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Analysis of the Laplacian on the Complete Riemannian Manifold

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1. INTRODUCTION

To what extent can the classical analysis on Euclidean space be carried over to the setting of a complete Riemannian manifold? This is the problem we address in this paper. We are particularly interested in the noncompact case, since analysis on compact manifolds—with or without the Riemannian structure—is quite well understood. Also we strive to avoid making unnecessary hypotheses on the manifold.

We will show that much of the classical theory of he Laplacian remains valid for the Laplace–Beltrami operator on a complete Riemannian manifold. This includes the essential self-adjointness, properties of the heat semi-group $e^{t\Delta}$, the Bessel potentials $(I-\Delta)^{-\alpha/2}$, the Riesz potentials $(-\Delta)^{-\alpha/2}$, and the Sobolev spaces based on Bessel potentials. Our results are complementary to those of Aubin [2, 3], who studies Sobolev spaces defined by covariant derivatives, and Yau [35] who studies the heat semi-group under the assumption that the Ricci curvature is bounded on both sides. Several other recent papers [6–8, 33] study more detailed properties of the heat semi-group under special assumptions on the curvature. The essential self-adjointness has previously been established in [9, 23].

We also give some generalizations of an inequality of McKean [19], which applies only to simply-connected manifolds with negative curvature (bounded above by a negative constant $-k$), and which implies that the spectrum of the Laplacian is bounded away from zero. We prove $\|f\|_p \leqslant (p/(n-1)\sqrt{k})\,\|\nabla f\|_p$ for $1 \leqslant p < \infty$ and f compactly supported, and we investigate under what circumstances $\nabla f \in L^p$ implies $f - c \in L^p$ for some

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constant c, without the assumption of compact support. Different sorts of extensions of McKean's work have been considered by Pinsky [20, 21].

An important open question which arises from our work concerns the L^p boundedness of the Riesz transforms $\nabla(-\Delta)^{-1/2}$ (the reader should be careful to distinguish between these Riesz transforms $\nabla(-\Delta)^{-1/2}$ and the Riesz potentials $(-\Delta)^{-\alpha/2}$, both named after Marcel Riesz). We are able to prove this ($1 < p < \infty$) only for rank-one symmetric spaces.

We hope that the results of this paper, taken as a whole, will convince analysts that a complete Riemannian manifold is a structure worthy of study, a home in many ways as comfortable and rich as Euclidean space. The techniques we use are quite eclectic, and often theorems are proved merely by combining ideas borrowed from different works. Only an elementary knowledge of differential geometry is required, but we will use the full range of modern analytic technique.

Notation. Let M_n denote an n-dimensional Riemannian manifold with Riemannian metric g_{jk}. We will assume for simplicity that the metric is of class C^∞, although for many of our results this can be relaxed considerably. We assume that the manifold is *complete*, and this is essential to everything we do. We will study functions (real or complex valued) on M_n, and more generally sections of various vector bundles over M_n, such as the k-forms Λ^k, and the tensors of rank (r, s), $T^{r,s}$. Functions can be regarded as 0-forms or as tensors of rank $(0, 0)$, but we will usually want to maintain the distinction between 1-forms and tensors of rank $(1, 0)$. We will use standard notation and concepts from the differential and integral calculus on M_n, including the exterior derivative d, the Hodge star operator, the co-derivative δ, the covariant and contravariant derivatives ∇_j and ∇^j, and the canonical measure $d\mu(x) = \sqrt{g}\, dx$. With respect to this measure we form L_p spaces, $1 < p \leqslant \infty$, of functions, forms or tensors, since the bundles Λ^k and $T^{r,s}$ possess a canonical norm at each point.

The fundamental operator we study is the Laplacian Δ. *We adopt the sign convention that makes Δ a negative definite operator.* For functions, Δ is the Laplace–Beltrami operator

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} \left(\sqrt{g}\, g^{jk} \frac{\partial}{\partial x_k} f \right).$$

For forms, Δ is the negative of the Hodge–de Rham operator, $\Delta = -(d\delta + \delta d)$. For tensors, $\Delta = \nabla^j \nabla_j$, the contraction of the second covariant derivative. The reader is reminded that the Laplacian on k-forms and rank $(k, 0)$ tensors are not the same; they differ by certain zero-order terms (the Weitzenbock formula). Although we only obtain complete results for the Laplacian on functions, we give as much information as we can about the general case.

2. Self-Adjointness of Δ

Let $\mathscr{D}$ denote the C^∞ sections of compact support. Then Δ is a negative-definite symmetric operator on $\mathscr{D}$. Let $\Delta_{\min}$ denote the L^2 closure of Δ on $\mathscr{D}$, and $\Delta_{\max}$ the adjoint of $\Delta_{\min}$. The domain $D(\Delta_{\min})$ of $\Delta_{\min}$ is the set of sections f such that there exists a sequence f_j in $\mathscr{D}$ such that $f_j \to f$ in L^2 and Δf_j converges to an element in L^2 which we can write Δf. It follows from elementary distribution theory that this element can be identified with the distribution Δf. The domain $D(\Delta_{\min})$ is the smallest domain we might consider for the Laplacian. The domain $D(\Delta_{\max})$ of $\Delta_{\max}$ consists of all functions f in L^2 such that the distribution Δf can be identified with an L^2 section, as can easily be verified from the definition of the adjoint. The domain $D(\Delta_{\max})$ is the largest domain we might consider for the Laplacian. We do not know a priori that these two domains coincide; also, although the negative definiteness implies that there exist self-adjoint extensions of $\Delta_{\min}$, we do not have a priori the existence of a unique self-adjoint extension. For incomplete manifolds with reasonable boundary there are many self-adjoint extensions, corresponding to different boundary conditions. In the case of a complete Riemannian manifold, $\Delta_{\min} = \Delta_{\max}$, so Δ is essentially self-adjoint. Proofs of this fact have appeared in [9, 23]. We will give a new proof which is well suited to generalization to L^p. The technique we use is the following criterion (see Reed and Simon [22, p. 136–137):

Lemma 2.1. *Let A be any closed negative-definite, symmetric, densely defined operator on a Hilbert space. Then $A = A^*$ if and only if there are no eigenvectors with positive eigenvalue in the domain of A^*.*

To apply this criterion we have to show the vanishing of all solutions of $\Delta u = \lambda u$ for $u \in L^2$ and $\lambda > 0$. This result is essentially due to Yau [34]. We will repeat the proof for the sake of completeness, and because there are some errors on p. 664 of [34]. We begin with the existence of approximations to unity.

Lemma 2.2 (cf, [2, 34]). *Let B_r denote the ball of radius r about a fixed point P on M_n. Then there exists a family of functions $\phi_{r,s}$ such that $\phi_{r,s}$ is* *one on B_r and zero outside B_s (take $s > r$), $\phi_{r,s}$ takes on values between zero and one, is Lipschitz continuous, and the estimate*

$$\|\nabla\phi_{r,s}\|_\infty \leqslant c(s-r)^{-1} \tag{2.1}$$

holds.

Proof. Take $\phi_{r,s}(x) = \Phi((s-r)^{-1}(d(x)+s-2r))$, where $d(x)$ denotes the distance of x to P and $\Phi(t)$ is a smooth function on the line which is one for $t \leqslant 1$ and zero for $t \geqslant 2$.

LEMMA 2.3 (Yau). *Let u be an L^2 function, form or tensor that satisfies $\Delta u = \lambda u$ for some $\lambda > 0$. Then u is identically zero.*

Proof. Consider first the case of tensors. Let ϕ be one of the functions $\phi_{r,s}$. Since $\phi^2 u$ has compact support we can integrate by parts to obtain

$$\begin{aligned}\lambda\langle\phi^2 u, u\rangle &= \langle\phi^2, \Delta u\rangle = -\langle\nabla^j(\phi^2 u), \nabla^j u\rangle \\ &= -\langle\phi^2\,\nabla^j u, \nabla^j u > -2\left\langle\frac{\partial\phi}{\partial x_j}\,u, \phi\,\nabla^j u\right\rangle.\end{aligned}$$

Note that u is a smooth tensor since it is the solution of an elliptic equation. Now $\lambda\langle\phi^2 u, u\rangle \geqslant 0$ so we have

$$\begin{aligned}\langle\phi^2\,\nabla^j u, \nabla^j u\rangle = \sum_j \|\phi\,\nabla^j u\|_2^2 &\leqslant 2\left|\left\langle\frac{\partial\phi}{\partial x_j}\,u, \phi\,\nabla^j u\right\rangle\right| \\ &\leqslant 2\,\|\nabla\phi\|_\infty\,\|u\|_2\left(\sum_j \|\phi\,\nabla^j u\|_2^2\right)^{1/2}\end{aligned}$$

by the Cauchy–Schwartz inequality, hence $\sum_j \|\phi\,\nabla^j u\|_2^2 \leqslant 4\,\|\nabla\phi\|_\infty^2\,\|u\|_2^2$. Now if we first fix r and let $s \to \infty$, then $\|\nabla\phi\|_\infty \to 0$ so $\sum_j \int_{B_r} |\nabla^j u|^2 = 0$. Since this holds for every r, we have $\nabla^j u \equiv 0$, hence $\Delta u \equiv 0$ hence $u = \lambda^{-1}\,\Delta u \equiv 0$.

Consider next the case of forms. We have now

$$\begin{aligned}\lambda\langle\phi^2 u, u\rangle = \langle\phi^2 u, \Delta u\rangle &= -\langle d(\phi^2 u), du\rangle - \langle\delta(\phi^2 u), \delta u\rangle \\ &= -\langle\phi^2 du, du\rangle - \langle\phi^2\,\delta u, \delta u\rangle - 2\langle\phi\,d\phi \wedge u, du\rangle \\ &\quad + 2\langle u, \phi\,d\phi \wedge \delta u\rangle\end{aligned}$$

hence

$$\begin{aligned}\|\phi\,du\|_2^2 + \|\phi\,\delta u\|_2^2 &\leqslant 2\,|\langle\phi\,d\phi \wedge u, du\rangle| + 2|\langle u, \phi\,d\phi \wedge \delta u\rangle| \\ &\leqslant 2\,\|d\phi\|_\infty\,\|u\|_2(\|\phi\,du\|_2 + \|\phi\,\delta u\|_2).\end{aligned}$$

By elementary estimates ($x^2 + y^2 \leqslant c(|x| + |y|)$ implies $|x| + |y| \leqslant 2c$) we have

$$\|\phi\, du\|_2 + \|\phi\, \delta u\|_2 \leqslant 4\, \|d\phi\|_\infty\, \|u\|_2$$

and we can complete the argument as before. Q.E.D.

THEOREM 2.4. *On a complete Riemannian manifold, the Laplacian Δ on functions, forms or tensors is essentially self-adjoint.*

Proof. Combine Lemmas 2.1 and 2.3. Q.E.D.

Remark. It suffices to assume the metric tensor g_{jk} is of smoothness class C^2 for the above proof. It would be interesting to know if the completeness of the manifold is necessary for the self-adjointness of Δ.

COROLLARY 2.5. *Suppose f and Δf are both in L^2. Then also ∇f is in L^2 (df and δf are in L^2 in the case of forms), and there exists a sequence f_j in $\mathscr{D}$ such that $f_j \to f$, $\Delta f_j \to \Delta f$ and $\nabla f_j \to \nabla f$ (or $df_j \to df$ and $\delta f_j \to \delta f$), all in the L^2 norm.*

Proof. If f is in $\mathscr{D}$, then

$$\|\nabla f\|^2 = \langle \nabla^j f, \nabla^j f\rangle = -\langle f, \Delta f\rangle \qquad \text{(tensors)} \tag{2.2}$$

or

$$\|df\|^2 + \|\delta f\|^2 = -\langle f, \Delta f\rangle \qquad \text{(forms)}$$

so

$$\|\nabla f\|_2^2 \leqslant \|f\|_2\, \|\Delta f\| \tag{2.3}$$

or

$$\|df\|_2^2 + \|\delta f\|_2^2 \leqslant \|f\|_2\, \|\Delta f\|_2.$$

By continuity these estimates continue to hold on $D(\Delta_{\min})$ hence on $D(\Delta_{\max})$ by the theorem. The existence of the sequence f_j is a consequence of the definition of $D(\Delta_{\min})$ and the above estimates. Q.E.D.

Remark. This result is essentially proved also in [1]. A direct approach to the corollary seems natural. One would hope to approximate f by sections of compact support ϕf, where ϕ is one of the functions in Lemma 2.1. The problem is that one of the terms in $\Delta(\phi f)$ is $(\Delta\phi)\cdot f$, and there seems to be no way to control $\|\Delta\phi\|_\infty$ without making some assumption on the curvature (cf. [2, 3]).

Now that we have a unique self-adjoint realization of Δ, we can use the spectral theorem to define various functions of Δ. In particular we will be interested in the heat kernel $e^{t\Delta}$, the Poisson kernel $e^{-t\sqrt{-\Delta}}$, the Bessel potentials $(I-\Delta)^{-t/2}$ and the Riesz potentials (possibly unbounded operators) $(-\Delta)^{-t/2}$.

COROLLARY 2.6 (L^2 boundedness of Riesz transforms). *Let $f \in L^2$. Then $(-\Delta)^{1/2} f$ is in L^2 if and only if ∇f is in L^2 (tensors) or df and δf are in L^2 (forms), and*

$$\begin{aligned} \|(-\Delta)^{1/2} f\|_2 &= \|\nabla f\|_2 \qquad (\textit{tensors}) \\ \|(-\Delta)^{1/2} f\|_2 &= (\|df\|_2^2 + \|\delta f\|_2^2)^{1/2} \qquad (\textit{forms}) \end{aligned} \tag{2.4}$$

Furthermore, for such f there exists a sequence f_j in $\mathscr{D}$ such that $f_j \to f$ and $(-\Delta)^{1/2} f_j \to (-\Delta)^{1/2} f$ in L^2 norm.

Proof. Suppose first f and ∇f are in L^2. Then it follows from Lemma 2.2 and the dominated convergence theorem that $\phi_{r,r+1} f \to f$ and $\nabla(\phi_{r,r+1} f) \to \nabla f$ as $r \to \infty$ in L^2 norm. Since $\phi_{r,r+1} f$ has compact support we may regularize locally and obtain a sequence f_j in $\mathscr{D}$ such that $f_j \to f$ and $\nabla f_j \to \nabla f$ in L^2 norm. Now (2.4) holds for $f_j \in \mathscr{D}$. In fact $D(-\Delta) \subseteq D((-\Delta)^{1/2})$ by spectral theory and $\|(-\Delta)^{1/2} f_j\|_2^2 = -\langle f_j, \Delta f_j \rangle = \|\nabla f_j\|_2^2$. Since $(-\Delta)^{1/2}$ is a closed operator we must have f in $D((-\Delta)^{1/2})$, and (2.4) holds in the limit. The argument for forms is analogous.

Conversely, if f is in $D((-\Delta)^{1/2})$, then there exists a sequence f_j in $D(\Delta)$ such that $f_j \to f$ and $(-\Delta)^{1/2} f_j \to (-\Delta)^{1/2} f$ in L^2 norm (we can take f_j to be the spectral projection onto the interval $[0, j]$). By Corollary 2.5 ∇f_j is in L^2 and (2.4) holds. Thus ∇f_j is a Cauchy sequence in L^2, hence by distribution theory ∇f is in L^2. Q.E.D.

We return to the L^p theory of Riesz transforms in Section 6.

3. THE HEAT SEMI-GROUP

We consider now the properties of the heat semi-group $e^{t\Delta}$. The main result is that these operators are L^p contractions and are unique (for $1 < p < \infty$). There are many approaches to obtaining the L^p estimates, but as far as we know only the present approach yields the uniqueness. We need the following generalization of Lemma 2.3, which is also essentially due to Yau [34].

LEMMA 3.1. *Let p and q be fixed numbers, $1 < p \leqslant q < \infty$. If u is a* *function in* $L^p + L^q$ *satisfying* $\Delta u = \lambda u$ *for some* $\lambda > 0$, *then* u *is identically zero.*

Proof. Let $h(t)$ be a smooth nonnegative function. Then from the identity

$$\begin{aligned}\lambda\langle\phi^2 h(|u|)u, u\rangle &= \langle\phi^2 h(|u|)u, \Delta u\rangle \\ &= -\langle\phi^2 h(|u|)\,\nabla u, \nabla u\rangle - \langle\phi^2 h'(|u|)\,\nabla u, \nabla u\rangle \\ &\quad - 2\langle\phi u h(|u|)\,\nabla\phi, \nabla u\rangle\end{aligned}$$

we obtain the estimate

$$\langle\phi^2(h(|u|) + |u|\,h'(|u|)\,\nabla u, \nabla u\rangle \leqslant 2\,|\langle\phi h(|u|)\,u\,\nabla\phi, \nabla u\rangle|. \tag{3.1}$$

Here we have assumed u is real valued, since we may take real and imaginary parts of the eigenvalue equation, in order to have $u\nabla\,|u| = |u|\,\nabla u$, and also the fact that u is differentiable, which is also a consequence of the eigenvalue equation. As before ϕ is one of the functions $\phi_{r,s}$ in Lemma 2.2.

We choose h so that $h(t) = t^{p-2}$ for $t \geqslant 1$ and $h(t) = (\varepsilon + t^2)^{(q-2)/2}$ for $t \leqslant 1 - \varepsilon$, where ε is a positive parameter that will eventually go to zero. In between, for $1 - \varepsilon \leqslant t \leqslant 1$, we arrange h so that $h + th' \geqslant ch$ for a fixed positive constant c independent of ε, this estimate clearly holding with $c = p - 1$ in $t \geqslant 1$ and $c = \min(q - 1, 1)$ in $t \leqslant 1 - \varepsilon$. We then obtain from (3.1) and the Cauchy–Schwartz inequality the estimate

$$\begin{aligned}&c\int \phi^2 h(|u|)\,|\nabla u|^2 \\ &\quad\leqslant \|\nabla\phi\|_\infty \left(\int \phi^2 h(|u|)|\,\nabla u|^2\right)^{1/2} \left(\int_{\operatorname{supp}\phi} h(|u|)\,|u|^2\right)^{1/2}.\end{aligned}$$

Since all the integrals are finite and nonnegative we may divide to obtain

$$\int \phi^2 h(|u|)\,|\nabla u|^2 \leqslant c^{-2}\,\|\nabla\phi\|_\infty^2 \int_{\operatorname{supp}\phi} h(|u|)\,|u|^2. \tag{3.2}$$

Next we let $\varepsilon \to 0$, so that

$$\begin{aligned}h(|u|)\,|u|^2 &= |u|^p \qquad \text{if} \quad |u| \geqslant 1, \\ &= |u|^q \qquad \text{if} \quad |u| \leqslant 1,\end{aligned}$$

which is globally integrable by the assumption $u \in L^p + L^q$. Thus the right side of (3.2) tends to zero if we let $\phi = \phi_{r,s}$ and let $s \to \infty$. Thus we obtain $\int_{B_r} h(|u|)\,|\nabla u|^2 = 0$ hence $\nabla u = 0$ on the open set where $u \neq 0$, hence $\Delta u = 0$ there, hence $u = \lambda^{-1}\,\Delta u = 0$ too. Q.E.D.

LEMMA 3.2. *Let p and q be fixed numbers, $1 < p \leqslant q < 3$. If u is a tensor in $L^p + L^q$ satisfying $\Delta u = \lambda u$ for some $\lambda > 0$, then u is identically zero.*

Proof. In this case we have the identity

$$\begin{aligned}\lambda\langle\phi^2 h(|u|)u, u\rangle &= \langle\phi^2 h(|u|)u, \Delta u\rangle \\ &= -\langle\phi^2 h(|u|)\,\nabla u, \nabla u\rangle - \langle\phi^2 h'(|u|)(\nabla\,|u|)\otimes u, \nabla u\rangle \\ &\quad - 2\langle\phi h(|u|)(\nabla\phi)\otimes u, \nabla u\rangle.\end{aligned}$$

We now require the estimate $|th'(t)| \leqslant \rho h(t)$, where ρ is a constant strictly less than one. Again this is true for $t \geqslant 1$ or $t \leqslant 1-\varepsilon$, because of the assumption that p and q are strictly less than 3, and can easily be arranged in between.

Now we claim $|\nabla\,|u|| \leqslant |\nabla u|$, this being clear in flat space, and then in curved space by computing in normal coordinates when both the metric and the covariant derivatives are the same as in flat space at the point in question. Thus $|\nabla\,|u|\otimes u| \leqslant |u|\,|\nabla u|$ and so

$$\begin{aligned}(1-\rho) &< \phi^2 h(|u|)\,\nabla u, \nabla u\rangle \\ &\leqslant \langle\phi^2 h(|u|)\,\nabla u, \nabla u\rangle + \langle\phi^2 h'(|u|)(\nabla\,|u|)\otimes u, \nabla u\rangle.\end{aligned}$$

Combining this with (3.3) gives

$$(1-\rho)\langle\phi^2 h(|u|)\,\nabla u, \nabla u\rangle \leqslant 2\,|\langle\phi h(|u|)(\nabla\phi)\otimes u, \nabla u\rangle|$$

from which we can again obtain (3.2) and the proof can be completed as before. Q.E.D.

Next we recall some facts from the theory of semigroups. If X is a Banach space and $x \in X$ a nonzero element, there exists an element x^* in the dual space X^* such that $\|x^*\| = \|x\|$ and $\langle x, x^*\rangle = \|x\|^2$. Such an element is called a normalized tangent functional, and its existence is given by the Hahn–Banach theorem. However, in the cases we will consider, X being L^p of real-valued functions or tensors for $1 < p < \infty$ and X^* being $L^{p'}$ with $(1/p) + (1/p') = 1$, we can easily write an explicit normalized tangent functional as $c\,|u|^{p-2}u$ for $c = \|u\|_p^{-p/p'}$. A densely defined operator L on X is said to be *dissipative* if for every x in the domain of L there exists a normalized tangent functional such that $\langle x^*, Lx\rangle \leqslant 0$. It is a theorem that the closure of a dissipative operator is also dissipative.

LEMMA 3.3. *A closed, densely defined operator L on a Banach space X is the infinitesimal generator of a contraction semigroup if and only if*

(a) *L is dissipative, and*

(b) *$\lambda - L$ maps domain of L* onto *X for some $\lambda > 0$.*

Proof. See [22, pp. 240, 330].

LEMMA 3.4. *The Laplacian on functions with domain $\mathscr{D}$ is dissipative on L^p for $1 < p < \infty$. For tensors the same is true for $1 < p < 3$.*

Proof. We need to show $\langle |u|^{p-2}u, \Delta u\rangle \leqslant 0$ for $u \in \mathscr{D}$. For functions we have $\langle |u|^{p-2}u, \Delta u\rangle = -\langle \nabla(|u|^{p-2}u), \nabla u\rangle = -(p-1)\langle |u|^{p-2}\, \nabla u, \nabla u\rangle$ which is clearly nonpositive. For tensors we have

$$-\langle \nabla(|u|^{p-2}u), \nabla u\rangle$$
$$= -\langle |u|^{p-2}\, \nabla u, \nabla u\rangle - (p-2)\langle |u|^{p-3}\, \nabla\, |u| \otimes u, \nabla u\rangle$$

and as in the proof of Lemma 3.2 we have $|\langle |u|^{p-3}\, \nabla\, |u| \otimes u, \nabla u\rangle| \leqslant \langle |u|^{p-2}\, \nabla u, \nabla u\rangle$. Thus $\langle |u|^{p-2}u, \Delta u\rangle \leqslant 0$ since $|p-2| < 1$. Q.E.D.

THEOREM 3.5. *Let $e^{t\Delta}$ denote the heat semi-group on L^2 of functions defined by the spectral theorem. There exists a* heat kernel *$H_t(x, y)$ satisfying*

(1) *$H_t(x, y)$ is a C^∞ real-valued function on $R^+ \times M_n \times M_n$.*

(2) *$H_t(x, y) = H_t(y, x)$.*

(3) *$\int |H_t(x, y)|\, d\mu(y) \leqslant 1$ for all x and $t > 0$, such that*

$$e^{t\Delta}u(x) = \int H_t(x, y)\, u(y)\, d\mu(y) \tag{3.4}$$

for all $u \in L^2$. We also have

(4) *$\|e^{t\Delta}u\|_p \leqslant \|u\|_p$ for all $t > 0$ and all $u \in L^2 \cap L^p$, $1 \leqslant p \leqslant \infty$, with $\|e^{t\Delta}u - u\|_p \to 0$ as $t \to 0$ if $1 \leqslant p < \infty$, and*

(5) *$\partial/\partial t\, e^{t\Delta}u = \Delta e^{t\Delta}$ for all $u \in L^2$, and these properties continue to hold for all $u \in L^p$, $1 \leqslant p \leqslant \infty$, if we define $e^{t\Delta}u$* by (3.4). *Moreover we have uniqueness of the semi-group for $1 < p < \infty$ in the following sense: if P_t is any strongly continuous contractive semigroup on L^p for fixed p, $1 < p < \infty$, such that $P_t u$ is a solution of the heat equation $\partial/\partial t\, P_t u = \Delta u$, then $P_t = e^{t\Delta}$.*

Proof. Fix a value of p, $1 < p < \infty$, and let L be the closure of Δ on $\mathscr{D}$ in the L^p graph norm. Then by Lemma 3.4 L is dissipative. To apply Lemma 3.3 we need to show also that $\lambda - L$ maps onto L^p, for $\lambda > 0$. If it did not there would exist a nonzero element $u \in L^{p'}$ such that $\langle u, (\lambda - L)v\rangle = 0$ for all $v \in \mathscr{D}$. But this means $\Delta u = \lambda u$, which is impossible by Lemma 3.1.

Thus there exists a strongly continuous contractive semigroup P_t on L^p

whose infinitesimal generator is L, which by the Hille–Yosida theorem is equivalent to the existence of the resolvent $(\lambda - L)^{-1}$ for $\lambda > 0$ and the estimate $\|(\lambda - L)^{-1}u\|_p \leqslant \lambda^{-1}\|u\|_p$ for all $u \in L^p$. Now we claim P_t and $e^{t\Delta}$ must be equal on $L^2 \cap L^p$. To prove this it suffices to show that the two resolvents $(\lambda - \Delta)^{-1}$ and $(\lambda - L)^{-1}$ are equal on $L^2 \cap L^p$, for we can recover the semigroup from the resolvent (for instance $e^{t\Delta}u = \lim_{n\to\infty}(I - (t/n)\Delta)^{-n}u$ and $P_t u = \lim_{n\to\infty}(I - (t/n)L)^{-n}u$, the first limit in L^2 and the second in L^p). But if $u \in L^2 \cap L^p$ and $(\lambda - \Delta)^{-1}u = v$ while $(\lambda - L)^{-1}u = w$, then $v \in L^2$ and $w \in L^p$ so $v - w$ is in $L^2 + L^p$ and satisfies $\Delta(v - w) = \lambda(v - w)$. Thus $v = w$ by Lemma 3.1. Thus $P_t = e^{t\Delta}$ on $L^p \cap L^2$.

Since P_t is a contraction semigroup on L^p we obtain part (4) for $u \in L^p \cap L^2$ and $1 < p < \infty$. Then we may let $p \to 1$ and obtain the same estimate for $u \in L^1 \cap L^2$ and $p = 1$. We note in passing that it appears to be necessary to obtain the L^1 theory by this limiting process because the methods used above for $p > 1$ break down when $p = 1$.

Now we are ready to construct the heat kernel. Let ψ_k be an approximation to the delta function at a fixed point x, so that $\psi_n \in \mathscr{D}$, $\int \psi_k = 1$ and ψ_k tends to zero uniformly and in L^1 as $k \to \infty$ in the complement of any neighbourhood of x. Then $e^{t\Delta}\psi_k$ for fixed t is a bounded sequence in L^1, and so, by passage to a subsequence if necessary, converges to a measure $d\nu$ (of norm at most one) in the weak-star topology,

$$\int (e^{t\Delta}\psi_k)u \to \int u\, d\nu$$

for every u which is continuous and vanishes at infinity; in particular if $u \in \mathscr{D}$. But since $e^{t\Delta}$ is symmetric on L^2 we have $\int (e^{t\Delta}\psi_k)u = \int \psi_k e^{t\Delta}u$ and this converges to $e^{t\Delta}u(x)$ because ψ_k approximates δ_x. Thus $e^{t\Delta}u(x) = \int u(y)\, d\nu_{t,x}(y)$ where we have explicitly exhibited the dependence of ν on t and x. But $e^{t\Delta}u(x)$ satisfies the heat equation and so must be a C^∞ function of (x, t). Also from the symmetry of $e^{t\Delta}$ we have $d\nu_{t,x}(y)\, d\mu(x) = d\nu_{t,y}(x)\, d\mu(y)$ so $d\nu$ must be absolutely with respect to $d\mu$, with $d\nu_{t,x}(y) = H_t(x, y)\, d\mu(y)$ exhibiting the symmetric C^∞ heat kernel $H_t(x, y)$. Property (3) follows from the fact that $\|d\nu\| \leqslant 1$, and this together with the symmetry implies (4) even for $p = \infty$. For each fixed y, the heat kernel $H_t(x, y)$ satisfies the heat equation and so we obtain (5).

Finally we come to the uniqueness. Suppose L is the infinitesimal generator of another contraction semigroup P_t on L_p, and let $(\lambda - L)^{-1}$ be its resolvent. We need to show $(\lambda - L)^{-1} = (\lambda - \Delta)^{-1}$. Now $(\lambda - L)^{-1}u = v$ means v is in the domain of L and $(\lambda - L)v = u$. But if v is in the domain of L we have $t^{-1}(P_t v - v) \to Lv$ in L^p, hence $t^{-1}(P_{s+t}v - P_s v) \to P_s Lv$ in L^p for any fixed $s > 0$, because P_s is a bounded operator. But if $P_t u$ satisfies the heat equation then $t^{-1}(P_{s+t}v - P_s v) \to \partial/\partial s\, P_s v = \Delta P_s v$ pointwise. Thus we have $P_s L v = \Delta P_s v$, and letting $s \to 0$ we obtain $Lv = \lim_{s\to 0} \Delta P_s v = \Delta v$ in the distribution sense. Thus v is an L^p function that satisfies $(\lambda - \Delta)v = u$. But if $(\lambda - \Delta)^{-1}u = w$, then $w \in L^p$ and $(\lambda - \Delta)w = u$, so $\Delta(v - w) = \lambda(v - w)$ with $v - w \in L^p$. By Lemma 2.1, $v = w$, so the resolvants are equal hence the semigroups are equal. Actually it is only necessary to assume that the semigroup P_t is bounded rather than contracive, for the above argument would show the equality of the resolvents for sufficiently large λ, then then for all λ because the resolvent is analytic in λ. Again we note that the assumption $1 < p < \infty$ is needed for the proof; we do not know whether uniqueness obtains for $p = 1$ or $p = \infty$. Q.E.D.

THEOREM 3.6. *The heat kernel $H_t(x, y)$ is strictly positive, $H_t(x, y) > 0$ for all x, y and $t > 0$.*

Proof. First we prove $H_t(x, y) \geqslant 0$, which is easily seen to be equivalent to the fact that $e^{t\Delta}$ is positivity preserving on L^2, $u \geqslant 0$ in L^2 implies $e^{t\Delta}u \geqslant 0$. But Simon [28] has shown that this is equivalent to Kato's inequality $\Delta |u| \geqslant \operatorname{sgn} u \, \Delta u$ for $u, \Delta u \in L^1_{\mathrm{loc}}$ in the distribution sense, together with the technical condition that u being in the form-domain $Q(\Delta)$ implies $|u|$ is in $Q(\Delta)$. But $Q(\Delta)$ is exactly the set of $u \in L^2$ such that $\nabla u \in L^2$, so this condition is satisfied since $\nabla |u| = \operatorname{sgn} u \, \nabla u$. Furthermore Kato's inequality is a purely local estimate, and the proof in the case of flat space (see [22, p. 183]) carries over to curved space.

Finally $H_t(x, y)$ can never vanish, because if it did it would attain its minimum, which a solution of the heat equation which is not constant cannot do. Q.E.D.

THEOREM 3.7. *Let $e^{t\Delta}$ denote the heat semigroup on L^2 of tensors. Then $\|e^{t\Delta}u\|_p \leqslant \|u\|_p$ for all $u \in L^p \cap L^2$ for $\frac{3}{2} \leqslant p \leqslant 3$, so $e^{t\Delta}$ extends to a contraction semigroup on those L^p spaces. The heat equation $\partial/\partial t \, e^{t\Delta}u = \Delta e^{t\Delta}u$ holds for all $u \in L^p$, and $e^{t\Delta}$ is the unique semigroup on L^p with these properties if $\frac{3}{2} < p < 3$.*

Proof. As in the proof of Theorem 3.5 we show that the closure of Δ on $\mathscr{D}$ in L^p, call it L, generates a contraction semigroup if $\frac{3}{2} < p < 3$. The restrictions $p < 3$ comes from Lemma 3.4, while the application of Lemma 3.2 to show the range of $\lambda - L$ is L^p requires $p' < 3$ hence $\frac{3}{2} < p$. Again Lemma 3.2 shows that the semigroups P_t and $e^{t\Delta}$ must agree on $L^2 \cap L^p$, which gives the contraction estimate $\|e^{t\Delta}u\|_p \leqslant \|u\|_p$ on the open interval $\frac{3}{2} < p < 3$, and then at the endpoints by a limiting argument. We already know that $e^{t\Delta}u$ satisfies the heat equation if u is in the domain of Δ, and since this domain is dense in L^p it follows that $e^{t\Delta}u$ satisfies the heat equation in the distribution sense, and hence pointwise, since the heat equation is hypoelliptic. Finally the proof of uniqueness is the same as in Theorem 3.5. Q.E.D.

THEOREM 3.8. *Suppose the volume of M_n is infinite. Then for any function $u \in L^p$, $1 < p < \infty$, $e^{t\Delta}u \to 0$ as $t \to \infty$ in L^p norm.*

Proof. For $p = 2$ the result follows from spectral theory provided there are no nonzero L^2 harmonic functions $\Delta u = 0$. But the proof of Lemma 2.3 shows that $\nabla u = 0$ so u is constant, and the hypothesis that M_n has infinite volume shows $u = 0$. For the general case is suffices to prove the result for a dense subset of L^p since $e^{t\Delta}$ are uniformly bounded. If $1 < p < 2$ we consider $u \in L^2 \cap L^1$ and use $\|e^{t\Delta}u\|_p \leqslant \|e^{t\Delta}u\|_2^s \|e^{t\Delta}u\|_1^{1-s}$, where $1/p = s/2 + (1-s)/1$ and the uniform boundedness of $\|e^{t\Delta}u\|_1$, while if $2 < p < \infty$ we use the analogous argument with L^∞ in place of L^1. Q.E.D.

Remark. All the previous results about the heat semi-group, except for the uniqueness, are valid for the Poisson semi-group $e^{-t\sqrt{-\Delta}}$. This follows by the principle of subordination,

$$e^{-t\sqrt{-\Delta}} = \pi^{-1/2} \int_0^\infty e^{(t^2/4s)\Delta}\, e^{-s}\, s^{-1/2}\, ds$$

(see [31]).

Although Theorem 3.5 gives the uniqueness of the heat semigroup $e^{t\Delta}$ in L_p, we can actually do better, showing that individual solutions of the heat equation are uniquely determined by the initial data if they are in $L^p(M_n)$ for each $t > 0$.

THEOREM 3.9. *Let $v(x, t)$ be a function satisfying the heat equation $\partial v/\partial t = \Delta v$ in $R_+ \times M_n$, where $v(\cdot, t) \in L^p$ for each $t > 0$ and $\|v(\cdot, t)\|_p \leqslant ce^{Mt}$ for some c and M and some p, $1 < p < \infty$. Then there exists $f \in L^p$ such that $v = e^{t\Delta}f$. More generally if $1 < p \leqslant q < \infty$ and $v(\cdot, t) \in L^p + L^q$ with $\|v(\cdot, t)\|_{L^p + L^q} \leqslant ce^{Mt}$, then there exists $f \in L^p + L^q$ such that $v = e^{t\Delta}f$.*

Proof. Let f be any weak star limit of $v(\cdot, t_k)$ for $t_k \to 0$. Then if $u = v - e^{t\Delta}f$ we have

$$\|u(\cdot, t)\|_p \leqslant ce^{Mt} \tag{3.5}$$

$$u(\cdot, t_k) \to 0 \quad \text{as} \quad t_k \to 0 \tag{3.6}$$

in the distribution sense, and u satisfies the heat equation. We need to show that these conditions imply $u = 0$. The idea is to form the Laplace transform of u, $w_\lambda(x) = \int_0^\infty e^{-t\lambda} u(t, x)\, dt$. It follows easily from (3.5) that if λ is sufficiently large then the integral defining $w_\lambda(x)$ converges absolutely for almost every x, and $w_\lambda \in L^p$. The next step is to show $\Delta w_\lambda = \lambda w_\lambda$, which is clear on a formal level.

For any test function $\phi \in \mathscr{D}$,

$$\langle \phi, \Delta w_\lambda \rangle = \langle \Delta\phi, w_\lambda \rangle = \int_0^\infty e^{-t\lambda} \langle \Delta\phi, u(t, \cdot) \rangle \, dt$$

with the double integral converging absolutely by (3.5) if λ is large. But by the heat equation $\langle \Delta\phi, u(t, \cdot)\rangle = \partial/\partial t \langle \phi, u(\cdot, t)\rangle$ and so

$$\begin{aligned}
\langle \phi, \Delta w_\lambda \rangle &= \int_0^\infty e^{-t\lambda} \frac{\partial}{\partial t} \langle \phi, u(\cdot, t)\rangle \, dt \\
&= \lim_{\substack{t_k \to 0 \\ N \to \infty}} \int_{t_k}^N e^{-t\lambda} \frac{\partial}{\partial t} \langle \phi, u(\cdot, t)\rangle \, dt \\
&= \lim_{\substack{t_k \to 0 \\ N \to \infty}} \lambda \int_{t_k}^N e^{-t\lambda} \langle \phi, u(\cdot, t)\rangle \, dt \\
&\quad + e^{-N\lambda} \langle \phi, u(\cdot, N)\rangle - e^{-t_k\lambda} \langle \phi, u(\cdot, t_k)\rangle \\
&= \lambda \int_0^\infty e^{-t\lambda} \langle \phi, u(\cdot, t)\rangle \, dt
\end{aligned}$$

because $e^{-N\lambda}\langle \phi, u(\cdot, N)\rangle \to 0$ by (3.5) and $e^{-t_k\lambda}\langle \phi, u(\cdot, t_k)\rangle \to 0$ by (3.6). Thus $\Delta w_\lambda = \lambda w_\lambda$ in the distribution sense.

Now we can apply Lemma 3.1 to conclude $w_\lambda = 0$. By the uniqueness of the Laplace transform we obtain $u(t, x) = 0$ for almost every x, hence $u = 0$. The proof of the $L^p + L^q$ case is similar. Q.E.D.

Remark. The uniqueness fails for $p = \infty$. There exist manifolds with unbounded curvature for which $e^{t\Delta}\, 1 \neq 1$. See [4].

4. Spaces of Bessel Potentials

One way of defining the Sobolev spaces L^p_α on $\mathbb{R}^n$ is as the image of L^p under the action of the Bessel potentials $(I - \Delta)^{-\alpha/2}$. This definition is available for generalization to functions on a complete Riemannian manifold M_n. Whether or not the spaces so obtained agree with other definitions, as is the case on $\mathbb{R}^n$, is an interesting question we will consider later.

The best way to define $(I-\Delta)^{-\alpha/2}$ is via the formal identity

$$(I-\Delta)^{-\alpha/2}=\Gamma\left(\frac{\alpha}{2}\right)^{-1}\int_0^\infty t^{(\alpha/2)-1}\,e^{-t}\,e^{t\Delta}\,dt. \tag{4.1}$$

Certainly for $\alpha>0$ (in fact for complex α with $\operatorname{Re}\alpha>0$) there is no difficulty in making this precise: Let

$$J_\alpha(x,y)=\Gamma\left(\frac{\alpha}{2}\right)^{-1}\int_0^\infty t^{(\alpha/2)-1}\,e^{-t}\,H_t(x,y)\,dt, \tag{4.2}$$

the integral converging absolutely for almost every x and y to a positive function symmetric in x and y such that

$$\int J_\alpha(x,y)\,d\mu(y)\leqslant 1. \tag{4.3}$$

Then

$$(I-\Delta)^{-\alpha/2}u(x)=\int J_\alpha(x,y)\,u(y)\,d\mu(y) \tag{4.4}$$

and we have the estimate $\|(I-\Delta)^{-\alpha/2}u\|_p\leqslant\|u\|_p$ for $\alpha>0$ and every p, $1\leqslant p\leqslant\infty$, as a consequence of (4.3). It is straightforward to verify the semigroup property, $(I-\Delta)^{-\alpha/2}(I-\Delta)^{-\beta/2}=(I-\Delta)^{-(\alpha+\beta)/2}$ and the identity $(I-\Delta)^k(I-\Delta)^{-\alpha/2}=(I-\Delta)^{-(\alpha-k)/2}$ if $\alpha>2k$, and this can be used to extend the definition to all α (or complex α) as a group of unbounded operators. It is also easy to verify that this definition agrees with that given by the spectral theorem for L^2 and when $\alpha=2$ with the resolvent for all L^p. We will primarily be interested in the case $1<p<\infty$, and we now make that restriction.

Definition 4.1. $L^p_\alpha(M_n)$ for $\alpha>0$ is the set of $u\in L^p$ such that $u=(I-\Delta)^{-\alpha/2}v$ for some $v\in L^p$, with the norm $\|u\|_{L^p_\alpha}=\|v\|_p$. For $\alpha<0$ we define L^p_α to be the set of distributions u of the form $u=(I-\Delta)^k v$, where $v\in L^p_{2k+\alpha}$, k being any positive integer such that $2k+\alpha>0$, and $\|u\|_{L^p_\alpha}=\|v\|_{L^p_{2k+\alpha}}$. We define $L^p_\infty=\bigcap L^p_\alpha$ and $L^p_{-\infty}=\bigcup L^p_\alpha$.

It is easy so see that L^p_α so defined are Banach spaces, and for $\alpha<0$ the definition is independent of k.

Theorem 4.2. *If $\alpha>\beta$, then L^p_α is contained in L^p_β, $\|u\|_{L^p_\beta}\leqslant\|u\|_{L^p_\alpha}$ and L^p_α is dense in L^p_β.*

Proof. The only nontrivial part is the density, for which we use semigroup theory. Let ϕ_j be a smooth approximate identity on the line supported in $t>0$, and set $u_j=\int\phi_j(t)\,e^{t\Delta}u\,dt$ for $u\in L^p_\beta$ (if $\beta<0$ we interpret this as $(I-\Delta)^k\int\phi_j(t)\,e^{t\Delta}v\,dt$, where $(I-\Delta)^k v=u$ and $2k+\beta>0$). Then u_j is in the domain of Δ^k for any k, and $\Delta^k u_j=\int((-\partial/\partial t)^k\phi_j(t))\,e^{t\Delta}u\,dt$, so $u_j\in L^p_{2k}$ hence $u_j\in L^p_\infty$. Now for $u\in L^p$ we have

$$\|u_j-u\|_p\leqslant\int\phi_j(t)\,\|e^{t\Delta}u-u\|_p\,dt\to 0$$

as $j\to\infty$ since $e^{t\Delta}$ is strongly continuous on L^p. Similarly we have $u_j\to u$ in L^p_β if $u\in L^p_\beta$. Q.E.D.

THEOREM 4.3. *$\mathscr{D}$ is dense in L^p_α for all real α and all p, $1<p<\infty$.*

Proof. Because of Theorem 4.2 it suffices to prove $\mathscr{D}$ is dense in L^p_{2k} for all nonnegative integers k. Suppose to the contrary. Then there exists a nonzero $v\in L^{p'}$ such that $\int u(I-\Delta)^k v=0$ for all $u\in\mathscr{D}$, hence $(I-\Delta)^k v=0$. We have already shown this is impossible for $k=1$. We prove the general case by induction. Thus assume $(I-\Delta)^k v=0$ for $v\in L^{p'}$ implies $v=0$, and consider the equation $(I-\Delta)^{k+1}v=0$ for $v\in L^{p'}$. The idea of the proof is that we would like to apply $(I-\Delta)^{-1}$ to this equation; however, we cannot immediately assert $(I-\Delta)^{-1}(I-\Delta)^{k+1}v=(I-\Delta)^k v$ so we must proceed more cautiously.

Let $\phi(t)$ be a smooth compactly supported function on the halfline $t>0$, and consider $w=\int\phi(t)\,e^{-t}\,e^{t\Delta}v\,dt$. We claim $(I-\Delta)^{k+1}w=0$. Indeed if $u\in\mathscr{D}$, then $\langle u,(I-\Delta)^{k+1}w\rangle=\langle(I-\Delta)^{k+1}u,w\rangle=\int\phi(t)\,e^{-t}\langle(I-\Delta)^{k+1}u,$ $e^{t\Delta}v\rangle\,dt$ since the double integral is absolutely convergent. But then $\langle(I-\Delta)^{k+1}u,e^{t\Delta}v\rangle=\langle e^{t\Delta}(I-\Delta)^{k+1}u,v\rangle=\langle(I-\Delta)^{k+1}\,e^{t\Delta}u,v\rangle$ because $e^{t\Delta}$ and Δ commute on the domain of Δ, and this vanishes because $(I-\Delta)^{k+1}v=0$.

But $\Delta w=-\int\partial/\partial t(\phi(t)\,e^{-t})\,e^{t\Delta}v\,dt=\int\phi(t)\,e^{-t}\,e^{t\Delta}v\,dt-\int\phi'(t)\,e^{-t}\,e^{t\Delta}v\,dt$ so $0=(I-\Delta)^{k+1}w=(I-\Delta)^k\int\phi'(t)\,e^{-t}\,e^{t\Delta}v\,dt$. Thus by the induction hypothesis $\int\phi'(t)\,e^{-t}e^{t\Delta}v\,dt=0$. By the appropriate choice of ϕ and a limiting process we obtain $e^{-a}\,e^{a\Delta}v-e^{-b}\,e^{b\Delta}v=0$ for all $a,b>0$. But $\|e^{-b}\,e^{b\Delta}v\|_{p'}\leqslant e^{-b}\,\|v\|_{p'}\to 0$ as $b\to\infty$ so $e^{-a}\,e^{a\Delta}v=0$, and by letting $a\to 0$ we obtain $v=0$ as desired. Q.E.D.

COROLLARY 4.3. *The pairing $\langle u,v\rangle$ for $u\in\mathscr{D}$ and $v\in\mathscr{D}$ extends to a bounded bilinear form for $u\in L^p_\alpha$, $v\in L^{p'}_{-\alpha}$ for $\alpha>0$ thus establishing isometric isomorphisms between L^p_α and the dual of $L^{p'}_{-\alpha}$, and between $L^{p'}_{-\alpha}$ and the dual of L^p_α.*

We omit the proof. Next we consider the relationship between the Bessel potentials and the Riesz potentials, which are the powers of $-\Delta$, hence in general are unbounded operators. Again we have in mind the formal identity

$$(-\Delta)^{-\alpha/2}=\Gamma\left(\frac{\alpha}{2}\right)^{-1}\int_0^\infty t^{(\alpha/2)-1}\,e^{t\Delta}\,dt,$$

but now we have problems with convergence as $t\to\infty$ when $\alpha\geqslant 0$ and as $t\to 0$ when $\alpha\leqslant 0$. Let us say that $u\in L^p$ belongs to the L^p domain of $(-\Delta)^{-\alpha/2}$ if $\Gamma(\alpha/2)^{-1}\int_\varepsilon^N t^{(\alpha/2)-1}\,e^{t\Delta}u\,dt$ converges as $\varepsilon\to 0$ and $N\to\infty$ in L^p to some element $v\in L^p$, and we write $v=(-\Delta)^{-\alpha/2}u$. It is not difficult to show that $(-\Delta)^{-\alpha/2}$ is a closed operator, and the definition agrees with that given by spectral theory when $p=2$.

THEOREM 4.4. *Let $\alpha>0$, $1<p<\infty$. Then $u\in L^p_\alpha$ if and only if u is in the L^p domain of $(-\Delta)^{\alpha/2}$, and $\|u\|_p+\|(-\Delta)^{\alpha/2}u\|_p$ is equivalent to the L^p_α norm.*

Proof. To show that $u\in L^p_\alpha$ implies u is in the L^p domain of $(-\Delta)^{\alpha/2}$ we use a method due to Stein, which is based on the identity $(t/(1+t))^{\alpha/2}=1+\sum_{k=1}^\infty c_k(1+t)^{-k}$ for $t>0$, where c_k are certain constants with $\sum_1^\infty |c_k|<\infty$ (see [30, p. 133] for a proof). A formal substitution of $-\Delta$ for t would show $(-\Delta)^{\alpha/2}(I-\Delta)^{-\alpha/2}$ is a bounded operator on L^p, since we know $(I-\Delta)^{-k}$ is an L^p contraction. In fact, we obtain the identity $(-\Delta)^{\alpha/2}(I-\Delta)^{-\alpha/2}v=v+\sum c_k(I-\Delta)^{-k}v$ for $v\in L^2$ by spectral theory, and then for $p\neq 2$ since all the operators involved are closed. If $u\in L^p_\alpha$, then $u=(I-\Delta)^{-\alpha/2}v$ for some $v\in L^p$, and so $(-\Delta)^{\alpha/2}v\in L^p$ with $\|(-\Delta)^{\alpha/2}v\|_p\leqslant c\,\|v\|_{L^p_\alpha}$.

For the converse we consider first the case when α is an even integer, $\alpha=2k$. Since $(I-\Delta)^k$ is a sum of lower powers of the Laplacian, we need to establish the estimate $\|\Delta^j u\|_p\leqslant c(\|\Delta^k u\|_p+\|u\|_p)$ for $j<k$ and all $u\in L^p$ with $\Delta^k u\in L^p$. Clearly it suffices by induction to prove this for $j=k-1$, for which we use the identities

$$\int \phi'(t)\,e^{t\Delta}\,\Delta^{k-1}u\,dt=-\int \phi(t)\,e^{t\Delta}\,\Delta^k u\,dt,$$

$$\int \psi(t)\,e^{t\Delta}\,\Delta^{k-1}u\,dt=(-1)^{k-1}\int \psi^{(k-1)}(t)\,e^{t\Delta}\,u\,dt,$$

for ϕ and ψ smooth compactly supported functions on $t>0$. Since we have the estimates

$$\left\|\int \phi(t)\,e^{t\Delta}\,\Delta^k u\,dt\right\|_p\leqslant\|\phi\|_1\,\|\Delta^k u\|_p,$$

$$\left\|\int \psi^{(k-1)}(t)\,e^{t\Delta}\,u\,dt\right\|_p\leqslant\|\psi^{(k-1)}\|_1\,\|u\|_p,$$

it suffices to find ϕ and ψ such that $\phi' + \psi$ approximates t^{-1} so that $\int (\phi'(t) + \psi(t))\, e^{t\Delta} \Delta^{k-1} u\, dt$ approximates $\Delta^{k-1}u$ and $\|\phi\|_1 + \|\psi^{(k-1)}\|_1$ remains bounded. But for this we need only take ϕ to approximate $\log t$ near $t = 0$ and $\psi(t)$ to approximate t^{-1} near infinity.

The above argument shows that there exist L^p bounded operators A and B that commute with all functions of Δ such that $(I - \Delta)^k u = Au + B(-\Delta)^k u$. We can use this to establish the analogous statement for $0 < \alpha < 2k$ as follows: We apply the bounded operator $(I - \Delta)^{(\alpha/2)-k}$ to both sides of the identity to obtain $(I - \Delta)^{\alpha/2} u = A'u + B'(-\Delta)^{\alpha/2} u$, where $A' = (I - \Delta)^{(\alpha/2)-k} A$ and $B' = (I - \Delta)^{(\alpha/2)-k} B(-\Delta)^{k-(\alpha/2)} = B(I - \Delta)^{(\alpha/2)-k}(-\Delta)^{k-(\alpha/2)}$. Clearly A is bounded, being the composition of bounded operators, and the same is true of B', the boundedness of $(I - \Delta)^{(\alpha/2)-k}(-\Delta)^{k-\alpha/2}$ having been established in the first part of the proof.

Q.E.D.

Remark. When appropriately formulated, this result remains true for $p = 1$. Also it can be generalized by replacing Δ by the infinitesmal generator of an L^p contraction semigroup.

We turn next to an application of the general Littlewood–Paley theory of Stein [31], as improved by Cowling [10, 11]. The heat semigroup satisfies all the axioms of [11], but not necessarily the axiom $e^{t\Delta}\ 1 = 1$ required in [31].

THEOREM 4.5. *If $\psi(t)$ is any L^∞ function on $t > 0$, the operator*

$$m(-\Delta) = -\Delta \int_0^\infty \psi(t)\, e^{t\Delta}\, dt \tag{4.5}$$

is bounded on L^p for all p, $1 < p < \infty$, with norm depending only on p and linearly on $\|\psi\|_\infty$. Here $m(\lambda) = \lambda \int_0^\infty \psi(t)\, e^{-t\lambda}\, dt$ is a bounded function so $m(-\Delta)$ is definable by spectral theory on L^2 and then by continuity, expression (4.5) *then being defined by a limiting process. In particular $(-\Delta)^{is}$ for real s is bounded on L^p for $1 < p < \infty$ with the norm growing at most exponentially in s, for each fixed p, and the same is true for $(I - \Delta)^{is}$.*

Proof. This is Theorem 3bis of [11], the special cases $(-\Delta)^{is}$ corresponding to the choice $\psi(t) = \Gamma(1 - is)^{-1} t^{-is}$. The choice $\psi(t) = \Gamma(1 - is)^{-1} e^{-t} t^{-is}$ corresponds to the operator $(I - \Delta)^{is} - (I - \Delta)^{is-1}$ and since we have already established the boundedness of $(I - \Delta)^{is-1}$ we obtain the boundedness of $(1 - \Delta)^{is}$.

Q.E.D.

Remark. Related results, under curvature hypotheses, are proved in [6].

COROLLARY 4.6. *For $1 < p_0, p_1 < \infty$ and any real α_0, α_1, the complex intermediate space $[L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1}]_t$ for $0 < t < 1$ can be identified with L^p_α, where $1/p = ((1-t)/p_0) + (t/p_1)$ and $\alpha = (1-t)\alpha_0 + t\alpha_1$.*

Proof. This is a routine consequence of the L^p boundedness of $(I-\Delta)^{is}$. Q.E.D.

Next we show that the Bessel potential operators $(I-\Delta)^{-\alpha/2}$ are pseudo-differential operators of order $-\operatorname{Re}\alpha$. This result is well known for compact manifolds (for example, see Seeley [26]), and is essentially known more generally, although it appears not to have been written down explicitly. The proof we give amounts to little more than putting together some results of Hörmander [17] with the results for the compact case. The reader will note that we use very little specific information about the Laplacian.

THEOREM 4.7. *Let M_n be a C^∞ Riemannian manifold (not necessarily complete), and let Δ denote any nonpositive self-adjoint extension of the Laplacee–Beltrami operator on $\mathscr{D}(M_n)$. Then the operator $(I-\Delta)^{-\alpha/2}$ defined by the spectral theorem is a pseudo-differential operator of order $-\operatorname{Re}(\alpha)$ with principal symbol $(-\sigma_2(\Delta))^{-\sigma/2}$, where $\sigma_2(\Delta)$ is a principal symbol of Δ. The same is true of the Riesz potential operators $(-\Delta)^{-\alpha/2}$.*

Proof. Let M_ϕ denote the operation of multiplication by ϕ. Acoording to the definition of pseudo-differential operator, we have to verify that $M_\psi(I-\Delta)^{-\alpha/2}M_\phi$ is a pseudo-differential operator with symbol $\psi(x)(-\theta_2(\Delta))^{-\alpha/2}\phi(x)$ for ϕ and ψ in $\mathscr{D}(M_n)$ with support in a compact coordinate patch $\Omega \subseteq M_n$. Now the idea is to consider a compact Riemannian manifold (without boundary) $\tilde{M}_n$ which contains a coordinate patch isometric to Ω (for notational simplicity we will denote this patch also Ω). The construction of $\tilde{M}_n$ is routine (see [27]). We then compare $M_\psi(I-\Delta)^{-\alpha/2}M_\phi$ with $M_\psi(I-\tilde{\Delta})^{-\alpha/2}M_\phi$, where $\tilde{\Delta}$ is the Laplace–Beltrami operator on $\tilde{M}_n$. Since we know from the compact case [26] that $M_\psi(I-\tilde{\Delta})^{-\alpha/2}M_\phi$ is a pseudo-differential operator of order $-\operatorname{Re}(\alpha)$ with principal symbol $\psi(x)(-\sigma_2(\Delta))^{-\alpha/2}\phi(x)$, and of course $\sigma_2(\Delta) = \sigma_2(\tilde{\Delta})$ on Ω, it suffices to show that the difference

$$M_\psi(I-\tilde{\Delta})^{-\alpha/2}M_\phi - M_\psi(I-\Delta)^{-\alpha/2}M_\phi$$

is an integral operator with C^∞ kernel.

Now we express the operators $(I-\Delta)^{-\alpha/2}$ for $\operatorname{Re}\alpha > 0$ in terms of the resolvents $((1-\lambda)I-\Delta)^{-1}$,

$$(I-\Delta)^{-\alpha/2} = \frac{i}{2\pi}\int_\Gamma \lambda^{-\alpha/2}((1-\lambda)I-\Delta)^{-1}\,d\lambda, \tag{4.6}$$

where Γ is a suitable contour in the complex plane (for example, from $-\infty$ to -1, then once around the unit circle clockwise, then back to $-\infty$, where $\lambda^{-\alpha/2}$ is split along the negative real axis). The validity of (4.6) is easily established by spectral theory (see [26]) and of course the same identity holds for $\tilde{\Delta}$. Now Hörmander [17, Proposition 4.8] gives an estimate for the difference of the resolvents $((1-\lambda)I-\Delta)^{-1}-((1-\lambda)I-\tilde{\Delta})^{-1}$. This difference is an integral operator whose kernel is bounded by a constant times $e^{-c|\lambda|^{1/m}}$, uniformly on any compact subset of $\Omega \times \Omega$ for $|\arg \lambda| > \varepsilon$. Combining this with (4.6) and obvious estimates, we conclude that $M_\psi(I-\Delta)^{-\alpha/2}M_\phi - M(I-\tilde{\Delta})^{-\alpha/2}M_\phi$ is an integral operator whose kernel $K_\alpha(x, y)$ is bounded.

Hörmander in [17] does not give estimates for the derivatives of the kernel of the differences of the resolvents. However, we can obtain these estimates rather easily. For example, since $\Delta = \tilde{\Delta}$ on Ω, we have

$$\begin{aligned} &M_\psi \Delta[((1-\lambda)I-\Delta)^{-1}-((1-\lambda)I-\tilde{\Delta})^{-1}] \\ &\quad = M_\psi[\Delta((1-\lambda)I-\Delta)^{-1}-\tilde{\Delta}((1-\lambda)I-\tilde{\Delta})^{-1}] \\ &= (1-\lambda)M_\psi[((1-\lambda)I-\Delta)^{-1}-((1-\lambda)I-\tilde{\Delta})^{-1}] \end{aligned}$$

and so the bound for the kernel of this integral operator is $|1-\lambda|e^{-c|\lambda|^{1/m}}$. The additional factor $|1-\lambda|$ is harmless, so we can continue the argument as before to show $\Delta_x K_\alpha(x, y)$ is bounded. Similarly we can show $\Delta_x^j \Delta_y^m K_\alpha(x, y)$ is bounded for every integer j and m, hence $K_\alpha(x, y)$ is C^∞. This completes the proof for $\alpha > 0$ and the result follows in general since $(I-\Delta)^{-\alpha/2} = (I-\Delta)^k(I-\Delta)^{-(\alpha+2k)/2}$.

Finally to deal with the Riesz potentials instead of the Bessel potentials, we compare $(-\Delta)^{-\alpha/2}$ with $(M_p-\tilde{\Delta})^{-\alpha/2}$, where p is a nonnegative C^∞ function on $\tilde{M}_n$, vanishing on Ω but not identically zero. Then $M_p-\tilde{\Delta}$ is a strictly positive operator on $L^2(\tilde{M}_n)$ and we may apply Hörmander's result as before. Q.E.D.

Remark. It seems very likely that this theorem is also true for the Laplacian on forms and tensors. The compact case is known for elliptic systems, but Hörmander's results are stated only for scalar operators.

5. McKean's Inequality

Now we make two additional assumptions on the manifold M_n: it is simply connected and all sectional curvatures are bounded above by a negative constant $-k$. Under these assumptions, McKean [19] has established the remarkable inequality.

$$\|f\|_2 \leqslant \frac{2}{(n-1)\sqrt{k}} \|\nabla f\|_2 \tag{5.1}$$

for all compactly supported functions f for which $\nabla f \in L^2$. There is a superficial resemblance between McKean's inequality and the well-known Poincaré inequality in Euclidean space, but the crucial difference is that the constant in (5.1) does not depend on the support of f. From this inequality, it follows immediately that the spectrum of the Laplacian is bounded above by $-(n-1)^2k/4$. Then by spectral theory it follows that all the Riesz transforms $(-\Delta)^z$ for $\operatorname{Re} z < 0$ are bounded operators on L^2.

In this section we will extend McKean's inequality to L^p, $1 \leqslant p < \infty$, and obtain the L^p boundedness of the Riesz potentials for $1 < p < \infty$. In the process will obtain a slight simplification in the proof of (5.1). We will also consider what can be said if we drop the assumption that f have compact support. This is closely related to the existence of nontrivial L^2 cohomology. While our results are not complete, they suggest that for $n \geqslant 3$ the condition $\nabla f \in L^p$ should imply $f - c \in L^p$ for some constant c, whereas for $n = 2$ this is not the case.

We now introduce the basic notation we will use throughout this section. We fix a point on M_n, and use the exponential map at this point to transfer the polar coordinates on $\mathbb{R}^n$ to the manifold. Thus we have a global polar geodesic coordinate system (r, u), where $r > 0$ and $u \in S^{n-1}$, in which the metric g_{jk} has the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right)$. We let $g = g(r, u) = \det g_{jk}$ in this particular coordinate system. The crucial estimate is given in Lemma 5.1. For the proof see McKean [19].

LEMMA 5.1. *For all* (r, u),

$$\frac{\partial g}{\partial r} \geqslant 2(n-1)\sqrt{k}\, g. \tag{5.2}$$

LEMMA 5.2. *Let f be a smooth function with compact support, vanishing near $r = 0$. Then*

$$\iint |f| \sqrt{g}\, dr\, du \leqslant \frac{1}{(n-1)\sqrt{k}} \iint \left|\frac{\partial f}{\partial r}\right| \sqrt{g}\, dr\, du. \tag{5.3}$$

Proof. The function $|f|$ is smooth away from the zeroes of f, and is everywhere Lipschitz continuous, and $|\partial |f|/\partial r| \leqslant |\partial f/\partial r|$ at points where $f \neq 0$. We can thus apply the integration by parts formula

$$\int_0^\infty \frac{\partial |f|}{\partial r} \sqrt{g}\, dr = -\frac{1}{2} \int_0^\infty |f| \frac{\partial g/\partial r}{\sqrt{g}}\, dr$$

with no boundary terms because of the support assumptions on f. Now using (5.2) we obtain

$$(n-1)\sqrt{k} \int_0^\infty |f| \sqrt{g}\, dr \leqslant \frac{1}{2} \int_0^\infty |f| \frac{\partial g/\partial r}{\sqrt{g}}\, dr$$
$$= \int_0^\infty \frac{\partial |f|}{\partial r} \sqrt{g}\, dr \leqslant \int_0^\infty \left|\frac{\partial f}{\partial r}\right| \sqrt{g}\, dr$$

from which (5.3) follows upon integration with respect to u. Q.E.D.

LEMMA 5.3. *Under the same hypotheses,*

$$\left(\iint |f|^p \sqrt{g}\, dr\, du\right)^{1/p} \leqslant \frac{p}{(n-1)\sqrt{k}} \left(\iint \left|\frac{\partial f}{\partial r}\right|^p \sqrt{g}\, dr\, du\right)^{1/p} \tag{5.4}$$

for any p, $1 \leqslant p < \infty$.

Proof. Applying the same argument to $|f|^p$ in place of $|f|$ we obtain

$$\iint |f|^p \sqrt{g}\, dr\, du \leqslant \frac{p}{(n-1)\sqrt{k}} \iint |f|^{p-1} \left|\frac{\partial f}{\partial r}\right| \sqrt{g}\, dr\, du.$$

We apply Hölder's inequality to the right side with exponent p for $|\partial f/\partial r|$ and p' for $|f|^{p-1}$. Since $p'(p-1) = p$ we obtain

$$\iint |f|^p \sqrt{g}\, dr\, du$$
$$\leqslant \frac{p}{(n-1)\sqrt{k}} \left(\iint |f|^p \sqrt{g}\, dr\, du\right)^{1/p'} \left(\iint \left|\frac{\partial f}{\partial r}\right|^p \sqrt{g}\, dr\, du\right)^{1/p}$$

and we may divide to obtain (5.4) since $\iint |f|^p \sqrt{g}\, dr\, du$ is a priori finite.

Q.E.D.

THEOREM 5.4. *Let M_n be a simply connected n-dimensional complete Riemannian manifold with all sectional curvatures bounded above by a* *negative constant* $-k$. *If* f *is any compactly supported function with* $\nabla f \in L^p$ *for some* p, $1 \leqslant p < \infty$, *then* $f \in L^p$ *and*

$$\|f\|_p \leqslant \frac{p}{(n-1)\sqrt{k}} \|\nabla f\|_p. \tag{5.5}$$

Proof. By choosing the origin of the coordinate system away from the support of f, we can always arrange that f vanishes near $r = 0$. Then if f is smooth, (5.5) follows from (5.4) and the fact that $|\partial f/\partial r| \leqslant |\nabla f|$ pointwise. The assumption that f is smooth can then be removed by a routine regularization argument. Q.E.D.

Remarks. The theorem remains true for tensors as well as functions. All that is required for this generalization is the pointwise estimate $|\nabla |f|| \leqslant |\nabla f|$ for tensors, which was established in the proof of Lemma 3.2. Thus we may iterate (5.5) to obtain

$$\|f\|_p \leqslant \left(\frac{p}{(n-1)\sqrt{k}}\right)^m \|\nabla^m f\|_p$$

for smooth, compactly supported functions or tensors.

THEOREM 5.5. *Under the same hypotheses on* M_n, *the Riesz potentials* $(-\Delta)^z$ *are bounded operators on* L^p *for* $\operatorname{Re} z < 0$ *and* $1 < p < \infty$.

Proof. Fix z with $\operatorname{Re} z < 0$ and consider the analytic family of operators $(wI - \Delta)^z$ for $\operatorname{Re} w > -(n-1)^2 \sqrt{k}/4$. This is well defined by spectral theory since the spectrum of Δ lies in the interval $(-\infty, -(n-1)^2 \sqrt{k}/4)$, and the operators are L^2 bounded. On the other hand, if $\operatorname{Re} w > 0$, these operators are bounded on L^1 and L^∞ (this was shown in Section 4 for $w = 1$, but the same proof works for $\operatorname{Re} w > 0$). The growth in norm of these operators as $\operatorname{Im} w \to \infty$ is at most exponential, so we can apply the analytic families interpolation theorem of Stein to obtain the L^p boundedness for $w = 0$. Q.E.D.

Remarks. The proof actually shows that $(wI - \Delta)^z$ is bounded on L^p for

$$0 > w > -\frac{(n-1)^2}{4} k \qquad \text{and} \qquad \left|\frac{1}{p} - \frac{1}{2}\right| \leqslant \frac{1}{2} + \frac{2w}{(n-1)^2 k}.$$

In the special case of constant curvature, the theorem is proved in Stanton and Tomas [29]. Essentially the same result is proved by Lohoué and Rychener [18a].

Next we investigate what happens if we drop the assumption that f have compact support. We show first that f must differ from an L^p function by a function of the angular variables alone. This requires no additional assumptions on n, p or the manifold.

LEMMA 5.6. *If f is a function such that $\partial f/\partial r \in L^p$ for some p, $1 \leqslant p < \infty$, then there exists a function $c_1(u)$ of the angular variables alone, such that $f(r,u) - c_1 \in L^p$ and*

$$\|f(r,u) - c_1(u)\|_p \leqslant \frac{p}{(n-1)\sqrt{k}} \left\| \frac{\partial f}{\partial r} \right\|_p.$$

Proof. Let $\tilde{f}(r,u) = -\int_r^\infty (\partial f/\partial r)(s,u)\,ds$. It follows easily from Fubini's theorem that $\tilde{f}$ is defined for almost every u, and since $\partial\tilde{f}/\partial r = \partial f/\partial r$ we have $\tilde{f}(r,u) = f(r,u) - c_1(u)$. Thus we need to show that $\tilde{f} \in L^p$.

Assume first $p > 1$. Now from (5.2) we have $g(r,u) \to \infty$ as $r \to \infty$ hence for any $\alpha > 0$, $g^{-\alpha}(r,u) = -\int_r^\infty (\partial/\partial s)(g^{-\alpha}(s,u))\,ds$. However, also by (5.2) we have

$$\frac{\partial}{\partial s}(g^{-\alpha}(s,u)) \leqslant 2\alpha(n-1)\sqrt{k}\; g^{-\alpha}(s,u),$$

so that

$$\int_r^\infty g^{-\alpha}(s,u)\,ds \leqslant \frac{1}{2\alpha(n-1)\sqrt{k}}\, g^{-\alpha}(r,u). \tag{5.6}$$

This estimate enables us to show that $\int |\tilde{f}(r,u)|^p\, g^{1/2}(r,u)\,du$ is uniformly bounded in r. In fact we have from the definition of $\tilde{f}$ and Hölder's inequality the estimate

$$|\tilde{f}(r,u)|^p \leqslant \left(\int_r^\infty g^{-p'/2p}(s,u)\,ds\right)^{p/p'} \times \int_r^\infty \left|\frac{\partial f}{\partial r}(s,u)\right| g^{1/2}(s,u)\,ds.$$

We use (5.6) with $\alpha = p'/2p$ and integrate to obtain

$$\int |\tilde{f}(r,u)|^p\, g^{1/2}(r,u)\,du \leqslant \frac{p}{p'(n-1)\sqrt{k}} \iint_r^\infty \left|\frac{\partial f}{\partial r}(s,u)\right|^p g^{1/2}(s,u)\,ds\,du. \tag{5.7}$$

Now, repeating the integration by parts argument in the proof of Lemma 5.3, we have

$$\iint_a^b |\tilde{f}(r,u)|^p \, g^{1/2}(r,u)\, dr\, du$$

$$\leqslant \frac{p}{(n-1)\sqrt{k}} \left(\iint_a^b |\tilde{f}(r,u)|^{p-1} \left| \frac{\partial f}{\partial r}(r,u) \right| g^{1/2}(r,u)\, dr\, du \right.$$

$$\left. + \int |\tilde{f}(b,u)|^p \, g^{1/2}(b,u)\, du + \int |\tilde{f}(a,u)|^p \, g^{1/2}(a,u)\, du \right)$$

for any interval (a, b). If we use Hölder's inequality in the double integral and (5.7) to estimate the single integrals this becomes $A(a,b) \leqslant c_1 A(a,b)^{1/p'} + c_2$, where we have introduced the abbreviation

$$A(a,b) = \iint_a^b |\tilde{f}(r,u)|^p \, g^{1/2}(r,u)\, dr\, du$$

and c_1 and c_2 are independent of a and b. It follows from this that $A(a,b)$ is uniformly bounded, hence $\tilde{f} \in L^p$.

The proof for $p = 1$ is even simpler. We have

$$\iint_0^\infty |\tilde{f}(r,u)| \, g^{1/2}(r,u)\, dr\, du$$

$$\leqslant \iint_0^\infty \int_0^s \left| \frac{\partial f}{\partial r}(s,u) \right| g^{1/2}(r,u)\, dr\, ds\, du$$

so it suffices to show

$$\int_0^s g^{1/2}(r,u)\, dr \leqslant c g^{1/2}(s,u).$$

However, this follows from (5.2) (as in the derivation of (5.6)) and the additional fact that $\lim_{r \to 0} g(r,u) = 0$ (this follows from the basic facts about geodesic coordinates).

Finally, once we have that $\tilde{f} \in L^p$, we can derive the desired estimate from (5.4) and a limiting argument. In fact if ϕ is any of the functions $\phi_{r,s}$ of Lemma 2.2, then $\phi\tilde{f}$ is compactly supported so

$$\|\phi\tilde{f}\|_p \leqslant \frac{p}{(n-1)\sqrt{k}} \left\| \frac{\partial}{\partial r}(\phi\tilde{f}) \right\|_p$$

by (5.4). But $\|\partial/\partial r(\phi\tilde{f})\|_p \leqslant \|\partial/\partial r\, \tilde{f}\|_p + \|\nabla\phi\|_\infty \|\tilde{f}\|_p$ and since we can make

$\|\nabla\phi\|_\infty \to 0$ by letting s go to ∞, we obtain $\|\tilde{f}\|_p \leqslant (p/(n-1)\sqrt{k})\,\|\partial f/\partial r\|_p$ by letting $r \to \infty$. Q.E.D.

To go further we need to make additional assumptions on the manifold. We will assume that the manifold is rotationally symmetric about the origin of coordinates. This means the submanifolds $r = \text{constant}$ are spheres of constant curvature, and the metric has the special form

$$g_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda(r)^2 h_{jk}(u) \end{pmatrix}, \tag{5.8}$$

where $h_{jk}(u)$ is the usual metric on the unit sphere S^{n-1}, $2 \leqslant j,\ k \leqslant n$.

LEMMA 5.7. *Suppose the metric has the form* (5.8) *and* $p \leqslant n-1$. *If* f *is a fuction such that* $\nabla f \in L^p$, *then there exists a function* $c_2(r)$ *of the radial variable alone such that*

$$\int |f(r,u) - c_2(r)|^p \lambda^{n-1-p}(r)\sqrt{h(u)}\,dr\,du \leqslant c\,\|\nabla f\|_p^p. \tag{5.9}$$

Proof. Because of the form of the metric,

$$|\nabla_u f(r,u)| \leqslant \lambda(r)\,|\nabla f(r,u)|,$$

where ∇_u denotes the gradient operator on the unit sphere. But the usual L^p Poincaré inequality on the sphere gives

$$\int |f(r,u) - c_2(r)|^p \sqrt{h(u)}\,du \leqslant c_p \int |\nabla_u f(r,u)|^p \sqrt{h(u)}\,du$$

for some constant $c_2(r)$, hence

$$\begin{aligned} &\iint |f(r,u) - c_2(r)|^p\,\lambda^{n-1-p}(r)\sqrt{h(u)}\,du\,dr \\ &\quad\leqslant c \iint |\nabla g(r,u)|^p\,\lambda^{n-1}(r)\sqrt{h(u)}\,du\,dr \\ &\quad= c\,\|\nabla f\|_p^p. \end{aligned}$$

Q.E.D.

THEOREM 5.8. *Let* M_n *satisfy the hypotheses of Theorem* 5.4, *and in addition assume the metric has the form* (5.8). *Let* $1 \leqslant p \leqslant n-1$. *If* f *is a function such that* $\nabla f \in L^p$, *then there exists a constant* c *such that* $f - c \in L^p$, *and*

$$\|f - c\|_p \leqslant \frac{p}{(n-1)\sqrt{k}}\,\|\nabla f\|_p. \tag{5.10}$$

Proof. It suffices to show that the function $c_1(u)$ in Lemma 5.6 is a costant. Now from (5.2) we have that $\lambda(r) \geqslant 1$ for r large enough, say $r \geqslant R$. Then combining Lemmas 5.6 and 5.7 we have

$$\iint_R^{\infty} |c_1(u) - c_2(r)|^p \sqrt{h}\, dr\, du < \infty$$

(here we have used $p \leqslant n-1$). Since the measure $\sqrt{h}\, dr\, du$ is infinite, this can only be true if $c_1(u) = c_2(r)$; in other words if both are constant. Q.E.D.

Remarks. Condition (5.8) on the form of the metric seems artificial, and it seems likely that the theorem is true without it. In the special case $p=2$ we can relax it considerably. If we denote by S_r the sphere of radius r about the origin of coordinates, then S_r is topologically a sphere and inherits a metric from M_n. The Poincaré inequality on S_r is then

$$\int |f(r,u) - c_2(r)|^2 \sqrt{g(r,u)}\, du$$

$$\leqslant \lambda_1(r)^{-1} \int |\nabla f(r,u)|^2 \sqrt{g(r,u)}\, du,$$

where $\lambda_1(r)$ is the first nonzero eigenvalue of the Laplace–Beltrami operator of S_r. If we merely assume $\operatorname{diam}(S_r) \leqslant c\sqrt{\operatorname{vol}(S_r)}$, then by using the lower bound for λ_1 in [18] we can carry out the same proof as before.

The condition $p \leqslant n-1$ is necessary, however, since it is easy to construct counterexamples when M_n is hyperbolic (constant negative curvature). We take a global coordinate system $x_1, \ldots, x_n$ for M_n with the metric $g_{jk} = \delta_{jk} - (x_j x_k / x_0^2)$, so that $g^{jk} = \delta_{jk} + x_j x_k$ and $g = x_0^{-2}$, where we have used the abbreviation $x_0 = \sqrt{1 + x_1^2 + \cdots + x_n^2}$. The function $f(x) = x_1/x_0$ does not belong to L^p for any $p > \infty$, but $\nabla f \in L^p$ if $p > n-1$. In fact a simple computation shows $\nabla_j f = (\delta_1^j x_0^2 - x_1 x_j) x_0^{-3}$ and $\nabla^j f = \delta_1^j x_0^{-1}$ so

$$|\nabla f|^2 = (x_0^2 - x_1^2)\, x_0^{-4} \leqslant x_0^{-2}$$

and

$$\int |\nabla f|^p x_0^{-1}\, dx \leqslant \int x_0^{-p-1}\, dx < \infty$$

if $p > n-1$.

In the case $p=2$, there is an immediate connection between (5.10) and L^2-cohomology in dimension one. If there exists nontrivial cohomology, there is a nontrivial harmonic 1-form F_1 in L^2. We also have the Kodaira decomposition of L^2 1-forms $F = F_0 + F_1 + F_2$, where F_0 is in the L^2 closure of $d\mathscr{D}_0$ and F_2 is in the L^2 closure of $\delta\mathscr{D}_2$ (here $\mathscr{D}_0$ and $\mathscr{D}_2$ are the Schwartz spaces of 0-forms and 2-forms), F_1 is harmonic ($dF_1 = 0$, $\delta F_1 = 0$), and the sum is an orthogonal direct sum. Thus $F_1 = df$ for some function f (the space is assumed simply connected). Now $\nabla f \in L^2$ but we cannot have $f - c \in L^2$, for then we would have $d(f-c) = df$ in the L^2 closure of $d\mathscr{D}_0$ (regularize $\phi(f-c)$, where ϕ is one of the functions $\phi_{r,s}$ of Lemma 2.2), contradicting the Kodaira decomposition. Thus we have a counterexample to (5.10).

Conversely, if the cohomology is trivial, the Kodaira decomposition is only $F = F_0 + F_2$. Applying this to df, we have $df = F_0$ because $\langle df, \delta\phi_2\rangle = \langle f, \delta^2\phi_2\rangle = 0$ for $\phi_2 \in \mathscr{D}_2$. Thus df is the L^2 limit of df_j, where $f_j \in \mathscr{D}_0$. But then by (5.5) the sequence $\{f_j\}$ must also have an L^2 limit, call it $\tilde{f}$. and since $d\tilde{f} = df$ we must have $\tilde{f} = f - c$, proving (5.10).

In the case of hyperbolic space, it is well known that the L^2-cohomology in dimension one is nontrivial if and only if $n = 2$ [5]. In that case the harmonic 1-forms realize certain discrete series representation of $SL(2, \mathbb{R})$, and in fact the counterexamples above originated with this observation.

Theorem 5.8 thus implies the vanishing of L^2-cohomology in dimension one for M_n provided $n \geqslant 3$, and the conjectured strengthening would imply the same for all simply connected complete M_n with curvature bounded above by a negative constant. The vanishing of L^2-cohomology is proved in somewhat greater generality by Dodziuk [12, 13].

6. Riesz Transforms

In Euclidean space, the Riesz transforms $\partial/\partial x_k(-\Delta)^{-1/2}$ are bounded on L^p for $1 < p < \infty$. They are the most basic examples of Calderón–Zygmund singular integral operators, and they play an important role in the theory of Hardy spaces. We can combine them into a single operator $\nabla(-\Delta)^{-1/2}$ which takes tensor values. This operator makes sense on a general Riemannian manifold. We can also consider more generally, with the same notation, a mapping from tensors $T^{r,s}$ to $T^{r+1,s}$, and the operators $d(-\Delta)^{-1/2}$ from k-forms to $(k+1)$-forms and $\delta(-\Delta)^{-1/2}$ from k-forms to $(k-1)$-forms. We will refer to all these operators as Riesz transforms.

Now the Riesz transforms are all bounded on L^2; in fact, they are isometric. We have the identities

$$\|\nabla(-\Delta)^{-1/2}f\|_2 = \|f\|_2 \qquad \text{for tensors,} \tag{6.1}$$

and

$$\|d(-\Delta)^{-1/2}f\|_2^2 + \|\delta(-\Delta)^{-1/2}f\|_2^2 = \|f\|_2^2 \tag{6.2}$$

for forms, by Corollary 2.6. A fundamental question is whether these operators are bounded in L^p, for $1 < p < \infty$. A routine argument involving polarization and duality shows that if we can prove

$$\|\nabla(-\Delta)^{-1/2}f\|_p \leqslant A_p \|f\|_p, \tag{6.3}$$

we automatically obtain the reverse inequality

$$\|f\|_p \leqslant A_p \|\nabla(-\Delta)^{-1/2}f\|_p \tag{6.4}$$

and hence the equivalence of the Sobolev spaces L_1^p defined by Riesz or Bessel potentials and the space of all $f \in L^p$ such that $\nabla f \in L^p$ (Aubin [2, 3] proves the density of $\mathscr{D}$ in these spaces). Similarly the estimates

$$\|d(-\Delta^{-1/2}f\|_p + \|\delta(-\Delta)^{-1/2}f\|_p \leqslant A_p \|f\|_p \tag{6.5}$$

imply the reverse estimates

$$\|f\|_p \leqslant A_p(\|d(-\Delta)^{-1/2}f\|_p + \|\delta(-\Delta)^{-1/2}f\|_p) \tag{6.6}$$

and the equivalence for forms of the conditions f, $(-\Delta)^{-1/2}f$ in L^p and f, df, δf in L^p. Another interesting consequence of these estimates is the boundedness in L^p of the projections giving the Kodaira decomposition of k-forms.

In the case of compact manifolds, the boundedness of all the Riesz transforms follows easily from the theory of pseudo-differential operators and Seeley [26]. We now show the boundedness of the Riesz transform on function for noncompact rank-one symmetric spaces. Our proof is facilitated by the detailed analysis of the Riesz potential operator $(-\Delta)^{-1/2}$ in Stanton and Tomas [29]. Incidentally, the Riesz transforms proposed by Stein [31] (Section 5.2) in this context are not the same as ours, and it is easy to see that they are not even L^2 bounded, for much the same reason that $\partial/\partial\theta(-\Delta)^{-1/2}$ is not L^2 bounded on $\mathbb{R}^2$.

THEOREM 6.1. *Let M_n be a rank-one symmetric space G/K, where G is a noncompact connected semi-simple Lie group of real rank one, and K a maximal compact subgroup. Then $\nabla(-\Delta)^{-1/2}$ is a bounded operator from L^p functions to L^p tensors of rank* $(1, 0)$ *for* $1 < p < \infty$, *and* (6.3) *and* (6.4) *hold.*

Proof. The operator $(-\Delta)^{-1/2}$ in this case is given by a spherical convolution,

$$(-\Delta)^{-1/2}f(x) = \int E(d(x, y))\, f(y)\, d\mu(y),$$

where E is a distribution on the line,

$$E(t) = \int_0^\infty (\lambda^2 + \rho^2)^{-1/2} \phi_\lambda(t) \, |c(\lambda)|^{-2} \, d\lambda.$$

Here ϕ_λ is the spherical function, $c(\lambda)$ is the Harish–Chandra c-function, and ρ is half the sum of the positive roots (see [20]). Let ψ be a smooth cut-off function that is one near zero and has compact support. Then

$$\nabla(-\Delta)^{-1/2} f(x) = \nabla_x \int \psi(d(x, y)) \, E(d(x, y)) \, f(y) \, d\mu(y)$$

$$+ \nabla_x \int (1 - \psi(d(x, y)) \, E(d(x, y)) \, f(y) \, d\mu(y),$$

the first term being the local part and the second term being the global part of the operator.

We claim the local part is bounded on L^p because it is a pseudo-differential operator, and because the manifold is homogeneous. Indeed by Theorem 4.7 the local part is a pseudo-differential operator of order zero, and these are locally bounded in L^p, $1 < p < \infty$. To pass from local to global estimates is then routine since the cut-off function is compactly supported and everything is invariant under the action of the group G (see [32]).

Finally the boundedness of the global part is essentially proved in Stanton and Tomas [29]. They consider the operator $(-\Delta)^{-1/2}$ so it is only necessary to examine the effect of the gradient on their estimates (Proposition 4.5 of [29]). The key observation is that $|\nabla_x E(d(x, y))| \leqslant |(d/dt) E(t)|$ at $t = d(x, y)$, which is obvious from geometric considerations since geodesics are globally unique. The derivative applies only to the spherical funcion $\phi_\lambda(t)$. Now the estimates in [29] involve a uniformly convergent series expansion $e^{(i\lambda - \rho)t} \sum \Gamma_k(\lambda) \, e^{-2kt}$, and the contribution corresponding to differentiating $e^{(i\lambda - \rho)t}$ vanishes because it leads to the integral of an odd function over the line. Thus the entire effect of the derivations is to replace $\Gamma_k(\lambda)$ by $-2k\Gamma_k(\lambda)$, and since $\Gamma_k(\lambda)$ already has exponential decay in k, this is a harmless change. Thus the global behavior of $\nabla_x(-\Delta)^{-1/2}$ is the same as that of $(-\Delta)^{-1/2}$, which is shown to be L^p bounded in [29]. Q.E.D.

Remarks. It seems likely that the analysis of the local part of the operator can also be carried out by the methods of Stanton and Tomas [29]. However, they base their local analysis ultimately on the Marcinkiewicz multiplier theorem, which is very closely related to the local L^p boundedness of pseudo-differential operators, so the two approaches are not so different.

It also seems likely that the L^p boundedness of Riesz transforms for higher tensors and forms on rank-one symmetric spaces can be established by similar methods.

It would be interesting to consider real-variable Hardy spaces on G/K defined by the conditions $f \in L^p$ and $\nabla(-\Delta)^{-1/2} f \in L^p$ for $p \leqslant 1$. For different approaches to the definition of Hardy spaces see [16, 24, 25].

We conclude with a proof that the heat semi-group for 1-forms on hyperbolic 2-space $(SL(2, \mathbb{R})/SO(2))$ is not an L^p contraction for all p. In fact consider the Kodaira decomposition $F = F_0 + F_1 + F_2$ for L^2 1-forms F, where F_1 is harmonic $F_0 = df_0$ for f_0 an L^2 0-form, and $F_2 = \delta f_2$ for f_2 an L^2 2-form. Since $F_0 = d(-\Delta)^{-1} \delta F$ and $(-\Delta)^{-1/2} \delta$ is the adjoint of $d(-\Delta)^{1/2}$, it follows from Theorem 6.1 that the projection $F \to F_0$ is bounded on L^p for $1 < p < \infty$. Since the projection $F \to F_2$ is essentially the same operator conjugated with the Hodge star operator, it is also bounded on L^p.

Suppose now that we had the estimate

$$\|e^{t\Delta} F\|_p \leqslant \|F\|_p \qquad \text{for all } L^p \text{1-forms.} \tag{6.7}$$

Writing $F = F_0 + F_1 + F_2$, all three summands are in L^p. We have $e^{t\Delta} F_1 = F_1$ because F_1 is harmonic. We claim $\|e^{t\Delta} F_0\|_p \to 0$ as $t \to \infty$. This is immediate from McKean's inequality when $p = 2$, and follows by interpolation with (6.7) in general. Again $\|e^{t\Delta} F_2\|_p \to 0$ as $t \to \infty$, for this is essentially the same. Thus (6.7) implies that the projection $F \to F_1$, which is the limit of $e^{t\Delta} F$ as $t \to \infty$, is an L^p contraction. If this were true for all $p > 1$, then by passing to the limit it would have to be true for $p = 1$.

But the projection $F \to F_1$ is not even bounded on L^1. Indeed by Theorem 5.8 there are no nonzero harmonic 1-forms in L^1 (this can also be seen directly in the unit disc model of hyperbolic 2-space). Since there must be 1-forms F in $L^1 \cap L^2$ for which $F_1 \neq 0$ (since $L^1 \cap L^2$ is dense in L^2 and there are nonzero L^2 harmonic 1-forms), we have a contradiction.

It would be interesting to know for which values of p the heat semi-group is contractive or merely bounded. Also it is interesting to compare the counterexample with the positive result (Theorem 3.7) for tensors for $\frac{3}{2} \leqslant p \leqslant 3$. Is the heat semi-group for tensors better behaved than the heat semi-group for forms? Finally, even in the case of compact manifolds these problems are open. Presumably one should be able to find closed form formulas for these heat semi-groups on spheres, and so answer these questions definitively in that case.

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