On a Duality for Crossed Products of $C^*$-Algebras

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Let $\mathcal{A}$ be a $C^*$-algebra, and $G$ be a locally compact abelian group. Suppose $\alpha$ is a continuous action of $G$ on $\mathcal{A}$. Then there exists a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ of $G$ on the $C^*$-crossed product $C^*(\mathcal{A}; \alpha)$ of $\mathcal{A}$ by $\alpha$ such that the $C^*$-crossed product $C^*(C^*(\mathcal{A}; \alpha); \hat{\alpha})$ is isomorphic to the tensor product $\mathcal{A} \otimes C(L^2(G))$ of $\mathcal{A}$ and the $C^*$-algebra $C(L^2(G))$ of all compact operators on $L^2(G)$.

1. Introduction

In the study of the structure of von Neumann algebras of type III, M. Takesaki [11] obtained a duality theorem for crossed products of von Neumann algebras. At the same time, he also conjectured about its $C^*$-algebra version that given a $C^*$-algebra $\mathcal{A}$ with a continuous action $\alpha$ of a locally compact abelian group $G$, the $C^*$-crossed product $C^*(\mathcal{A}; \alpha)$ of $\mathcal{A}$ by $\alpha$ has a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ of $G$ such that the second $C^*$-crossed product $C^*(C^*(\mathcal{A}; \alpha); \hat{\alpha})$ is isomorphic to the tensor product $\mathcal{A} \otimes C(L^2(G))$ of $\mathcal{A}$ and the $C^*$-algebra $C(L^2(G))$ of all compact operators on $L^2(G)$.

In this paper, it will be shown that the statement cited above is affirmative.

2. Crossed Products and Reduced Crossed Products of $C^*$-Algebras

Let $G$ be a locally compact group, and $\mathcal{A}$ be a $C^*$-algebra. We denote by Aut($\mathcal{A}$) the group of all $*$-automorphisms of $\mathcal{A}$. A mapping $\alpha$ of $G$ into Aut($\mathcal{A}$) is said to be a continuous action if it is a strongly

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continuous homomorphism in the sense that for any \( x \in \mathfrak{A} \), and \( \epsilon > 0 \), there exists a neighborhood \( U \) of the unit \( e \) of \( G \) such that
\[
\| \alpha_g(x) - x \| < \epsilon \quad \text{for every} \quad g \in U.
\]

We now define the crossed product \( C^*(\mathfrak{A}; \alpha) \) of \( \mathfrak{A} \) with a continuous action \( \alpha \) of \( G \) in the following way: it is the enveloping \( C^* \)-algebra of the Banach \( \ast \)-algebra \( L^1(\mathfrak{A}; \alpha) \) of all Bochner integrable \( \mathfrak{A} \)-valued measurable functions on \( G \) with respect to the left Haar measure \( dg \) of \( G \) with the \( \ast \)-algebraic structure given by
\[
(xy)(g) = \int_G x(h) \alpha_h[y(h^{-1}g)] \, dh
\]
and
\[
x^*(g) = \Delta(g)^{-1} \alpha_g[x(g^{-1})]^*,
\]
for each \( x, y \in L^1(\mathfrak{A}; G) \), and \( g \in G \), where \( \Delta(g) \) is the modular function of \( G \).

Due to S. Doplicher, D. Kastler, and D. W. Robinson [2] there is a one-to-one correspondence between a covariant representation \( (\rho, U) \) of \( (\mathfrak{A}, G) \) and a \( \ast \)-representation \( \Pi \) of \( C^*(\mathfrak{A}; \alpha) \) which is determined by
\[
\Pi(x) = \int_G \rho(x(g))U(g) \, dg, \quad x \in L^1(\mathfrak{A}; G).
\] (2.1)

Let \( \text{Cov rep}(A, G) \) be the set of all covariant representations of \( (A, G) \).

Following the construction in the discrete case due to Zeller-Meier [12], we shall next define the reduced crossed product \( C_{\gamma}^*(\mathfrak{A}; \alpha) \) of \( \mathfrak{A} \) by \( \alpha \) in the continuous case as follows.

Let \( (\tilde{\rho}, \lambda) \in \text{Cov rep}(\mathfrak{A}, G) \) induced by the trivial one \( (\rho, \iota) \in \text{Cov rep}(\mathfrak{A}, \{e\}) \) where \( \iota \) is the identity representation of \( \{e\} \) on \( \mathfrak{H}_\rho \) (cf. [9]). Let \( \text{Ind} \rho \) denote the representation of \( C^*(\mathfrak{A}; \alpha) \) corresponding to \( (\tilde{\rho}, \lambda) \). Making use of \( \text{Ind} \rho \), we define a norm \( \| \cdot \|_\gamma \) on \( L^1(\mathfrak{A}; G) \) by
\[
\| x \|_\gamma = \sup\{\|(\text{Ind} \rho)(x)\| : \rho \in \text{Rep} \mathfrak{A}\},
\] (2.2)
for each \( x \in L^1(\mathfrak{A}; G) \), where \( \text{Rep} \mathfrak{A} \) is the set of all \( \ast \)-representations of \( \mathfrak{A} \). It is seen that the completion \( C_{\gamma}^*(\mathfrak{A}; \alpha) \) of \( L^1(\mathfrak{A}; G) \) with respect to \( \| \cdot \|_\gamma \) is a \( C^* \)-algebra, which is nothing but the quotient \( C^* \)-algebras of \( C^*(\mathfrak{A}; \alpha) \) by the kernel of \( \text{Ind} \rho \) \( (\rho \in \text{Rep} \mathfrak{A}) \). We call \( C_{\gamma}^*(\mathfrak{A}; \alpha) \) the reduced crossed product of \( \mathfrak{A} \) by \( \alpha \), and \( \| \cdot \|_\gamma \), the reduced norm on \( C_{\gamma}^*(\mathfrak{A}; \alpha) \).
Now we extend some properties of C*-crossed products in the discrete case to the continuous case. One defines for a positive linear functional $\varphi$ of a C*-algebra $\mathcal{A}$, and $f_i \in K(G)$ $(i = 1, z)$,

$$\tilde{\varphi}_{f_i, f_j}(x) = \int_{G \times G} f_i(g^{-1}h) \overline{f_j(h)} \varphi \circ \alpha^{-1}_g(x(g)) \, dg \, dh,$$

(2.3)

for any $x \in L^1(\mathcal{A}; G)$, where $K(G)$ is the set of all continuous functions on $G$ with compact support. Identifying $L^2(\mathcal{H}_\varphi; G)$ with $L^2(G) \otimes \mathcal{H}_\varphi$ the right hand side of (2.3) is nothing but $\langle (\text{Ind } \pi_\varphi)(x) f_1 \otimes \xi_\varphi \mid f_2 \otimes \xi_\varphi \rangle$ where $(\pi_\varphi, \xi_\varphi)$ is the cyclic representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\varphi$ corresponding to $\varphi$. Hence we have that

$$\| (\text{Ind } \pi_\varphi)(x) \| = \sup \{ \langle \tilde{\varphi}_{f_i, f_j}(y^*x^*xy) \rangle^{1/2} / \tilde{\varphi}_{f_i, f_j}(y^*y) \}^{1/2} : y \in K(\mathcal{A}; G), \ f \in K(G) \},$$

(2.4)

for any $x \in L^1(\mathcal{A}; G)$, where $\tilde{\varphi}_{f_i, f_j}(y^*y) \neq 0$, and $K(\mathcal{A}; G)$ is the set of all $\mathcal{A}$-valued continuous functions on $G$ with compact support. Let $\Omega(y; f)$ for $y \in K(\mathcal{A}; G)$, $f \in K(G)$ be the set of all positive linear functionals $\varphi$ of $\mathcal{A}$ with $\tilde{\varphi}_{f_i, f_j}(y^*y) = 0$. Then one easily gets by (2.4) that

$$\| x \|_\varphi = \sup \{ \langle \tilde{\varphi}_{f_i, f_j}(y^*x^*xy) \rangle^{1/2} / \tilde{\varphi}_{f_i, f_j}(y^*y) \}^{1/2} : y \in K(\mathcal{A}; G), \ f \in K(G), \ \varphi \in \Omega(y; f) \},$$

(2.5)

for $x \in L^1(\mathcal{A}; G)$. Let $\mathcal{M}(\rho)$ be the set of all positive linear functional $\varphi$ of $\mathcal{A}$ such that $\pi_\varphi$ is weakly contained in $\rho \in \text{Rep } \mathcal{A}$. Then the same way as in the discrete case gives us that for $\rho \in \text{Rep } \mathcal{A}$, $x \in L^1(\mathcal{A}; G)$,

$$\| (\text{Ind } \rho)(x) \| = \sup \{ \langle \tilde{\varphi}_{f_i, f_j}(y^*x^*xy) \rangle^{1/2} / \tilde{\varphi}_{f_i, f_j}(y^*y) \}^{1/2} : y \in K(\mathcal{A}; G), \ f \in K(G), \ \varphi \in \Omega(y; f) \cap \mathcal{M}(\rho) \},$$

(2.6)

(cf., [12, Theor. 4.8]).

Remark. In the discrete case, it is clear that $\tilde{\varphi}_{\pi_\varphi, \pi_\varphi} = \varphi$, where $\tilde{\varphi}(x) = \varphi[x(e)]$ for $x \in L^1(\mathcal{A}; G)$.

We now state the following proposition.

**Proposition 2.1.** Let $G$ be a locally compact group, and $\mathcal{A}$ be a C*-algebra with a continuous action $\alpha$ of $G$. Then given a $\rho \in \text{Rep } \mathcal{A}$, the following properties are equivalent;

(i) $\sum_{g \in G} \alpha_g \cdot \rho$ is faithful on $\mathcal{A}$;

(ii) $\| (\text{Ind } \rho)(x) \| = \| x \|_\varphi$ for any $x \in L^1(\mathcal{A}; G)$ where $(\alpha_g \cdot \rho)(a) = \rho \cdot \alpha^{-1}_g(a)$ for $a \in \mathcal{A}$, $g \in G$.

(Compare with [12, Theor. 4.11].)
Proof. Since (ii) implies that \( \text{Ind} \rho \) is faithful on \( L^1(\mathfrak{A}; G) \), (i) is obtained by direct computation. Conversely, suppose \( \sum_{g \in G} \alpha_g \cdot \rho \) is faithful. Then it follows that \( \bigcap_{g \in G} \ker \alpha_g \cdot \rho = 0 \), which implies that any positive linear functional of \( \mathfrak{A} \) is a weak limit of some linear combinations of positive linear forms of \( \mathfrak{A} \) which are associate with \( \alpha_g \cdot \rho \) \( (g \in G) \). On the other hand, since \( (\alpha_g \cdot \rho, \lambda) \in \text{Cov rep}(\mathfrak{A}, G) \) induced by \( (\alpha_g \cdot \rho, \lambda) \in \text{Cov rep}(\mathfrak{A}, \{e\}) \) is unitarily equivalent to \( (\bar{\rho}, \lambda) \in \text{Cov rep}(\mathfrak{A}, G) \) induced by \( (\rho, \iota) \in \text{Cov rep}(\mathfrak{A}, \{e\}) \) (cf. [9, Theor. 8.1]), it is verified that \( \text{Ind}(\alpha_g \cdot \rho) \) is unitarily equivalent to \( \text{Ind} \rho \) for every \( g \in G \). Hence one gets that

\[
\| (\text{Ind} \rho)(x) \| = \sup \{ \| (\text{Ind} \alpha_g \cdot \rho)(x) \| : g \in G \}, \ x \in L^1(\mathfrak{A}; G). \quad (2.7)
\]

By (2.5)-(2.7), we have that

\[
\| (\text{Ind} \rho)(x) \| = \sup \{ (\tilde{\alpha}, f(x^*x)\tilde{f})^{1/2}/\tilde{\alpha}, f(x^*x)\tilde{f})^{1/2} : y \in K(\mathfrak{A}; G), f \in K(G), \varphi \in \Omega(y; f) \cap \mathfrak{W}(\alpha_g \cdot \rho), g \in G \}
\geq \sup \{ (\tilde{\alpha}, f(x^*x)\tilde{f})^{1/2}/\tilde{\alpha}, f(x^*x)\tilde{f})^{1/2} : y \in K(\mathfrak{A}; G), f \in K(G), \varphi \in \Omega(y; f) \}
= \| x \|_y.
\]

The following proposition gives a condition on the group under which reduced crossed products coincide with crossed products.

**Proposition 2.2.** Let \( G \) be a locally compact group, and \( \mathfrak{A} \) be a \( C^* \)-algebra with a continuous action \( \alpha \) of \( G \). If \( G \) is amenable as a topological group, then

\[
C^*(\mathfrak{A}; \alpha) = C^*_r(\mathfrak{A}; \alpha).
\]

*(Compare with [12, Theor. 5.1].)*

*Proof.* Let \( \Phi \) be a positive linear functional on \( C^*(\mathfrak{A}; \alpha) \) with \( \| \Phi \| \leq 1 \). It suffices to show that \( \Phi \) is continuous with respect to \( \| \cdot \|_\tau \)-norm. Since \( G \) is amenable, the function \( 1 \) on \( G \) is a limit point of \( f \ast f \) \( (f \in K(G)) \) with respect to compact open topology, where \( (\varphi \ast \psi)(g) = \int_G \varphi(h) \psi(h^{-1}g) \, dh \) and \( \tilde{\varphi}(g) = \overline{\varphi(g^{-1})} \) for \( \varphi, \psi \in K(G) \) (cf. [1, Sect. 8]). Let \( (\Pi_\varphi, \xi_\varphi) \) be the cyclic representation of \( C^*(\mathfrak{A}; \alpha) \) on a Hilbert space \( \mathcal{H}_\varphi \) corresponding to \( \Phi \). Then there is a unique \( (\rho, U) \in \text{Cov rep}(\mathfrak{A}, G) \) such that

\[
\Pi_\varphi(x) = \int_G \rho(x(g)) U(g) \, dg \quad \text{for} \ x \in L^1(\mathfrak{A}; G).
\]
Let us define a positive linear functional $\Phi_f$ on $L^1(\mathfrak{A}; G)$ by

$$
\Phi_f(x) = \int_G (f^*f)(g) \langle \rho[x(g)] U(g)\xi_\phi \mid \xi_\phi \rangle \, dg.
$$

In fact, since $(\rho, U) \in \text{Cov rep}(\mathfrak{A}, G)$, one has that

$$
\Phi_f(x) = \int_{G \times G} f(h^{-1}g) f(h^{-1}g) \langle \rho \circ \alpha_\lambda^{-1}[x(g)] U(h^{-1}) U(g)\xi_\phi \mid U(h^{-1})\xi_\phi \rangle \, dh \, dg
$$

for each $x \in L^1(\mathfrak{A}; G)$, where $(\tilde{\rho}, \lambda) \in \text{Cov rep}(\mathfrak{A}, G)$ induced by $(\rho, i) \in \text{Cov rep}(\mathfrak{A}, \{e\})$, and $\xi(g) = f(g^{-1}) U(g^{-1})\xi_\phi \in L^2(\mathfrak{A}; G)$.

Since $\|\Phi\| \leq 1$, we may assume that $\|\tilde{\rho}\| \leq 1$. Then we have by (2.8) that

$$
\|\Phi_f(x^*x)\| \leq \|x^*x\|_\gamma,
$$

for $x \in L^1(\mathfrak{A}; G)$. Since $\Phi_f$ weakly converges to $\Phi$ on $K(\mathfrak{A}; G)$, it follows by (2.9) that

$$
\Phi(x^*x) \leq \|x^*x\|_\gamma,
$$

for any $x \in K(\mathfrak{A}; G)$. Since $K(\mathfrak{A}; G)$ is dense in $C^*(\mathfrak{A}; \alpha)$ the inequality (2.10) also holds for $x \in C^*(\mathfrak{A}; \alpha)$. Therefore, $\Phi$ is continuous with respect to the reduced norm, which means that $C^*(\mathfrak{A}; \alpha) = C_{\gamma^*}(\mathfrak{A}; \alpha)$.

Q.E.D.

**Remark.** If $\mathfrak{A} = \mathbb{C}$, the above proposition says that

$$
C^*(G) = C_{\gamma^*}(G) \quad (\text{cf. [1, Sect. 18]}).
$$

Finally we state here the following two observations without proof for a later use though both are more or less known.

**Proposition 2.3.** Let $G$ be a locally compact group, and $\mathfrak{A}$ (respectively $\mathfrak{B}$) be a $C^*$-algebra with a continuous action $\alpha$ (respectively $\beta$) of $G$ on $\mathfrak{A}$ (respectively $\mathfrak{B}$). Suppose there exists a isomorphism $\Phi$ of $\mathfrak{A}$ onto $\mathfrak{B}$ such that

$$
\beta_g = \Phi \circ \alpha_\gamma \circ \Phi^{-1} \quad \text{for} \ g \in G,
$$

then $C^*(\mathfrak{A}; \alpha)$ is isomorphic to $C^*(\mathfrak{B}; \beta)$.
Remark. The above statement is also true for reduced crossed products.

**Proposition 2.4.** Let $G$ (respectively $H$) be a locally compact group, and $\mathcal{A}$ (respectively $\mathcal{B}$) be a $C^*$-algebra with a continuous action $\alpha$ (respectively $\beta$) of $G$ (respectively $H$). Then $C^*(\mathcal{A} \otimes_\alpha \mathcal{B}; \alpha \otimes \beta)$ is isomorphic to $C^*(\mathcal{A}; \alpha) \bar{\otimes} C^*(\mathcal{B}; \beta)$, where $\nu$ is the greatest $C^*$-cross norm (cf. [4, 5]).

Remark. It is true that given $\mathcal{A}$, $\mathcal{B}$, $G$, $H$, $\alpha$, $\beta$ as above, $C^*_\nu(\mathcal{A} \otimes_\alpha \mathcal{B}; \alpha \otimes \beta)$ is isomorphic to $C^*_\nu(\mathcal{A}; \alpha) \bar{\otimes} C^*_\nu(\mathcal{B}; \beta)$, where $*$ is Turumaru’s cross norm.

3. **Duality**

In this section, we show a duality for crossed products of $C^*$-algebras by a locally compact abelian group. Let $G$ be a locally compact abelian group, and $\mathcal{A}$ be a $C^*$-algebra with a continuous action $\alpha$ of $G$. First of all, we define a continuous action $\delta_\alpha$ of the dual group $\hat{G}$ of $G$ on $L^1(\mathcal{A}; G)$ by

$$\delta_\alpha(x)(g) = \langle g, p \rangle x(g),$$

for $x \in L^1(\mathcal{A}; G)$, $p \in \hat{G}$, and $g \in G$. Indeed, one can compute by definition that

$$(\delta_\alpha(x) \delta_\alpha(y))(g) = \int_G \delta_\alpha(x)(h) \alpha_h(\delta_\alpha(y)(h^{-1}g)) \, dh$$

$$= \int_G \langle h, p \rangle x(h) \langle h^{-1}g, p \rangle \alpha_h(y(h^{-1}g)) \, dh$$

$$= \langle g, p \rangle \int_G x(h) \alpha_h(y(h^{-1}g)) \, dh$$

$$= \delta_\alpha(xy)(g)$$

and

$$\delta_\alpha(x^*)(g) = \langle g, p \rangle x^*(g) = \langle g, p \rangle \alpha_p[x(g^{-1})]^*$$

$$= \alpha_p[\langle g^{-1}, p \rangle x(g^{-1})]^* = \alpha_p[\delta_\alpha(x)(g^{-1})]^*$$

$$= \delta_\alpha(x)^*(g),$$

for each $x, y \in L^1(\mathcal{A}; G)$, $p \in \hat{G}$, and $g \in G$. So $\delta_\alpha$ is a $*$-homomorphism on $L^1(\mathcal{A}; G)$.
Moreover, we have that

\[ \|A_\rho(x)\| = \sup \{ \|P[A_\rho(x)]\| : P \in \text{Rep } L^1(\mathcal{A}; G) \} \]

\[ = \sup \left\{ \left\| \int_{\sigma} \rho([A_\rho(x)](g)) \ U(g) \ dg \right\| : (\rho, U) \in \text{Cov rep } (\mathcal{A}, G) \right\} \]

\[ = \sup \left\{ \left\| \int_{\sigma} \langle g, \tilde{p} \rangle \ \rho[x(g)] \ U(g) \ dg \right\| : (\rho, U) \in \text{Cov rep } (\mathcal{A}, G) \right\}. \]

Let \( U_p(g) = \langle g, \tilde{p} \rangle \ U(g) \) for \( \tilde{p} \in \tilde{G}, g \in G \). Then it is easily seen that \( (\rho, U_p) \in \text{Cov rep } (\mathcal{A}, G) \) for \( (\rho, U) \in \text{Cov rep } (\mathcal{A}, G) \). This implies that \( \|A_\rho(x)\| \leq \|x\| \). Since \( A_\rho^{-1}(x)(g) = \langle g, \tilde{p} \rangle \ x(g) \), the inequality \( \|A_\rho^{-1}(x)\| \leq \|x\| \) holds similarly. So \( \hat{A}_\rho \) is isometric on \( L^1(\mathcal{A}; G) \) with respect to the enveloping \( C^* \)-norm, which means that \( \hat{A}_\rho \) can be extended to an automorphism on \( C^*(\mathcal{A}; \alpha) \). Denoting it by the same symbol \( \hat{A}_\rho \) (\( \tilde{p} \in \tilde{G} \)), one can show that \( \hat{A}_\rho \) is a continuous action of \( \mathcal{A} \) on \( C^*(\mathcal{A}; \alpha) \). In fact, one estimates the following:

\[ \| \hat{A}_\rho(x) - x \| \leq \| \hat{A}_\rho(x) - x \|_1 = \int_{\tilde{G}} |\langle g, \tilde{p} \rangle - 1| \|x(g)\| \ dg, \]

for any \( x \in K(\mathcal{A}; G) \), where \( \| y \|_1 = \int_{G} \|y(g)\| \ dg \ (y \in L^1(\mathcal{A}; G)) \). Since \( \text{supp } x \) is equicontinuous as a family of functions on \( G \) by compactness, given an \( \epsilon > 0 \), there exists a neighborhood \( W \) of the unit 1 of \( G \) such that

\[ |\langle g, \tilde{p} \rangle - 1| < \epsilon \quad \text{for } g \in \text{supp } x, \tilde{p} \in W. \]

Therefore, one gets that

\[ \| \hat{A}_\rho(x) - x \| \leq \epsilon \int_{\text{supp } x} \|x(g)\| \ dg \]

\[ = \epsilon \| x \|_1 \quad \text{for } \tilde{p} \in W. \]

Since \( K(\mathcal{A}; G) \) is dense in \( C^*(\mathcal{A}; \alpha) \), \( \hat{A}_\rho \) is a continuous action of \( \mathcal{A} \) on \( C^*(\mathcal{A}; \alpha) \). Thus we can construct the crossed product \( C^*(C^*(\mathcal{A}; \alpha) \otimes \mathcal{A}) \) of \( C^*(\mathcal{A}; \alpha) \) by \( \hat{A}_\rho \).

We shall show that \( C^*(C^*(\mathcal{A}; \alpha) \otimes \mathcal{A}) \) constructed above is isomorphic to \( \mathcal{A} \otimes \mathcal{C}(L^2(G)) \), the tensor product of \( \mathcal{A} \) and the \( C^* \)-algebra \( \mathcal{C}(L^2(G)) \) of all compact operators on \( L^2(G) \). Let \( \Pi \) be a faithful representation of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \), and let \( \Pi \Pi \) be the representation of \( C^*(\mathcal{A}; \alpha) \) induced by \( \Pi \). Since \( G \) is abelian, Proposition 2.2 tells us that \( C^*(\mathcal{A}; \alpha) = C^*(\mathcal{A}; \alpha) \). Moreover, we consider
the representation \( \text{Ind}(\text{Ind} \, \Pi) \) of \( C^*_{\gamma}(C^*(\mathcal{A}; \alpha); \alpha) \) induced by \( \text{Ind} \, \Pi \), which will be denoted by \( \tilde{\Pi} \). Then it can be verified that

\[
\| \tilde{\Pi}(x) \| = \| x \|, \quad \text{for} \quad x \in L^1(C^*(\mathcal{A}; \alpha); \mathcal{G}).
\] (3.2)

Actually, since \( \Pi \) is faithful, \( \sum_{g \in G} \alpha_g \cdot \Pi \) is also faithful. Hence it follows from Proposition 2.1 that \( \text{Ind} \, \Pi \) is faithful on \( C^*_{\gamma}(\mathcal{A}; \alpha) \). Thus it is faithful on \( C^*(\mathcal{A}; \alpha) \). Similarly applying Proposition 2.1 to \( C^*(\mathcal{A}; \alpha) \) and \( \text{Ind} \, \Pi \), the equality (3.2) holds. So \( \tilde{\Pi} \) is a faithful representation of \( C^*_{\gamma}(C^*(\mathcal{A}; \alpha); \alpha) \). By applying Proposition 2.2 again to \( \mathcal{A} \) and \( \alpha \), we conclude that \( C^*_{\gamma}(C^*(\mathcal{A}; \alpha); \alpha) = C^*_{\gamma}(C^*(\mathcal{A}; \alpha); \alpha) \).

We shall study the structure of \( \tilde{\Pi}[C^*(C^*(\mathcal{A}; \alpha); \alpha)] \). First of all, we compute an operator \( \tilde{\Pi}(x) \) for \( x \in K(\mathcal{A}; \mathcal{G} \times \mathcal{G}) \). By construction, the operator \( \tilde{\Pi}(x) \) is acting on \( L^2(\mathcal{G}_\Pi; \mathcal{G} \times \mathcal{G}) \). Then we have that

\[
(\tilde{\Pi}(x)\xi)(g, p) = \int_G (\text{Ind} \, \Pi \, [\xi(q)]) \lambda(q) \xi(g, p) \, dq
\]

\[
= \int_G (\text{Ind} \, \Pi \circ \gamma_q^{-1}[\xi(q)]) \xi\, (g, q^{-1}p) \, dq
\]

\[
\int_G \left\langle \Pi(\gamma_q^{-1}[\xi(q)](h)) \lambda(h) \xi\, (g, q^{-1}p) \right\rangle \, dh \, dq
\]

\[
= \int_{\mathcal{G} \times \mathcal{G}} \left\langle h, p \right\rangle \Pi \circ \gamma_q^{-1}[\xi(h, q)] \xi(h^{-1}g, q^{-1}p) \, dh \, dq,
\] (3.3)

for \( x \in K(\mathcal{A}; \mathcal{G} \times \mathcal{G}) \), \( \xi \in L^2(\mathcal{G}_\Pi; \mathcal{G} \times \mathcal{G}) \), \( p \in \mathcal{G} \), and \( g \in \mathcal{G} \), where \( (\text{Ind} \, \Pi, \lambda) \in \text{Cov rep}(C^*(\mathcal{A}; \alpha), \mathcal{G}) \) induced by \( (\text{Ind} \, \Pi, \iota) \in \text{Cov rep}(C^*(\mathcal{A}; \alpha), \{1\}) \), and \( (\Pi, \lambda) \in \text{Cov rep}(\mathcal{A}, \mathcal{G}) \) induced by \( (\Pi, \iota) \in \text{Cov rep}(\mathcal{A}, \{e\}) \). So \( \tilde{\Pi}(x) \) can be described as follows:

\[
\tilde{\Pi}(x) = \int_{\mathcal{G} \times \mathcal{G}} x(g, p) \, u(g) \, v(p) \, dg \, dp,
\] (3.4)

where

\[
(a\xi)(g, p) = \Pi \circ \gamma_q^{-1}(a) \xi(g, p) \quad (a \in \mathcal{A}),
\]

\[
(u(h)\xi)(g, p) = \left\langle h, p \right\rangle \xi(h^{-1}g, p) \quad (h \in \mathcal{G}),
\]

\[
(v(q)\xi)(g, p) = \xi(g, q^{-1}p) \quad (q \in \mathcal{G}),
\] (3.5)

for \( \xi \in L^2(\mathcal{G}_\Pi; \mathcal{G} \times \mathcal{G}) \), \( p \in \mathcal{G} \), and \( g \in \mathcal{G} \). Define a unitary operator \( J \) of \( L^2(\mathcal{G}_\Pi; \mathcal{G} \times \mathcal{G}) \) onto \( L^2(\mathcal{G}_\Pi; \mathcal{G} \times \mathcal{G}) \) by

\[
(J\xi)(g, p) = \overline{\xi(g, p)}.
\]
for \( \xi \in L^2(\mathcal{H}; G \times \hat{G}) \), \( \rho \in \hat{G} \), and \( g \in G \). Since \( (J^* \eta)(g, \rho) = \langle \rho, g \rangle \eta(\rho, g) \) for \( \eta \in L^2(\mathcal{H}; \hat{G} \times G) \), we have by (3.4), (3.5) that

\[
J \tilde{\Pi}(x)^* = \int_{G \times G} x(g, \rho) u(g) v(\rho) \, dg \, d\rho, \quad (3.6)
\]

where

\[
(a') \eta)(g, \rho) = \Pi \circ \alpha^\rho_\rho (a) \eta(\rho, g) \quad (a \in \mathfrak{A}),
\]

\[
(u(h)' \eta)(g, \rho) = \eta(p, \rho^{-1}g) \quad (h \in G), \quad (3.7)
\]

\[
(v(\xi)' \eta)(p, \rho) = \overline{\xi(p^{-1} \rho, g)} \quad (\xi \in \hat{G}),
\]

for \( \eta \in L^2(\mathcal{H}; \hat{G} \times G) \), \( \rho \in \hat{G} \), and \( g \in G \).

Keeping (3.6) and (3.7) in mind, we shall discuss another crossed product based on the trivial action \( \iota_G \) of \( \hat{G} \) on \( \mathfrak{A} \). Namely, let us consider the crossed product \( C^*(\mathfrak{A}; \iota_G) \) of \( \mathfrak{A} \) by \( \iota_G \). Then it can be defined a continuous action \( \beta \) of \( G \) on \( C^*(\mathfrak{A}; \iota_G) \) by

\[
\beta_\rho(x)(\rho) = \langle \rho, \rho \rangle \alpha_\rho[x(\rho)] \quad (3.8)
\]

for \( x \in L^1(\mathfrak{A}; \hat{G}) \), \( \rho \in \hat{G} \), and \( g \in G \). In fact, one has that

\[
(\beta_\rho(x) \beta_\rho(y))(\rho) = \int_G \beta_\rho(x)(\rho) \beta_\rho(y)(\rho^{-1}g) \, dq
\]

\[
= \langle \rho, \rho \rangle \int_G \alpha_\rho[x(\rho)] \alpha_\rho[y(\rho^{-1}g)] \, dq
\]

\[
= \beta_\rho(xy)(\rho)
\]

and

\[
\beta_\rho(x^*)(\rho) = \langle \rho, \rho \rangle \alpha_\rho[x^*(\rho)] = \langle \rho, \rho \rangle \alpha_\rho[x(\rho^{-1})]^*
\]

\[
= \langle \rho, \rho \rangle \alpha_\rho[x(\rho^{-1})]^* = \beta_\rho(x)(\rho^{-1})^*
\]

\[
= \beta_\rho(x)^*(\rho),
\]

for each \( x, y \in L^1(\mathfrak{A}; \hat{G}) \), \( \rho \in \hat{G} \), and \( g \in G \). Moreover, the norm estimation for \( \beta_\rho \) is given as follows:

\[
\| \beta_\rho(x) \| = \sup \{ \| \rho[\beta_\rho(x)] \| : \rho \in L^1(\mathfrak{A}; \hat{G}) \}
\]

\[
= \sup \left\{ \left\| \int_G L[\beta_\rho(x)(\rho)] \, dp \right\| : (L, \nabla) \in \text{Cov rep} (\mathfrak{A}, \hat{G}) \right\}
\]

\[
= \sup \left\{ \left\| \int_G \langle \rho, \rho \rangle L \circ \alpha_\rho[x(\rho)] \, dp \right\| : (L, \nabla) \in \text{Cov rep} (\mathfrak{A}, \hat{G}) \right\}
\]
Let $V_g(p) = \langle g, p \rangle \in V(p)$. Since $(L, \nabla) \in \text{Cov rep}(\mathcal{A}, \hat{G})$, it follows that $(\alpha^{-1}_g \cdot L, \nabla_g) \in \text{Cov rep}(\mathcal{A}, \hat{G})$. This implies that $||\beta^*_g(x)|| \leq ||x||$ for $g \in G$. Since $\beta^*_g = \beta^*_g (g \in G)$, $||\beta^*_g(x)|| = ||x||$ for $x \in L^1(\mathcal{A}, \hat{G})$, $g \in G$. By a similar reasoning as for $\hat{e}$, $\hat{e}^*$ is a continuous action of $G$ on $C^*(\mathcal{A}; \tau_0)$. Then we can also construct the crossed product $C^*(C^*(\mathcal{A}; \tau_0); \beta)$ of $C^*(\mathcal{A}; \tau_0)$ by $\beta$. We now show the following proposition which is essential to prove the duality.

**Proposition 3.1.** Let $G$ be a locally compact abelian group, and $\mathcal{A}$ be a $C^*$-algebra with a continuous action $\alpha$ of $G$. Then there exists a continuous action $\hat{\alpha}$ (respectively, $\beta$) of $\hat{G}$ (respectively, $G$) on $C^*(\mathcal{A}; \alpha)$ (respectively $C^*(\mathcal{A}; \tau_0)$) such that $C^*(C^*(\mathcal{A}; \alpha); \hat{\alpha})$ is isomorphic to $C^*(C^*(\mathcal{A}; \tau_0); \beta)$.

**Proof.** Let $\hat{\Pi}$ be the representation of $C^*(C^*(\mathcal{A}; \alpha); \beta)$ induced by $\Pi$, where $\Pi$ is as before. By the same argument as $\hat{\Pi}$, $\hat{\Pi}$ is faithful on $C^*(C^*(\mathcal{A}; \tau_0); \beta)$. Moreover, the similar computation as (3.3) gives us that

\[
(\hat{\Pi}(x)\xi)(p, g) = \int_G \left( \text{Ind } \Pi \left[ x(h) \right] \lambda(h) \xi(p, g) \right) dh \\
= \int_G \left( \text{Ind } \Pi \circ \beta^{-1}_g [x(h)] \xi(\cdot, h^{-1}g) \right) (p) dh \\
= \int_G \int_G \left\{ \Pi(\beta^{-1}_g [x(h)](q)) \lambda(q) \xi(\cdot, h^{-1}g) \right\} (p) dq dh \\
= \int_{\mathcal{G} \times G} \left\langle g, q \right| \Pi \circ \alpha^{-1}_g [x(q, h)] \xi(q^{-1}p, h^{-1}g) dq dh \\
= \int_{\mathcal{G} \times G} \left\langle g, q \right| \Pi \circ \alpha^{-1}_g [x(q, h)] \xi(q^{-1}p, h^{-1}g) dq dh,
\]

for $x \in K(\mathcal{A}; \hat{G} \times G)$, $\xi \in L^2(\mathcal{G}; \hat{G} \times G)$, $p \in \mathcal{G}$, and $g \in G$, where $(\text{Ind } \Pi, \lambda) \in \text{Cov rep}(C^*(\mathcal{A}; \tau_0), \hat{G} \times G)$ induced by $(\text{Ind } \Pi, \iota) \in \text{Cov rep}(C^*(\mathcal{A}; \tau_0), \{e\})$, and $(\Pi, \lambda) \in \text{Cov rep}(\mathcal{A}, \hat{G})$ induced by $(\Pi, \iota) \in \text{Cov rep}(\mathcal{A}, \{1\})$. Therefore, one can describe the operator $\hat{\Pi}(x)$ as follows:

\[
\hat{\Pi}(x) = \int_{\mathcal{G} \times G} x(q, h) \varphi(q) u(h) df dq, \quad (3.9)
\]

for $x \in K(\mathcal{A}; \hat{G} \times G)$, where $\varphi, \varphi(q)$, $u(h)$ is as (3.7) ($a \in A$, $q \in \hat{G}$,
Since \( v(q') u(h') = \langle h, q \rangle u(h') v(q') \) for \( q \in \hat{G}, h \in G \), it follows by (3.9) that

\[
\hat{I}(x) = \int_{G \times G} \langle h, q \rangle x(q, h) u(h') v(q') \, dh \, dq
\]

\[
= \int_{G \times G} \langle h, q \rangle (x(q, h))' u(h') v(q') \, dh \, dq
\]

\[
= \int_{G \times G} y(h, q) u(h') v(q') \, dh \, dq,
\]

where \( y(h, q) = \langle h, q \rangle x(q, h) \) \((h \in G, q \in \hat{G})\). Comparing (3.10) with (3.6), we obtain that

\[
\hat{I}(x) = J\hat{I}(y)J^*,
\]

which implies that

\[
\hat{I}[C^*(C^*(\mathfrak{A}; \iota_G); \beta)] = J\hat{I}[C^*(\mathfrak{A}; \alpha); \varnothing]]J^*.
\]

This means that \( C^*(C^*(\mathfrak{A}; \alpha); \varnothing) \) is isomorphic to \( C^*(C^*(\mathfrak{A}; \iota_G); \beta) \).

Q.E.D.

Now we can see without difficulty what is the structure of \( C^*(C^*(\mathfrak{A}; \iota_G); \beta) \). Actually, let us first define a Banach *-algebra which is isomorphic to \( L^1(\mathfrak{A}; \hat{G}) \) with the trivial action \( \iota_G \) as follows: consider the algebraic tensor product \( \mathfrak{A} \otimes L^1(\hat{G}) \) of \( \mathfrak{A} \) and \( L^1(\hat{G}) \) with a *-algebraic structure defined by

\[
\left( \sum_{i=1}^{n} a_i \otimes f_i \right) \left( \sum_{j=1}^{m} a'_j \otimes f'_j \right) = \sum_{i,j=1}^{n,m} a_ia'_j \otimes (f_i \ast f'_j)
\]

and

\[
\left( \sum_{i=1}^{n} a_i \otimes f_i \right)^* = \sum_{i=1}^{n} a_i^* \otimes f_i^*,
\]

for each \( a_i, a'_j \in \mathfrak{A}, \) and \( f_i, f'_j \in L^1(\hat{G}). \) Then, the completion \( A \otimes_\gamma L^1(\hat{G}) \) of \( \mathfrak{A} \otimes L^1(\hat{G}) \) with respect to the greatest cross norm \( \gamma \), is a Banach *-algebra which is isomorphic to \( L^1(\mathfrak{A}; \hat{G}) \) with \( \iota_G \) (cf. [3]). The isomorphism \( \Phi \) between them is determined by

\[
\Phi \left[ \sum_{i=1}^{n} f_i(\cdot) a_i \right] = \sum_{i=1}^{n} a_i \otimes f_i,
\]
for \( a_\in \mathfrak{A}, f_i \in L^1(\hat{G}) \), where \((f_i(\cdot) a_\in)(p) = f_i(p) a_\in (p \in \hat{G})\). Furthermore, \( \Phi \) can be extended to a isomorphism of \( C^*(\mathfrak{A}; \iota_G) \) onto the enveloping \( C^\ast \)-algebra \( C^*(\mathfrak{A} \otimes L^1(\hat{G})) \) of \( \mathfrak{A} \otimes L^1(\hat{G}) \) which is also denoted by \( \Phi \). Then, one can easily check that

\[
\Phi \circ \beta_g \circ \Phi^{-1} = \alpha_g \otimes L_{\tau_g}, \quad \text{for } g \in G,
\]

where \((L_{\tau_g}f)(p) = \chi_g(p)f(p) = \langle g, p \rangle f(p) (f \in L^1(\hat{G}), p \in \hat{G})\). On the other hand, due to Guichardet–Okayasu (cf. [4, 7]),

\[
C^*(\mathfrak{A} \otimes L^1(\hat{G})) = \mathfrak{A} \otimes_v C^\ast(\hat{G})
\]

where \( v \) is the greatest \( C^\ast \)-norm. By applying Proposition 2.3, one gets that \( C^*(C^*(\mathfrak{A}; \iota_G); \beta') \) is isomorphic to \( C^*(\mathfrak{A} \otimes_v C^\ast(\hat{G}); \alpha \otimes L_\tau) \) where \((\alpha \otimes L_{\tau_g})_g = \alpha_g \otimes L_{\tau_g} (g \in G)\). Denoting by \( C_0(G) \) the \( C^\ast \)-algebra consisting of all continuous functions vanishing at infinity, it easily follows that there exists an isomorphism \( \Psi \) of \( \mathfrak{A} \otimes_v C^\ast(\hat{G}) \) onto \( A \otimes_v C_0(G) \) such that

\[
\Psi \circ (\alpha \otimes L_\tau)_g \circ \Psi^{-1} = \alpha_g \otimes \tau_g, \quad \text{for } g \in G,
\]

where \((\tau_g f)(h) = f(g^{-1}h) (f \in C_0(G))\). Applying Proposition 2.3 again, we obtain that \( C^*(\mathfrak{A} \otimes_v, C^\ast(\hat{G}); \alpha \otimes \tau) \) is isomorphic to \( C^*(\mathfrak{A} \otimes_v C_0(G); \alpha \otimes \tau) \) where \((\alpha \otimes \tau)_g = \alpha_g \otimes \tau_g (g \in G)\). By a reasoning similar to the one above, there is an isomorphism \( \Phi' \) of \( \mathfrak{A} \otimes_v C_0(G) \) onto the \( C^\ast \)-algebra \( C_0(\mathfrak{A}; G) \) consisting of all continuous \( \mathfrak{A} \)-valued functions vanishing at infinity such that

\[
\Phi' \circ (\alpha \otimes \tau)_g \circ (\Phi')^{-1} = \beta'_g, \quad \text{for } g \in G,
\]

where \( \beta'_g(x)(h) = \alpha_g[x(g^{-1}h)] \), for \( x \in C_0(\mathfrak{A}; G) \). So we have that \( C_0(\mathfrak{A} \otimes_v C_0(G); \alpha \otimes \tau) \) is isomorphic to \( C^*(C_0(\mathfrak{A}; G); \beta') \). However, since \( C_0(G) \) is abelian, \( \mathfrak{A} \otimes_v C_0(G) = \mathfrak{A} \otimes_v C_0(G) \) (cf. [8]). Therefore, \( C^*(\mathfrak{A} \otimes_v C_0(G); \alpha \otimes \tau) \) is isomorphic to \( C^*(C_0(\mathfrak{A}; G); \beta') \). Let \( \Psi'(x)(g) = \alpha^{-1}_g[x(g)] \) for \( x \in C_0(\mathfrak{A}; G) \). Then it is verified that \( \Psi' \) is an isomorphism on \( C_0(\mathfrak{A}; G) \) such that

\[
\Psi' \circ \beta'_g \circ (\Psi')^{-1} = \tau'_g, \quad \text{for } g \in G,
\]

where \( \tau'_g(x)(h) = x(g^{-1}h) \), for \( x \in C_0(\mathfrak{A}; G) \). This means that \( C^*(C_0(\mathfrak{A}; G); \beta') \) is isomorphic to \( C^*(C_0(\mathfrak{A}; G); \tau') \). Using \( \Phi' \) again, one gets that

\[
(\Phi')^{-1} \circ \tau'_g \circ \Phi' = \iota_{\mathfrak{A}} \otimes \tau_g, \quad \text{for } g \in G,
\]
where \( \psi \) is the trivial automorphism of \( \mathcal{A} \). Thus, we have that
\[
C^*(\mathcal{A}, C_0(G); \tau') \text{ is isomorphic to } C^*(\mathcal{A} \otimes \mathcal{A}, C_0(G); \psi \otimes \tau),
\]
where \( \psi \) is the trivial action of \( \{ e \} \) on \( \mathcal{A} \). Summing up the argument discussed above, we get the following.

**Proposition 3.2.** Let \( G \) be a locally compact abelian group, and \( \mathcal{A} \) be a \( C^* \)-algebra with a continuous action \( \alpha \) of \( G \). Then \( C^*(\mathcal{A} \otimes \mathcal{A}, C_0(G); \alpha \otimes \tau) \) is isomorphic to \( C^*(\mathcal{A}, C_0(G); \psi \otimes \tau) \), where \( \psi \) is the trivial action of \( \{ e \} \) on \( \mathcal{A} \).

Applying now Proposition 2.4 to \( C^*(\mathcal{A} \otimes \mathcal{A}, C_0(G); \psi \otimes \tau) \), one obtains that it is isomorphic to \( C^*(\mathcal{A}, C_0(G); \psi \otimes \tau) \) which is equal to \( \mathcal{A} \otimes \mathcal{A} \).

Finally we prove the following which is of independent interest.

**Proposition 3.3.** Let \( G \) be a locally compact group, and \( C_0(G) \) be the \( C^* \)-algebra of all continuous functions on \( G \) vanishing at infinity. Then the crossed product \( C^*(C_0(G), \tau) \) of \( C_0(G) \) by the translation \( \tau \) of \( G \) is isomorphic to the \( \mathcal{C}(L^2(G)) \) of all compact operators on \( L^2(G) \).

**Proof.** Let \( \Pi \) be an irreducible representation of \( C^*(C_0(G); \tau) \) and \( (\rho, \mathcal{V}) \in \text{Cov rep}(C_0(G), G) \) corresponding to \( \Pi \). Then \( \rho \) is faithful. In fact, let \( F \) be the closed subset of \( G \) corresponding to \( \ker \rho \). Since \( (\rho, \mathcal{V}) \in \text{Cov rep}(C_0(G), G) \), it follows that \( gF = F \) for any \( g \in G \). Hence \( F = G \), which implies that \( \rho \) is faithful. Consider a bounded linear functional \( F_{\xi, \eta}(f) = \langle \rho(f) \xi | \eta \rangle \) on \( C_0(G) \) for each \( \xi, \eta \in \mathcal{S}_G \). By Riesz–Markov’s theorem, there exists a unique regular Borel measure \( \mu_{\xi, \eta} \) which represents \( F_{\xi, \eta} \). Let \( \Sigma_G \) be the set of all Baire sets of \( G \). Since \( (\xi, \eta) \mapsto \mu_{\xi, \eta}(E) \) is a bounded sesquilinear form on \( \mathcal{S}_{\rho} \times \mathcal{S}_{\rho} \) for each \( E \in \Sigma_G \), there is a unique bounded linear operator \( P(E) \) on \( \mathcal{S}_{\rho} \) such that \( \langle P(E) \xi | \eta \rangle = \mu_{\xi, \eta}(E) \). Since \( \rho \) is a faithful representation on \( C_0(G) \), \( P(E) \) is a projection and \( P(E) \neq 0 \) for every non-empty open set \( E \in \Sigma_G \). Moreover, the mapping \( E \mapsto P(E) \) is a Boolean \( \sigma \)-homomorphism from \( \Sigma_G \) onto an abelian family of projections on \( \mathcal{S}_{\rho} \). Since \( (\rho, \mathcal{V}) \in \text{Cov rep}(C_0(G), G) \), the uniqueness of \( P(E) \) gives us that \( \mathcal{V}(g) P(E) \mathcal{V}(g)^* = P(gE) \) for \( E \in \Sigma_G \), and \( g \in G \). Therefore, since \( (P, \mathcal{V}) \) is irreducible, it follows by Loomis [6] that \( (P, \mathcal{V}) \) is unitarily equivalent to \( (L, \lambda) \), where \( (L, \lambda)(f)(g) = \lambda_{\xi, \eta}(g)f(g) \) and \( (L, \lambda)(h)f)(g) = f(h^{-1}g) \) for \( f \in L^2(G) \), \( E \in \Sigma_G \). Then the pair \( (\rho, \mathcal{V}) \) is unitarily equivalent to \( (L, \lambda) \). This implies that \( \Pi \) is unitarily equivalent to \( \text{Ind} \delta \), where \( \delta \) is the character of \( C_0(G) \) given
by \( \delta(f) = f(e) \). Thus, \( \text{Ind} \delta \) is faithful on \( C^*(C_0(G); \tau) \). Computing \( (\text{Ind} \delta)(x) \) for \( x \in K(G \times G) \), one has that

\[
[(\text{Ind} \delta)(x)\xi](g) = \int_G x(g, h) \xi(h^{-1}g) \, dh
\]

\[
= \int_G x(g, gh^{-1}) \Delta(h)^{-1} \xi(h) \, dh
\]

\[
= \int_G y(g, h) \xi(h) \, dh, \quad \xi \in L^2(G),
\]

where \( y(g, h) = x(g, gh^{-1}) \Delta(h^{-1}) \). Since \( y \) is square integrable on \( G \times G \), \( (\text{Ind} \delta)(x) \) is an operator of Hilbert–Schmidt class. So \( (\text{Ind} \delta)(x) \in \mathcal{B}(L^2(G)) \). Therefore, since \( K(G \times G) \) is dense in \( C^*(C_0(G); \tau) \), \( (\text{Ind} \delta)[C^*(C_0(G); \tau)] \subset \mathcal{B}(L^2(G)) \). Since \( \text{Ind} \delta \) is irreducible, we have by [1, Cor 4.1.11] that \( (\text{Ind} \delta)[C^*(C_0(G); \tau)] = \mathcal{B}(L^2(G)) \). Thus, \( C^*(C_0(G); \tau) \) is isomorphic to \( \mathcal{B}(L^2(G)) \). Q.E.D.

We shall now state our main theorem as follows.

**Theorem 3.4.** Let \( G \) be a locally compact abelian group, and \( \mathcal{A} \) be a \( C^* \)-algebra with a continuous action \( \alpha \) of \( G \). Then there exists a continuous action \( \delta \) of the dual group \( \hat{G} \) of \( G \) on the crossed product \( C^*(A; \alpha) \) of \( \mathcal{A} \) by \( \alpha \) such that the crossed product \( C^*(C^*(\mathcal{A}; \alpha); \delta) \) of \( C^*(\mathcal{A}; \alpha) \) by \( \delta \) is isomorphic to the tensor product \( \mathcal{A} \otimes \mathcal{L}(L^2(G)) \) of \( \mathcal{A} \) and the \( C^* \)-algebra \( \mathcal{L}(L^2(G)) \) of all compact operators on \( L^2(G) \).

**Proof.** By Propositions 3.1–3.3, \( C^*(C^*(\mathcal{A}; \alpha); \delta) \) is isomorphic to \( \mathcal{A} \otimes \mathcal{L}(L^2(G)) \). However, since \( \mathcal{L}(L^2(G)) \) is of type I, \( \mathcal{A} \otimes \mathcal{L}(L^2(G)) = \mathcal{A} \mathcal{L}(L^2(G)) \) (cf. [8]). Q.E.D.

**Remark.** Let \( G_n \) be the cyclic group of order \( P_n \) for \( n = 1, 2, \ldots, \) and \( G \) denote the product group \( \prod_{n=1}^\infty G_n \) with the weak product topology. Then the dual group \( \hat{G} \) of \( G \) can be identified as the restricted product group \( \bigcup_{n=1}^\infty G_n \) with the discrete topology. Consider the action \( \tau \) of \( \hat{G} \) on \( C(G) \) by \( (\tau f)(g) = f(gs) \) where \( C(G) \) is the \( C^* \)-algebra of all continuous functions on \( G \); Then, the above theorem says that there exists a continuous action \( \tilde{\tau} \) of \( \hat{G} \) on \( C^*(C(G); \tau) \) such that \( C^*(C^*(C(G); \tau); \tilde{\tau}) \) is isomorphic to \( C(G) \mathcal{L}(L^2(G)) \) which is exactly the theorem in [10].
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