Sorgenfrey line and continuous separating families

Wei-Xue Shi *, Yin-Zhu Gao

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

Received 2 September 2003; accepted 13 January 2004

Abstract

We study continuous separating families on linearly ordered extensions of the Sorgenfrey line S. Let \( \mathbb{R} \) be the set of all real numbers, \( \mathbb{Z} \) the set of all integers, and \( S^* = \mathbb{R} \times \{ n \in \mathbb{Z} : n \leq 0 \} \) with the lexicographical ordering \( \preceq \) and with the usual interval topology defined by \( \preceq \). Then \( S^* \) is a linearly ordered extension of \( S \). We prove that, in ZFC, \( S^* \) does not admit a continuous separating families and that any linearly ordered extension of \( S \) does not admit a continuous separating family. Two problems posed by H. Bennett and D. Lutzer are answered.

© 2004 Elsevier B.V. All rights reserved.

MSC: 54F05; 54E52; 54D35; 54C10; 54C30

Keywords: Sorgenfrey line; Linearly ordered topological spaces; Generalized ordered spaces; Continuous separating families

1. Introduction

A continuous separating family is a continuous mapping

\[ \Phi : X^2 \setminus \Delta \to C_u(X), \]

where \( \Delta = \{ (x, x) : x \in X \} \) and \( C_u(X) \) is the space of continuous real-valued functions on \( X \) with the uniform convergence topology, such that if \( f_{x,y} = \Phi(x, y) \), then \( f_{x,y}(x) \neq f_{x,y}(y) \), and we also say the collection \( \{ f_{x,y} : (x, y) \in X^2 \setminus \Delta \} \) is a continuous separating family for \( X \), which was introduced by Stepanova. It is shown that a paracompact p-space...
(i.e., the preimage of a metric space under a perfect mapping) is metrizable if and only if it has a continuous separating family \([6,7]\). A linearly ordered space is metrizable if and only if it has a \(\sigma\)-closed-discrete dense subset and a continuous separating family \([2]\).

Recall that a linearly ordered topological space (LOTS) is a linearly ordered set with its usual open interval topology. A generalized ordered space (GO-space) is a linearly ordered set equipped with the topology which is \(T_1\) and has a base consisting of convex sets. It is well known that a GO-space is precisely a subspace of a LOTS. If \(X\) is a GO-space and \(Y\) is a LOTS containing \(X\) as a subspace, and the ordering on \(X\) is inherited from the ordering on \(Y\), then \(Y\) is called a linearly ordered extension of \(X\). Let \(X\) be a GO-space on which the topology is \(\tau\), and \(\lambda\) the interval topology on \(X\). Put

\[
R = \{x \in X: [x, \rightarrow) \in \tau \setminus \lambda\} \quad \text{and} \quad L = \{x \in X: (-, x] \in \tau \setminus \lambda\}
\]

and

\[
X^* = (X \times \{0\}) \cup (R \times \{n \in \mathbb{Z}: n < 0\}) \cup (L \times \{n \in \mathbb{Z}: n > 0\}), \quad (\bigodot)
\]

where \(\mathbb{Z}\) is the set of all integers. Regard \(X^*\) as the subset of \(X \times \mathbb{Z}\) with the lexicographical ordering. Then \(X^*\) with the order topology is a linearly ordered extension of \(X\). It often happens that a GO-space \(X\) has a topological property \(P\) if and only if its linearly ordered extension \(X^*\) has property \(P\) (for instance, \(P\) is paracompactness, metrizability, Lindelöfness or quasi-developability). Now suppose \(P\) is the property having a continuous separating family. Consider the Michael line \(M\) and the Sorgenfrey line \(S\). Each has a continuous separating family since each has a weaker metric topology \([4]\). Let \(M^*\) and \(S^*\) be the linearly ordered extensions of \(M\) and \(S\), respectively defined as above. In \([2]\), it is shown that \(M^*\) has a continuous separating family, and under the set-theoretic axiom CH, \(S^*\) does not admit a continuous separating family. The following two questions were posed by H. Bennett and D. Lutzer.

**Question 1** \([1,2]\). In ZFC, does \(S^*\) have a continuous separating family?

**Question 2** \([1]\). In ZFC, is there an example of a GO-space \(X\) that has a continuous separating family, but whose linearly ordered extension \(X^*\) does not?

In this paper, we prove that, in ZFC, \(S^*\) does not admit a continuous separating family. This result answers the above two questions. Moreover, we prove that any linearly ordered extension of \(S\) does not admit a continuous separating family. For undefined terminology we refer to \([3]\).

2. **Main theorem**

In this section, we consider the linearly ordered extension \(S^*\) of the Sorgenfrey line \(S\). Note that when \(X = S\) in \((\bigodot)\),

\[
X^* = S^* = \mathbb{R} \times \{n \in \mathbb{Z}: n \leq 0\}.
\]

**Theorem 1.** In ZFC, the space \(S^*\) does not admit a continuous separating family.
Proof. Let $\preceq$ be the lexicographical ordering on $S^*$. For contradiction, suppose that there is a continuous separating family for $S^*$, say \( \{f_{(x,i),(y,j)}: (x,i), (y,j) \in (S^*)^2 \setminus \Delta\} \). By Proposition 2.1 of [2], we may assume that if \((y,j) < (x,i)\), then

\[
f_{(x,i),(y,j)}((z,k)) = \begin{cases} 
0, & \text{whenever } (z,k) \preceq (y,j), \\
1, & \text{whenever } (x,i) \preceq (z,k).
\end{cases}
\]

(†)

Consider the subspace $X = \{ (x,i) \in S^*: i \in \{0, -1\} \}$ of $S^*$. Then by restricting the continuous separating family to the subspace $X$, we obtain a continuous separating family for $X$.

For every $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, let

\[
B_k = \{ (x,y) \in \mathbb{R}^2: y < x \text{ and } |f_{(x,0),(y,-1)}((0,0))| \geqslant 1/k \},
\]

\[
B_k(y) = \{ x \in \mathbb{R}: (x,y) \in B_k \},
\]

and

\[
Y_k = \{ y \in \mathbb{R}: B_k(y) \text{ is dense in some open interval} \}
\]

in the sense of the usual topology of the real line.

Claim. For each $k \in \mathbb{N}$, $Y_k$ is countable.

Suppose that for some $k_0 \in \mathbb{N}$, $Y_{k_0}$ is not countable. For each $y \in Y_{k_0}$, we can find two real numbers $a(y)$ and $b(y)$ such that $y \leqslant a(y) < b(y)$ and $B_{k_0}(y)$ is dense in the open interval $(a(y), b(y))$ in the sense of the usual topology of the real line. For each $i \in \mathbb{N}$, put

\[
Y_{k_0,i} = \{ y \in Y_{k_0}: b(y) - a(y) \geqslant 1/i \}.
\]

Then $Y_{k_0} = \bigcup \{ Y_{k_0,i}: i \in \mathbb{N} \}$, and there is an $i_0 \in \mathbb{N}$ such that $Y_{k_0,i_0}$ is uncountable. It is easy to see that there must be a natural number $N$ such that

\[
Y_{k_0,i_0}^N = \{ y \in Y_{k_0,i_0}: b(y) \leqslant N \}
\]

is uncountable. It follows that there is a $y_0 \in \mathbb{R}$ such that for any $\varepsilon > 0$, $(y_0, y_0 + \varepsilon)$ contains uncountably many points of $Y_{k_0,i_0}^N$. For each $y \in Y_{k_0,i_0}^N$, define $x(y) = (a(y) + b(y))/2$. Thus if $y > y_0$ and $y \in Y_{k_0,i_0}^N$, then $y_0 + 1/2i_0 < y + 1/2i_0 \leqslant x(y) < N$. Hence $\{ x(y): y > y_0 \}$ and $y \in Y_{k_0,i_0}^N \subset \{ y_0 + 1/2i_0, N \}$.

Subclaim. There must be an $x_0 > y_0$ such that any neighborhood (in the sense of the usual topology of the Euclidean plane) of the point $(x_0, y_0)$ contains infinitely many points of the set $\{(x(y), y): y > y_0 \text{ and } y \in Y_{k_0,i_0}^N\}$.

Assume that for each $x \in [y_0 + 1/2i_0, N]$, there is an open neighborhood $U(x)$ (in the Euclidean plane) of $(x, y_0)$ such that $U(x)$ contains at most finite points of $\{(x(y), y): y > y_0 \text{ and } y \in Y_{k_0,i_0}^N\}$. Then there is a finite subset $\{x_1, x_2, \ldots, x_m\}$ of $[y_0 + 1/2i_0, N]$ such that $\bigcup \{ U(x_i): i = 1, 2, \ldots, m \} \supset \{ y_0 + 1/2i_0, N \} \times \{ y_0 \}$ since $[y_0 + 1/2i_0, N] \times \{ y_0 \}$ is compact in the Euclidean plane. It follows that there is an $\varepsilon_0 > 0$ such that $(y_0, y_0 + \varepsilon_0) \cap Y_{k_0,i_0}^N$ is finite. Thus we get a contradiction. So the proof of Subclaim is completed.
Since \( f(x_0, y_0)(y_0, 0) = 0 \) and \( f(x_0, y_0, 0) \in \mathcal{C}_u(X) \), there is a neighborhood \( V = \{(y_0, y_0 + \delta_1) \times \{0, -1\} \setminus \{(y_0, -1)\} \} \) of \( (y_0, 0) \) in \( X \), where \( \delta_1 \) is small enough such that for \( (z, l) \in V \),
\[
|f(x_0, y_0, 0)(z, l)| < 1/2k_0.
\]
Moreover since \( \Phi(x, i, (y, j)) = f(x, i, (y, j)) \in \mathcal{C}_u(X) \) is continuous with respect to the uniform convergence topology on \( \mathcal{C}_u(X) \), there is a neighborhood \( W = \{(x_0, x_0 + \varepsilon) \times \{0, -1\} \setminus \{(x_0, -1)\} \} \times \{(y_0, y_0 + \delta_2) \times \{0, -1\} \setminus \{(y_0, -1)\} \} \) of \( (x_0, 0), (y_0, 0) \) in \( X^2 \setminus \Delta \) such that for \( (x, i), (y, j) \in W \),
\[
\sup_{(z, l) \in X} |f(x, i, (y, j)) - f(x_0, y_0, 0)(z, l)| < 1/2k_0.
\]
Take \( \delta = \min\{\delta_1, \delta_2\} \), and
\[
O = \{(x_0, x_0 + \varepsilon) \times \{0, -1\} \setminus \{(x_0, -1)\} \} \times \{(y_0, y_0 + \delta) \times \{0, -1\} \setminus \{(y_0, -1)\} \}.
\]
Then for \( (x, i), (y, j) \in O \) and \( (z, l) \in ((y_0, y_0 + \delta) \times \{0, -1\} \setminus \{(y_0, -1)\}) \), we have
\[
|f(x, i, (y, j))(z, l)| \\
\leq |f(x, i, (y, j))(z, l)| - f(x_0, y_0, 0)(z, l)| + |f(x_0, y_0, 0)(z, l)| \\
< 1/2k_0 + 1/2k_0 = 1/k_0.
\]
In particular, whenever \( (\xi, \eta) \in (x_0, x_0 + \varepsilon) \times (y_0, y_0 + \delta) \), we have
\[
\eta < \xi \quad \text{and} \quad |f(\xi, 0, (\eta, -1))(\xi, 0)| < 1/k_0.
\]
Therefore
\[
((x_0, x_0 + \varepsilon) \times (y_0, y_0 + \delta)) \cap B_{k_0} = \emptyset.
\]
Take an \( \varepsilon_1 > 0 \) satisfying \( 0 < \varepsilon_1 < \min\{\varepsilon, 1/2k_0, \delta\} \). Let
\[
B((x_0, y_0), \varepsilon_1) = \{(x, y) \in \mathbb{R}: \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon_1\}.
\]
Then by the definition of \( x_0 \) and \( y_0 \) in the above subclaim, \( B((x_0, y_0), \varepsilon_1) \cap \{(x(y), y): y > y_0 \text{ and } y \in Y_{k_0,i_0}^N\} \neq \emptyset \). Take a point \( y_1 \in Y_{k_0,i_0}^N \) such that \( (x(y_1), y_1) \in B((x_0, y_0), \varepsilon_1) \cap \{(x(y), y): y > y_0 \text{ and } y \in Y_{k_0,i_0}^N\} \). It follows that \( |x(y_1) - x_0| < \varepsilon_1 \leq 1/2k_0 \) and thus \( x_0 \in (a(y_1), b(y_1)) \). Therefore there is an \( x' \in B_{k_0}(y_1) \) such that \( x' \in (x(y_1, x_0 + \varepsilon) \text{ since } B_{k_0}(y_1) \text{ is dense in } (a(y_1), b(y_1)). \) Also since \( y_0 < y_1 < y_0 + \varepsilon_1 \leq y_0 + \delta \), \( x', y_1 \in B_{k_0} \cap ((x_0, x_0 + \varepsilon) \times (y_0, y_0 + \delta)) \). This contradicts \( (*) \). So we have proved that \( Y_k \) is countable for every \( k \in \mathbb{N} \). Thus the proof of \textit{Claim} is finished.

Let \( Y = \bigcup_{k \in \mathbb{N}} Y_k \). Then \( Y \) is countable, and if \( y \notin Y_k \), then \( B_k(y) \) is nowhere dense in the real line. Moreover if \( y \notin Y \), then \( B(y) = \bigcup_{k \in \mathbb{N}} B_k(y) \) is a first category set in the real line.
Take a point \( y' \in \mathbb{R} \setminus Y \). There is a decreasing monotone sequence \( \{x_n\} \) contained in \( \mathbb{R} \setminus B(y') \) such that \( \{x_n\} \) converges to \( y' \) in the space \( S \).

In \( X \), we have \( \lim_{n \to \infty} (x_n, 0) = (y', 0) \). Hence

\[
\lim_{n \to \infty} f((x_n, 0), (y', -1)) = f((y', 0), (y', -1)).
\]

Therefore

\[
\lim_{n \to \infty} f((x_n, 0), (y', -1)) (\langle y', 0 \rangle) = f((y', 0), (y', -1)) (\langle y', 0 \rangle).
\]

Since for each \( k \in \mathbb{N} \), \( (x_n, y') \notin B_k \), \( |f((x_n, 0), (y', -1)) (\langle y', 0 \rangle)| < 1/k \) for any natural number \( k \), so that for all \( n \in \mathbb{N} \), \( f((x_n, 0), (y', -1)) (\langle y', 0 \rangle) = 0 \). It follows that

\[
\lim_{n \to \infty} f((x_n, 0), (y', -1)) (\langle y', 0 \rangle) = 0.
\]

But by (†), \( f((y', 0), (y', -1)) (\langle y', 0 \rangle) = 1 \). This is a contradiction. Thus the proof of the theorem is completed. \( \Box \)

It is obvious that Theorem 1 gives a negative answer to Question 1 and an affirmative answer to Question 2. In the proof of Theorem 1, it is worth to note that if \( C \) is a countable subset of \( \mathbb{R} \), then \( (\mathbb{R} \setminus C) \setminus B(y) \) is also dense in \( \mathbb{R} \). So by a slightly modification of the proof, we can prove

**Corollary 2.** If \( T \) is a subspace of the Sorgenfrey line \( S \) by removing a countable subset from \( S \), then \( T^* \) does not admit a continuous separating family.

### 3. General results

In Section 2 we have proved that the linearly ordered extension \( S^* \) of \( S \) does not admit a continuous separating family. A natural question is: “if a GO-space \( X \) has a continuous separating family, is there a linearly ordered extension \( Y \) of \( X \) such that \( Y \) also has a continuous separating family?” In this section, we answer this question negatively by proving that any linearly ordered extension of the Sorgenfrey line \( S \) does not admit a continuous separating family. Recall that a LOTS is said to be a linearly ordered dense extension of a GO-space \( (X, \tau, \leq) \) if \( Y \) contains \( X \) as a dense subspace and the ordering on \( Y \) extends the ordering \( \leq \) on \( X \).

**Lemma 3.** Let \( X \) be an uncountable subspace of the Sorgenfrey line \( S \). Then any linearly ordered dense extension \( Y \) of \( X \) does not admit a continuous separating family.

**Proof.** It is easy to see that \( X \) is separable so that \( Y \) is separable since \( X \) is dense in \( Y \). If \( Y \) admitted a continuous separating family, by [2, Proposition 2.6], \( Y \) would be metrizable and so would be \( X \). But \( X \) is not metrizable since \( X \) is separable and has no countable base. \( \Box \)

**Theorem 4.** Any linearly ordered extension of the Sorgenfrey line \( S \) does not admit a continuous separating family.
Proof. Let \(Y\) be a linearly ordered extension of \(S\). By the discussion in [5, Section 3], we can classify \(S\) into two disjoint sets according to the extension type of each point of \(S\) for \(Y\). Simply explaining the case for \(S\), for a point \(x \in S\), if the set of points of \(Y\) which lie between \((\leftarrow, x)\) and \([x, \to)\) has a minimum point, then we say \(x\) has the left extension type \(-1\) for \(Y\); otherwise we say \(x\) has the left extension type \(-\infty\) for \(Y\). Put

\[
A_1 = \{x \in S: x \text{ has the left extension type } -1 \text{ for } Y\};
A_2 = \{x \in S: x \text{ has the left extension type } -\infty \text{ for } Y\}.
\]

Then \(S = A_1 \cup A_2\) and \(A_1 \cap A_2 = \emptyset\). Put

\[
E(S) = \left( A_1 \times \{0, -1\} \right) \cup \left( A_2 \times \{n \in \mathbb{Z}: n \leq 0\} \right).
\]

Orderize \(E(S)\) by the lexicographical ordering and topologize \(E(S)\) by the linearly ordered topology. In [5], it is proved that \(E(S)\) is homeomorphic to a subspace of \(Y\). So it suffices to prove that \(E(S)\) does not admit a continuous separating family. If \(A_1\) is countable, then \(A_2 = S \setminus A_1\). By Corollary 2, the subspace \(A_2 \times \{n \in \mathbb{Z}: n \leq 0\}\) of \(E(S)\) does not admit a continuous separating family. If \(A_1\) is uncountable, then the subspace \(A_1 \times \{0, -1\}\) of \(E(S)\) is a dense linearly ordered extension of \(A_1\), by Lemma 3, \(A_1 \times \{0, -1\}\) does not admit a continuous separating family. \(\square\)

References