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Revolutionaries and spies: Spy-good and spy-bad graphs

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ABSTRACT

We study a game on a graph G played by r *revolutionaries* and s *spies*. Initially, revolutionaries and then spies occupy vertices. In each subsequent round, each revolutionary may move to a neighboring vertex or not move, and then each spy has the same option. The revolutionaries win if m of them meet at some vertex having no spy (at the end of a round); the spies win if they can avoid this forever.

Let $\sigma(G, m, r)$ denote the minimum number of spies needed to win. To avoid degenerate cases, assume $|V(G)| \geq r - m + 1 \geq \lfloor r/m \rfloor \geq 1$. The easy bounds are then $\lfloor r/m \rfloor \leq \sigma(G, m, r) \leq r - m + 1$. We prove that the lower bound is sharp when G has a rooted spanning tree T such that every edge of G not in T joins two vertices having the same parent in T. As a consequence, $\sigma(G, m, r) \leq \gamma(G) \lfloor r/m \rfloor$, where $\gamma(G)$ is the domination number; this bound is nearly sharp when $\gamma(G) \leq m$.

For the random graph with constant edge-probability p, we obtain constants c and c' (depending on m and p) such that $\sigma(G, m, r)$ is near the trivial upper bound when $r < c \ln n$ and at most c' times the trivial lower bound when $r > c' \ln n$. For the hypercube Q_d with $d \geq r$, we have $\sigma(G, m, r) = r - m + 1$ when $m = 2$, and for $m \geq 3$ at least $r - 39m$ spies are needed.

For complete k-partite graphs with partite sets of size at least $2r$, the leading term in $\sigma(G, m, r)$ is approximately $\frac{k}{k-1}\frac{r}{m}$ when $k \geq m$. For $k = 2$, we have $\sigma(G, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ and $\sigma(G, 3, r) = \lfloor r/2 \rfloor$, and in general $\frac{3r}{2m} - 3 \leq \sigma(G, m, r) \leq \frac{(1+1/\sqrt{3})r}{m}$.

1. Introduction

We study a pursuit game involving two teams on a graph. The first team consists of r *revolutionaries*; the second consists of s *spies*. The revolutionaries want to arrange a one-time meeting of m revolutionaries free of oversight by spies. Initially, the revolutionaries take positions at vertices, and then the spies do the same. In each subsequent round, each revolutionary may move to a neighboring vertex or not move, and then each spy has the same option. All positions are known by all players at all times.

The revolutionaries win if at the end of a round there is an *unguarded meeting*, where a *meeting* is a set of (at least) m revolutionaries on one vertex, and a meeting is *unguarded* if there is no spy at that vertex. The spies win if they can prevent this forever. Let RS(G, m, r, s) denote this game played on the graph G by s spies and r revolutionaries seeking an unguarded meeting of size m.

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The spies trivially win if $s \geq |V(G)|$ or $r < m$. If $\lfloor r/m \rfloor < |V(G)|$, then the revolutionaries can form $\lfloor r/m \rfloor$ meetings initially, and hence at least $\lfloor r/m \rfloor$ spies are needed to avoid losing immediately. On the other hand, the spies win if $s \geq r - m + 1$; they follow $r - m + 1$ distinct revolutionaries, and the other $m - 1$ revolutionaries cannot form a meeting. To avoid degenerate or trivial games, henceforth in this paper **we always assume**

$$|V(G)| \geq r - m + 1 \geq \lfloor r/m \rfloor \geq 1.$$

Let $\sigma(G, m, r)$ denote the minimum s such that the spies win the game $\mathrm{RS}(G, m, r, s)$.

The game of Revolutionaries and Spies was invented by Jozef Beck in the mid-1990s (unpublished). Smyth promptly showed that $\sigma(G, m, r) = \lfloor r/m \rfloor$ when G is a tree, achieving the trivial lower bound (a later proof appears in [2]). Howard and Smyth [4] studied the game when G is the infinite 2-dimensional integer grid with one-step horizontal, vertical, and diagonal edges. They observed that the spy wins $\mathrm{RS}(G, m, 2m - 1, 1)$ (the spy stays at the median position), and hence $\sigma(G, m, r) \leq r - 2m + 2$ when $r \geq 2m - 1$ (note that always $\sigma(G, m, r) \leq \sigma(G, m, r - 1) + 1$). For $m = 2$, they proved that $6 \lfloor r/8 \rfloor \leq \sigma(G, 2, r) \leq r - 2$ when $r \geq 3$; they conjectured that the upper bound is the correct answer.

Cranston, Smyth, and West [2] showed that $\sigma(G, m, r) \leq \lceil r/m \rceil$ when G has at most one cycle. Furthermore, let G be a unicyclic graph consisting of a cycle of length ℓ and t vertices not on the cycle. They showed that if $m \nmid r$ (and as usual $|V(G)| > r/m$ to avoid degeneracies), then $\sigma(G, m, r) = \lfloor r/m \rfloor$ if and only if $\ell \leq \max\{\lfloor r/m \rfloor - t + 2, 3\}$.

Our objective in this paper is to advance the systematic study of this game. We show that the trivial lower and upper bounds on $\sigma(G, m, r)$ each may be sharp on various classes of graphs. Furthermore, we obtain classes where neither bound is asymptotically sharp and yet still $\sigma(G, m, r)$ can be determined or closely approximated.

Say that G is *spy-good* if $\sigma(G, m, r)$ equals the trivial lower bound $\lfloor r/m \rfloor$ for all m and r such that $r/m < |V(G)|$. In Section 2, we prove that every webbed tree is spy-good, where a *webbed tree* is a graph G containing a rooted spanning tree T such that every edge of G not in T joins vertices having the same parent in T. For example, every graph having a dominating vertex u is a webbed tree (rooted at u).

Section 3 considers general bounds. Always $\sigma(G, m, r) \leq \gamma(G) \lfloor r/m \rfloor$, where $\gamma(G)$ is the domination number of G (the minimum size of a set S such that every vertex outside S has a neighbor in S). Since always $\lfloor r/m \rfloor \geq (r - m + 1)/m$, this upper bound is nontrivial only when $\gamma(G) < m$. In that case, it is nearly sharp: for $t, m, r \in \mathbb{N}$ with $t < m$, we construct a graph with domination number t such that $\sigma(G, m, r) > t(r/m - 1)$.

In contrast to spy-good graphs, a graph G is *spy-bad for r revolutionaries and meeting size m* if $\sigma(G, m, r)$ equals the trivial upper bound $r - m + 1$. Section 3 constructs chordal graphs (and bipartite graphs) that are spy-bad (for given r and m).

In Section 4 we study hypercubes, showing first that the d-dimensional hypercube Q_d is spy-bad when $d \geq r$ and $m = 2$. Also, the winning strategy for the revolutionaries uses only vertices near a fixed vertex. By splitting the revolutionaries into disjoint groups who play this strategy around vertices far apart, it follows that when $d < r \leq 2^d/d^8$, the revolutionaries win against $(d - 1) \lfloor r/d \rfloor$ spies on Q_d (for $m = 2$). For general m, we show that hypercubes are nearly spy-bad by proving $\sigma(Q_d, m, r) \geq r - 39m$ for $d \geq r \geq m$. (For small m, the bound $\sigma(Q_d, m, r) \geq r - \frac{3}{4}m^2$ when $d \geq r \geq m$ is better.)

In these examples of spy-bad graphs, there are few revolutionaries compared to the number of vertices. Similar behavior holds for the random graph with constant edge-probability (Section 5); the threshold for spies to win depends on the relationship between r and the number of vertices, n. Via fairly simple arguments, we obtain constants c and c' (depending on m) such that almost always $r - m + 1$ spies are needed when $r < c \ln n$, while a multiple of r/m spies are enough when $r > c' \ln n$. Using more intricate structural characteristics of the random graph and a more complex strategy for the spies, Mitsche and Prałat [5] proved that $\sigma(G, m, r) = (1 + o(1))r/m$ spies suffice when r grows faster than $(\log n)/p$ (here also p may depend on n).

A complete k-partite graph is *r-large* if each part has at least $2r$ vertices, which is as many vertices as the players might want to use. In Section 6, we prove $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{r}{m} + k$. Also $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{r}{m+c} - k$ when $k \geq m$ and $c = \frac{1}{k-1}$.

Section 7 focuses on complete bipartite graphs and contains our most delicate results. When G is an r-large complete bipartite graph, we obtain $\sigma(G, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ and $\sigma(G, 3, r) = \lfloor r/2 \rfloor$. For larger m we do not have the complete answer; we prove

$$\left(\frac{3}{2} - o(1)\right)\frac{r}{m} - 2 \leq \sigma(G, m, r) \leq \left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m} < 1.58\frac{r}{m},$$

where the upper bound requires $\frac{r}{m} \geq \frac{1}{1 - 1/\sqrt{3}}$. We conjecture that $\sigma(G, m, r)$ is approximately $\frac{3r}{2m}$ when 3 divides m, but in other cases the revolutionaries do a bit better. That advantage should fade as m grows, with $\sigma(G, m, r) \sim \frac{3r}{2m}$.

Upper bounds for $\sigma(G, m, r)$ are proved using strategies for the spies. We define a notion of *stable position* in the game. Proving that a particular number of spies can win involves showing that in a stable position all meetings are guarded and that for any move by the revolutionaries from a stable position, the spies can reestablish stability. This technique is used for graphs with dominating vertices and for webbed trees in Section 2, for random graphs in Section 5, and for complete multipartite and complete bipartite graphs in Sections 6 and 7. Each setting uses its own definition of stability tailored to the graphs under study.

Lower bounds are proved by strategies for the revolutionaries, which usually are much simpler. Most of our winning strategies for revolutionaries take at most two rounds, but on hypercubes they take $m - 1$ rounds. In [2], strategies for revolutionaries proving that $\sigma(C_n, m, r) = \lceil r/m \rceil$ (when $r/m < n$) may take many rounds.

Many questions remain open, such as a characterization of spy-good graphs. In all known spy-good graphs, the spies can ensure that at the end of each round the number of spies at any vertex v is at least $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries at v. Existence of such a strategy is preserved when vertices expand into a complete subgraph. Also, Howard and Smyth [4] observed that $\sigma(G, m, r)$ is preserved by taking the distance power of a graph. Hence every graph obtained from some webbed tree via some sequence of distance powers or vertex expansions is spy-good, but these are not the only spy-good graphs.

It would also be interesting to bound $\sigma(G, m, r)$ in terms of other graph parameters, such as treewidth. Generalizations of the game are also possible, such as by allowing players to travel farther in a move or by requiring more spies to guard a meeting. One can also consider analogous games on directed graphs.

2. Dominating vertices and webbed trees

We begin with graphs having a *dominating vertex* (a vertex adjacent to all others); we then apply this result to webbed trees. Let $N(v)$ denote the neighborhood of a vertex v. Also $N[v] = N(v) \cup \{v\}$, and $N(S) = \bigcup_{v \in S} N(v)$.

Definition 2.1. For a graph G having a dominating vertex u, a position in the game RS(G, m, r, s) is *stable* if, for each vertex v other than u, the number of spies at v is exactly $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries at v. The other spies, if any, are at u.

Theorem 2.2. *If a graph G has a dominating vertex, then $\sigma(G, m, r) = \lfloor r/m \rfloor$.*

Proof. Let u be a dominating vertex in G, and let $s = \lfloor r/m \rfloor$. Since $s = \lfloor r/m \rfloor$, a stable position will have a spy at u if there is a meeting at u. Hence a stable position has no unguarded meeting. When $s = \lfloor r/m \rfloor$, there are enough spies to establish a stable position after the initial round. We show that the spies can reestablish a stable position at the end of each round.

Consider a stable position at the start of round t. Let X be a maximal family of disjoint sets of m revolutionaries such that each set is located at one vertex other than u. Let Y be such a maximal family after the revolutionaries move in round t. In X or Y, more than one set may be located at a single vertex in G. For example, a vertex v having $pm + q$ revolutionaries at the start of round t (where $0 \le q < m$) corresponds to p elements of X, and there are p spies at v at that time.

Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_{k'}\}$. Let $X' = \{x_{k+1}, \dots, x_s\}$, representing the excess spies waiting at u at the start of round t. Define an auxiliary bipartite graph H with partite sets $X \cup X'$ and Y. For $x_i \in X$ and $y_j \in Y$, put $x_iy_j \in E(H)$ if some revolutionary from meeting x_i is in meeting y_j (note that x_i and y_j may be the same set). Also make all of X' adjacent to all of Y. If some matching in H covers Y, then the spies can move so that every vertex other than u having $p'm + q'$ revolutionaries at the end of round t (where $0 \le q' < m$) has exactly p' spies on it (and the remaining spies are at u).

The existence of such a matching follows from Hall's Theorem. For $S \subseteq Y$, always $X' \subseteq N(S)$, so $|N(S)| = |X'| + |N(S) \cap X|$. Consider the $m|S|$ revolutionaries in the meetings corresponding to S. Such revolutionaries came from meetings in $|N(S) \cap X|$ or were not in any of the k meetings indexed by X. Hence $m|S| \le m|N(S) \cap X| + (r - km)$. Since $|X'| = s - k$ and $s = \lfloor r/m \rfloor$,

$$|N(S)| \ge |X'| + |S| - (\lfloor r/m \rfloor - k) = s - k + |S| - (\lfloor r/m \rfloor - k) = |S|,$$

so Hall's Condition holds. □

Corollary 2.3. *Fix n, m, r with $n \ge r/m$. For $0 \le k \le \binom{n}{2}$, there is an n-vertex graph G with k edges such that $\sigma(G, m, r) = \lfloor r/m \rfloor$.*

Proof. For $k \ge n$, form G by adding the desired number of edges joining leaves of an n-vertex star; Theorem 2.2 applies. For $k \le n - 1$, let G be a star plus isolated vertices; use Theorem 2.2 and $\lfloor a \rfloor + \lfloor b \rfloor \le \lfloor a + b \rfloor$. □

Definition 2.4. For any vertex v in a rooted tree, the *parent* of a non-root vertex v (written v^+) is the first vertex after v on the path from v to the root. The set of *children* of v (written $C(v)$) is the set of neighbors of v other than its parent, and the set of *descendants* of v (written $D(v)$) is the set of vertices whose path to the root contains v. A *webbed tree* is a graph G having a rooted spanning tree T such that every edge of G outside T joins two vertices having the same parent (called *siblings*). Fig. 1 shows a webbed tree, with the rooted spanning tree T in bold.

Trivially, every tree is a webbed tree, as is every graph having a dominating vertex. In fact, a 2-connected graph is a webbed tree if and only if it has a dominating vertex. Every webbed tree is a graph whose blocks have dominating vertices, but the converse does not hold. Consider the graph obtained from two 4-cycles with a common vertex by adding chords of the 4-cycles to create four vertices of degree 3; every block has a dominating vertex, but the graph is not a webbed tree.

Our main result in this section is that all webbed trees are spy-good. This conclusion is proved for trees in [2]. In that paper, an invariant defined in terms of the positions of the revolutionaries specifies how many spies should be placed on each vertex. The invariant guarantees that all meetings are covered, and a direct proof is given to show that the spies can restore the invariant after each round.

Here we use the same invariant to generalize the tree result to the class of webbed trees. Our method of proving that the invariant has the desired properties is different from that in [2]. Here we decompose the spies' response into independent responses in imagined games on subgraphs having a dominating vertex. After the revolutionaries move, the spies restore the invariant by applying the strategy in Theorem 2.2 independently to each graph induced by a vertex and its children in the spanning tree. Because we will apply Theorem 2.2, we do not use "stable" for positions satisfying the invariant in a webbed tree; instead, we reserve that term for positions in the auxiliary local games, whose graphs have dominating vertices.

In [2], the result on trees is extended in a different direction to determine the winner in $\mathrm{RS}(G, m, r, s)$ whenever G has at most one cycle. A similar extension is possible here for graphs obtained by adding a cycle through the roots of disjoint webbed trees, but the resulting family is not as natural as the family of unicyclic graphs.

Theorem 2.5. *If G is a webbed tree, then $\sigma(G, m, r) = \lfloor r/m \rfloor$.*

Proof. Let T be a rooted spanning tree in G such that every edge of G not in T joins sibling vertices in T. Let z be the root of T, and let $s = \lfloor r/m \rfloor$. The notation for children and descendants is as in Definition 2.4 with respect to T.

For each vertex v, let $r(v)$ and $s(v)$ denote the number of revolutionaries and spies on v at the current time, respectively, and let $w(v) = \sum_{u \in D(v)} r(u)$. The spies maintain the following invariant specifying the number of spies on each vertex at the end of any round:

$$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \quad \text{for } v \in V(G). \tag{1}$$

Since $\sum_{x \in C(v)} w(x) = w(v) - r(v)$, the formula is always nonnegative. Also, if $r(v) \geq m$, then $s(v) \geq \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w(v)-r(v)}{m} \right\rfloor \geq 1$. Hence (1) guarantees that every meeting is guarded.

To show that the spies can establish (1) after the first round, it suffices that all the formulas sum to $\lfloor r/m \rfloor$. More generally, summing over the descendants of any vertex v,

$$\sum_{u \in D(v)} s(u) = \left\lfloor \frac{w(v)}{m} \right\rfloor, \tag{2}$$

since $\lfloor w(u)/m \rfloor$ occurs positively in the term for u and negatively in the term for u^+, except that $\lfloor w(v)/m \rfloor$ occurs only positively. When $v = z$, the total is $\lfloor r/m \rfloor$, since $w(z) = r$.

To show that the spies can maintain (1), let $r(v)$ and $s(v)$ refer to the start of round t, let $r'(v)$ denote the number of revolutionaries at v after the revolutionaries move in round t, and let $w'(v) = \sum_{u \in D(v)} r'(v)$. The spies will move in round t to achieve the new values required by (1). To determine these moves, we will use Theorem 2.2 to obtain a stable position in each subgraph induced by a vertex and its children, independently. Let $G(v)$ denote the subgraph induced by $C(v) \cup \{v\}$; note that v is a dominating vertex in $G(v)$. We will play a round in an imagined "local" game on $G(v)$ for each vertex v.

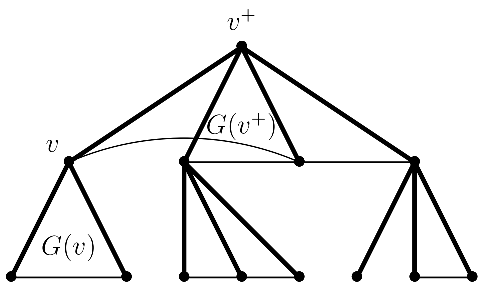

Fig. 1. Decomposition of a webbed tree.

To set up the local games, we partition the $s(v)$ spies at each vertex v into a set of $\check{s}(v)$ spies to be used in the local game on $G(v)$ and a set of $\hat{s}(v)$ spies to be used in the local game on $G(v^+)$, where $\check{s}(v)$ and $\hat{s}(v)$ sum to $s(v)$ (when the tree is drawn with the root z at the top, the accent indicates the direction of the relevant subgraph).

Let $D^*(v) = D(v) - \{v\}$. Let $w^*(v)$ be the number of revolutionaries that are in $D^*(v)$ at the start of round t *or* are there after the revolutionaries move in round t. Every revolutionary counted by $w^*(v)$ is also counted by $w(v)$, and every revolutionary counted by $\sum_{x \in C(v)} w(x)$ is also counted by $w^*(v)$. These statements also hold with w' in place of w. Hence

$$w(v) \geq w^*(v) \quad \text{and} \quad w^*(v) \geq \sum_{x \in C(v)} w(x). \tag{3}$$

By (3), $\hat{s}(v)$ and $\check{s}(v)$ are nonnegative when we define

$$\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor \quad \text{and} \quad \check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor. \tag{4}$$

By (1), $\hat{s}(v) + \check{s}(v) = s(v)$. Note also that if v is a leaf of T, then $\check{s}(v) = 0$ and $\hat{s}(v) = s(v)$.

For each non-leaf vertex v, the spies first imagine positions of revolutionaries in a game on the graph $G(v)$ that together with (4) for the spies form a stable position. After viewing the actual moves by revolutionaries within $G(v)$ as moves in this game, the spies reestablish stability as in Theorem 2.2. We will show that the resulting positions satisfy the global invariant. The spies imagine $\hat{r}(v)$ spies at v in $G(v^+)$ and $\check{r}(v)$ spies at v in $G(v)$, where

$$\hat{r}(v) = w(v) - m\left\lfloor \frac{w^*(v)}{m} \right\rfloor \quad \text{and} \quad \check{r}(v) = w^*(v) - \sum_{x\in C(v)} w(x). \tag{5}$$

By (3), the values of $\check{r}(v)$ and $\hat{r}(v)$ are nonnegative. Furthermore, we claim that if (4) and (5) hold at each vertex v, then the position on each subgraph induced by one parent and its children is stable. In $G(v)$ we use $\check{s}(v)$ and $\check{r}(v)$, and we use $\hat{s}(x)$ and $\hat{r}(x)$ for $x \in C(v)$. By definition, $\hat{s}(x) = \lfloor \hat{r}(x)/m \rfloor$. It remains only to check the sum. We compute the total number of revolutionaries in the local game:

$$\check{r}(v) + \sum_{x\in C(v)} \hat{r}(x) = w^*(v) - \sum_{x\in C(v)} w(x) + \sum_{x\in C(v)} w(x) - m\sum_{x\in C(v)} \left\lfloor \frac{w^*(x)}{m} \right\rfloor.$$

Dividing by m yields $\frac{w^*(v)}{m} - \sum_{x\in C(v)} \left\lfloor \frac{w^*(x)}{m} \right\rfloor$, whose floor is $\check{s}(v) + \sum_{x\in C(v)} \hat{s}(x)$, as desired.

The spies next view the actual moves by revolutionaries in the global game as moves by the revolutionaries in the imagined local games. Each such move occurs within the subgraph $G(v)$ for one vertex v. The local game can model these moves if the relevant value of $\hat{r}$ or $\check{r}$ is at least the number of real revolutionaries leaving this vertex and staying within this subgraph. The revolutionaries leaving v by edges in $G(v^+)$ are those that were in $D(v)$ and now are not; there are at most $w(v) - w^*(v)$ of them. By (5), $\hat{r}(v)$ is at least this large. Similarly, revolutionaries leaving v via $G(v)$ wind up in $D^*(v)$ but were not there previously, so the number of them is at most $w^*(v) - \sum_{x\in C(v)} w(x)$, which equals $\check{r}(v)$.

The net change in the actual number of revolutionaries at v is $r'(v) - r(v)$. Some of this change is due to moves in $G(v)$ and the rest to moves in $G(v^+)$. Moves in $G(v^+)$ enter or leave $D(v)$. Hence the net change in the number of revolutionaries at v due to such moves is $w'(v) - w(v)$. The remaining net change, due to moves between v and its children (in $G(v)$), is $(r'(v) - r(v)) - (w'(v) - w(v))$. Therefore, after executing the actual moves in the imagined local games, the new imagined distributions for the revolutionaries are given by

$$\hat{r}'(v) = \hat{r}(v) + w'(v) - w(v) \quad \text{and} \quad \check{r}'(v) = \check{r}(v) + (r'(v) - r(v)) - (w'(v) - w(v)). \tag{6}$$

The specification of $\hat{r}(v)$ in (5) and the change from $\hat{r}(v)$ to $\hat{r}'(v)$ in (6) immediately yield the formula for $\hat{r}'(v)$ in (7). To obtain $\check{r}'(v)$, start with the formula for $\check{r}'(v)$ in (5) and adjust by the definitions of $r(v) - r(v)$ and $w'(v) - r'(v)$, as indicated in (6). We compute

$$\begin{aligned}\check{r}'(v) &= \check{r}(v) + (w(v) - r(v)) - (w'(v) - r'(v))\\ &= w^*(v) - \sum_{x\in C(v)} w(x) + \sum_{x\in C(v)} w(x) - \sum_{x\in C(v)} w'(x) = w^*(v) - \sum_{x\in C(v)} w'(x).\end{aligned}$$

Thus

$$\hat{r}'(v) = w'(v) - m\left\lfloor \frac{w^*(v)}{m} \right\rfloor \quad \text{and} \quad \check{r}'(v) = w^*(v) - \sum_{x\in C(v)} w'(x). \tag{7}$$

The spies now respond in the local games. By Theorem 2.2, these positions are stable, so $\hat{s}'(x) = \lfloor \hat{r}'(x)/m \rfloor$ for $x \in C(v)$, and $\check{s}'(v)$ is the leftover amount for v in the local game on $G(v)$. By the same computation that earlier showed $\check{s}(v)$ was the correct needed amount of spies left for v in $G(v)$, also $\check{s}'(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x\in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor$.

Because each spy participated in exactly one local game, playing the local games independently ensures automatically that each spy moves at most once in round t. Hence the spy moves we have described are feasible. It remains only to show that (1) holds for the resulting distribution of spies; that is

$$\hat{s}'(v) + \check{s}'(v) = \left\lfloor \frac{w'(v)}{m} \right\rfloor - \sum_{x\in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor \quad \text{for } v \in V(G).$$

Since the terms involving w^* again cancel, we use (7) to show that $\hat{s}'(v) + \check{s}'(v)$ equals the desired value $s'(v)$ in the same way we used (5) to show that the invented values $\hat{s}(v)$ and $\check{s}(v)$ sum to $s(v)$. □

3. Spy-good vs. spy-bad

It is not true that all spy-good graphs are webbed trees. Given G, let G^k denote the graph defined by $V(G^k) = V(G)$ and $E(G^k) = \{uv\colon\ d_G(u,v) \le k\}$. The spies can simulate one round of the game on G^k by playing k rounds on G. Thus $\sigma(G^k, m, r) \le \sigma(G, m, r)$, as noted by Howard and Smyth [4]. This makes the square of a webbed tree spy-good, even though it is not generally a webbed tree (consider $G = P_n$, for example).

Say that a spy strategy is *conformal* if at the end of each round the number of spies at each vertex v is at least $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries there. For any conformal spy strategy on G, the strategy described above for G^k is also conformal. Another graph operation also preserves the existence of conformal strategies.

Proposition 3.1. *Obtain G' from a graph G by expanding a vertex of G into a clique. If $\lfloor r/m \rfloor$ spies win* RS(G, m, r, s) *by a conformal strategy, then the same holds for G'.*

Proof. Let Q be the clique into which vertex v of G is expanded to form G'. The spies play on G' by imagining a game on G. At each round, the revolutionaries on Q in G' are collected onto v in G, with $r(v)$ there after the previous round and $r'(v)$ after the revolutionaries move. For other vertices, the amounts before and after are as in the real game on G'.

Since $\sum \lfloor a_i \rfloor \le \lfloor \sum a_i \rfloor$, the spies on v at the end of the round in G suffice to cover the $r'(v)$ revolutionaries on Q in G and can move there, since all vertices of Q have the same neighbors outside Q that v has in G. Extra spies move to any vertex of Q. Movements of spies from v in G can also be matched by moves in the game on G'. Other movements are the same in G and G'. This produces a conformal strategy on G'. □

Proposition 3.2. *On a webbed tree G, the winning strategy in Theorem 2.5 is conformal.*

Proof. Let T be a rooted spanning tree such that edges outside T join siblings in T. After each round, the number of spies on vertex v is given by

$$\left\lfloor \frac{r(v) + \sum_{x \in C(v)} w(x)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor .$$

Since $\sum \lfloor a_i \rfloor \le \lfloor \sum a_i \rfloor$, the strategy is conformal. □

These results imply that graphs obtained from webbed trees by vertex expansions and distance powers are spy-good. For example, the square of a path is spy-good. This graph is not a webbed tree, since it is 2-connected but has no dominating vertex (when it has at least six vertices). On the other hand, it is an interval graph, where an *interval graph* is a graph representable by assigning each vertex v an interval on the real line so that vertices are adjacent if and only if their intervals intersect. An interval graph that is not a distance power and has no two vertices with the same closed neighborhood is obtained from the square of an 8-vertex path by adding an edge joining the third and sixth vertices.

Question 3.3. *Which graphs are spy-good?*

We believe that all interval graphs are spy-good, even though the class is not contained in the spy-good classes obtained above.

Although not all graphs are spy-good, Theorem 2.2 yields good upper bounds on $\sigma(G, m, r)$ for graphs with small dominating sets. A *dominating set* in a graph G is a set $S \subseteq V(G)$ such that every vertex outside S has a neighbor in S; the *domination number* $\gamma(G)$ is the minimum size of a dominating set in G.

Corollary 3.4. *$\sigma(G, m, r) \le \gamma(G) \lfloor r/m \rfloor$ for any graph G.*

Proof. Let S be a smallest dominating set. With each vertex $u \in S$, associate $\lfloor r/m \rfloor$ spies. Let G_u be the subgraph of G induced by $N[u]$; it has u as a dominating vertex. The spies associated with u stay in G_u, following the strategy of Theorem 2.2 on G_u. When there are fewer than r revolutionaries in G_u, the spies imagine that the missing ones are at u. When a real revolutionary comes to vertex v in G_u from outside G_u, a revolutionary in the imagined game moves from u to v to perform its moves. When the real revolutionary leaves G_u, the revolutionary tracking it in the game on G_u returns to u. These moves are possible, since u is a dominating vertex in G_u. Since the spies win each imagined game, the revolutionaries in the real game never make an unguarded meeting at the end of a round. □

As remarked in the introduction, Corollary 3.4 is of interest only when $\gamma(G) \le m$, because otherwise the trivial upper bound $r - m + 1$ is stronger. When $\gamma(G) \le m$, the bound in Corollary 3.4 cannot be improved. To motivate the proof, we first present a simple construction of spy-bad graphs.

A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set. A *chordal graph* is a graph in which every cycle of length at least 4 has a chord; split graphs clearly have this property. Recall that for fixed r and m a graph is spy-bad if the revolutionaries can beat $r - m$ spies ($r - m + 1$ spies trivially win).

Proposition 3.5. *Given $r, m \in \mathbb{N}$, there is a chordal graph G (in fact a split graph) such that $\sigma(G, m, r) = r - m + 1$.*

Proof. Let $G_{m,r}$ be the split graph consisting of a clique Q of size r and an independent set S of size $\binom{r}{m}$, with the neighborhoods of the vertices in S being distinct m-sets in Q. We show that $r - m$ spies cannot win.

The revolutionaries initially occupy each vertex of Q. Let s' be the number of vertices of Q initially occupied by spies. The number of threatened meetings that spies on Q are not adjacent to is $\binom{r-s'}{m}$. Protecting against such threats requires putting spies initially on the $\binom{r-s'}{m}$ vertices of S corresponding to these m-sets, but only $r - m - s'$ remaining spies are available, and $\binom{r-s'}{m} > r - m - s'$ when $r - s' \ge m$. □

Note that $\frac{r-m+1}{r/m}$ can be made arbitrarily large. When $r = 2m$, the ratio exceeds $m/2$. Letting m also grow, we observe that $\sigma(G, m, r)$ cannot be bounded by a constant multiple of r/m, even on split graphs. Furthermore, the strategy for revolutionaries in Proposition 3.5 does not use any edges within the clique, so the statement remains true also for the bipartite graph obtained by deleting those edges.

When m grows, the degrees of all vertices in $G_{m,r}$ also grow. If the degrees in the independent set are bounded, then the spies can do better. We state the next result without proof, because the proof is a bit technical and the class of graphs is somewhat specialized. The technique is as usual for upper bounds: defining stable positions and showing that the spies can reestablish a stable position after each round. The proof will appear in the thesis of the third author.

Theorem 3.6. *Let G be a split graph with clique Q and independent set S in which each vertex of S has degree at most d. If m is a multiple of d, then $\sigma(G, m, r) \le d\lceil r/m\rceil$.*

A construction like that of Proposition 3.5 enables us to show that Corollary 3.4 is nearly sharp. When $t = m$, the upper and lower bounds in this result are equal; when $m \mid r$, the difference between them is $t - 1$.

Theorem 3.7. *Given $t, m, r \in \mathbb{N}$ such that $t \le m \le r - m$, there is a graph G with domination number t such that $\sigma(G, m, r) > t(r/m - 1)$.*

Proof. First we construct a graph G. Begin with a copy of $K_{t,r}$ having partite sets T of size t and R of size r. Add an independent set U of size $t\binom{r}{m}$, grouped into sets of size t. With each m-set A in R, associate one t-set A' in U. Make all of A adjacent to all of A', and add a matching joining A' to T (see Fig. 2). Note that T is a dominating set.

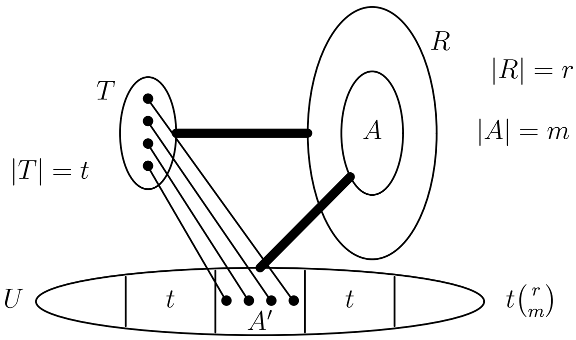

Fig. 2. Sharpness of the domination bound.

To show that $\gamma(G) = t$, let S be a smallest dominating set. For each m-set A in R, the t vertices in A' are adjacent only to A in R. Thus if $|S \cap R| < t \le r - m$, then some t-set A' in U is undominated by $S \cap R$. Outside of R, the closed neighborhoods of the vertices in A' are pairwise disjoint, so S needs t additional vertices to dominate them. Hence $|S| \ge t$.

Now, we give a strategy for the revolutionaries to win against $\lfloor t(r/m - 1)\rfloor$ spies on G. Let $s = \lfloor t(r/m - 1)\rfloor$. The revolutionaries initially occupy R, one on each vertex. A spy on a vertex u of U can protect all the same threats (and more) by locating at the neighbor of u in T instead. Hence we may assume (at least for the purpose of trying to survive the next round) that no spies locate initially in U.

Let v be a vertex of T having the fewest initial spies, and let $s(v)$ be the number of spies there. The revolutionaries will win by attacking the neighbors of v. Let s' be the number of spies initially in R, so $s(v) \le (s - s')/t$.

The revolutionaries want to form meetings at $s(v) + 1$ neighbors of v that are neighbors of no other vertices with spies. Let R' be the set of vertices in R that do not have spies; note that $|R'| \ge r - s'$. If $|R'| \ge m(s(v) + 1)$, then the revolutionaries win as follows. First, group vertices in R' into $s(v) + 1$ sets of size m. For each such set A, the revolutionaries on A move to the unique vertex $u_{A,v}$ in the associated subset A' of U that is adjacent to v in T. For each such vertex, the only neighbor having a spy is v, so the meetings cannot all be guarded and the revolutionaries win.

It thus suffices to show that $r - s' \ge m(s(v) + 1)$. Since v has the fewest spies among vertices of T, we have $ts(v) \le s - s' \le t(r/m - 1) - s'$. Multiplying by m/t and adding m yields $m(s(v) + 1) \le r - s'(m/t) \le r - s'$, as desired, using $t \le m$ at the end. $\square$

Although the construction in Theorem 3.7 depends heavily on m, it does not depend much on r. Indeed, the construction works equally well whenever the number of revolutionaries is at most r, because the revolutionaries can use the strategy for a smaller number of revolutionaries on the appropriate subgraph of the graph constructed for r revolutionaries. The same comment applies to Proposition 3.5.

4. Hypercubes and retracts

For $d \in \mathbb{N}$, let $[d] = \{1, \ldots, d\}$. The d-dimensional hypercube Q_d is the graph with vertex set $\{v_S : S \subseteq [d]\}$ such that v_S and v_T are adjacent when the symmetric difference of S and T has size 1. The *weight* of the vertex v_S is $|S|$. For vertices of small weight, we write the subscripts without set brackets. We show first that Q_d is spy-bad for $m = 2$ when $d \ge r$. For larger m, we will later obtain a lower bound on $\sigma(Q_d, m, r)$ using the same basic idea.

Theorem 4.1. *If $G = Q_d$ and $d \geq r$, then $\sigma(G, 2, r) = r - 1$.*

Proof. The upper bound is trivial; we show that $r - 2$ spies cannot win. The revolutionaries begin by occupying $v_1, \ldots, v_r$, threatening meetings of size 2 at $\varnothing$ and at $\binom{r}{2}$ vertices of weight 2. Let t be the number of revolutionaries left uncovered by the initial placement of the spies. Threats at $\binom{t}{2}$ vertices must be watched by spies not on vertices of weight 1. A spy at a vertex of weight 2 can watch one such threat; spies at vertices of weight 3 can watch three of them. Hence $s \geq (r-t) + \frac{1}{3}\binom{t}{2}$ if the spies stop the revolutionaries from winning on the first round. This yields $s \geq r - 1$ if $t \geq 5$ or $t \leq 2$.

If $t = 4$ and $s = r - 2$, then the spies need to watch six threats at weight 2 using two spies at vertices of weight 3. A spy at a vertex of weight 3 watches the three pairs in its name. The four uncovered revolutionaries threaten meetings at six vertices of weight 3 corresponding to the edges of the complete graph K_4. A spy at weight 3 can watch three pairs corresponding to a triangle. Since the edges of K_4 cannot be covered with two triangles, $r - 2$ spies are not enough when $t = 4$.

If $t = 3$, then the counting bound yields $s \geq r - 2$ for spies to avoid losing on the first round. If the initial placement of $r - 2$ spies can watch all immediate threats, then they must cover $r - 3$ revolutionaries at vertices of weight 1 and occupy one vertex at weight 3. By symmetry, we may assume the spies locate at v_{123} and $v_4, \ldots, v_r$.

In the first round, revolutionaries at v_1 and v_2 move to $v_\varnothing$; the others wait where they are. To guard the meeting at $v_\varnothing$, a spy at some vertex of weight 1 must move there; let v_j be the vertex from which a spy moves to $v_\varnothing$.

In the second round, the revolutionaries at v_3 and v_j move to v_{3j}, winning. The distance from each spy to v_{3j} after round 1 is at least 3, except for the spy at v_j, so no other spy could have moved after round 1 to watch that threat. □

Extra spies on vertices of weight at least 5 cannot prevent the revolutionaries from winning with the strategy given in the proof of Theorem 4.1. This enables the revolutionaries to win against somewhat fewer spies when r is larger than the dimension.

A *code* with length d and distance k is a set of vertices in Q_d such that the distance between any two of them is at least k. Let $A(d, k)$ denote the maximum size of a code with distance k in Q_d, and let $B(d, k)$ be the number of vertices with distance less than k from a fixed vertex in Q_d. Note that $B(d, k) = \sum_{i=0}^{k-1} \binom{d}{i} < d^{k-1}$ when $k > 2$. If $M < 2^d/B(d, k)$, then any code of size M having distance k can be extended by adding some vertex, so $A(d, k) \geq 2^d/d^{k-1}$ when $k > 2$.

Corollary 4.2. *If $d < r \leq 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d - 1)\lfloor r/d \rfloor$.*

Proof. Let X be a code in Q_d with distance 9 and size at least $2^d/d^8$. The revolutionaries devote d revolutionaries to playing the strategy in the proof of Theorem 4.1 at each of $\lfloor r/d \rfloor$ vertices of X. If the ball of radius 4 at any such vertex has fewer than $d - 1$ spies in the initial configuration, then the revolutionaries win in that ball in two rounds, since any spy initially outside that ball is too far away to guard a meeting formed at distance 2 from the central point in round 2.

Since the code has distance 9, the balls of radius 4 are disjoint. Hence $(d - 1)\lfloor r/d \rfloor$ spies are needed to keep the revolutionaries from winning within two rounds. □

Theorem 4.1 and Corollary 4.2 together imply that at least $(d-1)\lfloor r/d \rfloor$ spies are needed to win against r revolutionaries on Q_d unless $d < \log_2 r + 7\log_2\log_2 r$. That many spies may not be enough, since three revolutionaries easily defeat one spy on Q_2 by starting initially at distinct vertices. Although four revolutionaries can threaten meetings at all eight vertices of Q_3, two spies can watch all those meetings and survive the next round. It appears that $\sigma(Q_3, 2, 4) = 2$, though we have not worked out a complete strategy for two spies against four revolutionaries. We have no nontrivial general upper bounds on $\sigma(Q_d, 2, r)$ when $r > d$.

Next we consider the game on hypercubes when $m > 2$. Again we use the threats made by revolutionaries placed initially at vertices of weight 1. However, for larger m we use a probabilistic argument instead of explicit counting. The probabilistic arguments are simpler and yield a stronger lower bound on $\sigma(Q_d, m, r)$ than the counting arguments would, but we no longer completely determine the threshold (and hence we separate this from the case $m = 2$). Again $V(Q_d) = \{v_S : S \subseteq [d]\}$, as specified as before Theorem 4.1.

Lemma 4.3. *For $v \in V(Q_d)$, a vertex u of weight m is within distance $m - 1$ of v if and only if $|u \cap v| \geq \frac{|v|+1}{2}$.*

Proof. The distance between any two vertices is their symmetric difference. Always the size of the symmetric difference is $|u| + |v| - 2|u \cap v|$. When $|u| = m$, it follows that $d_{Q_d}(u, v) \leq m - 1$ is equivalent to $|u \cap v| \geq \frac{|v|+1}{2}$. □

Our main tool for the game on Q_d is a lemma about families of sets.

Lemma 4.4. *Let S be a set of at most t vertices in Q_t, all having weight at least 2. If $t \geq 38.73m$, then Q_t has a vertex w of weight m such that $d_{Q_t}(v, w) \geq m$ for all $v \in S$.*

Proof. Fix $p \in (0, 1)$, to be determined later. Construct a random index set $I \subseteq [t]$ by independently including each element of $[t]$ with probability p. In light of Lemma 4.3, for $v \in S$ we say that I *avoids* v if $|v \cap I| < \frac{|v|+1}{2}$. Our goal is to show that with p chosen appropriately, with positive probability I avoids all of S and has size at least m. The desired vertex w can then be any vertex of weight m contained in such a set I. Our first task is to obtain a lower bound on $\mathbb{P}[A_v]$, where A_v is the event that I avoids v.

Let $\mathrm{Bin}(n, p)$ denote a random variable having the binomial distribution with n trials and success probability p. Let B be the event that $2k+1$ trials yield k successes in the first $2k-1$ trials plus two failures at the end. Let B' be the event that $2k+1$ trials yield $k-1$ successes in the first $2k-1$ trials plus two successes at the end. Canceling common factors yields $\mathbb{P}[B] > \mathbb{P}[B']$ if and only if $p < 1/2$. As a consequence, $\mathbb{P}[\mathrm{Bin}(2k+1, p) < k+1] > \mathbb{P}[\mathrm{Bin}(2k-1, p) < k]$ when $p < 1/2$. Note also that $\mathbb{P}[\mathrm{Bin}(2k-2, p) < k] \geq \mathbb{P}[\mathrm{Bin}(2k-1, p) < k]$.

Now let $k = \left\lceil \frac{|v|+1}{2} \right\rceil$, so $k \geq 2$ and $|v| \in \{2k-2, 2k-1\}$. For the event that I has fewer than k elements of v, our observations about the binomial distribution yield

$$\mathbb{P}[A_v] \geq \mathbb{P}[\mathrm{Bin}(2k-1, p) < k] \geq \mathbb{P}[\mathrm{Bin}(3, p) < 2] = (1-p)^2(1+2p).$$

Let $q = \min_v \mathbb{P}[A_v]$. Events of the form A_v are down-sets in the subset lattice. By the FKG inequality (see Theorem 6.2.1 of [1]), such events are positively correlated when $p < 1/2$, so

$$\mathbb{P}\left[\bigcap_{v\in S} A_v\right] \geq q^t = e^{t \ln q}.$$

Now let $X = |I|$. For $m \leq \alpha t p$ with $\alpha < 1$, Chernoff's Inequality yields

$$\mathbb{P}[X < m] = \mathbb{P}[X - tp < m - tp] \leq e^{-(m-tp)^2/(2tp)} = e^{-(1-\alpha)^2 tp/2}.$$

Our goal is to show $\mathbb{P}\left[\bigcap_{v\in S} A_v\right] > \mathbb{P}[X < m]$, which follows from

$$\ln[(1-p)^2(1+2p)] > -(1-\alpha)^2 p/2.$$

With $\alpha = .324722$ and $p = .079532$, the strict inequality holds, and we obtain $\alpha p \approx .0258259$. Hence when $d \geq m/(\alpha p) \geq 38.73m$, some m-set avoids all vertices in S. $\square$

Before we apply this lemma to the game on the hypercube, we prove a general result that relates the game on a graph and its retracts. The notion of retract appeared as early as [3], as a homomorphism fixing a subgraph. The variation from [6] that we use becomes the homomorphism version when loops are available at all vertices.

Definition 4.5. An induced subgraph H of a graph G is a *retract* of G if there is a map $f: V(G) \to V(H)$ such that (1) $f(v) = v$ for $v \in V(H)$, and (2) $uv \in E(G)$ implies that $f(u)$ and $f(v)$ are equal or adjacent.

Nowakowski and Winkler [6] proved a theorem for the classical cop-and-robber pursuit game that is analogous to our next result.

Theorem 4.6. *Let H be a retract of a graph G. If the revolutionaries win $\mathrm{RS}(H, m, r, s)$, then the revolutionaries win $\mathrm{RS}(G, m, r, s)$. Equivalently, $\sigma(G, m, r) \geq \sigma(H, m, r)$.*

Proof. Let $f: G \to H$ be as guaranteed in Definition 4.5. The revolutionaries play in G by playing exclusively on H, using the map f to play as if the spies in $V(G) - V(H)$ were actually in $V(H)$.

The revolutionaries take initial positions as specified by their winning strategy on H. They simulate a spy on $v \in V(G)$ by a spy on $f(v) \in V(H)$. Whenever a spy can legally move from u to v in G, the definition of retract guarantees that the simulated spy can move from $f(u)$ to $f(v)$ in H. Therefore, the simulated spies always play legal moves in the imagined game. The revolutionaries play their winning strategy against the simulated spies in H and eventually form an uncovered meeting at some vertex w. Since $f(w) = w$, the absence of a simulated spy on w means that there is no real spy on w, and the revolutionaries have won the "real game" in G. $\square$

Theorem 4.7. *If $s \leq r - 38.73m$ and $d \geq r$, then the revolutionaries win $\mathrm{RS}(Q_d, m, r, s)$.*

Proof. The revolutionaries initially occupy $v_1, \ldots, v_r$. The revolutionaries threaten meetings after $m-1$ steps at $\binom{r}{m}$ vertices of weight m. The vertices of weight m protected by a spy at v_i are precisely those whose corresponding sets contain i. Let t be the number of revolutionaries left uncovered by the initial placement of spies. By symmetry, we may assume that the uncovered revolutionaries are at $v_1, \ldots, v_t$. Let S be the set of spies initially on vertices having weight at least 2; only such spies can protect vertices in the set of $\binom{t}{m}$ vertices of weight m above uncovered revolutionaries. Note that $0 \leq |S| \leq s - (r-t) \leq t - 38.73m$, and hence $t \geq 38.73m$.

Every subcube of Q_d is a retract of Q_d, by projection. Hence by Theorem 4.6, we may assume that the spies in S are all in Q_t. We can therefore apply Lemma 4.4. With $t \geq 38.73m$ and $|S| \leq t - 38.73m < t$, some vertex of weight m in Q_t is too far from S to be reached by any spy within $m-1$ rounds, and the revolutionaries win. $\square$

Although $|S| \leq t - 38.73m$ in Theorem 4.7 while Lemma 4.4 allows $|S| \leq t$, generalizing the lemma to vary $|S|$ in terms of t does not noticeably strengthen the application.

When $t \geq 2m$, an explicit counting bound on the number of vertices of weight m in Q_t that are within distance $m-1$ of a given vertex of S leads to the following theorem.

Theorem 4.8. *If $d \geq r \geq m \geq 3$ and $s \leq r - \frac{3}{4}m^2$, then the revolutionaries win $\mathrm{RS}(Q_d, m, r, s)$, so $\sigma(Q_d, m, r) > r - \frac{3}{4}m^2$.*

Theorem 4.8 is stronger than Theorem 4.7 when $m \leq 52$. We omit the proof, because the proofs of this counting lemma and theorem are longer and more technical than those of Lemma 4.4 and Theorem 4.7, and because we believe that the revolutionaries may win against as many as $r - 2m$ spies. The proof will appear in the thesis of the third author.

As in Theorem 4.1, the revolutionaries in Theorem 4.7 play locally, winning by staying within distance m of a fixed vertex. Hence with general meeting size m we can apply the same coding theory argument as in Corollary 4.2. Given a code with distance $4m - 1$, the balls of radius $2m - 1$ are disjoint. Any vertex with distance more than $2m - 1$ from the central point has distance more than $m - 1$ from the threatened meetings and cannot reach them in $m - 1$ turns, which is the number of rounds the revolutionaries need to win in the strategy of Theorem 4.7. We thus have the following.

Corollary 4.9. *If $d < r \leq 2^d/d^{4m}$, then $\sigma(Q_d, m, r) > (d - 38m)\lfloor r/d \rfloor$.*

Finally, the hypercube result applies to more general cartesian products via the notion of retract. For $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of G induced by U.

Corollary 4.10. *Let $G = G_1 \square \cdots \square G_d$, where $G_1, \ldots, G_d$ are graphs with at least one edge. If the revolutionaries win $\mathrm{RS}(Q_d, m, r, s)$, then the revolutionaries win $\mathrm{RS}(G, m, r, s)$.*

Proof. By Theorem 4.6, it suffices to show that G contains a retract isomorphic to Q_d. Select $v_i w_i \in E(G_i)$ for each i, and let $U = \{v_1, w_1\} \times \cdots \times \{v_d, w_d\}$. Note that $G[U] \cong Q_d$.

To define $f : V(G) \to U$, first define $g_i : V(G_i) \to \{v_i, w_i\}$ by setting $g_i(x) = v_i$ if $x = v_i$ and $g_i(x) = w_i$ otherwise. Now let $f(x_1, \ldots, x_d) = (g_1(x_1), \ldots, g_d(x_d))$. Clearly f fixes U. If $xy \in E(G)$, then there exists exactly one i such that $x_i \neq y_i$; without loss of generality, $x_i \neq v_i$. If also $y_i \neq v_i$, then $g_i(x_i) = g_i(y_i) = w_i$, so $f(x) = f(y)$.

On the other hand, if $y_i = v_i$, then $g_i(x_i) = w_i$ and $g_i(y_i) = v_i$ while $g_j(x_j) = g_j(y_j)$ for all $j \neq i$, so $f(x)f(y) \in E(G[U])$ since $w_i v_i \in E(G)$. Therefore f satisfies the conditions in Definition 4.5, and $G[U]$ is a retract of G isomorphic to Q_d. □

5. Random graphs

In the Erdős–Renyi binomial model $G(n, p)$, the vertex set is $[n]$, pairs of vertices occur as edges independently with probability p, and we say that an event occurs *almost surely* if its probability tends to 1 as $n \to \infty$.

When the graph is randomly generated and there are not too many revolutionaries, the revolutionaries can play a strategy like that in Proposition 3.5 to defeat $r - m$ spies: the revolutionaries occupy vertices so that no matter where the spies are placed, any m uncovered vertices can meet at some vertex adjacent to no spy. When the number of revolutionaries is larger, also the allowed number of spies is larger; the revolutionaries no longer can find such a placement, and the number of spies needed is only a fraction of r.

Our main task in this section is to show that for constant edge-probability p, these two situations for the number of revolutionaries are surprisingly close together, differing only by a constant factor. In particular, when $r < \ln 2 \ln n$ the revolutionaries almost always win against $r - m$ spies, and when $r > cm \ln n$ almost always cr/m spies can win, where c is any constant greater than 4. The argument in the first setting also yields results when p depends on n.

Independently, Mitsche and Prałat [5] have proved that for G in $G(n, p)$, almost surely $\sigma(G, m, r) \leq \frac{r}{m} + 2(2 + \sqrt{2} + \epsilon) \log_{1/(1-p)} n$; here p can depend on n (they also obtain conditions under which $r - m + 1$ spies are needed). Their upper bound is sharp within an additive constant, but also they require r to grow faster than $(\log n)/p$. In comparison to our method, they use more intricate structural characteristics of the random graph and a more complex strategy for the spies. Our strategy for the spies is like that used elsewhere in this paper: introduce a notion of "stable position" that keeps the meetings covered, and show that the spies can maintain a stable position.

First we consider the range where $r - m + 1$ spies are needed. Motivated by Alon and Spencer [1], we say that G has the *r-extension property* if for any disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there is a vertex $x \in V(G)$ adjacent to all of T and none of U. We first show why this property makes the game easy for the revolutionaries.

Proposition 5.1. *If a graph G satisfies the r-extension property, and $m \leq r' \leq r$, then G is spy-bad for r' revolutionaries and meeting size m.*

Proof. The r' revolutionaries initially occupy any set of r' vertices in G. To see that $r' - m$ spies cannot prevent them from winning on the first round, let U be the set occupied by the spies, and let T be the set occupied by uncovered revolutionaries. The revolutionaries on T win by moving to the vertex x guaranteed by the r-extension property. □

Alon and Spencer [1, Theorem 10.4.5] present the result below for constant r, but the proof holds more generally.

Theorem 5.2. *Let $\epsilon = \min\{p, 1 - p\}$, where p is a probability that depends on n. If $r = o\left(\frac{n\epsilon^r}{\ln n}\right)$ and $n\epsilon^r \to \infty$, then $G(n, p)$ almost surely has the r-extension property (and hence is spy-bad for all m and r' with $m \leq r' \leq r$).*

Proof. Let G be distributed as $G(n, p)$. Given $T, U \subset V(G)$ with $|T| + |U| \leq r$, write $t = |T|$ and $u = |U|$. For $x \in V(G) - (T \cup U)$, let $A_{T,U,x}$ be the event that x is adjacent to all of T and none of U; note that $\mathbb{P}[A_{T,U,x}] = p^t(1 - p)^u \geq \epsilon^r$.

Let $A_{T,U}$ be the event that $A_{T,U,x}$ fails for all $x \in V(G) - (T \cup U)$. The events $A_{T,U,x}$ for different x are determined by disjoint sets of vertex pairs, so $\mathbb{P}[A_{T,U}] \leq (1 - \epsilon^r)^{n-r} \leq e^{-\epsilon^r(n-r)}$.

The r-extension property fails if and only if some event of the form $A_{T,U}$ occurs. Hence it suffices to show that the probability of their union tends to 0. There are 3^r ways to form T and U within a fixed r-set of vertices, since a vertex can be in either set or be omitted, and there are $\binom{n}{r}$ sets of size r. Hence the union consists of at most $(3n)^r$ events, each of whose probability is at most $e^{-\epsilon^r(n-r)}$. We compute

$$(3n)^r e^{-\epsilon^r(n-r)} = e^{r\ln(3n)-\epsilon^r(n-r)} = e^{r\ln 3+r\ln n-\epsilon^r(n-r)}.$$

Since $\epsilon \le 1/2$, the condition $r = o\left(\frac{n\epsilon^r}{\ln n}\right)$ implies $r = o(n)$, so the exponent is dominated by $-n\epsilon^r$ and tends to $-\infty$. Thus the bound on the probability of lacking the r-extension property tends to 0, and $G(n, p)$ almost surely satisfies this property. □

In particular, when p is constant, $G(n, p)$ is almost surely spy-bad for $r \ge m$ when $r \le c\ln n$, where $c < \ln(1/\epsilon)$. Similarly, when r is constant, $G(n, p)$ is almost surely spy-bad when p tends to 0 more slowly than $1/n^{1/r}$. With $p \le 1/2$, the key condition is $np^r \to \infty$.

Now we confine our attention to the realm of constant edge-probability p and consider well-known properties of the random graph that enable the spies to do well. For every vertex, the expected degree is $p(n-1)$, and for any two vertices the expected size of their common neighborhood is $p^2(n-2)$. Moreover, these random variables are so highly concentrated at their expectations that almost always the degrees of *all* vertices and the sizes of common neighborhoods of *all* pairs are within constant factors of their expected values. We begin by stating this formally; the proofs are standard and straightforward using the Chernoff Bound. We treat G as a sample from the model $G(n, p)$.

Lemma 5.3. *Fix p and γ with $0 < \gamma < p < 1$. In the random graph model $G(n, p)$, almost surely $(p-\gamma)n < d(v) < (p+\gamma)n$ and $(p^2-\gamma^2)n < |N(v) \cap N(w)| < (p^2+\gamma^2)n$ for all $v, w \in V(G)$.*

Lemma 5.4. *Fix p and γ with $0 < \gamma < p < 1$. In the random graph model $G(n, p)$, almost surely $\frac{|N(v)\cap N(w)|}{|N(v)|} \ge p-\gamma$ for all $v, w \in V(G)$.*

Proof. Using the lower bound on common neighborhood size and the upper bound on degree from Lemma 5.3, almost surely $\frac{|N(v)\cap N(w)|}{|N(v)|} \ge \frac{(p^2-\gamma^2)n}{(p+\gamma)n} = p-\gamma$ for all $v, w \in V(G)$. □

Definition 5.5. For $q \in (0, 1)$, a graph G is *q-common* if $\frac{|N(v)\cap N(w)|}{|N(v)|} \ge q$ for all $v, w \in G$.

We develop a strategy for spies that will be successful on q-common graphs under certain conditions. In a game position, we need to distinguish players occupied in forming or covering meetings from those who are not. These notions will also be important for spy strategies on complete multipartite or bipartite graphs.

Definition 5.6. Given a game position, say that m specified revolutionaries in a meeting and one spy covering them are *bound*. After designating the bound players for all vertices hosting meetings, the remaining spies and revolutionaries are *free*. A vertex having at least m revolutionaries has exactly m bound revolutionaries.

For a vertex subset U, let r_U and $\hat{r}_U$ denote the total number of revolutionaries and number of free revolutionaries on U. Similarly, let s_U and $\hat{s}_U$ denote the total number of spies and number of free spies on U. Write $\hat{r}$ and $\hat{s}$ for $\hat{r}_{V(G)}$ and $\hat{s}_{V(G)}$. A game position is *stable* if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \ge \hat{r}/m$ for all $v \in V(G)$.

As in Section 2, the name *stable* is motivated by permitting the game to continue.

Lemma 5.7. *On any graph G, if the position at the beginning of a round is stable, then the spies can respond to cover all meetings at the end of the round.*

Proof. Let the notation in Definition 5.6 refer to the counts at the beginning of round t, in a stable position. Let X be the set of distinct vertices hosting meetings after the revolutionaries move in round t. Let Y be the set of spies. Define an auxiliary bipartite graph H with partite sets X and Y. For $x \in X$ and $y \in Y$, put $xy \in E(H)$ if spy y can reach x from its position at the start of round t, being adjacent to x or already there. If some matching in H covers X, then the spies can move in round t to cover all the meetings.

It suffices to show that H satisfies Hall's Condition for a matching that covers X. Consider $S \subseteq X$. If $N_G[S]$ contains b vertices that hosted meetings at the start of round t, then $|S| \le \frac{\hat{r}+mb}{m}$, because revolutionaries who were in meetings not in $N_G[S]$ cannot reach S in one move. On the other hand, every free spy at a vertex of $N_G[S]$ can reach S in one move, as can every spy bound to a meeting in S. Choosing $x \in S$, we have

$$|N_H(S)| \ge \hat{s}_{N[x]} + b \ge \hat{r}/m + b \ge |S|.$$

Hence Hall's Condition is satisfied and the matching exists. □

The next lemma provides the second half of what the spies need to do.

Lemma 5.8. *Let G be a q-common graph with n vertices, and fix $\epsilon > 0$. Given a position in* RS(G, m, r, s) *such that (1) all meetings are covered, (2) $\hat{s} \ge \frac{1+\epsilon}{q}\frac{\hat{r}}{m}$, and (3) $\hat{s} \ge \frac{\ln n}{2(1-1/(1+\epsilon))^2q^2}$, the free spies can move to produce a stable position.*

Proof. We prove that if each free spy moves to a uniformly random vertex in the neighborhood of its current position, then with positive probability a stable position is produced.

For $v \in V(G)$, let X_v be the number of spies in $N[v]$ after the frees spies move. Since G is q-common, each free spy lands in $N[v]$ with probability at least q. Also, these events for individual spies are independent, so X_v is a sum of $\hat{s}$ independent indicator variables, each with success probability at least q. By the Chernoff Bound, $\mathbb{P}[X_v - \mathbb{E}[X_v] < -a] < e^{-2a^2/\hat{s}}$ for any positive a. Since $\mathbb{E}[X_v] \geq q\hat{s}$, taking $a = \left(1 - \frac{1}{1+\epsilon}\right) q\hat{s}$ yields

$$\mathbb{P}\left[X_v < \frac{1}{1+\epsilon} q\hat{s}\right] < e^{-2\left(1-\frac{1}{1+\epsilon}\right)^2 q^2 \hat{s}} \leq e^{-\ln n} = 1/n,$$

where the simplification of the exponent uses hypothesis (3).

Since G has n vertices, with positive probability each vertex receives at least $\frac{1}{1+\epsilon} q\hat{s}$ free spies in its neighborhood. By condition (2), this quantity is at least $\hat{r}/m$. Hence there is some move by the free spies after which each closed neighborhood has at least $\hat{r}/m$ free spies, making the position stable. □

Theorem 5.9. *Let G be a q-common graph with n vertices, and fix $\epsilon > 0$. If $s \geq \frac{1+\epsilon}{q} \frac{r}{m}$ and $s \geq \frac{r}{m} + \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2}$, then the spies win* RS$(G, m, r, s)$.

Proof. If they can produce a stable position via the initial placements, then the spies use the following strategy in each subsequent round to produce a stable position. In Phase 1, they cover all meetings by moving the fewest possible spies. In Phase 2, they move the spies who are then free to produce a stable position.

Since every spy moved in Phase 1 covers a meeting (by the condition of moving the fewest spies), this strategy never moves a spy twice in one round. Since the position at the beginning of the round is stable, Lemma 5.7 implies that spies can move to cover all meetings. Hence Phase 1 can be performed. (Also, in the initial placement the spies can start by covering all meetings, since $s \geq r/m$.)

If $\hat{s}$ is now large enough to satisfy the hypotheses of Lemma 5.8, then the free spies can complete Phase 2. This argument is also used to complete the initial placement: after covering the initial meetings, the free spies imagine being at an arbitrary vertex, and then Lemma 5.8 guarantees that they can "move" (that is, be placed) to satisfy the neighborhood requirement for stability.

Consider the position after Phase 1; all meetings are covered. Since at most r/m spies can be bound, the second assumed lower bound on s yields $\hat{s} \geq s - \frac{r}{m} \geq \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2}$.

Finally, we use the given lower bound $s \leq \frac{1+\epsilon}{q} \frac{r}{m}$ to obtain the needed lower bound $\hat{s} \leq \frac{1+\epsilon}{q} \frac{\hat{r}}{m}$ that completes the hypotheses of Lemma 5.8. Let $\bar{r}$ denote the number of bound revolutionaries at the start of the round. Since $q < 1 < 1 + \epsilon$, we have

$$\hat{s} = s - \frac{\bar{r}}{m} \geq \frac{1+\epsilon}{q} \frac{r}{m} - \frac{1+\epsilon}{q} \frac{\bar{r}}{m} = \frac{1+\epsilon}{q} \frac{\hat{r}}{m}.$$

We have shown that Phase 1 and Phase 2 can be completed to maintain a stable position after each round. □

Theorem 5.10. *Fix p and q with $0 < q < p < 1$. In the random graph model $G(n, p)$, almost always G has the following property for all $m \in \mathbb{N}$: if $s \geq \frac{1+\epsilon}{q} \frac{r}{m}$ and $s \geq \frac{r}{m} + \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2}$, then the spies win* RS$(G, m, r, s)$.

Proof. By Lemma 5.4, almost always G is q-common. By Theorem 5.9, the spies win in the given parameter range on every q-common graph. □

Since $1/q > 1$, the next hypotheses imply the hypotheses of Theorem 5.10.

Corollary 5.11. *For p, q, G as above, almost surely G has the following property for all $m \in \mathbb{N}$: if $s \geq \frac{1+\epsilon}{q} \frac{r}{m}$ and $r \geq \frac{(1+\epsilon)^2 m \ln n}{2\epsilon^3 q}$, then the spies win* RS$(G, m, r, s)$.

In particular, for the random graph with $p = 1/2$, setting $\epsilon = 1$ and letting q approach $1/2$ from below yields the following simply-stated corollary.

Corollary 5.12. *Almost every graph G has the following property for all $m \in \mathbb{N}$ and $c > 4$: if $s \geq c\frac{r}{m}$ and $r \geq cm \ln n$, then the spies win* RS(G, m, r, s).

For sparse graphs, as $p \to 0$, we also need $q \to 0$, and the needed number of revolutionaries to apply our method grows at a faster rate than $m \ln n$. Hence for sparse graphs we do not obtain the conclusion that the ranges for r where the needed number of spies behaves like cr/m or like $r - m$ are close together.

6. Complete *k*-partite graphs

In this section we obtain lower and upper bounds on $\sigma(G, m, r)$ when G is a complete k-partite graph. The lower bound requires partite sets large enough so that the revolutionaries can always access as many vertices in each part as they might want (enough to "swarm" to distinct vertices there that avoid all the spies). The upper bounds apply more generally; they do not require large partite sets, and they require only a spanning k-partite subgraph (if there are additional edges within parts, then spies will be able to follow revolutionaries along them when needed).

Definition 6.1. A complete k-partite graph G is r-*large* if every part has at least $2r$ vertices. At the revolutionaries' turn on such a graph, an i-*swarm* is a move in which the revolutionaries make as many new meetings of size m as possible in part i. All revolutionaries outside part i move to part i, greedily filling uncovered partial meetings to size m and then making additional meetings of size m from the remaining incoming revolutionaries. When G is r-large, sufficient vertices are available in part i to permit this.

Theorem 6.2. *Let G be an r-large complete k-partite graph. If $k \geq m$, then $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{k\lfloor r/k \rfloor}{m+c} - k$, where $c = 1/(k-1)$. When $k \mid r$ the bound simplifies to $\frac{k}{k-1}\frac{r}{m+c} - k$.*

Proof. We may assume that $k \mid r$, since otherwise the revolutionaries can play the strategy for the next lower multiple of k, ignoring the extra revolutionaries.

Let $t = r/k$. The revolutionaries initially occupy t distinct vertices in each part. Let s_i be the initial number of spies in part i. We may assume that they cover $\min\{s_i, t\}$ distinct revolutionaries, since each vertex of part i has the same neighborhood, and within part i these are the best locations. We compute the number of spies needed to avoid losing by a swarm on round 1.

Case 1: *$s_i > t$ for some i.* If the revolutionaries swarm to part i, then all revolutionaries previously in part i are covered, so new meetings consist entirely of incoming revolutionaries and are not coverable by spies from part i. Since $(k-1)t$ revolutionaries arrive, at least $\lfloor (k-1)t/m \rfloor$ spies must arrive from other parts to cover the new meetings. Thus

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t\left(1 + \frac{k-1}{m}\right) = \frac{k-1+m}{k}\frac{r}{m}.$$

Case 2: *$s_i \leq t$ for all i.* For each i, part i has $t - s_i$ partial meetings. Since $s_i \geq 0$, an i-swarm is guaranteed to fill them if $(k-1)t \geq t(m-1)$, which holds when $k \geq m$. Hence the new meetings include all revolutionaries except the s_i covered by spies in part i before the swarm. Spies from other parts must cover $\lfloor (r - s_i)/m \rfloor$ new meetings in part i. Summing $s - s_i \geq (r - s_i - m + 1)/m$ over all parts yields $(k - 1 + 1/m)s \geq k(r - m + 1)/m$, so

$$s \geq \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1}\frac{r}{m+c} - k.$$

The lower bound in Case 2 is smaller (better for spies) than the lower bound in Case 1, so the spies will prefer to play that way. The lower bound in Case 2 is thus a lower bound on $\sigma(G, m, r)$. □

As in Section 5, our strategy for spies maintains a "stable position", defined by invariants ensuring that the spies can cover all meetings and reestablish a stable position. Indeed, for complete multipartite graphs the notion of stable position is very similar to what it was in the random graph.

Definition 6.3. Define bound and free revolutionaries and spies as in Definition 5.6. Let $\hat{r}_i$ and $\hat{s}_i$ denote the numbers of free revolutionaries and free spies in part i in the current position of a game on a complete k-partite graph. Let $\hat{r}$ and $\hat{s}$ denote the total numbers of free revolutionaries and free spies. A game position is *stable* if (1) all meetings are covered, and (2) $\hat{s} - \hat{s}_i \geq \hat{r}/m$ for each part i.

Since the neighborhood of a vertex in a complete multipartite graph consists of all the partite sets not containing it, for such a graph G the condition for a stable position is the same as it was in Section 5.

Lemma 6.4. *Let G be a graph having a spanning complete k-partite subgraph G'. If the position at the start of round t is stable for G', then the revolutionaries cannot win in the current round on G. (As always, assume $s \geq \lfloor r/m \rfloor$.)*

Proof. Since the closed neighborhood of every vertex in a complete multipartite graph includes all vertices outside its partite set, we have $\hat{s}_{N[v]} \geq \hat{s} - \hat{s}_i$ for all $v \in V(G)$. Hence Lemma 5.7 applies. □

Theorem 6.5. *If a graph G has a spanning complete k-partite subgraph, then $\sigma(G, m, r) \leq \left\lceil \frac{k}{k-1}\frac{r}{m} \right\rceil + k$.*

Proof. Let G' be the specified subgraph, and let $s = \left\lceil \frac{k}{k-1}\frac{r}{m} \right\rceil + k$. It suffices to show that s spies can produce a stable position at the end of each round. First, after the revolutionaries have moved, the spies cover all newly created meetings, moving the fewest possible spies to do so. By Lemma 6.4, the spies can do this since the previous round ended in a stable position (also, $s \geq \lfloor r/m \rfloor$ guarantees that the spies can do this in the initial position).

Next, the spies that are now free distribute themselves equally among the k parts of G'. More precisely, with $\hat{s}$ being the total number of free spies after the new meetings are covered and $\hat{s}_i$ being the number of them in part i, we have $|\hat{s}_i - \hat{s}/k| < 1$ for all i.

It suffices to show that this second step produces a stable position. In order to have $\hat{s} - \hat{s}_i \geq \hat{r}/m$ for all i, it suffices to have $\hat{s}_j \geq \hat{r}/[m(k-1)]$ for each j. Since the free spies are distributed equally, it suffices for the average to be big enough: $\hat{s}/k \geq \hat{r}/[m(k-1)] + 1$. Multiplying by k, we require $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$.

We are given $s \geq \frac{k}{k-1}\frac{r}{m} + k$. The number of bound revolutionaries is exactly m times the number of bound spies; hence $s - \hat{s} = (r - \hat{r})/m$. Subtracting this equality from the given inequality yields

$$\hat{s} \geq \frac{1}{k-1}\frac{r}{m} + \frac{\hat{r}}{m} + k \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k,$$

where the last inequality uses $r \geq \hat{r}$. We now have the inequality that we showed suffices for a stable position. $\square$

7. Complete bipartite graphs

Finally, let G be an r-large bipartite graph. We give lower and upper bounds on $\sigma(G, m, r)$ for fixed m. The lower bounds use strategies for the revolutionaries that win after one or two rounds, while the upper bounds use more delicate strategies for the spies (maintaining invariants that prevent the revolutionaries from winning on the next round).

Since the lower bounds are much easier, we start with them, but first we compare all the bounds in Table 1. When $3 \mid m$, the lower bound is roughly $\frac{3}{2}r/m$. We believe that this is the asymptotic answer when $3 \mid m$. When $3 \nmid m$, the revolutionaries cannot employ this strategy quite so efficiently, which leaves an opening for the spies to do better. Indeed, for $m = 2$, the answer is roughly $\frac{7}{5}r/m$, a bit smaller. For larger m, the relative value of this advantage diminishes, and we expect the leading coefficient to tend to 3/2 as $m \to \infty$.

Table 1
Bounds on $\sigma(G, m, r)$.

Meeting size	Lower bound	Upper bound	References
2	$\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$	$\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$	Theorems 7.2 and 7.9
3	$\lfloor r/2 \rfloor$	$\lfloor r/2 \rfloor$	Theorems 7.3 and 7.10
$m \in \{4, 8, 10\}$	$\frac{1}{5}\left\lfloor \frac{7r}{m} - \frac{13}{2} \right\rfloor$		Corollary 7.4
m	$\left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$	$\left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m} + 1$	Corollary 7.4; Theorem 7.11

We first motivate the lower bounds by giving simple strategies for the revolutionaries when $m \in \{2, 3\}$. Henceforth call the partite sets X_1 and X_2.

Example 7.1. Initially place $\lfloor r/2 \rfloor$ revolutionaries in X_1 and $\lceil r/2 \rceil$ revolutionaries in X_2. Regardless of where the spies sit, swarming revolutionaries can form at least $\lfloor (r-1)/(2m) \rfloor$ new meetings on either side that can only be covered by spies from the other side, so the initial placement must satisfy $s_1 \geq \lfloor (r-1)/(2m) \rfloor$ and $s_2 \geq \lfloor r/(2m) \rfloor$, where s_i is the number of spies in X_i.

However, the uncovered revolutionaries can also be used to form meetings. If $m = 2$, then the revolutionaries can form $\lfloor (r - s_i)/2 \rfloor$ meetings when swarming to X_i, so the spies lose unless $s_{3-i} \geq \lfloor (r - s_i)/2 \rfloor$ for both i. Summing the inequalities yields $s_1 + s_2 \geq 2(r-1)/3$.

For $m = 3$, considering only r of the form $4k$, where $k \in \mathbb{N}$, we show that the revolutionaries win against $2k - 1$ spies. Initially there are $2k$ revolutionaries in each part, on distinct vertices. We may assume $s_1 \leq s_2$, so $s_1 \leq k-1$. Since there are only $2k - 1 - s_1$ spies in X_2, there are at least $s_1 + 1$ uncovered revolutionaries in X_2. Since $s_1 \leq k-1$, we can use $2(s_1 + 1)$ revolutionaries from X_1 to form meetings of size 3 with the uncovered revolutionaries in X_2. Since only s_1 spies are available to cover these meetings, the spies lose.

Thus $\sigma(G, 3, r) \geq r/2$ when $4 \mid r$. However, when $r = 4k + 2$, the revolutionaries cannot immediately win against $2k$ spies by this construction. With $2k + 1$ revolutionaries in each part and k spies sitting on revolutionaries in each part, swarming revolutionaries can only make k new meetings in either part, which can be covered by the spies. $\square$

The symmetric strategy in Example 7.1 is optimal when $m = 3$ and $4 \mid r$. However, when $m = 2$ and when $m = 3$ with $r = 4k+2$, the revolutionaries can do better using an asymmetric strategy that takes advantage of moving away from spies. When $m = 3$ and $r = 4k + 2$, this other strategy just increases the threshold by 1, to the value $\lfloor r/2 \rfloor$ that we will show is optimal for all r. For $m = 2$, however, the better strategy increases the leading term from $2r/3$ to $7r/10$.

Recall that the partite sets are X_1 and X_2 and that a vertex (or meeting) is *covered* if there is a spy there. Say that a spy is *lonely* when at a vertex with no revolutionary.

Theorem 7.2. *If G is an r-large complete bipartite graph, then $\sigma(G, 2, r) \geq \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$.*

Proof. We present a strategy for the revolutionaries and compute the number of spies needed to resist it. The revolutionaries start at r distinct vertices in X_1. In response, at least $\lfloor r/2\rfloor$ spies must start in X_1, since otherwise the revolutionaries can next make $\lfloor r/2\rfloor$ meetings at uncovered vertices in X_2 and win.

In the first round, $\lfloor r/2\rfloor$ revolutionaries move from X_1 to X_2, occupying distinct vertices. They leave from vertices of X_1 that are covered by spies (as much as possible), so after they move at least $\lfloor r/2\rfloor$ spies in X_1 are lonely. Now the spies move; let s_i be the number of spies in X_i after they move (for $i \in \{1, 2\}$). Let c be the number of revolutionaries in X_1 that are now covered by spies. Since at most s_2 spies leave X_1, there remain at least $\lfloor r/2\rfloor - s_2$ lonely spies in X_1. We conclude that $c \le s_1 - \lfloor r/2\rfloor + s_2$.

In round 2, the revolutionaries have the opportunity to swarm to X_1 or X_2. Since there are $\lfloor r/2\rfloor$ revolutionaries in X_2, there are at most $\lfloor r/2\rfloor + 1$ uncovered revolutionaries in X_1 (on distinct vertices), so swarming revolutionaries can make meetings with all but at most 1 uncovered revolutionary in X_1. The revolutionaries can therefore make $\lfloor (r-c)/2\rfloor$ new meetings in X_1. These meetings can only be covered by spies moving from X_2, so the spies lose unless $s_2 \ge \lfloor (r-c)/2\rfloor$.

If the revolutionaries swarm to X_2, then the new meetings there can only be covered by spies coming from X_1. At most s_2 revolutionaries in X_2 are covered by spies. Since $\lceil r/2\rceil$ revolutionaries come from X_1, they can make meetings with all uncovered revolutionaries in X_2, so the spies lose unless $s_1 \ge \lfloor (r-s_2)/2\rfloor$.

Adding twice the lower bound on s_1 to the lower bound on s_2 (with $c \le s_1 - \lfloor r/2\rfloor + s_2$),

$$s_2 + 2s_1 \ge \frac{\lfloor 3r/2\rfloor - s_1 - s_2 - 1}{2} + r - s_2 - 1.$$

The inequality simplifies to $5(s_1+s_2) \ge \lfloor 7r/2\rfloor - 3$, as desired. □

The general lower bound in Corollary 7.4 uses the formula for $m = 3$, which we study first. The key is that $r/2 - 1$ spies are not enough when $r \equiv 2 \mod 4$; we first sketch the idea in an easy case. Suppose that $r = 4k+2 \equiv 6 \mod 12$. The revolutionaries start at distinct vertices in X_1. Suppose that all s spies start in X_1 and that there are enough of them to win. In round 1, $2r/3$ revolutionaries move to X_2, leaving the spies in X_1 lonely. Let s_2 be the number of spies that move to X_2 after round 1, leaving s_1 spies in X_1. The revolutionaries in X_2 now can make $r/3$ meetings with the remaining $r/3$ revolutionaries in X_1, so $s_2 \ge r/3$. Since $s_2 \le 2k = r/2 - 1$, at least $r/6 + 1$ revolutionaries remain uncovered in X_2. The remaining $r/3$ revolutionaries in X_1 can make meetings with $r/6$ of them in round 2. Hence $s_1 \ge r/6$, and $s = s_1 + s_2 \ge r/2$.

The initial placement only requires $r/3$ spies in X_1, not $r/2$. We must allow for initial placement of x spies in X_2, where $0 \le x \le r/6$. The x spies originally in X_2 can move to X_1 in round 1 and cover revolutionaries there; this prevents the revolutionaries from threatening as many meetings by a swarm to X_1. In response, fewer than $2r/3$ revolutionaries move to X_2 in round 1, and yet we can guarantee more threatened meetings in the swarm to X_2.

Theorem 7.3. *If G is an r-large complete bipartite graph, then $\sigma(G, 3, r) \ge \lfloor r/2\rfloor$.*

Proof. Since $\lfloor r/2\rfloor = \lfloor (r+1)/2\rfloor$ when r is even, and having an extra revolutionary cannot reduce σ, it suffices to prove the lower bound when r is even. Example 7.1 proves it when $4 \mid r$, so only the case $r = 4k+2$ remains. We show that $4k+2$ revolutionaries can win against $2k$ spies. Suppose that the spies can survive for two full rounds after the initial placement.

The revolutionaries start at r distinct vertices of X_1, so at least $\lfloor r/3\rfloor$ spies must start in X_1. Let x be the initial number of spies in X_2, with $2k-x$ spies in X_1. Since X_1 contains at least $\lfloor r/3\rfloor$ spies, $x \le \lceil (2k-2)/3\rceil = \lceil r/6\rceil - 1$. Define j by $r - x \equiv j \mod 3$ with $j \in \{0, 1, 2\}$. In round 1, p revolutionaries move to X_2, where $p = 2(r-x-j)/3$. Note that $p \ge 2k - x$, so all spies in X_1 are now lonely. The number of revolutionaries remaining in X_1 is $r - p$, which equals $(r+2x+2j)/3$.

Let s_i be the number of spies in X_i after the spies respond in round 1. Since at most x spies move from X_2 to X_1 in round 1, the number of uncovered revolutionaries in X_1 is now at least $(r-x+2j)/3$. With $p = 2(r-x-j)/3$, there are enough revolutionaries in X_2 to threaten meetings at $(r-x-j)/3$ vertices in X_1 with revolutionaries who remained there. Hence $s_2 \ge (r-x-j)/3$.

Now consider a swarm to X_2 in round 2. Since there were $2k - x$ spies in X_1 initially, the number who moved to X_2 and covered revolutionaries after round 1 is at most $2k-x$. Hence round 2 starts with at least $p-2k+x$ uncovered revolutionaries in X_2. The $r - p$ revolutionaries remaining in X_1 move in pairs to generate meetings with uncovered revolutionaries in X_2. Note that $r - p = (r+2x+2j)/3$ and $p - 2k + x = (r+2x+6-4j)/6$. The number of meetings that can be made in X_2 (and can only be covered by the s_1 spies in X_1) depends on j.

When $j = 0$, the number of meetings made is $(r+2x)/6$, so $s_1 \ge (r+2x)/6$, and we obtain $s_2 + s_1 \ge \frac{r-x}{3} + \frac{r+2x}{6} = r/2$. When $j = 1$, the revolutionaries can make $p - 2k + x$ meetings in the swarm; hence $s_1 \ge (r+2x+2)/6$, and we obtain $s_2 + s_1 \ge \frac{r-x-1}{3} + \frac{r+2x+2}{6} = r/2$. Finally, when $j = 2$, the same computation yields only $s \ge \frac{r-x-2}{3} + \frac{r+2x-2}{6} = r/2 - 1$. However, equality holds only if all $2k - x$ spies initially in X_1 move to X_2 in round 1 to cover revolutionaries. Only x spies remain in X_1 to guard the swarm to X_2 that makes $(r+2x-2)/6$ meetings. The inequality $x \ge (r+2x-2)/6$ requires $x \ge (r-2)/4$, but guarding the initial position required $x < r/6$. □

Corollary 7.4. *If G is an r-large complete bipartite graph, then $\sigma(G, m, r) \ge \left\lfloor \frac{1}{2}\left\lfloor \frac{r}{\lceil m/3\rceil}\right\rfloor\right\rfloor$. If m is even, then $\sigma(G, m, r) \ge \frac{1}{5}\left\lfloor \frac{7r}{m} - \frac{13}{2}\right\rfloor$.*

Proof. Let $m' = \lceil m/3\rceil$. The revolutionaries group into cells of size m'; each cell moves together, modeling one player in a game with meeting size 3. When three of these cells converge to make an unguarded meeting, the revolutionaries win the original game. The r revolutionaries make $\lfloor r/m' \rfloor$ such cells and ignore extra revolutionaries. By Theorem 7.3, the number of spies needed to keep the revolutionaries from winning is at least $\lfloor \lfloor r/m' \rfloor /2 \rfloor$.

For even m, let $m' = m/2$. The revolutionaries can group into $\lfloor r/m' \rfloor$ cells of size m' and play a game with meeting size 2. In the lower bound of Theorem 7.2, we replace r by the number of cells in this imagined game, which is $\lfloor 2r/m \rfloor$. Dropping the outer ceiling function, the resulting lower bound is $\frac{1}{5}\left\lfloor \frac{7}{2}\left\lfloor \frac{2r}{m}\right\rfloor - 3\right\rfloor$. We use $\left\lfloor \frac{2r}{m}\right\rfloor > \frac{2r}{m} - 1$ to obtain the slightly simpler expression claimed. It improves on the bound above when $m \in \{4, 8, 10\}$. □

Finally, we consider upper bounds for $\sigma(G, m, r)$ when G is an r-large bipartite graph, proved by giving strategies for the spies.

Definition 7.5. Henceforth, always G is an r-large bipartite graph with partite sets X_1 and X_2, and we consider the game $\mathrm{RS}(G, m, r, s)$. Any statement that includes index j is considered for both $j = 1$ and $j = 2$. The numbers of revolutionaries and spies in part j at the beginning of the current round are denoted by r_j and s_j, respectively, and the number of revolutionaries in part j that are on vertices covered by spies is denoted by c_j. The corresponding counts at the end of the round are denoted by r'_j, s'_j and c'_j. The revolutionaries *swarm* X_j in a round if at the end of the round all revolutionaries are in X_j.

Definition 7.6. A *greedy migration strategy* is a strategy for the spies having the following properties. First, no vertex ever has more than one spy on it. Next, after the revolutionaries move during the current round and the spies compute the new desired distribution s'_1, s'_2 of spies on X_1 and X_2, they move to reach that distribution as follows. Since $s'_1 + s'_2 = s_1 + s_2$, by symmetry there is an index $i \in \{1, 2\}$ such that $s'_i \le s_{3-i}$. The spies reach their locations for the end of the round via the following steps.

(1) s'_i spies move away from X_{3-i}, iteratively leaving vertices that now have the fewest revolutionaries among those in X_{3-i}.

(2) *All* s_i spies previously on X_i leave X_i and move to uncovered vertices in X_{3-i}, iteratively covering vertices having the most revolutionaries.

(3) The s'_i spies that left X_{3-i} now move to uncovered vertices in X_i, iteratively covering vertices having the most revolutionaries.

Remark 7.7. At the end of round t under a greedy migration strategy, we designate each meeting or spy as "old" or "new". An *old meeting* is a meeting at a vertex where there was also a meeting at the start of round t; all other meetings at the end of round t are *new*. An *old spy* is a spy who did not move during round t; all spies who moved are *new spies*.

For $j \in \{1, 2\}$ either all spies that end round t in X_j are new (started round t in X_{3-j}), or all spies that started round t in X_{3-j} are new (end round t in X_j). In the specification of the movements in Definition 7.6, the former occurs when $j = i$, and the latter occurs when $j = 3 - i$. In the first case, round t ends with s'_j new spies in X_j; in the second case, it ends with s_{3-j} new spies in X_j. In particular, at least $\min\{s'_j, s_{3-j}\}$ spies in X_j are new.

Lemma 7.8. *A greedy migration strategy in* $\mathrm{RS}(G, m, r, s)$ *is a winning strategy for the spies if it prevents the revolutionaries from winning by swarming a part.*

Proof. As in Definition 7.5, Let r_j, s_j, r'_j, s'_j count the revolutionaries and spies at vertices of X_j at the start and end of round t, respectively, and define old and new meetings and spies as in Remark 7.7. We show that if a given greedy migration strategy for the spies keeps the revolutionaries from winning by swarming on round t or round $t + 1$, then all meetings are covered at the end of round t. Hence the revolutionaries never win.

By swarming to X_{3-j} in round t, the revolutionaries can produce at least $\lfloor r_j/m \rfloor$ new meetings there. Since these meetings can be covered only by spies in X_j at the start of round t, and the strategy prevents the revolutionaries from winning by this swarm, we obtain $s_j \ge \lfloor r_j/m \rfloor$ (and similarly $s_{3-j} \ge \lfloor r_{3-j}/m \rfloor$). Applying the same argument in round $t + 1$ yields $s'_j \ge \lfloor r'_j/m \rfloor$.

If all s'_j spies in X_j at the end of round t are new, then they cover all the meetings in X_j, since $s'_j \ge \lfloor r'_j/m \rfloor$ and greedy migration maximizes the coverage. Hence we may assume that some of these s'_j spies are old. Now Remark 7.7 implies that all s_{3-j} spies in X_{3-j} at the start of round t moved to X_j during round t. We consider two cases:

Case 1: *In round t every old meeting in X_j is covered by some old spy.* In this case it remains to show that at the end of round t, the s_{3-j} new spies in X_j cover all the new meetings there. We claim that otherwise the revolutionaries could have won in round t by swarming to X_j. A revolutionary who stayed in X_j or moved from X_{3-j} to X_j in the actual round t also would do so in a swarm to X_j. A revolutionary who moved from X_j to X_{3-j} would instead remain in X_j in the swarm, and a revolutionary who stayed in X_{3-j} in the actual round would move to X_j in the swarm. Thus the swarm produces at least as many new meetings, and the same number of old meetings, as the revolutionaries' actual moves in round t. The spies therefore cannot cover all of the new meetings formed by this swarm if their greeting migration does not cover all of the new meetings actually formed in X_j in round t.

Case 2: *At the end of round t some old meeting in X_j is not covered by an old spy*. Since greedy migration picks departing spies to minimize the number of revolutionaries uncovered, all old spies who remain in X_j are covering meetings. The new spies who move to X_j maximize coverage, so if there is an uncovered meeting in X_j at the end of round t, then every spy in X_j is covering a meeting. Since $s'_j \ge \lfloor r'_j/m \rfloor$, all the meetings are covered. □

Theorem 7.9. *If G is an r-large complete bipartite graph, then* $\sigma(G, 2, r) \leq \lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \rceil$.

Proof. Let $s = \lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \rceil$; we give a winning strategy for the spies in $\mathrm{RS}(G, 2, r, s)$. Let $\alpha = s - \lfloor r/2 \rfloor$ and $\beta = \lfloor (r - \alpha)/2 \rfloor$. Later we will use the following inequalities: $\alpha \leq \beta$, $\alpha + \beta \leq s$, and $\lfloor (r + \beta)/2 \rfloor \leq s$. These inequalities can be checked explicitly for each congruence class modulo 10. The first two are loose, since $\alpha \approx 2r/10$, $\beta \approx 4r/10$, and $s \approx 7r/10$, but the third is delicate, with equality holding except in two congruence classes and the floor function needed for correctness in four congruence classes.

During the game, if the revolutionaries swarm X_{3-j} in the current round, then they generate at most $\min\left\{r_j, \left\lfloor \frac{r - c_{3-j}}{2} \right\rfloor\right\}$ new meetings. The spy strategy will ensure

$$s_j \geq \min\left\{r_j, \left\lfloor \frac{r - c_{3-j}}{2} \right\rfloor\right\} \quad \text{for } j \in \{1, 2\}, \tag{A}$$

and hence it will keep the revolutionaries from winning by a swarm. The spies move by greedy migration after computing the new values s'_1 and s'_2 in response to r'_1 and r'_2. By Lemma 7.8, the spies win by a greedy migration strategy that keeps the revolutionaries from winning by swarm.

The spies determine s'_1 and s'_2 via three cases, using the first that applies. Always $s'_1 + s'_2 = s$.

Case 1: If $r'_i \leq \alpha$ for some $i \in \{1, 2\}$, then $s'_i = \alpha$.
Case 2: If $s_i \geq \min\{r'_{3-i}, \beta\}$ for some $i \in \{1, 2\}$, then $s'_{3-i} = \min\{r'_{3-i}, \beta\}$.
Case 3: Otherwise, $s'_i = s_{3-i}$ and $s'_{3-i} = s_i$.

It remains to prove (A). In order to do so, we first prove

$$s_j \geq \alpha \quad \text{for } j \in \{1, 2\}. \tag{B}$$

Trivially the spies can satisfy both (A) and (B) in round 0. Assuming that these invariants hold before the current round begins, we will show that they also hold when it ends.

Invariant (B) *is preserved.* In Case 1, $s'_i = \alpha$ and $s'_{3-i} = \lfloor r/2 \rfloor > \alpha$. In Case 3, $s'_j = s_{3-j} \geq \alpha$. In Case 2, $r'_{3-i} > \alpha$, so $s'_{3-i} = \min\{r'_{3-i}, \beta\} \geq \alpha$, and $s'_i = s - s'_{3-i} = s - \min\{r'_{3-i}, \beta\} \geq s - \beta \geq \alpha$.

Invariant (A) *is preserved.* In Case 1, $s'_i = \alpha \geq r'_i \geq \min\{r'_i, \lfloor \frac{r - c'_{3-i}}{2} \rfloor\}$ and $s'_{3-i} = \lfloor r/2 \rfloor \geq \lfloor \frac{r - c'_{3-i}}{2} \rfloor \geq \min\{r'_i, \lfloor \frac{r - c'_{3-i}}{2} \rfloor\}$.

In Case 2 with $s_i \geq \min\{r'_{3-i}, \beta\}$, first consider $j = 3 - i$. We have $s'_{3-i} = \min\{r'_{3-i}, \beta\}$. If $s'_{3-i} = r'_{3-i}$, then s'_{3-i} is already big enough, so suppose $s'_{3-i} = \beta$. By Remark 7.7, at least $\min\{s'_i, s_{3-i}\}$ spies in X_i are new. By (B), this quantity is at least α, and Case 2 requires $r'_i > \alpha$. Hence the new spies cover at least α revolutionaries, and $c'_i \geq \alpha$ yields $s'_{3-i} = \beta = \lfloor \frac{r - \alpha}{2} \rfloor \geq \min\{r'_{3-i}, \lfloor \frac{r - c'_i}{2} \rfloor\}$.

Now consider $j = i$. By Remark 7.7, at least $\min\{s_i, s'_{3-i}\}$ spies in X_i are new, and in Case 2 each of s_i and s'_{3-i} is at least $\min\{r'_{3-i}, \beta\}$. Since spies cover greedily, $c'_{3-i} \geq \min\{r'_{3-i}, \beta\} = s'_{3-i}$. Also $s'_{3-i} \leq \beta$, so

$$s'_i = s - s'_{3-i} \geq \left\lfloor \frac{r + \beta}{2} \right\rfloor - s'_{3-i} \geq \left\lfloor \frac{r - s'_{3-i}}{2} \right\rfloor \geq \left\lfloor \frac{r - c'_{3-i}}{2} \right\rfloor \geq \min\left\{r'_i, \left\lfloor \frac{r - c'_{3-i}}{2} \right\rfloor\right\}. \tag{8}$$

Finally, $s'_j = s_{3-j} < \min\{r'_j, \beta\}$ in Case 3, since Case 2 does not apply. Since all spies move and $s'_j \leq r'_j$, we have $c'_j \geq s'_j$. Hence for each j the computation in (8) is valid. □

The method for the upper bound when $m = 3$ is essentially the same.

Theorem 7.10. *If G is an r-large complete bipartite graph, then* $\sigma(G, 3, r) \leq \lfloor r/2 \rfloor$.

Proof. We present a greedy migration strategy for $\lfloor r/2 \rfloor$ spies that keeps the revolutionaries from winning by swarming; by Lemma 7.8 it is a winning strategy for the spies.

Define r_j, s_j, c_j at the start of a round and r'_j, s'_j, c'_j at the end of the round in the same way as before. Also, we need to know the maximum number of revolutionaries together on an uncovered vertex in X_j at the beginning and end of the round; let these values be u_j and u'_j. If the revolutionaries have not already won, then $u_j, u'_j \leq 2$. Let $s = \lfloor r/2 \rfloor$, $\alpha = \lfloor r/2 \rfloor - \lfloor r/3 \rfloor$, and $\beta = s - \lfloor (r - \alpha)/3 \rfloor$. We will want the inequalities $\beta \geq \lfloor \frac{r - 2\alpha}{3} \rfloor$ and $\beta \leq \lceil \frac{\lfloor r/2 \rfloor}{2} \rceil$. The latter is always satisfied (the left side is about $2r/9$ and the right side is about $r/4$), but both sides of the first inequality are about $2r/9$. Checking each congruence class modulo 18 shows that $\beta \geq \lfloor \frac{r - 2\alpha}{3} \rfloor$ except when $r \equiv 3 \mod 18$.

The values s'_1 and s'_2 that determine the movements of spies in this round under the greedy migration strategy are computed as follows, with $s'_{3-i} = s - s'_i$ always. Note that since $r'_1 + r'_2 = r$, when one of the cases below holds, it holds for exactly one index i unless $r'_1 = r'_2 = r/2$. In this case of equality, it does not matter which index we call i.

Case 1: If $r'_i \leq \alpha$ for some $i \in \{1, 2\}$, then $s'_i = \alpha$.
Case 2: If $\alpha < r'_i \leq \beta$ for some $i \in \{1, 2\}$, then $s'_i = r'_i$.

Case 3: If $\beta < r_i' \le 2\beta$ for some $i \in \{1, 2\}$, then $s_i' = \beta$, except that $s_i' = \beta + 1$ when $s_i = \alpha$ and $r \equiv 3 \mod 18$.
Case 4: If $2\beta < r_i' \le \lfloor r/2 \rfloor$ for some $i \in \{1, 2\}$, then $s_i' = \lfloor r_i'/2 \rfloor$.

Let $f_j = \min\{\lfloor \frac{r-c_{3-j}}{3} \rfloor, \lfloor \frac{r_j}{3-u_{3-j}} \rfloor\}$. During the game, if the revolutionaries swarm X_{3-j} in the current round, then they generate at most f_j new meetings. Hence it suffices to show that the strategy specified above always ensures

$$s_j \ge f_j \quad \text{for } j \in \{1, 2\}. \tag{A}$$

As in Theorem 7.9, in order to prove (A) we will also need

$$s_j \ge \alpha \quad \text{for } j \in \{1, 2\}. \tag{B}$$

Place the spies to satisfy (A) and (B) in round 0. In each Case of play, $\alpha \le s_i' \le \lfloor r/4 \rfloor \le s - \alpha$, so (B) is preserved. Now $s_1, s_2, s_1', s_2' \ge \alpha$, and we study (A).

With f_j' being the value of f_j at the end of the round, we need $s_j' \ge f_j'$. By Remark 7.7, each part receives at least α new spies in each round. In Cases 2–4 each part contains at least α revolutionaries, so $c_j' \ge \alpha$ in those cases. Also $s_j' \ge \lfloor r_j'/3 \rfloor$ in each Case. Since $s_j' \ge \lfloor r_j'/3 \rfloor = \lfloor r_j'/(3 - u_{3-j}') \rfloor$ when $u_{3-j}' = 0$, we may assume $u_j' \in \{1, 2\}$.

In addition, since the greedy strategy places new spies in X_j to maximize coverage, leaving an uncovered vertex with u_j' revolutionaries implies that each of the (at least) α new spies covers at least u_j' revolutionaries at its vertex. Hence $c_j' \ge u_j'\alpha$.

Invariant (A) *is preserved* (we separately prove both $s_i' \ge f_s'$ and $s_{3-i}' \ge f_{3-i}'$):
In Case 1, $s_i' = \alpha \ge r_i' \ge f_i'$ and $s_{3-i}' = s - \alpha \ge \lfloor r/3 \rfloor \ge f_{3-i}'$.

In Case 2, $s_i' = r_i' \ge f_i'$. Also, $c_i' \ge \alpha$ and $s_{3-i}' = s - r_i' \ge s - \beta = \lfloor \frac{r-\alpha}{3} \rfloor \ge \lfloor \frac{r-c_i'}{3} \rfloor \ge f_{3-i}'$.

In Case 3, we use $c_i' \ge \alpha$. In the nonexceptional case, $s_{3-i}' = s-\beta = \lfloor \frac{r-\alpha}{3} \rfloor \ge \lfloor \frac{r-c_i'}{3} \rfloor \ge f_{3-i}'$. If $s_i = \alpha$ and $r \equiv 3 \mod 18$, then $s_{3-i}' = s - \beta - 1$ and we must be a bit more careful. Since all α spies that were in X_i move to X_{3-i}, and $r_i' \ge \beta + 1$, we have $c_i' \ge \beta + 1$, and hence $\lfloor \frac{r-\alpha}{3} \rfloor - 1 \ge \lfloor \frac{r-c_i'}{3} \rfloor$.

In Case 3 or Case 4, if $u_{3-i}' = 1$, then $s_i' \ge \lfloor r_i'/2 \rfloor = \lfloor \frac{r_i'}{3-u_{3-i}'} \rfloor \ge f_i'$. If $u_{3-i}' = 2$, then $c_{3-i}' \ge 2\alpha$. Hence $s_i' \ge \beta \ge \lfloor \frac{r-2\alpha}{3} \rfloor \ge \lfloor \frac{r-c_{3-i}'}{3} \rfloor \ge f_i'$, with the exception that $\beta = \lfloor \frac{r-2\alpha}{3} \rfloor - 1$ when $r \equiv 3 \mod 18$. In this case either $s_i' > \beta$, which suffices, or $s_i > \alpha$. If $s_i > \alpha$, then X_{3-j} has more than α new spies, so $c_{3-i}' \ge 2\alpha + 2$, which fixes the problem for $r \equiv 3 \mod 18$.

In Case 4, if $u_i' = 1$, then $s_{3-i}' = s - \lfloor \frac{r_i'}{2} \rfloor \ge \lfloor \frac{r_{3-i}'}{2} \rfloor = \lfloor \frac{r_{3-i}'}{3-u_i'} \rfloor \ge f_{3-i}'$. If $u_i' = 2$, then $c_i' \ge 2\alpha$. Now $s_{3-i}' = s - \lfloor \frac{r_i'}{2} \rfloor \ge \lfloor \frac{r}{2} \rfloor - \lfloor \frac{\lfloor r/2 \rfloor}{2} \rfloor = \lceil \frac{\lfloor r/2 \rfloor}{2} \rceil \ge \lfloor \frac{r-2\alpha}{3} \rfloor \ge \lfloor \frac{r-c_i'}{3} \rfloor \ge f_{3-i}'$. □

Theorem 7.11. *If G is an r-large complete bipartite graph, then $\sigma(G, m, r) \le (1 + \frac{1}{\sqrt{3}})\frac{r}{m} + 1$.*

Proof. For $s \ge (1 + \frac{1}{\sqrt{3}})\frac{r}{m} + 1$, we present a greedy migration strategy for s spies that keeps the revolutionaries from winning by swarming. Suppose first that $\frac{r}{m} < \frac{1}{1-1/\sqrt{3}} < 2.5$. In this case, the revolutionaries can never make more than two meetings. We want to show that at most 4.75 spies suffice. In fact, four spies always suffice, because they can always arrange to keep two spies on each side to handle up to two new meetings on the other side. The greedy migration strategy that always sets $s_1 = s_2 = 2$ accomplishes this. Henceforth, we may assume $\frac{r}{m} \ge \frac{1}{1-1/\sqrt{3}}$.

As usual, r_j and s_j count the revolutionaries and spies in X_j to begin a round, r_j' counts the revolutionaries after they move, and s_j' is the number of spies to be computed for X_j to end the round. To determine s_1' and s_2', the spies compute x, α, u_1, and u_2 (not necessarily integers) such that

$$x \le \lfloor r/m \rfloor, \qquad x + r/m + 1 \le s, \quad \text{and} \tag{9}$$

$$\alpha = x + r/m - \frac{r - u_1 x}{m} = x + r/m - \frac{r_2'}{m - u_1} = \frac{r_1'}{m - u_2} = \frac{r - u_2 x}{m}. \tag{10}$$

We will show that such numbers always exist. Now s_1' and s_2' are computed as follows:

Case 1: If $\alpha \le x$, then $s_1' = \lceil x \rceil$ and $s_2' = s - s_1'$.
Case 2: If $\alpha > \lfloor r/m \rfloor$, then $s_1' = \lfloor r/m \rfloor$ and $s_2' = s - s_1'$.
Case 3: If $x < \alpha \le \lfloor r/m \rfloor$, then $s_1' = \lceil \alpha \rceil$ and $s_2' = s - s_1'$.

Since always $s_j' \ge x$, greedy migration moves at least $\lceil x \rceil$ new spies to each part in each round, by Remark 7.7. Consider a swarm. If all uncovered vertices in X_j have at most u_j revolutionaries, then swarming X_j generates at most $r_{3-j}'/(m - u_j)$ new meetings. If some uncovered vertex in X_j has more than u_j revolutionaries, then by greedy migration at least x spies in X_j

have covered more than u_j revolutionaries each, and swarming X_j forms at most $(r - u_j x)/m$ new meetings. Hence swarming X_j fails to win if

$$s'_{3-j} \geq \max\left\{\frac{r'_{3-j}}{m - u_j}, \frac{r - u_j x}{m}\right\}. \tag{11}$$

For $j = 2$, both quantities on the right in (11) equal α, so the condition is equivalent to $s'_1 \geq \alpha$, which holds in Cases 1 and 3. In Case 2, $s'_1 = \lfloor r/m \rfloor$, which always protects against swarming X_2 since at most $\lfloor r/m \rfloor$ meetings can be made.

For $j = 1$, both quantities on the right in (11) equal $x + r/m - \alpha$, so the condition is equivalent to $s'_2 \geq x + r/m - \alpha$. Since $s - 1 \geq x + r/m$, proving $s'_2 \geq s - 1 - \alpha$ shows that swarming X_1 is ineffective. In Case 1, $s'_2 > r/m$, which suffices. In Case 2 or 3, $s'_1 \leq \lceil \alpha \rceil$, so $s'_2 \geq s - \lceil \alpha \rceil > s - 1 - \alpha$, as desired.

It remains to show that such numbers exist. Solving (10) yields

$$\begin{aligned} x &= \frac{\sqrt{9r^2 + 12r'_1 r - 12{r'_1}^2}}{6m} \\ u_1 &= \frac{r + mx - \sqrt{r^2 + 2rxm + x^2m^2 - 4xr'_1 m}}{2x} \quad \text{and} \\ u_2 &= \frac{r + mx - \sqrt{r^2 - 2rxm + x^2m^2 + 4xr'_1 m}}{2x}. \end{aligned}$$

Since $x \leq r/(\sqrt{3}m)$, the inequalities in (9) hold when $\frac{r}{m} \geq \frac{1}{1-1/\sqrt{3}}$. $\square$

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