A Study of the Motion of Zeros of the Epstein Zeta Function Associated to $m^2 + y^2n^2$ as $y$ varies from 1 to $\sqrt{6}$

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Abstract—This paper is concerned with the special Epstein zeta function defined by

$$Z(s,y) = \frac{1}{2} \sum'_{m,n \in \mathbb{Z}} (m^2 + y^2n^2)^{-s}, \quad s = \sigma + it, \quad y \in \mathbb{R}^+$$

for $\sigma > 1$ and by analytic continuation in the entire $s$-plane. A numerical study of its zeros with $0 < t < 18$ is described and for a selection of values of $y$ these zeros are listed. Most of them are on the critical line $\sigma = \frac{1}{2}$, but for $y$ in certain intervals there also is a pair of zeros not on this line and in some cases exactly two such pairs are revealed. Our calculations are based on a new representation of $(y/\pi)^s \zeta(s)Z(s,y)$ by factorial series, which is derived in the paper and which makes numerical computation of zeros as functions of $y$ feasible on a medium sized computer.

1. INTRODUCTION

Analytic number theorists have long been interested in Epstein zeta functions and their role for the study of quadratic number fields. Certain types of Epstein zeta functions are equal to Dedekind zeta functions of quadratic number fields and therefore factor into Dirichlet $L$-Series and the Riemann zeta function. For the latter functions there is conjectured the extended Riemann hypothesis that all zeros in the upper complex half-plane are on the critical line $\sigma = \frac{1}{2}$. Other types of Epstein zeta functions do have zeros off the critical line. In fact, in 1935 Potter and Titchmarsh [1] announced that the first six zeros of the Epstein zeta function associated to $m^2 + 5n^2$ are on the critical line followed by a pair of zeros off the critical line but in the critical strip $0 < \sigma < 1$. In 1983, O’Leary [2] recalculated these zeros and one more in the region $-0.9 \leq \sigma \leq 1.9$ and $0 \leq t \leq 18$ with greater precision. This raised our interest in studying the behavior of this pair of zeros off the critical line as $y$ varies from $\sqrt{5}$ to 1 through the reals.

Some other investigations of zeros of Epstein zeta functions with binary quadratic forms are available in the literature. In particular, Bateman and Grosswald [3] in 1964 proved that the function defined for $\sigma > 1$ by

$$Z(s) = \frac{1}{2} \sum'_{m,n \in \mathbb{Z}} (am^2 + bmn + cn^2)^{-s}, \quad s = \sigma + it$$

and in the entire $s$-plane by analytic continuation, is negative for $s = \sigma \in (0,1)$ if $y^2 = \frac{4ac-b^2}{4a^2}$ is in $[.75, 49.77]$. In 1967, Stark [4] proved that all zeros of $Z(s)$ in $-1 < \sigma < 2$, $-2y \leq t \leq 2y$ are
on the line $\sigma = \frac{1}{2}$ and are simple, if $y > K$ for a suitably large $K$. This fact in a slightly weakened form was already proved by Deuring [5] in 1933. More recently Hejhal [6] used a supercomputer to investigate the distribution of zeros with large $t$.

In order to study the motion of the zeros off the critical line found for $y^2 = 5$ by Potter and Titchmarsh and confirmed by O’Leary, as $y^2$ varies from 5 to 1 (the Epstein zeta function associated to $m^2 + n^2$ is the zeta function of the quadratic number field $Q(\sqrt{-1})$), the focus of the present work is on the Epstein zeta function associated to $m^2 + y^2n^2$ for $1 \leq y \leq \sqrt{6}$. It turns out that the resulting motion shows a relationship between certain zeros on the critical line and those off the critical line. In particular, some other zeros off the critical line and not originating from those of Potter and Titchmarsh were found. They will be described in more detail in Section 8.

There is another motivation of the work presented in this paper. Previous computations involving Epstein zeta functions used one of three known representations: either by a series of modified Bessel functions of the third kind [3,7], or by a series of incomplete gamma functions [2], or by an analogue of the Riemann-Siegel formula developed for the Riemann zeta function [6]. Our computations are based on a new series representation of the Epstein zeta function which will be given in Section 4. Indeed, it is the goal of Section 4 to develop a series expansion where the terms of the series are rational functions of the variable $s$. This will improve the feasibility of numerical calculation of the zeros on a medium size computer and obviate a supercomputer.

In Sections 7 and 8, the actual computations of zeros on the critical line and off the critical line are described. Numerical methods for finding complex zeros are usually based on iterative techniques (like Newton’s Method) and approximation formulas of varying efficiency. Instead, in Section 8 we calculate these zeros by numerically evaluating the classical integral

$$\frac{1}{2\pi i} \int_C f(s) ds \quad (1.2)$$

around a suitable rectangular contour. This does not require any iteration and is a rather bold approach which is feasible only because in our application the functions $f(s)$ and $f'(s)$ have sufficiently simple representations (see Section 4) which make the numerical evaluation of the integral practical. This work was done on a Bull HN DPS8/49 CP-6 operating system with 36-bit capability; the second author wrote a Fortran 77 program that uses double precision. An accuracy check is maintained and described in detail in Section 6.

2. BASIC RESULTS

The Epstein zeta function is defined by

$$Z(s, F) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}}' F(m,n)^{-s} \quad (2.1)$$

for complex $s = \sigma + it$ ($\sigma > 1$), where $F(m,n) = am^2 + bmn + cn^2$ with real $a, b, c$ and discriminant $\Delta = b^2 - 4ac < 0$, $a > 0$. The prime indicates that the summation is taken over all $m$ and $n$ except $m = n = 0$. In this paper we study only the special case where $a = 1$, $b = 0$, $c = y^2$, $y \in \mathbb{R}^+$, i.e.,

$$Z(s, y) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}}' (m^2 + y^2n^2)^{-s}, \quad (2.2)$$

with $\Delta = -4y^2$, $y > 0$, $\sigma > 1$.

In the general case, Epstein’s zeta function has an analytic continuation to the whole $s$-plane with only a simple pole at $s = 1$ and it satisfies the functional equation $\Phi(s, F) = \Phi(1 - s, F)$
where $\Phi(s, F) = \left( \frac{\sqrt{\pi}}{2s} \right)^s \Gamma(s) Z(s, F)$ and $\Gamma(s)$ is the gamma function. In our special case, we have the functional equation $\Phi(s, y) = \Phi(1 - s, y)$ where

$$\Phi(s, y) = \left( \frac{y}{\pi} \right)^s \Gamma(s) Z(s, y). \quad (2.3)$$

Also,

$$\Phi(s, y) = y^{1/2 - 2s} \phi(1 - 2s) + y^{1/2 + 2s} \phi(1 + 2s) + 4y^{1/2} \sum_{k>0} \rho_k(v) K_v(2\pi ky), \quad (2.4)$$

where $\nu = s - \frac{1}{2}$, $\phi(s) = \pi^{-\nu} \Gamma(\frac{\nu}{2}) \zeta(s) = \phi(1 - s)$, $\rho_k(v) = \sum_{n=-k}^{k} \left( \frac{\nu}{2\pi} \right)^v$ and

$$K_v(2y) = \frac{1}{2} \int_{0}^{\infty} x^v e^{-y(x^s - 1)} dx = K_{-v}(2y)$$

is a Bessel function which is entire in $\nu$, and where $\zeta(s)$ is the Riemann zeta function. The series in (2.4) converges uniformly in any strip $\sigma_1 \leq \sigma \leq \sigma_2$. Equation (2.4) is proved by Bateman and Grosswald in [3], for instance.

Another expansion of $\Phi(s, y)$ which has been used by others [2] for numerical computations, is in terms of incomplete gamma functions. Instead of these we use

**Theorem 1.** $\Phi(s, y) = \frac{1}{2(s-1)} - \frac{1}{2s} + \Psi(s, y)$, where

$$\Psi(s, y) = \frac{1}{2} \int_{1}^{\infty} \left( x^{-s} + x^{s-1} \right) \left( \vartheta(x, y) - 1 \right) dx,$$

$$\vartheta(x, y) = \sum_{m,n \in \mathbb{Z}} e^{-\pi x(m^2 + n^2)} \quad \text{for } x, y > 0,$$

and $\Psi(s, y)$ is an entire function of $s$ for each $y > 0$.

**Proof.** With $\vartheta(x) = \sum_{m,n \in \mathbb{Z}} e^{-\pi x^2 m^2} = x^{-1/2} \vartheta(x^{-1})$ for $x > 0$, we have for $s > 1$ and $y > 0$ by absolute convergence

$$\Phi(s, y) = \frac{1}{2} y^s \int_{0}^{\infty} x^{s-1} \left( \sum_{m,n \in \mathbb{Z}} e^{-\pi x(m^2 + n^2)} - 1 \right) dx$$

$$= \frac{1}{2} y^s \int_{0}^{\infty} x^{s-1}[\vartheta(x) - \vartheta(xy^2)] \frac{dx}{x} = \frac{1}{2} y^s \left( \int_{0}^{y^{-1}} + \int_{y^{-1}}^{\infty} \right)$$

$$= \frac{1}{2} y^s \left( \int_{0}^{y^{-1}} x^{s-1} \vartheta \left( \frac{y^2}{x} \right) dx \frac{1}{x^2} + \frac{1}{2} y^s \int_{y^{-1}}^{\infty} x^{s-1}[\vartheta(x) - \vartheta(xy^2)] \frac{dx}{x} \right)$$

$$= \frac{1}{2} y^s \left( \int_{0}^{y^{-1}} x^{s-1} \vartheta \left( \frac{y^2}{x} \right) \frac{dx}{x} \right) + \frac{1}{2} y^s \int_{1}^{\infty} u^{s-1} \sum_{m,n \in \mathbb{Z}} e^{-\pi uv(m^2 + n^2)} du$$

$$= \frac{1}{2} \int_{1}^{\infty} u^{-s-1} \sum_{m,n \in \mathbb{Z}} e^{-\pi uv(m^2 + n^2)} du + \frac{1}{2(s-1)} - \frac{1}{2s} + \frac{1}{2} \int_{1}^{\infty} u^{s-1} \sum_{m,n \in \mathbb{Z}} e^{-\pi uv(m^2 + n^2)} du$$

$$- \frac{1}{2(s-1)} - \frac{1}{2s} + \Psi(s, y), \quad \text{(for all } s \in \mathbb{C}, y > 0). \quad \square$$
By switching summation and integration in the last two integrals one would obtain an expansion in series of incomplete gamma functions.

A third expansion that has been used to compute values of \( Z(s, F) \) is an analogue of the Riemann-Siegel formula developed for \( \zeta(s) \). Hejhal [61] used this for calculations on a supercomputer.

Each of these expansions has some drawbacks if one wants to numerically calculate the function. Their usefulness depends on the effectiveness of algorithms for computations. According to [8], in the case of the modified Bessel function \( K_{-\frac{1}{2}}(2\pi ky) \) the difficulty is that its numerical complex value is absolutely very small and massive cancellation occurs in the integral listed under (2.4). When dealing with incomplete gamma functions, there are hardly any satisfactory algorithms available for complex \( s \) with large \( t \). In Section 4 of this paper, we present a further expansion of \( \Phi(s, y) \) and use it thereafter to compute its values and to find zeros of \( Z(s, y) \).

3. EPSTEIN ZETA FUNCTIONS AND QUADRATIC NUMBER FIELDS

When \( y^2 \) is a rational number, \( Z(s, y) \) is essentially the zeta function of an ideal class of an order in some imaginary quadratic number field. In particular, if \( y^2 \) is integral and \( \Delta = -4y^2 \) is a quadratic discriminant \( d \), this is the zeta function of the principal ideal class in \( K = \mathbb{Q}(\sqrt{d}) \); if also the ideal class number \( h(d) \) of \( K \) equals one, it is the zeta function of \( K \). For the latter, it is known that

\[
\zeta_K(s) = \zeta(s)L(s; d), \quad \text{where } L(s; d) = \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) n^{-s} \quad (\sigma > 0)
\]

and \( (d/n) \) is the Kronecker symbol for the quadratic discriminant \( d \). We list here some relevant identities for \( Z(s, y) \):

\[
\begin{align*}
y^2 &= 1, \quad \Delta = -4 = d, \quad h(d) = 1, \quad Z(s, 1) = 2\zeta_K(s); \\
y^2 &= 2, \quad \Delta = -8 = d, \quad h(d) = 1, \quad Z(s, \sqrt{2}) = \zeta_K(s); \\
y^2 &= 3, \quad \Delta = -12, \quad d = -3, \quad h(d) = 1, \quad \{1, \omega\} \text{ with } \omega = \frac{1 + \sqrt{-3}}{2} \text{ is the maximal order in } K \text{ which contains the order } \{1, \sqrt{-3}\} = \{1, 2\omega\}, \text{ so}
\end{align*}
\]

\[
\zeta_K(s) = \frac{1}{6} 4^s \sum_{m,n \in \mathbb{Z}}' [(2m + n)^2 + 3n^2]^{-s} = \frac{1}{3} 4^s \left[ Z(s, \sqrt{3}) - \frac{1}{2} \sum_{k \not\equiv n \mod 2} (k^2 + 3n^2)^{-s} \right];
\]

\[
\begin{align*}
y^2 &= 4, \quad \Delta = -16, \quad d = -4, \quad h(d) = 1, \quad Z(s, 2) = \zeta_K(s)(1 - 2^{-s} + 2^{1-2s}); \\
y^2 &= 5, \quad \Delta = -20 = d, \quad h(d) = 2, \quad Z(s, \sqrt{5}) = \zeta_K(s) - \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (2m^2 + 2mn + 3n^2)^{-s}; \\
y^2 &= 6, \quad \Delta = -24 = d, \quad h(d) = 2, \quad Z(s, \sqrt{6}) = \zeta_K(s) - \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (2m^2 + 3n^2)^{-s}.
\end{align*}
\]

Observe that the factor \( 1 - 2^{-s} + 2^{1-2s} \) appearing in \( Z(s, 2) \) has its zeros precisely on \( \sigma = \frac{1}{2} \).

These special cases originating from the connection between integral positive definite binary quadratic forms and ideal classes of quadratic number fields connect Epstein zeta functions to analytic number theory. In what follows, we are also concerned with other quadratic forms, especially when \( y^2 \) is real rather than integral and treated as a continuously varying parameter.
4. TRANSFORMATION OF THE
THETA INTEGRAL REPRESENTATION OF $\Phi(s, y)$

We begin with transforming the integral in the representation of $\Phi(s, y)$ in Theorem 1. For this purpose we study the mapping $w \rightarrow z = (1 - w)^{-\alpha}$, $0 < \alpha < 1$, for $w$ in the unit disk. We have $|z| = |1 - w|^{-\alpha}$ and $\text{arc}^*z = -\alpha \text{arc}^*(1 - w)$ where $\text{arc}^*$ means the principal arcus in $(-\pi, \pi)$. For $|w| \leq 1$, $w \neq 1$ we have $\frac{-\alpha \pi}{2} < \text{arc}^*(1 - w) < \frac{\alpha \pi}{2}$ and therefore

$$\frac{-\alpha \pi}{2} < \text{arc}^*z < \frac{\alpha \pi}{2},$$

and thus $\text{Re}z > 0$. So $z$ as a function of $w$ is holomorphic in $|w| \leq 1$, $w \neq 1$, and as $w \rightarrow 1$, $|z| \rightarrow \infty$. We also need that

$$x - \text{Re}z = |1 - w|^{-\alpha} \cos[\alpha \text{arc}^*(1 - w)] > |1 - w|^{-\alpha} \cos \frac{\alpha \pi}{2} > 0.$$

Hence $x \rightarrow +\infty$ as $w \rightarrow 1$ in $|w| \leq 1$.

We make the substitution $x = (1 - u)^{-\alpha}$, $0 \leq u < 1$, $dx = \frac{\alpha}{2} u^{\alpha - 1} du$ in the integral appearing in Theorem 1 and get for $s \in \mathbb{C}$

$$\Psi(s, y) = \frac{1}{2} \int_{1}^{\infty} (x^{-s} + x^{s-1})[\vartheta(x, y) - 1] dx$$

$$= \frac{\alpha}{2} \int_{0}^{1} [(1 - u)^{\alpha s} + (1 - u)^{-\alpha s + \alpha}][\vartheta(1 - u)^{-\alpha}, y] - 1](1 - u)^{-1 - \alpha} du. \quad (4.1)$$

Now set $T(w, y) = \vartheta((1 - w)^{-\alpha}, y)$ and observe that this is a homomorphic function of $w$ in $|w| \leq 1$, $w \neq 1$, since $\text{Re}(1 - w)^{-\alpha} > 0$ as shown above. Therefore $T(w, y)$ has a Taylor series expansion in $|w| < 1$ of the form

$$T(w, y) = \sum_{r=0}^{\infty} C_r(y, \alpha) w^r, \quad C_r = \frac{1}{r!} T^{(r)}(0, y). \quad (4.2)$$

We now show that the series converges uniformly on $|w| \leq 1$, so that we can later substitute it into the integral (4.1). We employ the following coefficient estimate which is proven in [9]:

**Theorem 2.** Let $\phi(w) = \sum_{n=0}^{\infty} d_n w^n$ have radius of convergence 1. Let $w = 1$ on $|w| = 1$ be the only singular point of $\phi(w)$, i.e., $\phi(w)$ is continuatable over every other boundary point. Also let $\phi(w) = o(1/|1 - w|^\mu)$ ($\mu > 1$) when $w \rightarrow 1$ in $|w| \leq 1$. Then $d_n = o(n^{\mu - 1})$ for $n \rightarrow +\infty$ (i.e., $\lim_{n \rightarrow +\infty} d_n/n^{\mu-1} = 0$).

Above we have already shown that $T(w, y)$ is homomorphic on $|w| \leq 1$ except at $w = 1$. Thus its derivatives by $w$ are also holomorphic there. We now show that for any $\epsilon > 0$

$$|T^{(4)}(w, y)| |1 - w|^{1+\alpha} < \epsilon \quad (0 < \alpha < 1) \text{ for } |w - 1| < \delta(\epsilon), \quad |w| \leq 1.$$  

Namely,

$$T^{(4)}(w, y) = \sum_{k=1}^{4} c_k(\alpha)(1 - w)^{-\alpha - 4} \vartheta^{(k)}(z, y), \quad \vartheta^{(k)}(z, y) = \frac{\partial^k \vartheta}{\partial z^k},$$

where $c_k(\alpha)$ are constants and $z$ and $y$ are independent variables (recall $z = (1 - w)^{-\alpha}$). Since $\vartheta(z, y) = \sum_{m, n \in \mathbb{Z}} e^{-\pi(z^m y^n - n \alpha)}$ we have

$$|\vartheta^{(k)}(z, y)| \leq (-1)^k \vartheta^{(k)}(x, y) \quad \text{ for } k \geq 1, \quad x = \text{Re}z$$

$$|\vartheta^{(k)}(x, y)| \leq \vartheta^{(4)}(x, y) \quad \text{ for } k \leq 4, \quad 1 \leq y \leq \pi.$$
So in order to estimate $T^{(4)}(w, y)$, it suffices to estimate $\vartheta^{(4)}(x, y)$. Now

$$\vartheta^{(4)}(x, y) = \sum_{m,n \in \mathbb{Z}} \pi^4 (m^2 y^{-1} + n^2 y)^4 e^{-\pi x (m^2 y^{-1} + n^2 y)},$$

and so for $y \in [1, 3]$ and $x \geq 1$ say,

$$\vartheta^{(4)}(x, y) \leq C \sum_{m,n \in \mathbb{Z}} (m^2 + n^2)^4 e^{-\pi x (m^2 + n^2)} \leq C_1 e^{-\pi x}.$$

Now recall that $x = \Re z > |1 - w| \cdot \cos \frac{\alpha \pi}{2} \to +\infty$ as $w \to 1$ in $|w| \leq 1$. Then

$$|T^{(4)}(w, y)| \leq C_2(\alpha) \frac{|\vartheta^{(4)}(x, y)|}{|1 - w|^{4\alpha + 4}} \leq C_3 \frac{e^{-C_0 |1 - w|^{-\alpha}}}{|1 - w|^{4\alpha + 4}}$$

and

$$|T^{(4)}(w, y)(1 - w)^{\alpha + 1}| \leq \frac{C_3 e^{-C_0 |1 - w|^{-\alpha}}}{|1 - w|^{3\alpha + 3}} = C_3 e^{-|1 - w|^{-\alpha} (C_0 + (3\alpha + 3)|1 - w|^{\alpha} \ln |1 - w|)} \to 0$$
as $w \to 1$. So by Theorem 2 applied to $\phi(w) = w^4 T^{(4)}(w, y)$ from (4.2), and with $\mu = 1 + \alpha$ we have for $r \in \mathbb{Z}^+$,

$$r(r - 1)(r - 2)(r - 3) C_r(y, \alpha) \ll r^\alpha,$$

**Theorem 3.**

$$C_r(y, \alpha) \ll r^\alpha \quad \text{as} \quad r \to \infty.$$

Now we have

$$\sum_{r=1}^{\infty} |C_r(y, \alpha)| \ll \sum_{r=1}^{\infty} r^\alpha < \infty,$$

since $0 < \alpha < 1$. Thus $\sum_{r=0}^{\infty} C_r(y, \alpha) w^r$ converges absolutely on $|w| \leq 1$ and so also uniformly. Therefore,

$$T(u, y) = \sum_{r=0}^{\infty} C_r(y, \alpha) w^r \quad (4.3)$$

converges uniformly on $0 \leq u \leq 1$ and we can substitute this into the integral (4.1).

Thus we have for $s \in \mathbb{C}$,

$$\Psi(s, y) = \frac{\alpha}{2} \int_0^1 (1 - u)^{-\alpha s - 1} (T(u, y) - 1) du + \frac{\alpha}{2} \int_0^1 (1 - u)^{-\alpha(1-s) - 1} (T(u, y) - 1) du$$

$$= \Psi_1(s, y) + \Psi_2(s, y). \quad (4.4)$$

For $\sigma < 0$ we can integrate termwise in $\Psi_1(s, y)$:

$$\Psi_1(s, y) = \frac{\alpha}{2} \int_0^1 (1 - u)^{-\alpha s - 1} \left( \sum_{r=0}^{\infty} C_r(y, \alpha) w^r \right) du$$

$$= \frac{\alpha}{2} \sum_{r=0}^{\infty} C_r(y, \alpha) \int_0^1 u^r (1 - u)^{-\alpha s - 1} du - \frac{\alpha}{2} \int_0^1 (1 - u)^{-\alpha s - 1} du$$

$$= \frac{\alpha}{2} \sum_{r=0}^{\infty} C_r(y, \alpha) B(r + 1, -\alpha s) + \frac{1}{2s}, \quad (\sigma < 0), \quad (4.5)$$

where $B(x, \beta) = \int_0^\infty t^{x-1} (1 - t)^{\beta-1} dt$ for $\Re x > 0$, $\Re \beta > 0$. 

Now using $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ we have

$$B(r+1, -\alpha s) = \frac{r!}{\prod_{k=0}^{\infty} (k - \alpha s)} = P(r, s), \quad r \geq 0, \quad r \in \mathbb{Z}. \quad (4.6)$$

Thus

$$\Psi_1(s, y) = \sum_{r=0}^{\infty} C_r(y, \alpha)P(r, s) + \frac{1}{2s}, \quad \text{for } \sigma < 0. \quad (4.7)$$

It also follows that $P(r, s) \ll r^{\sigma}$ as $r \to \infty$ by the known asymptotic behavior of the gamma function, and by Theorem 3 we have

$$C_r(y, \alpha)P(r, s) \ll r^{\alpha + \sigma} \quad \text{as } r \to \infty.$$ 

Therefore,

$$\sum_{r=0}^{\infty} |C_r(y, \alpha)P(r, s)| \leq C_4 \sum_{r=0}^{\infty} r^{\alpha + \sigma - 4} < \infty \quad \text{if } \sigma < \frac{3 - \alpha}{\alpha}$$

and so $\sum_{r=0}^{\infty} C_r(y, \alpha)P(r, s)$ converges absolutely for $\sigma \leq 2 < \frac{3}{\alpha} - 1$ and the series represents the analytic continuation of $\Psi_1(s, y)$ from $\sigma < 0$ to $\sigma \leq 2$.

Next we observe $\Psi_2(s, y) = \Psi_1(1 - s, y)$ and therefore the series

$$\Psi_2(s, y) = \frac{\alpha}{2} \sum_{r=0}^{\infty} C_r(y, \alpha)P(r, 1 - s) + \frac{1}{2(1-s)} \quad (4.8)$$

converges absolutely in $\sigma \geq -1$ and represents the analytic continuation of the second integral in (4.4).

Now by Theorem 1, (4.4), (4.7) and (4.8) we finally have the following theorem.

**Theorem 4.** $\Phi(s, y) = \frac{\alpha}{2} \sum_{r=0}^{\infty} C_r(y, \alpha)[P(r, s) + P(r, 1-s)]$ and the series converges absolutely and uniformly for $\sigma \in [-1, 2]$ except near the poles $s = 0$ and $s = 1$ and near the points $s = \frac{1}{\alpha}$ and $s = 1 - \frac{1}{\alpha}$.

5. **Recursion Formulas for the Expansion Coefficients**

Recall

$$T(w, y) = \vartheta((1-w)^{-\alpha}, y) = \sum_{m,n \in \mathbb{Z}} e^{-\pi(1-w)^{-\alpha}(m^2y^{-1}+n^2y)} = \sum_{r=0}^{\infty} C_r(y, \alpha)w^r \quad \text{in } |w| < 1.$$ 

Fix $y$ in $[\frac{2}{\alpha}, \frac{5}{2}]$ and set

$$f(w) = e^{-\pi(1-w)^{-\alpha}(m^2y^{-1}+n^2y)+\pi(m^2y^{-1}+n^2y)} = \sum_{r=0}^{\infty} c_r w^r, \quad |w| < 1, \quad c_0 = 1.$$ 

Then

$$f'(w) = -\alpha \pi (m^2y^{-1}+n^2y)(1-w)^{-\alpha-1}f(w).$$
Now
\[(1 - w)^{-\beta} = \sum_{k=0}^{\infty} \binom{-\beta}{k} (-w)^k, \quad |w| < 1\]
and
\[(-1)^k \binom{-\beta}{k} = \frac{\Gamma(k + \beta)}{k! \Gamma(\beta)} = \frac{(\beta)_k}{k!}.
\]
Thus
\[\sum_{r=1}^{\infty} r c_r w^{r-1} = -\alpha \pi (m^2 y^{-1} + n^2 y) \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k}{k!} w^k \sum_{\ell=0}^{\infty} c_{\ell} w^\ell \]
and
\[(r + 1)c_{r+1} = -\alpha \pi (m^2 y^{-1} + n^2 y) \sum_{k=0}^{r} \frac{(\alpha + 1)_k}{k!} c_{r-k}, \quad r > 0, \quad c_0 = 1.
\]
Therefore
\[e^{-\pi(1-w)-\alpha(m^2 y^{-1} + n^2 y)} = e^{-\pi(m^2 y^{-1} + n^2 y)} \sum_{r=0}^{\infty} c_r w^r, \quad |w| < 1,
\]
and we have the following theorem.

**THEOREM 5.** \(C_r(y, \alpha) = C e^{-\pi(m^2 y^{-1} + n^2 y)} G(r(m^2 y^{-1} + n^2 y)),\) where the \(c_r\) follow recursively uniquely from
\[c_{r+1} = \frac{-\alpha \pi (m^2 y^{-1} + n^2 y)}{r + 1} \sum_{k=0}^{r} \frac{(\alpha + 1)_k}{k!} c_{r-k}, \quad r > 0, \quad c_0 = 1 \quad (5.1)
\]
as polynomials in \(m^2 y^{-1} + n^2 y\).

**THEOREM 6.** \(\sum_{r=0}^{\infty} C_r(y, \alpha) = 1.\)

**PROOF.** First by Theorem 1.
\[\lim_{s \to 0} s \Phi(s, y) = \frac{-1}{2}.
\]
On the other hand, by Theorem 4. and (4.6)
\[\lim_{s \to 0} s \Phi(s, y) = \frac{-1}{2} \sum_{r=0}^{\infty} C_r(y, \alpha).
\]

We now have our new representation of \(\Phi(s, y)\) in Theorem 4, which is suitable for numerical computations since the terms of the series consist only of rational functions of \(s\) (rather than incomplete gamma functions or \(K\)-Bessel functions). In the next sections, we describe and present results obtained by these computations.

### 6. COMPUTATION OF ZEROS OF \(Z(s, y)\)

First, we recall the well known Argument Principle: Let \(\gamma\) be a simple closed contour in the \(s\)-plane, described in the positive sense, and let \(f\) be a function which is analytic inside and on \(\gamma\), except possibly for poles interior to \(\gamma\). Also, let \(f\) have no zeros on \(\gamma\). Then
\[\frac{1}{2\pi} \Delta_{\gamma} \arg f(s) = N - P\]
where \(\Delta_{\gamma} \arg f(s)\) denotes the change in argument of \(f(s)\) along \(\gamma\), and \(N\) and \(P\) are the number of zeros and the number of poles of \(f\) respectively, counting multiplicities, interior to \(\gamma\).
We turn our attention to \( f(s) = \Phi(s, y) \), which by Theorem 1 is holomorphic in the entire \( s \)-plane except for two simple poles at \( s = 0 \) and \( s = 1 \). The zeros of \( Z(s, y) \) in the half-plane \( t > 0 \), \( (s = \sigma + it) \), are precisely the zeros of \( \Phi(s, y) \) by (2.3) and since \( \Gamma(s) \) is never zero and has no poles there. From the functional equation it follows that \( Z(s, y) \) has zeros at the negative integers. By Bateman and Grosswald [3] it is known that \( Z(s, y) \neq 0 \) for \( s = \sigma \in (0, 1) \) if \( y^2 \) is in \([1, \sqrt{6}]\), certainly. For \( s = \sigma > 1 \), \( Z(s, y) \neq 0 \) since the Dirichlet series defining \( Z(s, y) \) is positive there. By the functional equation we get \( Z(s, y) \neq 0 \) for \( s = \sigma < 0 \) and \( \sigma \notin \mathbb{Z} \). Thus \( Z(s, y) \neq 0 \) on the line \( t = 0 \) except for the negative integers.

As a contour \( \gamma \) we take the boundary of the rectangle with corners \( 1.9 \pm iT \) and \(-0.9 \pm iT \) and suitable \( T > 0 \).

Notice that there are no zeros of \( \Phi(s, y) \) on the real line in the rectangle. Since the only poles are \( s = 0 \) and \( s = 1 \), \( \Phi(s, y) \) is holomorphic on \( \gamma \). By estimating the Dirichlet series for \( Z(s, y) - \zeta(2s) \) on the line \( \sigma = 1.9 \), it follows that there are no zeros of \( \Phi(s, y) \) on that line. We assume that \( \Phi(s, y) \) has no zeros on the segment \( t = T, -0.9 \leq \sigma \leq 1.9 \). The following theorem, based on the functional equation and symmetry of \( \Phi(s, y) \) in \( s \), is presented without proof.

**Theorem 7.** \( \Delta_\gamma \arg \Phi(s, y) = 4\Delta_\gamma \arg \Phi(s, y) \), where \( \gamma_1 \) is the part of \( \gamma \) from \( s = 1.9 \) to \( s = 1.9 + iT \) to \( s = 1.9 + iT \).

Applying the argument principle to \( \Phi(s, y) \) now gives \( N = \frac{\pi}{2} \Delta_\gamma \arg \Phi(s, y) + 2 \). Since \( \Phi(s, y) \) is symmetric about the real axis, \( N \) is even, and we obtain for the number \( N_1 \) of zeros of \( Z(s, y) \) with \(-0.9 \leq \sigma \leq 1.9 \), \( 0 \leq t \leq T \)

\[
N_1 = \frac{1}{2} N = \frac{1}{2} \pi \Delta_\gamma \arg \Phi(s, y) + 1 = N_1(T, y). 
\]

(6.1)

To utilize this formula, we calculated the complex values of \( \Phi(s, y) \) for \( s = 1.9 + it \), \( 0 \leq t \leq 18 \) in increments of \( \Delta t = 0.1 \), and then for \( s = x + 18i \), \( 1.9 \geq x \geq 1.9 \) in increments of \( \Delta x = -0.1 \). This was done for different values of \( y \) in \( 1 \leq y \leq \sqrt{6} \). The resulting complex function values (for fixed \( y \)) were scaled and graphed in order to count the corresponding multiples of \( \pi \) of the argument change of \( \Phi(s, y) \) along \( \gamma_1 \). We describe the numerical computation of \( \Phi(s, y) \) in the next section. In Table 1, we show the results of computer computation in the cases \( y = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5} \) for \( T = 18 \).

<table>
<thead>
<tr>
<th>( N_1 )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>8</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>( \sqrt{5} )</td>
</tr>
</tbody>
</table>

Table 1. Total number of zeros of \( Z(s, y) \) with \(-0.9 \leq \sigma \leq 1.9 \), \( 0 \leq t \leq 18 \).

Computation of zeros and determining the number \( N_1 \) of zeros by (6.1) both depend on accurate calculations of \( \Phi(s, y) \) for \( s \) in the chosen region and \( y \) in the chosen interval. A Fortran 77 program was written to perform these calculations. They were done on a Bull HN DPS8/48 CP-6 operating system with 36 bit capability using double precision real arithmetic (double precision complex arithmetic was not available).

The representation of \( \Phi(s, y) \) in Theorem 4 was used. The first stage of the project consisted of developing a program to calculate and store the values of \( C_r(y, \alpha) \) for each chosen value of \( y \) and \( \alpha \). Two crucial choices were involved: selecting an appropriate value for \( \alpha \) and deciding how many terms of the series to use for best accuracy. In [2], the values of all zeros when \( y^2 = 5 \), \( 0 \leq t \leq 18 \), \(-0.9 \leq \sigma \leq 1.9 \) are listed. Also, for \( y^2 = 1 \), the zeros of \( Z(s, y) \) are the zeros of \( \zeta(s) \) and \( L(s, -4) \); these zeros also are zeros for \( y^2 = 4 \) as are those \( s \) with \( 1 - 2^{-s} + 2^{1 - 2s} = 0 \) (they
are on $\sigma = \frac{1}{2}$.) All these zeros for $t \leq 18$ are numerically known and thus it was possible to check the accuracy of our calculations of these values. And, since Theorem 6 says $\sum_{r=0}^{\infty} C_r(y, \alpha) - 1$, we have an additional way of determining after what $r = R$ to truncate this series and the series in Theorem 4.

Our calculations were performed for $0.1 \leq \alpha \leq 0.9$ in increments of $0.1$. The sum $\sum_{r=0}^{R} C_r(y, \alpha)$ and the zeros of $\Phi_R(s, y)$, given by the correspondingly truncated series from Theorem 4, for $y = 1, 2$ and $\sqrt{5}$, and $t \leq 18$ were calculated. Finally,

$$\alpha = 0.8 \quad \text{and} \quad R = 701$$

gave the best accuracy. All subsequent computations were done with this $\alpha$ and $R$.

7. ZEROS ON THE CRITICAL LINE

$\Phi(s, y)$ is real-valued on $\sigma = \frac{1}{2}$ by the functional equation and symmetry. Thus, finding the zeros of $\Phi(s, y)$ on this line is a matter of calculating $\Phi(s, y)$ for $s = \frac{1}{2} + it$, $0 \leq t \leq 18$, and observing where $\Phi(s, y)$ changes sign. We used increments of $\Delta t = 0.1$ to locate intervals where $\Phi(s, y)$ changes sign and then used smaller increments of $t$ on those intervals to further approximate the location of the zero.

The individual zeros on $\sigma = \frac{1}{2}$ were computed for various $y$'s, $1 \leq y \leq \sqrt{6}$ by calculating $\Phi(s, y) = e^{\pi t} \cdot \Phi(s, y)$, $s = \frac{1}{2} + it$, $0 \leq t \leq 18$ in increments of $\Delta t = 0.1$. The scale factor $e^{\pi t}$ is applied since $|\Phi(s, y)|$ is very small. These values were then graphed for each $y$ chosen, and some interesting variations of the graphs as $y$ increased were observed; e.g., a change in $y$ produced a change in the locations of the zeros of $\Phi(s, y)$ and in the size of the local maxima or minima of $\Phi(s, y)$.

8. ZEROS OFF THE CRITICAL LINE

According to our calculations, for $y = \sqrt{5}$ there are ten zeros of $Z(s, y)$ in our specified region. But our calculation of the zeros on $\sigma = \frac{1}{2}$ yields only eight such zeros. Assuming for the moment that these are simple zeros, this leaves two zeros with $\sigma \neq \frac{1}{2}$. This situation is expected from our discussion in Section 1.

In general, there may be other values of $y$ for which there are zeros off the line $\sigma = \frac{1}{2}$. Because of the functional equation of $\Phi(s, y)$, these zeros with $t > 0$ would always occur in pairs. A numerical method for finding such zeros is needed. We chose to employ the following fact.

**Theorem 8.** Let $f(s)$ be holomorphic on and in the interior of a simple closed contour $C$, $f(s) \neq 0$ on $C$ and let $s_1$ be the only zero of $f(s)$ interior to $C$. Then

$$\frac{1}{2\pi i} \oint_C s \frac{f'(s)}{f(s)} \, ds = s_1 \quad \text{and} \quad \frac{1}{2\pi i} \oint_C f'(s) \, ds = 1.$$ 

Numerically calculating both integrals constitutes a method for finding $s_1$, (which is different from methods usually employed in the literature; the latter mainly use approximation formulas involving iterative processes).

In order to use Theorem 8 a representation of $\Phi'(s, y)$ is needed. We obtain from Theorem 4 by termwise differentiation

$$\Phi'(s, y) = \frac{\alpha}{2} \sum_{r=0}^{\infty} C_r(y, \alpha)$$

$$\times \left[ \frac{1}{r + 1} \left( \sum_{\ell=0}^{r} \frac{\alpha}{\ell - as} \right) \prod_{j=0}^{r} \frac{j + 1}{j - \alpha s} \right] \left( \sum_{\ell=0}^{r} \frac{\alpha}{\ell - \alpha(1 - s)} \right) \prod_{j=0}^{r} \frac{j + 1}{j - \alpha(1 - s)},$$

which is justified by uniform convergence.
To calculate the two integrals in Theorem 8, we used a rectangular contour:

\[ \oint_C g(s) ds = i \int_{T_1}^{T_2} g(\sigma_2 + it) dt - \int_{\sigma_1}^{\sigma_2} g(\sigma + iT_2) d\sigma - i \int_{T_1}^{T_2} g(\sigma_1 + it) dt + \int_{\sigma_1}^{\sigma_2} g(\sigma + iT_1) d\sigma, \tag{8.2} \]

where \( \frac{1}{2} \leq \sigma_1 \leq \sigma_2 \leq 2, \ 0 < T_1 < T_2 < 20. \)

Each of the four integrals was then evaluated using Simpson's Rule on 64 subintervals. The integral with \( g(s) = \Phi'(s, y)/\Phi(s, y) \) was used to verify that the chosen \( C \) contains exactly one zero. Once a zero was calculated using \( g(s) = s\Phi'(s, y)/\Phi(s, y) \), it was recalculated several times using different values of \( \sigma_1, \sigma_2, T_1, \) and \( T_2 \) to establish its accuracy.

Consider again the case \( y = \sqrt{5} \). In [2], O'Leary gives the zeros \( .0670 + 15.66835 \) and \( .9330 + 15.6683i \). We found \( .0742 + 15.6682i \) and \( .9258 + 15.6682i \) with our method (and much better agreement for the other eight zeros, on \( \sigma = \frac{1}{2} \)).

Since the emphasis of this work is to study the motion of the zeros of \( Z(s, y) \) as \( y \) changes, we decided to “track” this pair of zeros. As \( y \) decreases from \( \sqrt{5} \), these zeros move toward the line \( \sigma = \frac{1}{2} \) while their common imaginary part increases. They eventually intersect on the line \( \sigma = \frac{1}{2} \) and at that particular \( y \) value (approximately \( y = \sqrt{4.006} \) and \( t = 16.37 \)) collapse to a double zero. They then split apart again, but on the line \( \sigma = \frac{1}{2} \), one of them with an increasing imaginary part, the other decreasing.

As \( y \) increased from \( y = \sqrt{5} \), these two zeros exhibit similar behavior, moving toward the line \( \sigma = \frac{1}{2} \) but with decreasing imaginary parts. They eventually intersect on \( \sigma = \frac{1}{2} \) with an imaginary part of approximately 14.8 and \( y \) near \( \sqrt{6.3} \). They then split apart again, one moving up the line, the other down (on \( \sigma = \frac{1}{2} \)).

Further calculations for the remaining \( y \) values in \( [1, \sqrt{6}] \) revealed similar behavior on four other \( y \)-intervals. The zeros move together on the line \( \sigma = \frac{1}{2} \), then split off onto twin arcs moving away from and then back to the line with decreasing common imaginary part as \( y \) increases. Tables 2–22 list all zeros of \( Z(s, y) \) in the rectangle \(-1 < \sigma < 2, \ 0 \leq t \leq 18 \) for the various values of \( y \) given in the label of each table.

For example, when \( y = 1 \) all five zeros are on the critical line \( \sigma = \frac{1}{2} \); and when \( y = \sqrt{1.15} \) there are three zeros on the line \( \sigma = \frac{1}{2} \) and a pair of zeros of the form \( \sigma_1 + it_1, \ (1 - \sigma_1) + it_1 \) with \( \sigma_1 \approx 1.16 \) and \( t_1 \approx 13.42 \). Tables 8, 16 and 17 show two pairs of zeros off the line \( \sigma = \frac{1}{2} \).
Tables 11–14 indicate some interesting motions of zeros. The two zeros off the critical line for $y = \sqrt{1.9}$ collapse onto the critical line and then separate to approximately $z_1 = \frac{1}{2} + 14.1i$ and $z_2 = \frac{1}{2} + 14.5i$ as $y$ increases to $\sqrt{2}$. Then for $y = \sqrt{2.1}$, $z_1$ has moved down the critical line to about $\frac{1}{2} + 12.9i$ toward another zero at about $\frac{1}{2} + 12.6i$. These two zeros then collapse and move off the critical line when $y$ increases to $\sqrt{2.2}$.
Tables 2, 12, 18, and 19 show that in the cases $y = 1, \sqrt{2}, \sqrt{3}, 2$ all zeros are on the critical line. When $y = \sqrt{4.1}$, a pair of zeros has moved off the critical line. These are the zeros that give rise to the ones found by Potter and Titchmarsh for $y = \sqrt{5}$. As mentioned above, these zeros collapse back onto the critical line for $y$ near $\sqrt{6.3}$.

One pattern that can be observed is that as the $y$ values get larger, the zeros move less far away from the line $\sigma = \frac{1}{2}$. On the first arc, the $\sigma$ value on the right side increases to almost 2.0. But on the last arc, the $\sigma$ values on the right side increase to only about .93. This behavior is consistent with the results of Deuring or Stark mentioned in the introduction.

REFERENCES