{k(w_0,w_0)\operatorname{Im}(w_0)}}(z-\overline{w_0})k(z,w_0), \tag{3.1}$$

where w_0 is some fixed complex number with $\operatorname{Im}(w_0) > 0$. It is customary and often useful to decompose $e(z)$ into its real and imaginary parts. Define

$$a(z) := \frac{e(z)+e^{\#}(z)}{2}, \qquad b(z) := i\frac{e(z)-e^{\#}(z)}{2}$$

then $e(z) = a(z) - ib(z)$. Both $a(z)$ and $b(z)$ are real entire functions in the sense that they obey the identity $f^{\#}(z) = f(z)$.

The reproducing kernel can be expressed in terms of the function $e(z)$ (see for instance [5, Section 5]) as follows

$$k(z,w) = \begin{cases} \frac{e^{\#}(z)e(\overline{w}) - e(z)e^{\#}(\overline{w})}{2\pi i(z-\overline{w})}, & z \neq \overline{w}, \\ \frac{i}{2\pi}[e^{\#'}(z)e(z) - e'(z)e^{\#}(z)], & z = \overline{w}. \end{cases}$$

Notice that

$$k(z,z)=\frac{|e(z)|^2-|e^\#(z)|^2}{4\pi\,\mathrm{Im}(z)}>0,\quad z\in\mathbb{C}^+,$$

whereby it follows that $e(z)$ given in (3.1) is indeed a Hermite–Biehler function. From (3.1) we also conclude that $k(z,w)$ is always different from zero whenever $z,w\in\mathbb{C}^+$.

An entire function $g(z)$ is said to be associated to a de Branges space $\mathcal{B}$ if for every $f(z)\in\mathcal{B}$ and $w\in\mathbb{C}$,

$$\frac{g(z)f(w)-g(w)f(z)}{z-w}\in\mathcal{B}.$$

The set of associated functions is denoted $\operatorname{Assoc}\mathcal{B}$. It is well known that

$$\operatorname{Assoc}\mathcal{B}=\mathcal{B}+z\mathcal{B};$$

see [2, Theorem 25] and [5, Lemma 4.5] for alternative characterizations. In passing, let us note that $e(z)\in\operatorname{Assoc}\mathcal{B}(e)\setminus\mathcal{B}(e)$; this fact follows easily from [2, Theorem 25]. Let us also recall that $\operatorname{Assoc}\mathcal{B}(e)$ contains an important family of entire functions. They are given by

$$s_\beta(z):=\frac{i}{2}\left[e^{i\beta}e(z)-e^{-i\beta}e^\#(z)\right]=-a(z)\sin\beta+b(z)\cos\beta,\quad \beta\in[0,\pi).\tag{3.2}$$

These real entire functions determine the selfadjoint extensions of the multiplication operator; see below. Notice that $a(z)=-s_{\pi/2}(z)$ and $b(z)=s_0(z)$. Also, every $s_\beta(z)$ has only real zeros of multiplicity one and the (sets of) zeros of any pair $s_\beta(z)$ and $s_{\beta'}(z)$ are always interlaced.

Some of the results obtained in this paper are related to the problem of telling when a de Branges space contains the function 1. In connection with this, there is a result [19, Theorem 1.1], which was recently generalized in various directions [20], that characterizes de Branges spaces for which 1 is an associated function. This result is of interest to us because it relies on the distribution of the zero-sets of the functions $s_\beta(z)$. The aforementioned theorem may be stated as follows.

Theorem 3.1 *(Woracek). Assume $e(x)\neq 0$ for $x\in\mathbb{R}$ and $e(0)=1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of zeros of the function $s_{\pi/2}(z)$. Also, let $\{x_n^+\}_{n\in\mathbb{N}}$ and $\{x_n^-\}_{n\in\mathbb{N}}$ be the sequences of positive, respectively negative, zeros of $s_{\pi/2}(z)$, arranged according to increasing modulus. Then an everywhere non-zero, real function belongs to* $\operatorname{Assoc}\mathcal{B}(e)$ *if and only if the following conditions hold true*:

(C1) *The limit*

$$\lim_{r\to\infty}\sum_{0<|x_n|\leqslant r}\frac{1}{x_n}$$

exists;

(C2) $$\lim_{n\to\infty}\frac{n}{x_n^+}=\lim_{n\to\infty}\frac{n}{x_n^-}<\infty;$$

(C3) *Assuming that $\{b_n\}_{n\in\mathbb{N}}$ are the zeros of $s_\beta(z)$, define*

$$h_\beta(z):=\begin{cases}\lim_{r\to\infty}\prod_{|b_n|\leqslant r}(1-\frac{z}{b_n}) & \text{if 0 is not a root of } s_\beta(z),\\ z\lim_{r\to\infty}\prod_{0<|b_n|\leqslant r}(1-\frac{z}{b_n}) & \text{otherwise.}\end{cases}$$

The series $\sum_{n\in\mathbb{N}}|\frac{1}{x_n^2 h_0(x_n)h'_{\pi/2}(x_n)}|$ is convergent.

Furthermore, $s_{\pi/2}(z)/h_{\pi/2}(z)\in\operatorname{Assoc}\mathcal{B}(e)$.

Remark.

1. We warn the reader about the precise meaning of condition (C2), which has been formulated under the implicit assumption that $\{x_n\}_{n\in\mathbb{N}}$ is not semibounded. If the sequence is semibounded, say from below, then (C2) means

$$\lim_{n\to\infty}\frac{n}{x_n^+}=0.$$

(The other limit is meaningless.)

2. The assumption that $e(z)$ has no real zeros amounts to no loss of generality (see Lemma 2.4 of [7]); by the same token, it has been assumed that $e(0)=1$. The latter was done to ensure that 0 is not a root of $s_{\pi/2}(z)=-a(z)$, otherwise the proof of the theorem in [19] would have been more complicated.
3. Due to the interlacing property mentioned above, $s_\beta(0)=0$ for only one value of β.

Theorem 3.1 has the following consequence.

Corollary 3.2. *Let $\{x_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{x_n^{(2)}\}_{n\in\mathbb{N}}$ be two unbounded interlaced sequences of real numbers, with no finite accumulation points. There exists a de Branges space $\mathcal{B}(e)$ such that these sequences are the zero sets of $s_{\pi/2}(z)$ and $s_0(z)$, respectively, and $1\in\operatorname{Assoc}\mathcal{B}$ if and only if $\{x_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{x_n^{(2)}\}_{n\in\mathbb{N}}$ satisfy*

(C0) $\min\{x_n^{(2)}\}<\min\{x_n^{(1)}\}$ *if the sequences are semibounded from below, and* $\max\{x_n^{(1)}\}>\max\{x_n^{(2)}\}$ *if the sequences are semibounded from above,*

along with conditions (C1), (C2) *and* (C3) *of Theorem* 3.1, *where in* (C3) *one substitutes the zeros of $s_{\pi/2}(z)$ by $\{x_n^{(1)}\}$ and the zeros of $s_0(z)$ by $\{x_n^{(2)}\}$.*

Proof. In view of Theorem 3.1, one direction of the statement has already been established. Note that by [4, Chapter VII, Theorem 1] the zero sets of $s_{\pi/2}(z)$ and $s_0(z)$ should satisfy (C0) when the sequences are semibounded. In order to prove the other direction, we assume w.l.o.g. that the sequences have been arranged so that $x_k^{(2)}<x_k^{(1)}<x_{k+1}^{(2)}$. Let us define

$$e(z):=-h_1(z)-ih_2(z),$$

where $h_i(z)$ is the function defined in (C3) by means of $\{x_n^{(i)}\}$. It will suffice to show that $e(z)$ is Hermite–Biehler which, in our case, reduces to showing that

$$\operatorname{Im}\left[-\frac{h_2(z)}{h_1(z)}\right]>0,\quad z\in\mathbb{C}^+.$$

This follows directly from [4, Chapter VII, Theorem 1] taking into account (C0) if the sequences are semibounded. Noting that $a(z)=-h_1(z)$ and $b(z)=h_2(z)$, and recalling Theorem 3.1, the assertion follows. □

Remark.

1. It is clear that, in Corollary 3.2, the phrase "$1\in\operatorname{Assoc}\mathcal{B}$" can be replaced by "$\operatorname{Assoc}\mathcal{B}$ contains a zero-free real entire function". See [7, Lemma 2.4].
2. The proof above also tells us that conditions (C1) and (C2) are sufficient for the existence of a de Branges space $\mathcal{B}(e)$, having the two sequences as zero sets of the associated functions $s_0(z)$ and $s_{\pi/2}(z)$. However, we cannot assure in this case that $\operatorname{Assoc}\mathcal{B}(e)$ contains a zero-free real entire function. Indeed, if (C3) is not satisfied then necessarily $\operatorname{Assoc}\mathcal{B}(e)$ does not contain such kind of entire functions.
3. Corollary 3.2 together with the results of the next section suggest a method for dealing with two-spectra inverse problems. This matter will be treated in a forthcoming paper.

In view of the interlacing property, it seems reasonable to believe that the zero-sets of essentially any pair $s_\beta(z)$ and $s_{\beta'}(z)$ bear similar properties as those of $s_0(z)$ and $s_{\pi/2}(z)$. Thus, one may expect that Theorem 3.1 can be refined accordingly. This refinement is the aim of the following results.

Lemma 3.3. *Suppose $e(z)$ belongs to the Hermite–Biehler class having no real zeros. For $\gamma\in(0,\pi)$, let $\breve{e}(z)=-s_\gamma(z)-is_0(z)$. Then $\breve{e}(z)$ also belongs to the Hermite–Biehler class and has no real zeros. Moreover, $\breve{s}_0(z)=s_0(z)$ and $\breve{s}_{\pi/2}(z)=s_\gamma(z)$.*

Proof. A short computation leads to the last assertion. Also, it is obvious that $\breve{e}(z)$ is not constant and has no real zeros. By the properties of the functions $s_\beta(z)$, it follows that the quotient

$$-\frac{\breve{a}(z)}{\breve{b}(z)}=\frac{\breve{s}_0(z)}{\breve{s}_\gamma(z)}$$

is real and analytic in $\mathbb{C}\setminus\mathbb{R}$. Moreover,

$$-\operatorname{Im}\left[\frac{\breve{a}(z)}{\breve{b}(z)}\right]=-(\sin\gamma)\operatorname{Im}\left[\frac{a(z)}{b(z)}\right]>0$$

for $z\in\mathbb{C}^+$. Thus, by a standard criterion for Hermite–Biehler functions [4, Chapter VII], the first assertion follows. □

Lemma 3.4. *As sets, $\mathcal{B}(\breve{e}) = \mathcal{B}(e)$, therefore* $\operatorname{Assoc}\mathcal{B}(\breve{e}) = \operatorname{Assoc}\mathcal{B}(e)$.

Proof. It suffices to show the two-sided estimate

$$r_1\left|e(x+iy)\right| \geqslant \left|\breve{e}(x+iy)\right| \geqslant r_2\left|e(x+iy)\right|$$

for all $y \geqslant 0$ and some positive constants r_1 and r_2.

A short computation leads to the identity

$$\breve{e}(z) = e(z)\left[\frac{1-e^{i(\gamma+\frac{\pi}{2})}}{2}\right]\left[1 - \frac{1-e^{-i(\gamma-\frac{\pi}{2})}}{1-e^{i(\gamma+\frac{\pi}{2})}}\frac{e^{\#}(z)}{e(z)}\right].$$

The second factor on the r.h.s. is a (γ-dependent) constant whose absolute value lies in $(\sqrt{2}/2, 1)$ because $\gamma \in (0,\pi)$. Similarly,

$$\left|\frac{1-e^{-i(\gamma-\frac{\pi}{2})}}{1-e^{i(\gamma+\frac{\pi}{2})}}\right| \in [0,1).$$

Finally, $|e(x+iy)| \geqslant |e^{\#}(x+iy)|$ whenever $y \geqslant 0$. The required inequalities now follows easily. $\square$

Proposition 3.5. *Suppose $e(x) \neq 0$ for $x \in \mathbb{R}$ and $e(0) = (\sin\gamma)^{-1}$ for some fixed $\gamma \in (0,\pi)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of zeros of the function $s_\gamma(z)$. Also, let $\{x_n^+\}_{n\in\mathbb{N}}$ and $\{x_n^-\}_{n\in\mathbb{N}}$ be the sequences of positive, respectively negative, zeros of $s_\gamma(z)$, arranged according to increasing modulus. Then a zero-free, real entire function belongs to* $\operatorname{Assoc}\mathcal{B}(e)$ *if and only if* (C1) *and* (C2) *of Theorem* 3.1 *hold true along with the additional condition*:

($\mathrm{C3}^\flat$) *For $h_\beta(z)$ defined as in* (C3)*, assume*

$$\sum_{n\in\mathbb{N}}\left|\frac{1}{x_n^2 h_0(x_n)h_\gamma'(x_n)}\right| < \infty.$$

Also, $s_\gamma(z)/h_\gamma(z) \in$ $\operatorname{Assoc}\mathcal{B}(e)$.

Proof. Combine Lemma 3.3, Lemma 3.4 and Theorem 3.1. $\square$

Obviously, multiplication of a Hermite–Biehler function by a (non-zero) complex number induces both a unitary transformation between de Branges spaces and a bijection between the respective sets of associated functions.

Corollary 3.6. *Proposition* 3.5 *remains valid if one replaces 0 and γ by β and β' such that $0 \leqslant \beta < \beta' < \pi$ and assumes that $e(z)$ has no real zeros.*

Proof. Apply Proposition 3.5 to $\mathring{e}(z) = [e(0)\sin\gamma]^{-1}e(z)$ with $\gamma = \beta' - \beta$. $\square$

Remark. The statement of Lemma 3.4 readily generalizes to the sets of N-associated functions introduced in [20]. Consequently, Theorem 3.2 of [20] can be accordingly refined (at least for the case of de Branges Hilbert spaces). We will not discuss this matter further here; nonetheless, see the remark at the end of Section 5.

The operator of multiplication in $\mathcal{B}$ is defined by

$$\operatorname{Dom}(S) := \left\{f(z) \in \mathcal{B}\colon\ zf(z) \in \mathcal{B}\right\}, \quad (Sf)(z) = zf(z). \tag{3.3}$$

This operator is symmetric, simple, regular and has deficiency indices $(1,1)$, although it is not necessarily densely defined; see [5, Proposition 4.2, Corollaries 4.3 and 4.7]. The following characterization of the domain is useful; see [2, Theorem 29] and [5, Corollary 6.3].

Theorem 3.7. $\overline{\operatorname{Dom}(S)} \neq \mathcal{B}$ *if and only if there exists $\gamma \in [0,\pi)$ such that $s_\gamma(z) \in \mathcal{B}$. Furthermore,* $\operatorname{Dom}(S)^\perp = \operatorname{Span}\{s_\gamma(z)\}$.

In this paper we shall deal only with cases where S has domain dense in $\mathcal{B}$. In passing we note that this assumption is fulfilled, for instance, when the polynomials are dense in $\mathcal{B}$. Conditions for this to happen have been studied in [1,6].

There exists an explicit relation between the selfadjoint extensions of S and the entire functions $s_\beta(z)$ defined by (3.2); see [5, Propositions 4.6 and 6.1].

Proposition 3.8. *The selfadjoint extensions of S are in one-to-one correspondence with the set of entire functions $s_\beta(z)$, $\beta \in [0,\pi)$. They are given by*

$$\operatorname{Dom}(S_\beta) = \left\{ g(z) = \frac{f(z) - \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0)}{z - z_0},\ f(z) \in \mathcal{B},\ z_0\colon s_\beta(z_0) \neq 0 \right\},$$

$$(S_\beta g)(z) = zg(z) + \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0). \tag{3.4}$$

Moreover, $\operatorname{Spec}(S_\beta) = \{x_n^{(\beta)} \in \mathbb{R}\colon s_\beta(x_n^{(\beta)}) = 0\}$. *The associated eigenfunctions (up to normalization) are* $g_n^{(\beta)}(z) := \frac{s_\beta(z)}{z - x_n^{(\beta)}}$.

Remark. The derivation of the expression for the selfadjoint extensions discussed in [5] is done in the broader context of selfadjoint relations (so it is valid even when the codimension of $\overline{\operatorname{Dom}(S)}$ is one). However, it might be beneficial to see the connection between the characterization of selfadjoint extensions given by Proposition 3.8 and the standard way *à la* von Neumann. We first note that

$$n(z, w_0) := \frac{k(z, w_0)}{\sqrt{k(w_0, w_0)}} \in \operatorname{Ker}\big(S^* - \overline{w_0} I\big), \qquad \operatorname{Im}(w_0) > 0.$$

The choice on the phase of w_0 is related to the one made on (3.1) so it is not an actual restriction. Also, $\|n(\cdot, w_0)\| = 1$. Since S has deficiency indices $(1,1)$, by the standard theory its selfadjoint extensions are given by

$$\operatorname{Dom}(S_\beta) = \big\{ g(z) = h(z) + pe^{-i\beta} n(z, \overline{w_0}) + pe^{i\beta} n(z, w_0),\ h(z) \in \operatorname{Dom}(S),\ p \in \mathbb{C} \big\},$$

$$(S_\beta g)(z) = zh(z) + pe^{-i\beta} w_0 n(z, \overline{w_0}) + pe^{i\beta}\, \overline{w_0} n(z, w_0),$$

where $\beta \in [0,\pi)$. Due to (A3), $n(z, \overline{w_0}) = \overline{n(\bar{z}, w_0)}$, so, by (3.1), doing some rearrangement and then using (3.2), we obtain

$$\operatorname{Dom}(S_\beta) = \left\{ g(z) = h(z) - ire^{-i\beta} \frac{e^\#(z)}{z - w_0} + ire^{i\beta} \frac{e(z)}{z - \overline{w_0}},\ h(z) \in \operatorname{Dom}(S),\ r \in \mathbb{C} \right\},$$

$$(S_\beta g)(z) = zh(z) - ire^{-i\beta} w_0 \frac{e^\#(z)}{z - w_0} + ire^{i\beta}\, \overline{w_0} \frac{e(z)}{z - \overline{w_0}} = zg(z) - 2rs_\beta(z). \tag{3.5}$$

Finally, define

$$f(z) := (z - w_0)h(z) + 2r \operatorname{Im}(w_0) e^{i\beta} \frac{e(z)}{z - \overline{w_0}}.$$

The function $f(z)$ is entire as long as either is $h(z)$; notice that $e(\overline{w_0}) = 0$ so the last term above is entire. It is not difficult to verify that $f(z) \in \mathcal{B}(e)$ if and only if $h(z) \in \operatorname{Dom}(S)$. Moreover, $f(w_0) = -2rs_\beta(w_0)$. We obtain (3.4) after substituting $h(z)$ by $f(z)$ in (3.5).

In passing, and for the sake of completeness, we note that a computation in a similar vein yields the following description of the adjoint operator:

$$\operatorname{Dom}\big(S^*\big) = \left\{ g(z) = \frac{f(z) - \frac{e(z)}{e(z_0)} f(z_0)}{z - z_0} + \frac{h(z) - \frac{e^\#(z)}{e^\#(\overline{z_0})} h(\overline{z_0})}{z - \overline{z_0}}\colon f(z), h(z) \in \mathcal{B},\ \operatorname{Im}(z_0) > 0 \right\},$$

$$\big(S^* g\big)(z) = zg(z) + \frac{e(z)}{e(z_0)} f(z_0) + \frac{e^\#(z)}{e^\#(\overline{z_0})} h(\overline{z_0}).$$

For the selfadjoint extension S_β of S, the spectral measure is given by

$$m_\beta(x) = \sum_{\substack{x_n \in \operatorname{Spec}(S_\beta) \\ x_n \leqslant x}} \big\|k(\cdot, x_n)\big\|^{-2}. \tag{3.6}$$

These spectral measures allow us to compute the inner product of $\mathcal{B}$ in different ways. We note that $\{k(z, x_n)\|k(\cdot, x_n)\|^{-1}\}$ is an orthonormal basis of $\mathcal{B}$ when $\{x_n\} = \operatorname{Spec}(S_\beta)$ thus one readily obtain

$$\big\langle f(\cdot), g(\cdot)\big\rangle_{\mathcal{B}} = \int_{-\infty}^{\infty} \overline{f(x)} g(x)\, dm_\beta(x). \tag{3.7}$$

Notice that

$$\left\|k(\cdot, x_n)\right\|^2 = k(x_n, x_n) = -\frac{1}{\pi} s_{\gamma+\frac{\pi}{2}}(x_n) s'_\gamma(x_n), \tag{3.8}$$

for $x_n \in \operatorname{Spec}(S_\gamma)$ and $\gamma, \gamma + \frac{\pi}{2} \in \mathbb{R}/[0,\pi)$, where the second equality comes from a standard expression for $k(z,w)$ [7, Equation 2.1].

Remark.

1. A straightforward consequence of (3.7) is that a function in $\mathcal{B}$ restricted to the real line is in $L^2(m_\beta)$. Moreover, from the simplicity of S_β it follows that $\mathcal{B}$ fills $L^2(m_\beta)$ (see [2, Problem 69]) in the sense that, given $\varphi(x) \in L^2(m_\beta)$, there exists $f(z) \in \mathcal{B}$ such that $\|\varphi(\cdot) - f(\cdot)\|_{L^2(m_\beta)} = 0$.
2. Also, by [2, Problem 69], if $g(z) \in \operatorname{Assoc}\mathcal{B}$ satisfies $\|g(\cdot)\|_{L^2(m_\beta)} = 0$ then $g(z)$ is a constant multiple of $s_\beta(z)$ (the converse statement is obvious).

The following result is analogous to Theorem 3.1, although it provides necessary and sufficient conditions for a subclass of the spaces considered in that theorem.

Proposition 3.9. *Let $\mathcal{B}(e)$ be a de Branges space such that S is densely defined. Then a real entire zero-free function lies in $\mathcal{B}(e)$ if and only if condition* (C1) *and* (C2) *of Theorem* 3.1 *hold along with*

$$\text{(C3}^\sharp\text{)} \qquad \sum_{x_n \in \operatorname{Spec}(S_\gamma)} \left| \frac{1}{h_0(x_n) h'_\gamma(x_n)} \right| < \infty, \quad \gamma \in (0,\pi),$$

where $h_\beta(z)$ is defined in (C3) *of the theorem already mentioned.*

Proof. We will prove the proposition for $\gamma = \pi/2$; by an argumentation like the one used in Lemma 3.3, the proof extends to arbitrary γ.

Suppose that a real entire zero-free function $g(z)$ is in $\mathcal{B}(e)$. Then $1 \in \mathcal{B}(\breve{e})$, where $\breve{e}(z) = e(z)/g(z)$. Consider the operator of multiplication $\breve{S}$ in $\mathcal{B}(\breve{e})$. Clearly, $\breve{S}$ is densely defined together with S, so one has

$$\|1\|^2_{\mathcal{B}(\breve{e})} = -\pi \sum_{x_n \in \operatorname{Spec}(S_{\frac{\pi}{2}})} \frac{1}{\breve{s}_0(x_n) \breve{s}'_{\frac{\pi}{2}}(x_n)} < \infty, \tag{3.9}$$

where (3.8) has been used together with the fact that $\operatorname{Spec}(S_\gamma) = \operatorname{Spec}(\breve{S}_\gamma)$. Since $1 \in \operatorname{Assoc}\mathcal{B}(\breve{e})$, then $\breve{s}_{\frac{\pi}{2}}(z)$ is of bounded type in the upper half-plane by [19, Lemma 2.1]. Thus, from [19, Proposition 2.2] we obtain that (C1), (C2), and the identities

$$\breve{s}_0(z) = c h_0(z), \qquad \breve{s}_{\frac{\pi}{2}}(z) = -h_{\frac{\pi}{2}}(z), \tag{3.10}$$

with $c > 0$, hold. These expressions for $\breve{s}_0$ and $\breve{s}_{\frac{\pi}{2}}$, in conjunction with (3.9), imply (C3$^\sharp$).

Note that Proposition 2.2 of [19] is based on [11, Theorem 3] (see also [16, Theorem 6.17]) and [4, Chapter 5 Theorem 11], and, by the latter, one would have obtained a factor e^{ikz} ($k \in \mathbb{R}$) on the r.h.s of each equality in (3.10). Due to the fact that the entire functions $\breve{s}_0(z)$ and $\breve{s}_{\frac{\pi}{2}}(z)$ are real, k must be zero.

Now, assume that (C1), (C2), and (C3$^\sharp$) are fulfilled. Then, $h_0(z)$ and $h_{\frac{\pi}{2}}(z)$ are well defined. Set

$$\tilde{e}(z) := -h_{\frac{\pi}{2}}(z) - i\tilde{c} h_0(z)$$

for $\tilde{c} > 0$ to be chosen later. As in the proof of Corollary 3.2, one establishes that $\tilde{e}(z)$ is Hermite–Biehler. By an appropriate choice of $\tilde{c}$, this Hermite–Biehler function and the function $e(z) = -s_{\frac{\pi}{2}}(z) - is_0(z)$ satisfies $\tilde{e}(z) = j(z)e(z)$, where $j(z)$ is a real entire and zero-free function. Hence, in view of Lemma 2.4 of [7], the operator of multiplication $\tilde{S}$ is densely defined in $\mathcal{B}(\tilde{e})$, thus $\tilde{s}_{\frac{\pi}{2}}(z) = h_{\frac{\pi}{2}}(z) \in \operatorname{Assoc}\mathcal{B}(\tilde{e}) \setminus \mathcal{B}(\tilde{e})$.

On the basis of (3.6), (3.7), (3.8) and condition (C3$^\sharp$), it follows that

$$\|1\|^2_{L^2(\tilde{m}_{\frac{\pi}{2}})} = -\frac{\pi}{\tilde{c}} \sum_{x_n \in \operatorname{Spec}(S_{\frac{\pi}{2}})} \frac{1}{h_0(x_n) h'_{\frac{\pi}{2}}(x_n)} < \infty.$$

By the first item of the last remark above, there exists $f(z) \in \mathcal{B}(\tilde{e})$ such that the $L^2(\tilde{m}_{\frac{\pi}{2}})$-norm of $g(z) := 1 - f(z)$ equals zero. By the second item of the same remark, and noting that necessarily $g(z) \in \operatorname{Assoc}\mathcal{B}(\tilde{e})$, it turns out that $g(z) = w\tilde{s}_{\frac{\pi}{2}}(z)$ for some $w \in \mathbb{C}$. Recalling (3.2), a computation yields

$$\frac{1 + w\tilde{s}_{\frac{\pi}{2}}(x)}{\tilde{e}(x)} = \frac{1}{|\tilde{e}(x)|} - w\cos\varphi(x), \quad x \in \mathbb{R},$$

where $\varphi(x)$ is the phase function associated to $\tilde{e}(z)$ given by the identity $\tilde{e}(x) = e^{-i\varphi(x)}|\tilde{e}(x)|$, $x \in \mathbb{R}$. By [2, Problem 48], $\varphi(x)$ is a continuous (indeed, differentiable) strictly increasing function; note that the zeros of $\tilde{s}_{\frac{\pi}{2}}(z)$ satisfies the identity $\varphi(x_n) = \pi/2 \bmod \pi$. Since $f(z) \in \mathcal{B}(\tilde{e})$, it follows that

$$\frac{1}{|\tilde{e}(x)|} - w\cos\varphi(x) \to 0 \quad \text{as } |x| \to \infty. \tag{3.11}$$

This implies that necessarily $w \in \mathbb{R}$. Furthermore, since the zero-set of $\tilde{s}_\beta(z)$ is unbounded, $\varphi(x)$ is unbounded so there exists an unbounded set $\{y_n\}_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$,

$$w\cos\varphi(y_n) = -|w|.$$

But this condition collides with (3.11) unless $w = 0$. We therefore conclude that $f(z) \equiv 1$, in other words, $1 \in \mathcal{B}(\tilde{e})$. Finally, recalling the relation between $\tilde{e}(z)$ and $e(z)$, we obtain that $j(z)^{-1} \in \mathcal{B}(e)$. □

Remark. After we submitted the present paper for publication, we learned of [20] which gives necessary and sufficient conditions for a real zero-free function to be N-associated to a de Branges Pontryagin space. This result contains as a particular case Theorem 3.1 and, in conjunction with Lemma 3.3, it also contains Proposition 3.9. Our alternative approach to this proposition provides a simpler proof than the one of the more general case treated in [20], in part due to the fact that we are dealing with Hilbert spaces (rather than Pontryagin spaces), but also because of the techniques of spectral theory we used.

Concluding this section, we note that if one substitutes (C3) by (C3$^\sharp$) in Corollary 3.2, one obtains necessary and sufficient conditions for two sequences to generate a de Branges space containing a real entire zero-free function.

4. Representation by symmetric and entire operators

It is known that every entire operator A generates a representation of $\mathcal{H}$ as a de Branges space in which A becomes the operator of multiplication by the independent variable [3, Chapter 2]. This assertion is in fact true for any operator in $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$. Moreover, recent results [13] show that for every regular, closed, simple symmetric operator A (not necessarily densely defined) with deficiency indices $(1,1)$, there exists a de Branges space such that the operator of multiplication in it is unitarily equivalent to A.

The representation proposed here not only provides an alternative approach, for the case of densely defined operators, to the theory developed in [13] but also allows to construct explicitly the isometry that associates every element of $\mathcal{H}$ with the corresponding entire function in the de Branges space. Furthermore, this representation reduces to the one derived from Krein's work for the case of entire operators and, for that reason, it is suitable for the purpose of the present work. It is also worth remarking that our representation makes possible the treatment of generalized entire operators in a straightforward way (see Section 5).

The representation proposed here is realized by associating to the operator A a certain nowhere-zero, vector-valued entire function

$$\mathbb{C} \ni z \mapsto \xi(z) \in \mathrm{Ker}\big(A^* - zI\big).$$

Fix an operator $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ and one of its selfadjoint extensions A_γ, with $\gamma \in [0,\pi)$. Let $g^{(\gamma)}(z)$ be a real entire function such that its zero-set coincides with $\mathrm{Spec}(A_\gamma)$. Since $\mathrm{Spec}(A_\gamma)$ has no finite accumulation points, there always exists such an entire function.

Consider the function $\psi^{(\gamma)}(z)$ introduced at the beginning of Section 2 (see (2.2)). By [3, Section 2, Theorem 7.1], there is a complex conjugation C that commutes with A, hence with all its selfadjoint extensions. Therefore,

$$C\psi^{(\gamma)}(z) = \psi^{(\gamma)}(\bar{z}). \tag{4.1}$$

Define

$$\xi^{(\gamma)}(z) := g^{(\gamma)}(z)\psi^{(\gamma)}(z). \tag{4.2}$$

Lemma 4.1. *$\xi^{(\gamma)}(z)$ is entire, everywhere non-zero and has image in $\mathrm{Ker}(A^* - zI)$ for every $z \in \mathbb{C}$. Moreover,*

$$C\xi^{(\gamma)}(z) = \xi^{(\gamma)}(\bar{z}).$$

Proof. The function $\xi^{(\gamma)}(z)$ is entire and everywhere non-zero by rather obvious reasons. Also, we already know that $\xi^{(\gamma)}(z) \in \operatorname{Ker}(A^* - zI)$ for $z \notin \operatorname{Spec}(A_\gamma)$ so we must verify the assertion only for $z = x_n \in \operatorname{Spec}(A_\gamma)$. In a neighborhood of x_n,

$$\psi^{(\gamma)}(z) = (z - x_n)^{-1}\kappa_n + \text{a function analytic at } x_n,$$

where the residue κ_n can be shown to lie in $\operatorname{Ker}(A^* - x_n I)$ (see the proof of Proposition 2 in [18]).

The second part of the assertion directly follows from (4.1) and the fact that $g^{(\gamma)}(z)$ is real. □

Remark. Note that, although the zero-set of $g^{(\gamma)}(z)$ is determined by the choice of the selfadjoint extension, one has still the freedom of multiplying it by an arbitrary zero-free, real entire function. This implies that $\xi^{(\gamma')}(z) = j(z)\xi^{(\gamma)}(z)$, where $j(z)$ is a zero-free real entire function. That is, up to multiplication by a zero-free real entire function, $\xi^{(\gamma)}(z)$ does not depends on the parameter γ. In view of this, the function $\xi^{(\gamma)}(z)$ will be denoted just as $\xi(z)$.

Now define

$$(\Phi\varphi)(z) := \langle \xi(\bar{z}), \varphi \rangle, \quad \varphi \in \mathcal{H}, \tag{4.3}$$

Φ maps $\mathcal{H}$ onto a certain linear manifold $\widehat{\mathcal{H}}$ of entire functions. Since A is simple, it follows that Φ is injective. A generic element of $\widehat{\mathcal{H}}$ will be often denoted by $\widehat{\varphi}(z)$, as a reminder of the fact that it is the image under Φ of a unique element $\varphi \in \mathcal{H}$.

The linear space $\widehat{\mathcal{H}}$ is turned into a Hilbert space by defining

$$\langle \widehat{\eta}(\cdot), \widehat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle.$$

Clearly, Φ is an isometry from $\mathcal{H}$ onto $\widehat{\mathcal{H}}$.

Proposition 4.2. *$\widehat{\mathcal{H}}$ is an axiomatic de Branges space.*

Proof. We shall verify (A1)–(A3) in that order.

(1) Define $k(z, w) := \langle \xi(\bar{z}), \xi(\overline{w}) \rangle$. Then, for every $\widehat{\varphi} \in \widehat{\mathcal{H}}$,

$$\langle k(\cdot, w), \widehat{\varphi}(\cdot) \rangle = \langle \xi(\overline{w}), \varphi \rangle = \widehat{\varphi}(w).$$

(2) Assume now that $\widehat{\varphi}(z)$ has a non-real zero $w \in \mathbb{C}$. That is, $\langle \xi(\overline{w}), \varphi \rangle = 0$ which implies that $\varphi \in \operatorname{Ran}(A - wI)$. Thus, it makes sense to set $\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi$, with image $\widehat{\eta}(z) \in \widehat{\mathcal{H}}$ under Φ. Furthermore, a rather straightforward computation [18] shows that

$$\widehat{\eta}(z) = \frac{z - \overline{w}}{z - w}\widehat{\varphi}(z).$$

Since η and φ are related by a Cayley transform, it follows that $\|\widehat{\eta}(\cdot)\| = \|\widehat{\varphi}(\cdot)\|$.

(3) Given any $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}$, consider the entire function

$$\widehat{\varphi}^{\#}(z) := \overline{\widehat{\varphi}(\bar{z})}.$$

Since by Lemma 4.1 we have

$$\overline{\widehat{\varphi}(\bar{z})} = \overline{\langle \xi(z), \varphi \rangle} = \langle C\xi(z), C\varphi \rangle = \langle \xi(\bar{z}), C\varphi \rangle,$$

it follows that $\widehat{\varphi}^{\#}(z)$ also belongs to $\widehat{\mathcal{H}}$ and has the same norm as $\widehat{\varphi}(z)$. □

The reproducing kernel in $\widehat{\mathcal{H}}$ is given by

$$k(z, w) = g^{(\gamma)}(z)g^{(\gamma)}(\overline{w})\langle \psi^{(\gamma)}(\bar{z}), \psi^{(\gamma)}(\overline{w}) \rangle.$$

Although the choice of z_0 and $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$ that enters in the definition of $\psi^{(\gamma)}(z)$ (see (2.2)) is arbitrary, w.l.o.g. we can conveniently assume that $\operatorname{Im}(z_0) > 0$ and $\|\psi_0\| = 1$. In that case, a computation yields

$$k(z, z_0) = g^{(\gamma)}(z)g^{(\gamma)}(\overline{z_0})\big[1 + (z - z_0)\langle \psi_0, (A_\gamma - zI)^{-1}\psi_0 \rangle\big].$$

Therefore, recalling (3.1),

$$e(z) = -i\left[\frac{\pi}{\operatorname{Im}(z_0)}\right]^{\frac{1}{2}} \frac{g^{(\gamma)}(\overline{z_0})}{|g^{(\gamma)}(\overline{z_0})|} g^{(\gamma)}(z)(z - \overline{z_0})\big[1 + (z - z_0)\langle \psi_0, (A_\gamma - zI)^{-1}\psi_0 \rangle\big].$$

It is known that the growth of $\mathcal{B}(e)$ coincides with the growth of $e(z)$; see [7]. The last expression shows that the growth of $e(z)$, hence of $\widehat{\mathcal{H}}$, is governed by the growth of $g^{(\gamma)}(z)$.

The next statement can easily be proved (cf. [3, Chapter 1, Theorem 2.2]). We leave the details to the reader.

Proposition 4.3. *Let S be the multiplication operator on $\widehat{\mathcal{H}}$ given by* (3.3)*. Then,*

1. $S = \Phi A \Phi^{-1}$ *and* $\operatorname{Dom}(S) := \Phi \operatorname{Dom}(A)$ *(thus S is densely defined)*;
2. *the selfadjoint extensions of S are in one-to-one correspondence with the selfadjoint extensions of A.*

The following assertions give explicit characterizations of when an operator is entire. We emphasize that Theorem 4.5 rests entirely upon conditions on the spectra of selfadjoint extensions.

Proposition 4.4. $A \in \operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$ *is entire if and only if $\widehat{\mathcal{H}}$ contains a real zero-free entire function.*

Proof. Let $g(z) \in \widehat{\mathcal{H}}$ be the function whose existence is assumed. Clearly there exists (a unique) $\mu \in \mathcal{H}$ such that $g(z) \equiv \langle \xi_1(\bar{z}), \mu \rangle$. Therefore, μ is never orthogonal to $\operatorname{Ker}(A^* - zI)$ for all $z \in \mathbb{C}$. That is, μ is an entire gauge for the operator A.

The necessity is established by noting that the image of the entire gauge under Φ is a zero-free function. □

Theorem 4.5. *For $A \in \operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$, consider the selfadjoint extensions A_0 and A_γ, with $0 < \gamma < \pi$. Then A is entire if and only if* $\operatorname{Spec}(A_0)$ *and* $\operatorname{Spec}(A_\gamma)$ *obey conditions* (C1), (C2) *and* (C3$^\sharp$) *of Proposition* 3.9.

Proof. Apply Proposition 3.9 along with Proposition 4.4. □

Proposition 4.6. *Assume* $1 \in \widehat{\mathcal{H}}$*. Then there exists $\mu \in \mathcal{H}$ such that*

$$g^{(\gamma)}(z) = \frac{1}{\langle \psi^{(\gamma)}(\bar{z}), \mu \rangle}$$

and $C\mu = \mu$. Moreover, μ is the unique entire gauge of A modulo a real scalar factor.

Proof. Necessarily, $1 \equiv \langle \xi(\bar{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By (4.2), and taking into account the occurrence of C, one obtains the stated expression for $g^{(\gamma)}(z)$. By the same token, the reality of μ is shown.

Suppose that there are two real entire gauges μ and μ'. The discussion in Paragraph 5.2 of [3] shows that $(\Phi_\mu \mu')(z) = ae^{ibz}$ with $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Due to the assumed reality, one concludes that $b = 0$ and $a \in \mathbb{R}$. □

Remark. Proposition 4.6 shows that Krein's theory of representation by entire operators is a particular case of the representation proposed here.

5. Operators entire in the generalized sense

In this section we give a spectral characterization of operators in $\operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$ that are entire with respect to a generalized gauge [3, Chapter 3, Section 9]. This section was added as a result of a suggestion of the reviewer.

Given $A \in \operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$, let $\mathcal{H}_+$ be the set $\operatorname{Dom}(A^*)$ equipped with the graph norm

$$\|\varphi\|_+^2 := \|\varphi\|^2 + \left\|A^*\varphi\right\|^2, \quad \varphi \in \operatorname{Dom}\bigl(A^*\bigr).$$

Let $\mathcal{H}_-$ be the completion of $\mathcal{H}$ under the norm

$$\|\eta\|_- := \sup_{\varphi \in \mathcal{H}_+} \frac{|\langle \eta, \varphi \rangle|}{\|\varphi\|_+}, \quad \eta \in \mathcal{H};$$

the elements of $\mathcal{H}_-$ are the continuous linear functionals on $\mathcal{H}_+$. In this way one obtains the scale of Hilbert spaces $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ associated to A^*, where the embeddings are dense and continuous. Sticking to the standard notation, for $\eta \in \mathcal{H}_-$ and $\varphi \in \mathcal{H}_+$ we define $\langle \eta, \varphi \rangle := \eta(\varphi)$ so accordingly $\langle \varphi, \eta \rangle := \overline{\eta(\varphi)}$.

Given a selfadjoint extension A_β of A and $z \notin \operatorname{Spec}(A_\beta)$, let $R_z^{(\beta)}$ be the extension of $(A_\beta - zI)^{-1}$ from $\mathcal{H}$ to $\mathcal{H}_-$. This operator satisfies the identity

$$\bigl\langle R_z^{(\beta)} \eta, \varphi \bigr\rangle = \bigl\langle \eta, (A_\beta - \bar{z}I)^{-1} \varphi \bigr\rangle, \quad \eta \in \mathcal{H}_-,\ \varphi \in \mathcal{H}.$$

It is straightforward to verify that $R_z^{(\beta)}$ maps $\mathcal{H}_-$ into $\mathcal{H}$. It also satisfies the extended resolvent identity

$$R_z^{(\beta)} - R_w^{(\beta)} = (z - w)(A_\beta - zI)^{-1} R_w^{(\beta)} = (z - w)(A_\beta - wI)^{-1} R_z^{(\beta)}.$$

A complex conjugation on $\mathcal{H}$ is extended to $\mathcal{H}_-$ by defining

$$\langle C\eta, \varphi\rangle := \overline{\langle \eta, C\varphi\rangle}, \quad \eta \in \mathcal{H}_-,\ \varphi \in \mathcal{H}_+.$$

We say that $\eta \in \mathcal{H}_-$ is real if $C\eta = \eta$.

Let $\xi(z)$ be the entire vector-valued function defined by (4.2). Let us recall that $\xi(z) \in \operatorname{Ker}(A^* - zI) \subset \operatorname{Dom}(A^*)$, therefore the linear map Φ defined by (4.3) on $\mathcal{H}$ can be extended to $\mathcal{H}_-$ in the obvious manner.

Proposition 5.1. $\operatorname{Assoc}\widehat{\mathcal{H}} = \Phi\mathcal{H}_- := \{\widehat{\eta}(z) = \langle \xi(\bar{z}), \eta\rangle\colon\ \eta \in \mathcal{H}_-\}$.

Proof. Given some arbitrary $w \in \mathbb{C}$, choose a selfadjoint extension A_β such that $w \notin \operatorname{Spec}(A_\beta)$. Let us recall that

$$\xi(z) = g^{(\beta)}(z)\psi^{(\beta)}(z),$$

where $\psi^{(\beta)}(z)$ is given by (2.2). By assumption $g^{(\beta)}(w) \neq 0$ (and also $g^{(\beta)}(\overline{w}) \neq 0$—this fact is used below). A computation involving the first resolvent identity yields the equality

$$\frac{\xi(z) - \xi(w)}{z - w} = (A_\beta - wI)^{-1}\xi(z) + \frac{g^{(\beta)}(z) - g^{(\beta)}(w)}{(z - w)g^{(\beta)}(w)}\xi(w). \tag{5.1}$$

Now consider $\widehat{\eta}(z) = \langle \xi(\bar{z}), \eta\rangle$ with $\eta \in \mathcal{H}_-$ and $\widehat{\varphi}(z) = \langle \xi(\bar{z}), \varphi\rangle$ with $\varphi \in \mathcal{H}$. Then $\widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta \in \mathcal{H}_-$. Moreover,

$$\begin{aligned} \frac{\widehat{\eta}(w)\widehat{\varphi}(z) - \widehat{\eta}(z)\widehat{\varphi}(w)}{z - w} &= \frac{1}{z - w}\big\langle \xi(\bar{z}), \widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta\big\rangle \\ &= \left\langle \frac{\xi(\bar{z}) - \xi(\overline{w})}{\bar{z} - \overline{w}}, \widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta \right\rangle \\ &= \big\langle (A_\beta - \overline{w})^{-1}\xi(\bar{z}), \widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta\big\rangle \\ &= \big\langle \xi(\bar{z}), \tau\big\rangle, \end{aligned}$$

where $\tau := R_w^{(\beta)}[\widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta] \in \mathcal{H}$. In the last computation we have used (5.1) and the fact that $\langle \xi(\overline{w}), \widehat{\eta}(w)\varphi - \widehat{\varphi}(w)\eta\rangle = 0$. It follows that $\Phi\mathcal{H}_- \subset \operatorname{Assoc}\widehat{\mathcal{H}}$.

Next, consider $g(z) \in \operatorname{Assoc}\widehat{\mathcal{H}}$. Then there exist two functions $f(z), h(z) \in \widehat{\mathcal{H}}$ such that $g(z) = f(z) + zh(z)$. Since the multiplication operator S is densely defined, there exists a sequence $\{h_n(z)\}_{n\in\mathbb{N}} \subset \operatorname{Dom}(S)$ that is $\widehat{\mathcal{H}}$-norm convergent to $h(z)$. Moreover, $f(z) + zh_n(z)$ converges to $g(z)$ uniformly on compact subsets. For every $n \in \mathbb{N}$ we have $h_n(z) = \langle \xi(\bar{z}), \eta_n\rangle$ for some unique $\eta_n \in \operatorname{Dom}(A)$, and also $f(z) = \langle \xi(\bar{z}), \psi\rangle$ for $\psi \in \mathcal{H}$. Set $\tau_n = \psi + A\eta_n$. Since

$$\|\tau_m - \tau_n\|_- = \sup_{\varphi\in\mathcal{H}_+} \frac{|\langle(\eta_m - \eta_n), A^*\varphi\rangle|}{\|\varphi\|_+} \leqslant \sup_{\varphi\in\mathcal{H}_+} \frac{\|\eta_m - \eta_n\|\|A^*\varphi\|}{\|\varphi\|_+} \leqslant \|\eta_m - \eta_n\|,$$

it follows that the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ converges to some $\tau \in \mathcal{H}_-$ which in turn satisfies $g(z) = \langle \xi(\bar{z}), \tau\rangle$. Therefore, $\operatorname{Assoc}\widehat{\mathcal{H}} \subset \Phi\mathcal{H}_-$. □

Definition 5.2. An operator $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ is said to be *entire with respect to a generalized gauge*, or just *entire in the generalized sense*, if there exists $\mu \in \mathcal{H}_-$ such that $\langle \xi(\bar{z}), \mu\rangle \neq 0$ for all $z \in \mathbb{C}$.

Note that this definition becomes equivalent to Definition 2.3 when the linear functional μ can be identified with an element in $\mathcal{H}$. The following assertion is an obvious consequence of Proposition 5.1.

Proposition 5.3. *An operator $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ is entire in the generalized sense if and only if $\operatorname{Assoc}\widehat{\mathcal{H}}$ contains a real zero-free entire function.*

Finally, we have:

Theorem 5.4. *For $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, consider the selfadjoint extensions A_0 and A_γ, with $0 < \gamma < \pi$. Then A is entire in the generalized sense if and only if $\operatorname{Spec}(A_0)$ and $\operatorname{Spec}(A_\gamma)$ obey conditions* (C1), (C2) *and* ($C3^\flat$)*.*

A statement analogous to Proposition 4.6 can also be formulated for generalized entire operators. We leave the details to the reader.

Remark. The generalized notion of entire operator discussed above may be extended to a notion of an operator having a generalized entire gauge in a suitable defined Hilbert space $\mathcal{H}_{-N}$, expected to be the space of linear functionals of an appropriate Gelfand triplet associated with $\mathcal{H}$. Since such space is likely to be related with the set of functions N-associated to the de Branges space $\widehat{\mathcal{H}}$, Theorem 3.2 of [20] should provide a spectral characterization of such kind of operators. This matter will be treated elsewhere.

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