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Discrete Mathematics 269 (2003) 93–112
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Finite BL-algebras

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Received 24 June 2002; accepted 22 July 2002

Abstract

BL-algebras were introduced by Hájek as algebraic structures of Basic Logic. The aim of this paper is to analyze the structure of finite BL-algebras. Extending the notion of ordinal sum, we characterize a class of finite BL-algebras, actually BL-comets. Then, just using BL-comets, we can represent any finite BL-algebra as a direct product of BL-comets. Furthermore we define a class of labelled trees, which can be one-to-one mapped onto finite BL-algebras.

Keywords: BL-algebra; BL-comet; Labelled tree

1. Introduction

BL-algebras were introduced by Hájek [4] as algebraic structures of Basic Logic. A BL-algebra is an algebra $A=(A,\wedge,\vee,\odot,\rightarrow,0,1)$ such that:

1. $(A,\wedge,\vee,0,1)$ is a lattice with 0 as least element and 1 as greatest element,
2. $(A,\odot,1)$ is a commutative monoid,
3. the following statements hold for every $x,y,z\in A$:
   (a) $z\leqslant x\rightarrow y$ iff $x\odot z\leqslant y$,
   (b) $x\wedge y=x\odot(x\rightarrow y)$,
   (c) $(x\rightarrow y)\vee(y\rightarrow x)=1$.

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PII: S0012-365X(02)00754-9

By former operations a negation operation $*$ is defined in the following way: $x^* = x \to 0$, for every $x \in A$. We will write x^p instead of $\underbrace{x \odot \cdots \odot x}_{p}$.

The set of all BL-algebras is a variety, whose subvariety defined by the further axiom $(x^*)^* = x$, for every $x \in A$, coincides with the variety of MV-algebras. Let A be a BL-algebra and $\mathrm{MV}(A) = \{x \in A | x^{**} = x\}$. MV($A$) is a subalgebra of A. It is the greatest subalgebra of A that is an MV-algebra [4].

We say that the BL-algebra A is totally ordered or that the BL-algebra A is a chain (shortly BL-chain) if the lattice $(A, \wedge, \vee, 0, 1)$ is totally ordered. Every BL-algebra is a subdirect product of BL-chains [4].

We say that the BL-algebra A is finite if the cardinal of the set A is finite.

The aim of this paper is to analyze the structure of finite BL-algebras. In the case of MV-algebras their structure is already well known. Every finite BL-chain is a finite ordinal sum whose components are finite MV-chains (see [1, Theorem 3.6]). Extending the notion of ordinal sum of BL-algebras (see Preliminaries), we characterize a class of finite BL-algebras, actually BL-comets (see Section 4) which can be seen as a generalization of finite BL-chains. Then, just using BL-comets, we can represent any finite BL-algebra A as a direct product of BL-comets. This result can be understood as a generalization of the representation of finite MV-algebras as a direct product of MV-chains (see [3]). Furthermore, in Section 5 we define a class of labelled trees, which can be one-to-one mapped onto finite BL-algebras. The class of all finite BL-algebras will be denoted by **FBL**. For any unexplained notion on MV-algebras see [3], on BL-algebras see [4].

2. Preliminaries

Let A be a finite BL-algebra, subdirect product of the BL-chains $C_1, C_2, \ldots, C_n$. We say that the chain C_i, $i \in I_n = \{1, 2, \ldots, n\}$, is *essential* in the representation of A iff A is not a subdirect product of $C'_1, \ldots, C'_{i-1}, C'_{i+1}, \ldots, C'_n$, with $C'_t \cong C_t$ for every $t \in \{1, \ldots, i-1, i+1, \ldots, n\}$. Let us assume that every chain C_i is essential in the representation of A.

Definition 1 (Agliano and Montagna [1]). Let $\mathscr{A}_i = (A_i, \wedge_i, \vee_i, \odot_i, \to_i, 0_i, 1)$ be BL-chains for $i \in \{1, \ldots, r-1\}$ and a BL-algebra for $i = r$. Assume:

1. $A_i \cap A_j = \{1\}$, for $i \neq j$.

Then the ordinal sum $\biguplus_{i=1}^{r} A_i = (\bigcup_{i=1}^{r} A_i, \wedge, \vee, \odot, \to, 0, 1)$ is a new BL-algebra whose operations $\wedge, \vee, \odot$ coincide with those of A_i, when applied on pairs of elements of A_i, $i = 1, \ldots, r$, and on the rest of pairs are defined as follows, for $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$:

1. $x \wedge y = y \wedge x = x$,
2. $x \vee y = y \vee x = y$,
3. $x \odot y = y \odot x = x$.

Finally, the operation $\to$ is defined by

$$x \to y = \begin{cases} 1 & \text{if } x \leqslant y, \\ x \to_i y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i, y \in A_j \text{ and } i > j. \end{cases} \tag{1}$$

For $i \in \{1, \ldots, n\}$ denote by $0_i = \alpha_{i,0} < \alpha_{i,1} < \cdots < \alpha_{i,n_i} < \alpha_{i,n_i+1} = 1_i$ the chain of the idempotent elements of C_i, shortly $\mathbb{I}(C_i, n_i + 1)$. By [1] Theorem 3.6, C_i is an ordinal sum of finite MV-chains, i.e. finite BL-chains that are MV-algebras, in symbols $C_i = \biguplus_{h=1}^{n_i+1} M(\alpha, i, h)$, where $M(\alpha, i, h) = [\alpha_{i,h-1}, \alpha_{i,h}[\cup \{1_i\}$. Then for every $i \in I_n$ and $h \in \{1, \ldots, n_i + 1\}$ the restrictions of the operations defined on C_i to the subset $M(\alpha, i, h)$ make it into a finite MV-chain, hence it is isomorphic to the MV-algebra $S_p = \{0, 1/p, \ldots, (p-1)/p, 1\}$, for some $p \geqslant 1$. From now on, every time we will deal with a finite BL-algebra A, we will use the above notations to give a subdirect representation of A by finite BL-chains and the decomposition of such chains as ordinal sum. Furthermore, in the sequel, every finite MV-chain will be identified with the subalgebra of $[0, 1]$, which it is isomorphic to. For every $f \in A$, denote by f_i the ith component of f. Moreover, for every $x \in C_i$, set $\alpha(x) = \max\{y \in \mathbb{I}(C_i, n_i + 1) \mid y \leqslant x\}$ and for every $f \in A$, set $\alpha(f) = (\alpha(f_1), \ldots, \alpha(f_n))$. If $f \in A$, then $\alpha(f) \in A$. Indeed, let $f_i \in M(\alpha, i, h_i) \cong S_{p_i}$ and $\mu = \max\{p_i, i \in I_n\}$; then $f^\mu = \alpha(f)$. In the sequel we will mean $\alpha_{i,h} = \alpha_{j,k}$ if $h = k, h, k < \min\{n_i + 1, n_j + 1\}$. Moreover, we will denote by α_h the n-tuple $(\alpha_{1,h}, \ldots, \alpha_{n,h})$, for every $h \leqslant \min\{n_i + 1, i \in I_n\}$, by $\mathbf{1}$ the n-tuple, having its ith component equal to 1_i and by $\mathbf{0}$ the n-tuple, having its ith component equal to 0_i, for every $i \in I_n$.

With the above notations we get.

Proposition 2. *Let $A \in \mathbf{FBL}$ and $f, g \in A$ such that for some $(i, j) \in I_n^2$:*

1. $f_i = \alpha_{i,h}$ *and* $g_i = \alpha_{i,k}$, $h = 0, 1, \ldots, n_i + 1$, $k = 0, 1, \ldots, n_i + 1$, $h \neq k$,
2. $f_j = g_j = \alpha_{j,m}$, $m = 0, 1, \ldots, n_j + 1$.

Then, for every $(x, y) \geqslant (\alpha_{i,t}, \alpha_{j,m})$, $t = \min\{h, k\}$, there is an element $l \in A$ such that $(l_i, l_j) = (x, y)$.

Proof. Assume $h < k$ and $(x, y) \geqslant (\alpha_{i,h}, \alpha_{j,m})$. Let u be an element of A, having its ith component equal to x and v be an element of A, having its jth component equal to y. Then $l = ((g \to f) \vee u) \wedge (((g \to f) \to f) \vee v)$ verifies the claim. □

Proposition 3. *Let $A \in \mathbf{FBL}$ and $f \in A$ such that for some $(i, j) \in I_n^2$:*

$$(f_i, f_j) = (\alpha_{i,h}, \alpha_{j,k}),\ h = 0, 1, \ldots, n_i + 1,\ k = 0, 1, \ldots, n_j + 1,\ h \neq k.$$

Then there is $g \in A$ such that $(g_i, g_j) = (\alpha_{i,t}, \alpha_{j,t})$, $t = \min\{h, k\}$.

Proof. Assume $h < k$. If $h = 0$, then it is trivial. For otherwise let w be an element of A, having its jth component equal to $\alpha_{j,h}$. Set $\alpha(w_i) = \alpha_{i,h_1}$ for a suitable h_1.

If $h_1 \geqslant h$, then $g = w \wedge f \in A$ verifies the claim.

If $h_1 < h$, we choose an element $w^1 \in A$, having its jth component equal to α_{j,h_1}. Set $\alpha(w_i^1) = \alpha_{i,h_2}$. If $h_2 \geqslant h_1$, then $w^1 \wedge \alpha(w) \in A$, moreover $w_i^1 \wedge \alpha(w_i) = \alpha_{i,h_1}$ and $w_j^1 \wedge \alpha(w_j) = \alpha_{j,h_1}$. Hence, the conclusion follows from Proposition 2. If $h_2 < h_1$, proceeding as above, the thesis shall be attained as soon as we find an element w^r such that $\alpha(w_j^r) = \alpha_{j,h_r}$, $\alpha(w_i^r) = \alpha_{i,h_{r+1}}$ and $h_{r+1} \geqslant h_r$. If the last condition is never verified, then we get a strictly decreasing sequence $k > h > h_1 > h_2 > \cdots > h_r > \cdots$ of natural numbers; consequently there must ultimately be an s, such that $h_s = 0$. Hence we will find an element w^{s-1} such that $(\alpha(w_i^{s-1}), \alpha(w_j^{s-1})) = (0_i, \alpha_{j,h_{s-1}})$, where $\alpha_{j,h_{s-1}} \neq 0_j$. Also in this case the claim follows by Proposition 2. □

Corollary 4. *Let $A \in \mathbf{FBL}$ and $f \in A$ such that for some $(i,j) \in I_n^2$:*

$$(f_i, f_j) = (\alpha_{i,h}, \alpha_{j,k}),\ h \text{ and } k \text{ run from } 0, 1, \ldots, \text{ like in Proposition 3.}$$

Then for every $(x,y) \geqslant (\alpha_{i,t}, \alpha_{j,t})$, with $t = \min\{h,k\}$, there is an element $l \in A$, such that $(l_i, l_j) = (x,y)$.

Proof. It follows by Propositions 2 and 3. □

Corollary 5. *Let $A \in \mathbf{FBL}$. Then, for every $h \leqslant \min\{n_i + 1, i \in I_n\}$, $\alpha_h = (\alpha_{1,h}, \ldots, \alpha_{n,h}) \in A$.*

Proof. We proceed by induction on n. Let $n = 2$, $h \leqslant \min\{n_1 + 1, n_2 + 1\}$, and $x, y \in A$ such that $x_1 = \alpha_{1,h}$ and $y_2 = \alpha_{2,h}$. If x and y are incomparable, then either $x \vee y = \alpha_h$ or $x \wedge y = \alpha_h$. Otherwise, consider $\alpha(x_2) = \alpha_{2,t}$ and $\alpha(y_1) = \alpha_{1,l}$. If $t = h$ or $l = h$, then either $\alpha(x) = \alpha_h$ or $\alpha(y) = \alpha_h$. Assume either $l > h$ or $t > h$; in both cases the conclusion follows from Proposition 3.

Suppose now the corollary true for $n-1$. Set $I_n^i = I_n \backslash \{i\}$ and let $A_{\upharpoonright \{I_n^i\}}$ be the set of the restrictions of all the elements of A to I_n^i; $A_{\upharpoonright \{I_n^i\}}$ is a finite BL-algebra and it is, up to isomorphism, a subdirect product of $\{C_i, i \in I_n^i\}$. Fix $h \leqslant \min\{n_i + 1, i \in I_n\}$. By induction, for every $i \in I_n$, there is an element $x^i \in A$ such that $(x^i)_j = \alpha_{j,h}$ for $j \neq i$. If two of the elements x^1, x^2 and x^3 are incomparable, say x^1 and x^2, then either $x^1 \vee x^2$ or $x^1 \wedge x^2$ satisfies the claim. Otherwise x^1, x^2 and x^3 are comparable. We safely can set $x^1 \leqslant x^2 \leqslant x^3$. Then we have $\alpha_{2,h} \leqslant x_2^2 \leqslant \alpha_{2,h}$, that is $x_2^2 = \alpha_{2,h}$. From that we get $x^2 = \alpha_h \in A$. □

Lemma 6. *Let $A \in \mathbf{FBL}$, $i, j \in I_n$, $h = 1, \ldots, n_i + 1$, $k = 1, \ldots, n_j + 1$ and $h \neq k$. Then the following are equivalent:*

1. *there exists $f \in A$ such that $(f_i, f_j) \in M(\alpha, i, h) \times M(\alpha, j, k) \backslash \{(1_i, 1_j)\}$,*
2. *for every $(x,y) \geqslant (\alpha_{i,h-1}, \alpha_{j,k-1})$ there is an element $g \in A$ such that $(g_i, g_j) = (x,y)$.*

Proof. Let $f \in A$ such that $(f_i, f_j) \in M(\alpha, i, h) \times M(\alpha, j, k) \backslash \{(1_i, 1_j)\}$. Then

$$(\alpha(f_i), \alpha(f_j)) = \begin{cases} (\alpha_{i,h-1}, \alpha_{j,k-1}) & \text{if } \alpha(f_i) \neq 1_i \text{ and } \alpha(f_j) \neq 1_j, \\ (\alpha_{i,h-1}, 1_j) & \text{if } \alpha(f_j) = 1_j, \\ (1_i, \alpha_{j,k-1}) & \text{if } \alpha(f_i) = 1_i. \end{cases} \tag{2}$$

Thus, the conclusion follows by Corollary 4. Vice versa is obvious. □

Lemma 7. *Let* $A \in \mathbf{FBL}$, $h \leqslant \min\{n_i+1, n_j+1 : i,j \in I_n\}$, $M(\alpha,i,h) \cong S_p$, $M(\alpha,j,h) \cong S_q$ *and* $p \neq q$.

Then for every $(x,y) \geqslant (\alpha_{i,h-1}, \alpha_{j,h-1})$, *there is an element* $g \in A$ *such that* $(g_i, g_j) = (x,y)$.

Proof. Let $p < q$ and $f \in A$ such that $f_j = (q-1)/q \in M(\alpha,j,h)$.

If $f_i < \alpha_{i,h-1}$, we get $(\alpha(f_i), \alpha(f_j)) = (\alpha_{i,k}, \alpha_{j,h-1}), k < h-1$.
If $\alpha_{i,h-1} \leqslant f_i < \alpha_{i,h}$, then $(f^p \to \alpha_{h-1})^q$ is an element of A, having its ith component equal to 1_i and its jth component equal to $\alpha_{j,h-1}$.
Finally, if $f_i \geqslant \alpha_{i,h}$, we get $(\alpha(f_i), \alpha(f_j)) = (\alpha_{i,l}, \alpha_{j,h-1}), l > h-1$.

In any case the claim follows by Corollary 4. □

Lemma 8. *Let* $A \in \mathbf{FBL}$, $h \leqslant \min\{n_i+1, n_j+1 : i,j \in I_n\}$.
If there is an element $f \in A$ *such that*:

1. $(f_i, f_j) \in M(\alpha,i,h) \times M(\alpha,j,h)$,
2. $f_i \neq f_j$,

then for every $(x,y) \geqslant (\alpha_{i,h-1}, \alpha_{j,h-1})$, *there is an element* $g \in A$ *such that* $(g_i, g_j) = (x,y)$.

Proof. If $M(\alpha,i,h) \cong S_p$, $M(\alpha,j,h) \cong S_q$ and $p \neq q$, it follows from Lemma 7. Therefore, it shall be understood $M(\alpha,i,h) \cong M(\alpha,j,h) \cong S_p$.

By Corollary 4, we get the claim in the following cases:

1. $f_i = 1_i$ or $f_j = 1_j$,
2. $f_i = \alpha_{i,h-1}$, hence the element $l = (f \to \alpha_{h-1})^p$ has its ith component equal to 1_i and its jth component equal to $\alpha_{j,h-1}$,
3. $f_j = \alpha_{j,h-1}$, hence the element $m = (f \to \alpha_{h-1})^p$ has its jth component equal to 1_j and its ith component equal to $\alpha_{i,h-1}$.

Assume now $f_i = r/p \in M(\alpha,i,h)$, $f_j = s/p \in M(\alpha,j,h)$ and $0 < r < s < p$. Then we get
Case 1: $s = p-1$. Then the element $f^{p-1} \in A$ has its ith component equal to $\alpha_{i,h-1}$ and its jth component equal to $1/p \in M(\alpha,i,h)$. Then we proceed as in 2.
Case 2: $s < p-1$.
Let $g \in A$ such that $g_j = (p-1)/p \in M(\alpha,j,h)$.

(a) If $g_i \leqslant f_i$, then the element $h = ((f \vee g) \to f)^p$, has its ith component equal to 1_i and its jth component equal to $\alpha_{j,h-1}$.
(b) If $g_i \geqslant \alpha_{i,h}$, then the element $k = ((f \to f \odot g)^p$, has its ith component equal to 1_i and its jth component equal to $\alpha_{j,h-1}$.
(c) Finally, if $f_i < g_i < \alpha_{i,h}$ then the element $d = (g^r \odot f) \to \alpha_h$ has its ith component equal to 1_i and its jth component equal to $\alpha_{j,h-1}$.

Again the conclusion follows from Corollary 4. □

Proposition 9. *Let* $A \in \mathbf{FBL}$ *and* $J = \{i_1, \ldots, i_r\} \subseteq I_n$.

If, for every $i_p, i_q \in J$ *and for every pair* $(x_{i_p}, x_{i_q}) \geqslant (\alpha_{i_p,h_p}, \alpha_{i_q,h_q})$, *there is an element* $g^{p,q} \in A$ *such that* $(g^{p,q}_{i_p}, g^{p,q}_{i_q}) = (x_{i_p}, x_{i_q})$, *then for every r-tuple* $(x_{i_1}, \ldots, x_{i_r}) \geqslant (\alpha_{i_1,h_1}, \ldots, \alpha_{i_r,h_r})$, *there is an element* $g \in A$ *such that* $(g_{i_1}, \ldots, g_{i_r}) = (x_{i_1}, \ldots, x_{i_r})$.

Proof. The proposition is true for $r = 2$. Let $r \geqslant 3$ and let us proceed by induction on the cardinality of J. Assume the proposition is true for $r-1$. Let $x = (x_{i_1}, \ldots, x_{i_r}) \geqslant (\alpha_{i_1,h_1}, \ldots, \alpha_{i_r,h_r})$.

For every $i_u \in J$, denote by x^u the $(r-1)$-tuple obtained from $(x_{i_1}, \ldots, x_{i_r})$, by deleting the i_uth component of x. By induction, there is an element $g^u \in A$ such that $g^u_{i_m} = x_{i_m}$, for every $m \in \{1, \ldots, u-1, u+1, \ldots, r\}$. If the restrictions to J of two among these elements, say g^{u_1} and g^{u_2}, are not comparable, then we have: either

1. $(g^{u_1})_{u_1} \leqslant (g^{u_2})_{u_1} = x_{u_1}$ and $x_{u_2} = (g^{u_1})_{u_2} \geqslant (g^{u_2})_{u_2}$

or

2. $(g^{u_1})_{u_1} \geqslant (g^{u_2})_{u_1} = x_{u_1}$ and $x_{u_2} = (g^{u_1})_{u_2} \leqslant (g^{u_2})_{u_2}$.

In the former case $(g^{u_1} \vee g^{u_2})_{i_m} = x_{i_m}$, for every $m \in \{1, \ldots, r\}$. In the latter case x is given by the restriction to J of $g^{u_1} \wedge g^{u_2}$.

Assume that the restrictions to J of all the elements g^u are each other comparable, then we safely can write $g^1 \leqslant g^2 \leqslant g^3$. From that $x_{i_2} = (g^1)_{i_2} \leqslant (g^2)_{i_2} \leqslant (g^3)_{i_2} = x_{i_2}$. Then it is $(g^2)_{i_m} = x_{i_m}$ for every $m \in \{1, \ldots, r\}$. □

Corollary 10. *Let* $A \in \mathbf{FBL}$, $M(\alpha, i, 1) \cong S_{p_i}$ *and* $S_{p_i} \neq S_{p_j}$ *for every* $(i,j) \in I_n^2$. *Then* $A = C_1 \times \cdots \times C_n$.

Proof. This follows by Lemma 7 and Proposition 9. □

Corollary 11. *Let* $A \in \mathbf{FBL}$. *Then the following implication holds*:

(*for every* $i \in I_n$ *there is an element* $f^i \in A$ *such that* $\alpha(f^i) \neq \mathbf{0}$ *and* $\alpha(f^i_i) = 0_i \Rightarrow (A = C_1 \times \cdots \times C_n)$.

Proof. This follows by Propositions 2 and 9. □

3. Direct decomposition

Let $A \in \mathbf{FBL}$. It is known that $\mathrm{MV}(A) = A \cap \prod_{i=1}^n M(\alpha, i, 1)$ [1]. Define on I_n the following equivalence relation:

$i \equiv j$ iff for every $f \in \mathrm{MV}(A)$, $f_i = f_j$.

Let $\pi = \{J_1, \ldots, J_r\}$ be the partition of I_n yielded by this relation.

Remark 12. The above definition is equivalent to the following: $i \equiv' j$ iff $f_i = f_j$ for every $f \in A$ such that $(f_i, f_j) \in M(\alpha, i, 1) \times M(\alpha, j, 1)$.

Indeed, let $i\equiv j$, $f\in A$, and $(f_i,f_j)\in M(\alpha,i,1)\times M(\alpha,j,1)$. Since $f^{**}\in \mathrm{MV}(A)$, we get $f_i=f_i^{**}=f_j^{**}=f_j$, hence $i\equiv' j$.

For a positive integer k denote by A_{J_k} the set of the restrictions to J_k of all the elements of A; A_{J_k} is a BL-algebra and it is, up to isomorphism, subdirect product of $\{C_i, i\in J_k\}$. In the sequel the restriction of $f\in A$ to J_k will be denoted by f_{J_k}.

The following result is crucial:

Theorem 13. *Let $A\in$* **FBL**. *Then A is isomorphic to the direct product $A_{J_1}\times\cdots\times A_{J_r}$.*

Proof. The map $\phi: f\in A\to(f_{J_1},\dots,f_{J_r})\in A_{J_1}\times\cdots\times A_{J_r}$ is a homomorphism. Now we shall prove that ϕ is bijective.

Claim 1. *ϕ is injective.*

Indeed, if $f\neq g$, then $f_i\neq g_i$ for some i. Let $i\in J_k$, then $f_{J_k}\neq g_{J_K}$ hence $\phi(f)\neq\phi(g)$.

Claim 2. *ϕ is surjective.*

We will prove the surjectivity of ϕ by induction on the cardinal of the set π. It is trivial if $|\pi|=1$. Assume that it is true for $|\pi|=r-1$ and set:

$\phi': f\in A_{J_1\cup\cdots\cup J_{r-1}}\to(f_{J_1},\dots,f_{J_{r-1}})\in A_{J_1}\times\cdots\times A_{J_{r-1}}$. By induction, ϕ' is surjective.

Let $(f^1_{J_1},\dots,f^r_{J_r})\in(A_{J_1}\times\cdots\times A_{J_{r-1}})\times A_{J_r}$ and $f\in A$ such that $\phi'(f_{J_1\cup\cdots\cup J_{r-1}})=(f^1_{J_1},\dots,f^{r-1}_{J_{r-1}})$; moreover let $g\in A$ such that $g_{J_r}=f^r_{J_r}$.

Fix a subset J of I_n containing exactly a single representative element from each class of π, that is $J=\{i_1,\dots,i_r\}$ and $i_m\in J_m$, $m\in\{1,\dots,r\}$. By Lemmas 7 and 8 and Proposition 9, for every r-tuple $(x_{i_1},\dots,x_{i_r})\in M(\alpha,i_1,1)\times\cdots\times M(\alpha,i_r,1)$, there is an element $g'\in A\cap\prod_{i=1}^n M(\alpha,i,1)$, such that $(g'_{i_1},\dots,g'_{i_r})=(x_{i_1},\dots,x_{i_r})$.

Therefore, let $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ such that:

1. $(a_{i_1},\dots,a_{i_r})$, $(b_{i_1},\dots,b_{i_r})\in M(\alpha,i_1,1)\times\cdots\times M(\alpha,i_r,1)$,
2. $a_{i_h}<b_{i_h}$ for $h=1,\dots,r-1$,
3. $a_{i_r}>b_{i_r}$.

Then we have

$$(a\to b)_i=\begin{cases}1_i & \text{if } i\in J_1\cup\cdots\cup J_{r-1},\\ a^*_{i_r}\oplus b_{i_r}\in M(\alpha,i,1) & \text{if } i\in J_r.\end{cases}\tag{3}$$

Set $c=\alpha(a\to b)$. Then $c_i=1_i$ for $i\in J_1\cup\cdots\cup J_{r-1}$ and $c_i=0_i$ for $i\in J_r$. On other side,

$$(b\to a)_i=\begin{cases}b^*_i\oplus a_i\in M(\alpha,i,1)\backslash\{1_i\} & \text{if } i\in J_1\cup\cdots\cup J_{r-1},\\ 1_i & \text{if } i\in J_r.\end{cases}\tag{4}$$

Let $w=\alpha(b\to a)$. Then $w_i=1_i$ for $i\in J_r$ and $w_i=0_i$ for $i\in J_1\cup\cdots\cup J_{r-1}$. Thus $(c\odot f)\vee(w\odot g)\in A$ and $\phi((c\odot f)\vee(w\odot g))=(f_{J_1},\dots,f_{J_r})$. $\square$

Corollary 14. *Let* $A \in \mathbf{FBL}$. *Then the following are equivalent*:

1. *for every* (i,j) *and* $i \neq j$ *there is an element* $f \in \mathrm{MV}(A)$ *such that* $f_i \neq f_j$,
2. $A = \prod_{i=1}^{n} C_i$.

Proof. Assume that for every (i,j) and $i \neq j$ there is an element $f \in \mathrm{MV}(A)$ such that $f_i \neq f_j$. Then $\pi = I$. Let $A_{\restriction \{i\}}$ be the BL-algebra of the restrictions to $\{i\}$ of all the elements in A. The $A_{\restriction \{i\}} \cong C_i$, for every $i \in \{1,2,\dots,n\}$. Hence the conclusion follows by Theorem 13. Vice versa is obvious. □

4. BL-comet

In this section we will introduce the concept of BL-comet and we will prove the main result according to any finite BL-algebra is a direct product of BL-comets (Corollary 28). To this aim we hold to describe the structure of the algebra A_{J_h}, that is the structure of a finite BL-algebra $A = (A, \wedge, \vee, \odot, \rightarrow, \mathbf{0}, \mathbf{1})$, having the further following property:

$$\text{every } f \in A \cap \prod_{i=1}^{n} M(\alpha, i, 1) = \mathrm{MV}(A) \text{ is constant on } I_n. \tag{5}$$

Such an algebra will be denoted by A_c and the class of all the algebras A_c will be denoted by $\mathbf{A_c}$.

Lemma 15. *Let* $A_c \in \mathbf{A_c}$. *Then, for every* $(i,j) \in I_n^2$, $M(\alpha, i, 1) \cong M(\alpha, j, 1)$.

Proof. Suppose there is $(i,j) \in I_n^2$ such that $M(\alpha, i, 1) \ncong M(\alpha, j, 1)$. We can safely assume $|M(\alpha, i, 1)| < |M(\alpha, j, 1)|$. By Lemma 7, we find an element $g \in A_c$ such that $g_i = 0_i$ and $g_j \in M(\alpha, j, 1) \setminus \{0_j\}$. Hence $g^* \in \mathrm{MV}(A)$ and it is not constant on I_n, absurd. □

Set $M(\alpha, i, h_1^m) = \biguplus_{h=1}^{m} M(\alpha, i, h)$ and, for every A_c, define:

$\delta_{A_c} = \max\{m \in N \mid \text{for every } f \in A_c \cap \prod_{i=1}^{n} M(\alpha, i, h_1^m), f \text{ is constant on } I_n\}$. We get $1 \leqslant \delta_{A_c} \leqslant \min\{n_i + 1 \mid i \in I_n\}$.

With the above notations we have:

Lemma 16. *Let* $A_c \in \mathbf{A_c}$. *Then, for every* $(i,j) \in I_n^2$ *and for every* $1 \leqslant m \leqslant \delta_{A_c}$, $M(\alpha, i, h_1^m) \cong M(\alpha, j, h_1^m)$.

Proof. By Lemma 15 it is true for $m = 1$, then we proceed by induction. Assume $M(\alpha, i, h_1^{m-1}) \cong M(\alpha, j, h_1^{m-1})$, for every $(i,j) \in I_n^2$. Suppose there is (i,j) such that $M(\alpha, i, m) \ncong M(\alpha, j, m)$. Arguing as in Lemma 15, we find an element $g \in A_c$ such that $g_i = \alpha_{i,m-1}$ and $g_j \in M(\alpha, j, m) \setminus \{\alpha_{j,m-1}\}$. Hence $(g \rightarrow \alpha_{m-1}) \in \prod_{i=1}^{n} M(\alpha, i, h_1^{\delta_{A_c}})$ and it is not constant on I_n, absurd. Since $M(\alpha, i, h_1^m) \cong M(\alpha, i, h_1^{m-1}) \uplus M(\alpha, i, m)$, for every $i \in I_n$, the desired conclusion immediately follows. □

In the sequel, when there is no misunderstanding, we will denote δ_{A_c} simply by δ.

Lemma 17. *Let $A_c \in \mathbf{A_c}$. Then $A_c \backslash (\prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \backslash \{1_i\}) = \{x \in A_c \mid x \geqslant \alpha_\delta\}$.*

Proof. The inclusion $\{x \in A_c \mid x \geqslant \alpha_\delta\} \subseteq A_c \backslash (\prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \backslash \{1_i\})$ is immediate.

Assume now $x \in A_c \backslash (\prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \backslash \{1_i\})$ and $x \not\geqslant \alpha_\delta$.

Then the subsets:

$I_1 = \{i \in I_n \mid x_i \geqslant \alpha_{i,\delta}\}$,
$I_2 = \{i \in I_n \mid x_i < \alpha_{i,\delta}\}$

are not empty and

$$(\alpha_\delta \to x)_i = \begin{cases} 1_i & \text{if } i \in I_1, \\ x_i & \text{if } i \in I_2. \end{cases} \tag{6}$$

Consequently $\alpha_\delta \to x \in \prod_{i=1}^{n} M(\alpha, i, h_1^{\delta})$, but $\alpha_\delta \to x$ is not a constant function on I_n, absurd. □

Corollary 18. *Let $A_c \in \mathbf{A_c}$. Then $A_c \cap \prod_{i=1}^{n} M(\alpha, i, h_1^{\delta})$ is a totally ordered subalgebra of A_c and it is isomorphic to $M(\alpha, i, h_1^{\delta})$ for every $i \in I_n$.*

Proof. By subdirect product properties and by Lemma 17 it follows that, for every $i \in I_n$, $p_i : f \in A_c \cap \prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \to f_i$ is a bijective map from $A_c \cap \prod_{i=1}^{n} M(\alpha, i, h_1^{\delta})$ on $M(\alpha, i, h_1^{\delta})$. Indeed p_i is the claimed isomorphism. □

Remark 19. As a consequence of Lemma 17 and Corollary 18 $\delta_{A_c} < v = \min\{n_i + 1 \mid i \in I_n\}$. Indeed set $I_v = \{i \in I_n \mid n_i + 1 = v\}$. If $\delta_{A_c} = v$, then, for every $x \in A_c \backslash (\prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \backslash \{1_i\})$, $x_i = 1_i$, for each $i \in I_v$. Whence the function p_v mapping any element f to its restriction to $I_n \backslash I_v$, $f_{I_n \backslash I_v}$ is an isomorphism between A_c and $(A_c)_{I_n \backslash I_v}$. Whereas, under our assumptions, any chain $C_i, i \in I_n$, has to be essential in the representation of A_c.

Proposition 20. *Let $A_c \in \mathbf{A_c}$. Set*

1. $B = A_c \backslash (\prod_{i=1}^{n} M(\alpha, i, h_1^{\delta}) \backslash \{1_i\})$,
2. $0_B = \alpha_\delta$,
3. $1_B = \mathbf{1}$,
4. $\odot_B$ *be the restriction of the product of A_c to B,*
5. $\to_B$ *be the restriction of the operation $\to$ of A_c to B.*

Then $B = (B, \wedge, \vee, \odot_B, \to_B, 0_B, 1_B)$ is a BL-*algebra.*

Proof. By Lemma 17, $B = (B, \wedge, \vee, 0_B, 1_B)$ is a lattice with 0_B as least element and 1_B as greatest element. Moreover

if $x \geqslant \alpha_\delta$ and $y \geqslant \alpha_\delta$ then $x \odot y \geqslant \alpha_\delta$,
if $f \geqslant \alpha_\delta$ and $g \geqslant \alpha_\delta$ then $f \to g \geqslant g \geqslant \alpha_\delta$. □

Let A be a BL-algebra. By $\mathbb{I}(A)$ we denote the set of all idempotent elements of A. We remark that $\mathbb{I}(A_c) \neq \{\mathbf{0},\mathbf{1}\}$ for every finite BL-algebra A that is not a MV-chain. For otherwise A is locally finite, hence it is an MV-chain [5].

The above remark suggests the following considerations:

Let $A \in \mathbf{FBL}$, for $x \in \mathbb{I}(A)$, denote by $\mathbb{C}(x)$ the subset of $\mathbb{I}(A)$ whose elements are comparable with x. Define $K(A) \subseteq \mathbb{I}(A)$ as follows:

$x \in K(A)$ iff the following conditions are satisfied:

1. $\mathbb{C}(x) = \mathbb{I}(A)$;
2. $\{y \in \mathbb{I}(A) | y \leqslant x\}$ is a chain.

We stress that $K(A)$ is not empty: indeed $\mathbf{0} \in K(A)$.

The above notations and remarks allow us to introduce the main following definitions:

Definition 21. Let A be a nontrivial element of **FBL**. Then A is called BL-*comet* if $\max K(A) \neq \mathbf{0}$.

Definition 22. Let A be is a BL-comet, then $\max K(A)$ is called pivot of A and it will be denoted by pivot(A).

Set $\rho = \max\{n_i + 1, i \in I_n\}$. For every $h \leqslant \rho$ we will denote by $\alpha_{(h)}$ the n-tuple $(\alpha_1, \ldots, \alpha_n)$ where

$$\alpha_i = \begin{cases} 1_i & \text{if } h \geqslant n_i + 1, \\ \alpha_{i,h} & \text{if } h < n_i + 1. \end{cases} \tag{7}$$

With above notations we introduce the following:

Definition 23. Let $A \in \mathbf{FBL}$ and $\beta \in \mathbb{I}(A)$. β is called *pseudoconstant* on I_n if there is $h \leqslant \rho$ such that $\beta = \alpha_{(h)}$.

By (7) every idempotent $\alpha_h \in A$, constant on I_n, is pseudoconstant on I_n; moreover $\alpha_{(h)} = \mathbf{1}$ iff $h = \rho$.

Lemma 24. *Let $A \in \mathbf{FBL}$. Then, for every $h \leqslant \rho$, $\alpha_{(h)} \in A$.*

Proof. If $h \leqslant v$ (see Remark 19), the claim is already proved (see Corollary 5). Then we can safely assume $v < h < \rho$. Suppose $n = 2$ and $n_1 + 1 < h < n_2 + 1$. Let $x \in A$ such that $x_2 = \alpha_{2,h}$. Set $\alpha(x_1) = \alpha_{1,k}$, for some $k \leqslant n_1 + 1$. Then, by applying Corollary 5, $(\alpha(x) \to \alpha_k) \vee x = \alpha_{(h)} \in A$. Proceeding by induction, let the lemma be true for $n-1$. Analogously to Corollary 5, for every $i \in I_n$, we find an element $x^i \in A$ such that for $j \neq i$

$$(x^i)_j = \begin{cases} 1_j & \text{if } h \geqslant n_j + 1, \\ \alpha_{j,h} & \text{if } h < n_j + 1. \end{cases} \tag{8}$$

If two of the elements x^1, x^2 and x^3 are incomparable, say x^1 and x^2, then either $x^1 \vee x^2$ or $x^1 \wedge x^2$ satisfies the claim. For otherwise x^1, x^2 and x^3 are comparable. We safely can set $x^1 \leqslant x^2 \leqslant x^3$. Then by (8) we have

$$\text{if } h \geqslant n_2 + 1, \text{ then } 1_2 = (x^1)_2 \leqslant (x^2)_2, \text{ hence } (x^2)_2 = 1_2;$$
$$\text{if } h < n_2 + 1, \text{ then } (x^1)_2 = \alpha_{2,h} \leqslant x_2^2 \leqslant (x^3)_2 = \alpha_{2,h}, \text{ hence } x_2^2 = \alpha_{2,h}.$$

In both cases $x^2 = \alpha_{(h)} \in A$. □

Lemma 25. *Let $A \in$ **FBL** and $\mathbb{I}(A)$ be a chain. Then for every $x \in A$ there exists $h \leqslant \rho$ such that*:

$$x_i = \begin{cases} 1_i & \textit{if } h \geqslant n_i + 1, \\ \in M(\alpha, i, h+1) \backslash \{1_i\} & \textit{if } h < n_i + 1. \end{cases} \tag{9}$$

Consequently $\mathbb{I}(A)$ is the set of all the pseudoconstant elements of A.

Proof. Let $x \in A \backslash \{\mathbf{1}\}$ and

$$I_1 = \{i \in I_n \mid x_i = 1_i\},$$
$$I_2 = \{i \in I_n \mid x_i < 1_i\}.$$

If for some $(i,j) \in I_2^2$, $x_i \in M(\alpha, i, h+1) \backslash \{1_i\}$, $x_j \in M(\alpha, j, k+1) \backslash \{1_j\}$ and $h < k$, then, by applying Corollary 4 for $f = \alpha(x)$, we find $a, b \in A$ such that $(a_i, a_j) = (1_i, \alpha_{j,h})$ and $(b_i, b_j) = (\alpha_{i,h}, 1_j)$. So $\alpha(a)$ and $\alpha(b)$ have to be two incomparable elements of $\mathbb{I}(A)$, absurd. Consequently, there is an $h < \rho$ such that $x_i \in M(\alpha, i, h+1) \backslash \{1_i\}$, for every $i \in I_2$. Let now $I_1 \neq \emptyset$ and $h < n_i + 1$ for some $i \in I_1$. By Lemma 24, $\alpha(x) \to \alpha_{(h)} \in \mathbb{I}(A)$, but it is not comparable with $\alpha(x)$. This contradiction shows that x verifies (9). □

Proposition 26. *Let A be a nontrivial element of* **FBL**. *Then the following are equivalent*:

1. *A is a* BL-*chain*,
2. *A is a* BL-*comet and pivot*$(A) = \mathbf{1}$.

Proof. $1 \Rightarrow 2$ is trivial. In order to show $2 \Rightarrow 1$ set, for every $x \in A$, $I_x = \{i \in I_n \mid x_i = 1_i\}$.

Claim 1. *The family $(I_x)_{x \in A}$ is totally ordered by inclusion.*

Actually let $x, y \in A$, $x \neq y$, $i \in I_x \backslash I_y$ and $j \in I_y \backslash I_x$. Then $(\alpha(x_i), \alpha(x_j)) = (1_i, \alpha_{j,h}) < (1_i, 1_j)$ and $(\alpha(y_i), \alpha(y_j)) = (\alpha_{i,k}, 1_j) < (1_i, 1_j)$, for suitable h and k. Consequently $\alpha(x)$ and $\alpha(y)$ are two incomparable elements of $\mathbb{I}(A)$, which contradicts the hypothesis *pivot*$(A) = \mathbf{1}$.

Claim 2. $I_x \subsetneq I_y \Rightarrow x < y$.

We can safely assume $y < \mathbf{1}$. Then by Lemma 25 there are suitable $h, k < \rho$ such that:

for every $i \in I_n \backslash I_x$, $x_i \in M(\alpha, i, h+1) \backslash \{1_i\}$
and
for every $i \in I_n \backslash I_y$, $y_i \in M(\alpha, i, k+1) \backslash \{1_i\}$.

Let now $j \in I_y \backslash I_x$; by (9) $h < n_j + 1 \leqslant k$, whence $x < y$.

Claim 3. *$I_x = I_y \Rightarrow x$ and y comparable.*

If $x = \mathbf{1}$ or $y = \mathbf{1}$, the claim is trivial. Then assume $x < \mathbf{1}$ and $y < \mathbf{1}$. Let h and k be as in the previous claim.

If $h < k$, then $x < y$,
If $k < h$, then $y < x$.

Assume now $h = k$. Since $\mathbb{I}(A)$ is a chain, as a consequence of Lemma 8 the restrictions of x and y to $I_n \backslash I_y$ are constant, which implies x and y comparable.

The conclusion now follows from Claims 1–3. □

Theorem 27. *Let A be a nontrivial element of* **FBL**. *Then the following are equivalent*:

1. *A is a* BL-*comet*,
2. $A \in \mathbf{A_c}$.

Proof. $1 \Rightarrow 2$: If $\text{pivot}(A) = \mathbf{1}$, then the implication follows by Proposition 26. Then assume $\beta = \text{pivot}(A) < \mathbf{1}$. Rejecting the thesis, by Lemma 8 there is $f \in A$ such that $(f_i, f_j) = (0_i, \mathbf{1}_j)$, for some $(i,j) \in I_n^2$. We can safely assume $f \in \text{MV}(A)$. Since $\alpha(f)$ and $\alpha(f^*)$ are two incomparable elements of $\mathbb{I}(A)$, necessarily we get $\alpha(f), \alpha(f^*) \geqslant \beta$. From that $\mathbf{0} = \alpha(f) \wedge \alpha(f^*) \geqslant \beta$, a contradiction.

$2 \Rightarrow 1$: By Lemma 17 and Corollary 18 it follows $\alpha_\delta \in K(A)$, whence $\alpha_\delta \leqslant \max K(A) = \text{pivot}(A)$. By definition $\alpha_\delta > \mathbf{0}$, so $\max K(A) \neq \mathbf{0}$. □

Corollary 28. *Let A be a nontrivial element of* **FBL**. *Then A is isomorphic to a direct product of* BL-*comets.*

Proof. It follows by Theorems 13 and 27. □

Proposition 29. *Let $A_c \in \mathbf{A_c}$. Then pivot$(A_\mathbf{c}) = \alpha_\delta$.*

Proof. If $\text{pivot}(A_c) = \mathbf{1}$, it follows by Proposition 26. Assume $\text{pivot}(A_c) < \mathbf{1}$. In the proof of Theorem 27 ($2 \Rightarrow 1$) it is proved that $\alpha_\delta \leqslant \text{pivot}(A_c)$. On other hand by definition of δ and by Lemma 8 we can find $f \in A_c$ such that for some $(i,j) \in I_n^2$, $(f_i, f_j) = (\alpha_{i,\delta}, 1_j)$. Since $\alpha(f)$ and $\alpha(f) \to \alpha_\delta$ are two incomparable elements of $\mathbb{I}(A)$, it follows $\alpha(f), \alpha(f) \to \alpha_\delta \geqslant \text{pivot}(A_c)$. Hence $\text{pivot}(A_c) \leqslant \alpha(f) \wedge (\alpha(f) \to \alpha_\delta) = \alpha_\delta$. □

Corollary 30. *Let $A_c \in \mathbf{A_c}$ and* $\text{pivot}(A_c) < \mathbf{1}$. *Then A_c is the ordinal sum of a finite* BL-*chain and a finite* BL-*algebra that is not a* BL-*comet.*

Proof. It follows by Corollary 18, and Propositions 20 and 29. □

5. Labelled trees

Now we recall some definitions about partially ordered sets.

Definition 31. A partial ordered set $(T,\leqslant)$ is called *tree* if T has a minimum element T_0 and, for every $x\in T$, the set $T_x=\{y\in T:\ y\leqslant x\}$ is totally ordered. The elements of a tree are called *nodes*.

Definition 32. Let $(T,\leqslant)$ be a finite tree, $x\in T$ and $x\neq T_0$. The greatest element of $T_x\backslash\{x\}$ is called the *previous element of* x and it shall be denoted by $pr(x)$.

Definition 33. Let $(T,\leqslant)$ be a finite tree, the elements $x,y\in T$. We say that *y covers x* if $pr(y)=x$. In this case we write $x\prec y$.

Definition 34. Let $(T,\leqslant)$ be a finite tree and $x\in T$. We say that x is a *simple node* if there is exactly one element covering x. If x is not simple or if $x=T_0, x$ will be called a *multiple node*.

Definition 35. Let $(T,\leqslant)$ be a finite tree. We call *height* of an element $x\in T$, in symbols $l(x)$, the cardinal of the set of all multiple nodes of the chain $]T_0,x]$.

Definition 36. Let $(T,\leqslant)$ be a finite tree. We call *height* of T, in symbols $l(T)$, the non negative integer equal to $\max\{l(x):\ x\in T\}$.

Definition 37. Let $(T,\leqslant)$ be a finite tree, $x\in T$ and $x\neq T_0$. The greatest multiple node of $T_x\backslash\{x\}$ is called *multiple node previous of* x, and it shall be denoted by $prm(x)$.

Let N be the set of all the positive integers; then we set:

$$\mathbf{N}=\{0\}\cup\left(\bigcup_{r\in N}N^r\right)$$

and, for every integer positive number p, $\mathbf{N}_p=(\{0\}\cup(\bigcup_{r\in N}N^r))^p$

Definition 38. A *labelled tree* is a triple $(T,\leqslant,h)$, verifying the following:

$(T,\leqslant)$ is a finite tree,
h is a map from T to $\bigcup_{p\in N}\mathbf{N}_p$,
$h(x)=0$ iff $x=T_0$.

If $h(T)\subseteq\{0\}\cup N$, then $(T,\leqslant,h)$ is called a *simply labelled tree*.

By definition, a *simply labelled tree* is a tree, having every node marked by an integer number m. Such a number m represents the MV-chain with $m+1$ elements.

Our aim now is to map finite *simply labelled trees* on finite BL-algebras.

Let $(T,\leqslant,h)$ be a *simply labelled tree* and $(T_f,\leqslant)$ the subtree of $(T,\leqslant)$ of all *multiple nodes*. Define the map $h_f:T_f\to\mathbf{N}_1=\{0\}\cup(\bigcup_{r\in N}N^r)$ as follows:

$$h_f(x)=\begin{cases}(h(x_1),\dots,h(x_r),h(x)) & \text{if } x \text{ is a multiple node different from } T_0 \text{ and}\\ & (x_1,\dots,x_r,x)=]\mathrm{prm}(x),x],\subseteq T,\\ 0 & \text{if } x=T_0.\end{cases}$$

Then the triple $(T_f, \leqslant, h_f)$ is a labelled finite tree. Each (multiple) *node* is marked by h_f with a finite sequence of positive integers $n_1, \ldots, n_t$. Such a sequence represents the BL-chain which is a finite ordinal sum whose components are the finite MV-chains with $n_1 + 1, \ldots, n_t + 1$ elements, respectively: $h_f(x) = S_{n_1} \uplus \cdots \uplus S_{n_t}$.

Now denote by

$\mathbf{T_{s,1}}$ the set of all finite simply labelled trees,
and
$\mathbf{T_{m,1}}$ the set of all finite labelled trees $(T, \leqslant, h)$ such that:
every $x \in T$ is multiple,
$h(T) \subseteq \mathbf{N}_1$.

With the above notations and arguments we can claim the following theorem:

Theorem 39. *The map f, defined by* $f(T, \leqslant, h) = (T_f, \leqslant, h_f)$, *is a bijective map between* $\mathbf{T_{s,1}}$ *and* $\mathbf{T_{m,1}}$.

Proof. It is obvious. □

In the sequel, when there is no misunderstanding, we will denote $f(T, \leqslant, h)$ by $f(T)$ or T_f.

Next we will define a function σ, mapping every element of $\mathbf{T_{m,1}}$ on a finite BL-algebra.

First let $(T, \leqslant, h) \in \mathbf{T_{m,1}}, l(T) = 1$ and $T_1 = T \backslash \{T_0\}$. Then we define:

$$\sigma(T) = \begin{cases} h(T_1) & \text{if } |T_1| = 1, \\ \prod_{x \in T_1} h(x) & \text{if } |T_1| > 1. \end{cases} \tag{10}$$

Assume now $l(T) = n > 1$ and set:

$T_i = \{x \in T : l(x) = i\},\ i = 1, \ldots, n,$
$T^r = \bigcup_{i=0}^{n-r} T_i,\ r = 1, \ldots, n-1,$
and
M equal to the set of all maximal elements of T.

Define a mapping $h^1 : T^1 = \bigcup_{i=0}^{n-1} T_i \to \bigcup_{p \in N} \mathbf{N}_p$, by

$$h^1(x) = \begin{cases} (h(x), (h(y), x \prec y)) & \text{if } l(x) = n-1 \text{ and } x \notin \mathsf{M}, \\ h(x) & \text{otherwise.} \end{cases} \tag{11}$$

In the labelled tree $(T^1, \leqslant, h^1)$, every (multiple) *node*, such that $l(x) = n - 1$ and $x \notin \mathsf{M}$, is marked by h^1 with a pair: $(h(x), (h(y), x \prec y))$. $h(x)$ is a sequence of positive integers, representing the BL-algebra $h(x) = S_{n_1} \uplus \cdots \uplus S_{n_t}$. The second component is a finite family of sequence of positive integers $(h(y), x \prec y)$, representing the BL-algebra $(h(y), x \prec y) = \prod_{x \prec y} h(y)$. The pair $h^1(x)$ shall represent the finite BL-algebra which is an ordinal sum of BL-algebras: $h^1(x) = h(x) \uplus (h(y), x \prec y) = (S_{n_1} \uplus \cdots \uplus S_{n_t}) \uplus \prod_{x \prec y} h(y)$.

Define now an application $h^2 : T^2 = \bigcup_{i=0}^{n-2} T_i \to \bigcup_{p\in N} \mathbf{N}_p$, as follows:

$$h^2(x) = \begin{cases} (h^1(x), (h^1(y), x \prec y)) & \text{if } l(x) = n-2 \text{ and } x \notin \mathsf{M}, \\ h^1(x) & \text{otherwise.} \end{cases} \tag{12}$$

In the tree $(T^2, \leqslant, h^2)$, every (multiple) *node* x, such that $l(x) = n-2$ and $x \notin \mathsf{M}$, is marked by h^2 with a pair: $(h^1(x), (h^1(y), x \prec y))$. The pair $h^2(x)$ shall represent the finite BL-algebra that is an ordinal sum of BL-algebras: $h^2(x) = h^1(x) \uplus \prod_{x\prec y} h^1(y)$.

Proceeding as above, at step $(n-1)$th, we get a map $h^{n-1} : T^{n-1} = T_1 \cup \{T_0\} \to \bigcup_{p\in N} \mathbf{N}_p$, by

$$h^{n-1}(x) = \begin{cases} (h^{n-2}(x), (h^{n-2}(y), x \prec y)) & \text{if } x \notin \mathsf{M}, \\ h^{n-2}(x) & \text{otherwise.} \end{cases} \tag{13}$$

Finally we define

1. $\sigma(T) = h^{n-1}(T_1)$ if $|T_1| = 1$,
2. $\sigma(T) = \prod_{x\in T_1} h^{n-1}(x)$, otherwise.

Theorem 40. *There is a map Γ from $\mathbf{T_{s,1}}$ to* **FBL**.

Proof. It is sufficient to set $\Gamma = \sigma \circ f$. Then Γ furnishes the claimed map. □

6. Idempotent irreducible elements

Let $A \in \mathbf{FBL}$. In the lattice $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ an element x is called *irreducible* if $x = u \vee v$ implies $x = u$ or $x = v$. Denote by $\mathrm{Irr}(\mathbb{I}(A))$ the ordered set of all idempotent irreducible elements of A.

Proposition 41. *Let $A \in$* **FBL** *and $x \in \mathrm{Irr}(\mathbb{I}(A))$. Then the set $A_x = \{y \in A : y \leqslant x\}$ is a chain of irreducible elements.*

Proof. Let $x \in \mathrm{Irr}(\mathbb{I}(A))$ and $h, k \in A$ such that $h < x$ and $k < x$. Then we have $x = x \wedge \mathbf{1} = x \wedge ((h \to k) \vee (k \to h)) = (x \wedge (h \to k)) \vee (x \wedge (k \to h))$. By hypothesis we get either $x = (x \wedge (h \to k))$ or $x = (x \wedge (k \to h))$. Assume $x = (x \wedge (h \to k))$, then $x \leqslant h \to k$ and $h = h \odot x \leqslant h \odot (h \to k) \leqslant k$. So h and k are comparable. Analogously if $x = x \wedge (k \to h)$. □

From the above proposition we immediately obtain:

Corollary 42. *Let $A \in$* **FBL**. *The ordered set* $(\mathrm{Irr}(\mathbb{I}(A)), \leqslant)$ *is a finite tree, having* **0** *as least element.*

Proposition 43. *Let A be a* BL-*comet. Then* pivot(A) *is a multiple node of* $(\mathrm{Irr}(\mathbb{I}(A)), \leqslant)$.

Proof. By Theorem 27, Proposition 29 and Corollary 18 $\alpha_\delta = \text{pivot}(A) \in \text{Irr}(\mathbb{I}(A))$. To show that α_δ is a *multiple node*, we observe that, by definition of δ and by Lemma 8, we can find $f \in A$ such that for some $(i,j) \in I_n^2, (f_i, f_j) = (\alpha_{i,\delta}, 1_j)$. Then $\alpha(f) \to \alpha_\delta$ and $(\alpha(f) \to \alpha_\delta) \to \alpha_\delta$ are incomparable and both greater than α_δ. Moreover $(\alpha(f) \to \alpha_\delta) \wedge [(\alpha(f) \to \alpha_\delta) \to \alpha_\delta] = \alpha_\delta$. Whence α_δ is a multiple node. □

Proposition 44. *Let* $A \in \mathbf{FBL}, \alpha \in \text{Irr}(\mathbb{I}(A)) \backslash \{\mathbf{0}\}$. *Set*

1. $C = [\text{pr}(\alpha), \alpha]$,
2. $0_C = \text{pr}(\alpha)$,
3. $1_C = \alpha$,
4. $\odot_C$ *be the restriction to* C *of the product defined on* A,
5. $x^{*_C} = \alpha \odot (x \to \text{pr}(\alpha))$, *for every* $x \in C$.

Then $C = (C, \odot_C, *_C, 0_C, 1_C)$ *is an* MV-*chain.*

Proof. Indeed,

if $\text{pr}(\alpha) \leqslant x \leqslant \alpha$ and $\text{pr}(\alpha) \leqslant y \leqslant \alpha$, then $\text{pr}(\alpha) \leqslant x \odot y \leqslant \alpha$,
$0_C^{*_C} = \alpha \odot (\text{pr}(\alpha) \to \text{pr}(\alpha)) = \alpha = 1_C$,
and $1_C^{*_C} = \alpha \odot (\alpha \to \text{pr}(\alpha)) = \alpha \wedge \text{pr}(\alpha) = \text{pr}(\alpha) = 0_C$.

Since for every $i \in \{1, \ldots, n\}$ either $(\text{pr}(\alpha))_i = \text{pr}(\alpha_i)$ or $(\text{pr}(\alpha))_i = \alpha_i$, it follows that $\text{pr}(\alpha) \leqslant x \leqslant \alpha$ implies $\text{pr}(\alpha) \leqslant x^{*_C} \leqslant \alpha$ and $(x^{*_C})^{*_C} = x$. □

Remark 45. *By the above proposition, we get* $[\text{pr}(\alpha), \alpha] \cong S_m$, *for some* $m \in N$.

Let $i : \text{Irr}(\mathbb{I}(A)) \to N$ be the map defined by: $i(\mathbf{0}) = 0$ and $i(x) = m$, if $x \neq \mathbf{0}$ and $[\text{pr}(x), x] \cong S_m$. Then $(\text{Irr}(\mathbb{I}(A)), \leqslant, i)$ is a simply labelled tree.

With above notations we have:

Theorem 46. *There is a map* Λ *from* **FBL** *to* $\mathbf{T_{s,l}}$.

Proof. Let A be a finite BL-algebra, set $\Lambda(A) = (\text{Irr}(\mathbb{I}(A)), \leqslant, i)$. Then Λ maps every finite BL-algebra into a simply labelled tree. □

Proposition 47. *Let* $A_i \in \mathbf{FBL}, i = 1, \ldots, r$ *and* $x = (x_1, \ldots, x_r) \in \prod_{i=1}^r A_i$. *Then the following are equivalent*:

1. $x \in \text{Irr}(\mathbb{I}(\prod_{i=1}^r A_i))$,
2. *there is* $i \in \{1, \ldots, r\}$ *such that* $x_i \in \text{Irr}(\mathbb{I}(A_i))$ *and* $x_j = 0_j$ *for every* $j \neq i$.

Proof. $1 \Rightarrow 2$: Let $x = (x_1, \ldots, x_r) \in \text{Irr}(\mathbb{I}(\prod_{i=1}^r A_i))$. Assume $x_{i_1} \neq 0_{i_1}$ and $x_{i_2} \neq 0_{i_2}$ for $i_1 \neq i_2$. Then choose two elements:

$y = (y_1, \ldots, y_r)$, setting $y_{i_1} = 0_{i_1}$ and $y_i = x_i$, for $i \neq i_1$,

and $z = (z_1, \ldots, z_r)$, setting $z_{i_2} = 0_{i_2}$ and $z_i = x_i$, for $i \neq i_2$.

Then we get $x \neq y$, $x \neq z$ and $x = y \vee z$, absurd. If x_i is the only non-zero component of x, it is obvious that $x_i \in \mathrm{Irr}(\mathbb{I}(A_i))$.

$2 \Rightarrow 1$: Let $x = (x_1, \ldots, x_r) \in \prod_{i=1}^{r} A_i$. Assume there is $i \in \{i = 1, \ldots, r\}$ such that $x_i \in \mathrm{Irr}(\mathbb{I}(A_i))$ and $x_j = 0_j$ for every $j \neq i$. Thus from $x = y \vee z$ it follows $x_i = y_i \vee z_i$ and $x_j = y_j = z_j = 0_j$ for every $j \neq i$. By hypothesis $x_i = y_i$ or $x_i = z_i$, whence $x = y$ or $x = z$, that is $x \in \mathrm{Irr}(\mathbb{I}(\prod_{i=1}^{r} A_i))$. □

Theorem 48. *Let A be a non-trivial element of* **FBL**. *If* $\mathrm{Irr}(\mathbb{I}(A))$ *is a chain, then A is a* BL-*chain.*

Proof. First we prove that A is a BL-comet.

By Corollary 28 $A = A_1 \times \cdots \times A_r$ and, $i \in \{1, \ldots, r\}, A_i \neq \{0_i\}$ is a BL-comet. Let $r > 1$ and $a = (a_1, 0, \ldots, 0)$ and $b = (0, b_2, 0, \ldots, 0)$ be two elements of $\mathrm{Irr}(\mathbb{I}(A))$. Then either $a_1 = 0_1$ or $b_2 = 0_2$. That is either $\mathrm{Irr}(\mathbb{I}(A_1)) = \{0_1\}$ or $\mathrm{Irr}(\mathbb{I}(A_2)) = \{0_2\}$, absurd. Thus $r = 1$ and A is a BL-comet.

Assume $\mathrm{pivot}(A) < 1$.

Then, by Corollary 30, $A = C \uplus B$, with C a BL-chain and B a finite BL-algebra that is not a BL-comet; hence $\mathrm{Irr}(\mathbb{I}(B)) \subseteq \mathrm{Irr}(\mathbb{I}(A))$. Consequently $\mathrm{Irr}(\mathbb{I}(B))$ has to be a chain and, by previous claim, B is a BL-comet, absurd. From that $\mathrm{pivot}(A) = \mathbf{1}$, therefore, by Proposition 26, A is a BL-chain. □

7. Dualizing BL-algebras and labelled trees

Following [2], we recall that, if $(C, \leqslant)$ is a finite chain and $(T, \leqslant')$ is a finite tree, C and T disjoint sets, then the ordinal sum of C and T, in symbols $C \dot{+} T$, is the finite tree, $(T'', \leqslant'')$, where $T'' = C \cup T \setminus T_0$ and $\leqslant''$ is defined by $x \leqslant'' y$ for every $x \in C$ and for every $y \in T$, while the order of the elements in C and the order of the elements in T are unchanged.

Proposition 49. *Let A be a finite* BL-*chain and $B \in$* **FBL**. *Then* $(\mathrm{Irr}(\mathbb{I}(A \uplus B), \leqslant) \cong (\mathrm{Irr}(\mathbb{I}(A)), \leqslant) \dot{+} (\mathrm{Irr}(\mathbb{I}(B)), \leqslant)$.

Proof. It follows by definitions. □

Let $(S, \leqslant)$ and $(T, \leqslant')$ be two trees. The direct product of S and T [2], in symbols $C \otimes T$, need not be a tree. Then we introduce the following definition:

Definition 50. We call *0-product of the two trees* $(S, \leqslant)$ and $(T, \leqslant')$, in symbols $C \odot T$, the ordered subset of $C \otimes T$, whose elements are the pairs (x, y) such that $x = S_0$ or $y = T_0$.

The above definition can be extended to a finite number of trees as follows:

Definition 51. We call *0-product* of the trees $(S^i, \leqslant^i), i = 1, \ldots, r$, in symbols $\bigodot_{i=1}^{r} (S^i, \leqslant^i)$, the ordered subset of $\bigotimes_{i=1}^{r} S^i$, whose elements are the r-tuples $(x_1, \ldots, x_r)$ such that there is $i_0 \in \{1, \ldots, r\}$ and $x_i = S_0^i$ for every $i \neq i_0$.

Remark 52. *There are natural embeddings* $g_i : S^i \to S = \bigodot_{i=1}^r S^i$. *Identifying* S^i *and* $g_i(S^i)$, *we get* $S^i \cap S^j = \{S_0\}$, *for* $i \neq j$ *and* $S = \bigcup_{i=1}^r S^i$.

Proposition 53. *The* 0-product *of a finite number of trees is a tree.*

Proof. It is a trivial. □

Proposition 54. *Let* $A_i \in \mathbf{FBL}, i = 1, \ldots, r$. *Then* $(\mathrm{Irr}(\mathbb{I}(\prod_{i=1}^r A_i)), \leqslant) \cong \bigodot_{i=1}^r)\mathrm{Irr}(\mathbb{I}(A_i)), \leqslant)$.

Proof. Let $x = (x_1, \ldots, x_r) \in \mathrm{Irr}(\mathbb{I}(\prod_{i=1}^r A_i))$. By Proposition 47, there is $i \in \{1, \ldots, r\}$ such that $x_i \in \mathrm{Irr}(\mathbb{I}(A_i))$ and $x_j = 0_j$ for every $j \neq i$. Thus, the map $f : \mathrm{Irr}(\mathbb{I}(\prod_{i=1}^r A_i)) \to \bigodot_{i=1}^r \mathrm{Irr}(\mathbb{I}(A_i))$, defined by $f((0_1, \ldots, x_i, \ldots, 0_r)) = (\mathrm{Irr}(\mathbb{I}(A_1))_0, \ldots, x_i, \ldots, \mathrm{Irr}(\mathbb{I}(A_r))_0)$, is the claimed order isomorphism. □

Definition 55. Let $(S, \leqslant, h)$ be a labelled chain and $(T, \leqslant', k)$ be a labelled tree. The *ordinal sum* of $(S, \leqslant, h)$ and $(T, \leqslant', k)$ is the labelled tree $(R, \leqslant'', d)$ such that $(R, \leqslant'')$ is the ordinal sum of $(S, \leqslant)$ and $(T, \leqslant')$ and d is defined by

$$\mathrm{d}(x) = \begin{cases} h(x) & \text{if } x \in S, \\ k(x) & \text{if } x \in T. \end{cases} \tag{14}$$

Definition 56. Let $(S, \leqslant, h)$ and $(T, \leqslant', k)$ be two labelled trees. The *labelled 0-product* of $(S, \leqslant, h)$ and $(T, \leqslant', k)$ is the labelled tree $(R, \leqslant'', d)$ such that $(R, \leqslant'')$ is the 0-product of $(S, \leqslant)$ and $(T, \leqslant')$ and d is defined by

$$\mathrm{d}(x, y) = \begin{cases} h(x) & \text{if } (x, y) = (x, T_0), \\ k(y) & \text{if } (x, y) = (S_0, y). \end{cases} \tag{15}$$

The above definition of a *labelled 0-product* can be extended to a finite number of trees in the obvious way. In the sequel we shall denote the ordinal sum and the 0-product of two labelled trees, S and T, by $S \dot{+} T$ or $S \odot T$, respectively.

Let f be defined as in Theorem 39. Then we get:

Proposition 57. *Let* $(T, \leqslant, h)$ *be a simply labelled chain and, for* $i \in \{1, \ldots, r\}$, *let* $(T^i, \leqslant^i, h^i) \in \mathbf{T_{s,l}}$. *Then*

1. $f(T \dot{+} T^i) = f(T) \dot{+} f(T^i)$, $i \in \{1, \ldots, r\}$,
2. $f(\bigodot_{i=1}^r T^i) = \bigodot_{i=1}^r f(T^i)$.

Proof. It follows by the definitions. □

With arguments and notations of Section 5, we get:

Proposition 58. *For* $i \in \{1, \ldots, r\}$, *let* $(T^i, \leqslant^i, h^i) \in \mathbf{T_{m,l}}$. *Then* $\sigma(\bigodot_{i=1}^r T^i) = \prod_{i=1}^r \sigma(T^i)$.

Proof. Set $\bigodot_{i=1}^{r} T^i = (S, \leqslant, d), l(S) = n$ and $l(T^i) = n_i$, for $i = 1, \dots, r$. If $x \in T^i$ and $x \neq S_0$, then $\{y \in S \mid x \prec y\} \subseteq T^i$. Hence $(h^i)^{n_i - 1}(x) = d^{n-1}(x)$, for $i \in \{1, \dots, r\}$ and $x \in T^i$. Therefore, $\sigma(S) = \prod_{x \in S_1} d^{n-1}(x) = \prod_{x \in (T^1)_1} (h^1)^{n-1}(x) \times \prod_{x \in (T^2)_1} (h^2)^{n-1}(x) \times \cdots \times \prod_{x \in (T^r)_1} (h^r)^{n-1}(x) = \sigma(T^1) \times \sigma(T^2) \times \cdots \times \sigma(T^r) = \prod_{i=1}^{r} \sigma(T^i)$. □

In the next theorems Γ and Λ are defined as in Theorems 40 and 46, respectively.

Theorem 59. *Let $(T, \leqslant, h) \in \mathbf{T_{s,1}}$. Then $\Lambda(\Gamma(T))$ is isomorphic to $(T, \leqslant, h)$.*

Proof. We will prove the theorem for induction on $l(T)$. Suppose $l(T) = 1$ and consider the set $T_1 = \{x \in T : l(x) = 1\}$.

Case 1: $|T_1| = 1$. Let $]T_0, T_1] = \{x_1, \dots, x_r\}, x_1 < x_2 \cdots < x_r$. By definition of $\sigma, \Gamma(T) = S_{n_1} \uplus \cdots \uplus S_{n_r}$, where $n_i = h(x_i)$, $i = 1, \dots, r$. Then $\Lambda(\Gamma(T)) = \mathrm{Irr}(\mathbb{I}(S_{n_1} \uplus \cdots \uplus S_{n_r})) \cong T$.

Case 2: $|T_1| = p > 1$. Set $T_1 = \{t_1, \dots, t_p\}$ and $]T_0, t_i] = \{x_{i,1}, \dots, x_{i,q_i}\}, x_{i,1} < x_{i,2} \dots < x_{i,q_i}$, for every $i = 1, \dots, p$. Then $T \cong \bigodot_{i=1}^{p} ([T_0, t_i] \leqslant^i, h^i)$, with $t_i \in T_1$ and $\leqslant^i, h^i$ restrictions of $\leqslant$ and h to $[T_0, t_i]$ respectively. By Propositions 57 and 58, $\Gamma(T) = \sigma(\bigodot_{i=1}^{p} f([T_0, t_i])) = \prod_{i=1}^{p} h_f(t_i) = \prod_{i=1}^{p} (S_{n_{i,1}} \uplus \cdots \uplus S_{n_{i,q_i}})$, where $n_{i,s} = h(x_{i,s}), x_{i,s} \in [T_0, t_i]$, for $i = 1, \dots, p$ and $s = 1, \dots, q_i$. From that and Proposition 54, $\Lambda(\Gamma(T)) = \mathrm{Irr}(\mathbb{I}(\prod_{i=1}^{p} (S_{n_{i,1}} \uplus \cdots \uplus S_{n_{i,q_i}}))) = \bigodot_{i=1}^{p} \mathrm{Irr}(\mathbb{I}(S_{n_{i,1}} \uplus \cdots \uplus S_{n_{i,q_i}}))$.

Using the arguments and the conclusion of the previous case, we get:

$\mathrm{Irr}(\mathbb{I}(S_{n_{i,1}})) \uplus \cdots \uplus S_{n_{i,q_i}}) = \Lambda(\Gamma([T_0, t_i]) \cong [T_0, t_i]$, hence $\Lambda(\Gamma(T)) \cong \bigodot_{i=1}^{p} [T_0, t_i] \cong T$.

Suppose now $l(T) = n > 1$.

If $|T_1| = 1$, then $T \backslash [T_0, T_1[$ is a *tree* and $T = [T_0, T_1] \dot{+} T \backslash [T_0, T_1[$. Since $l(T \backslash [T_0, T_1[) = n - 1$, by induction hypothesis and the above results we get: $\Lambda(\Gamma(T \backslash [T_0, T_1[)) \cong T \backslash [T_0, T_1[$ and $\Lambda(\Gamma([T_0, T_1])) \cong [T_0, T_1]$. By Proposition 57, $\Gamma(T) = \Gamma([T_0, T_1]) \uplus \Gamma(T \backslash [T_0, T_1[)$. Therefore, by Proposition 49, $\Lambda(\Gamma(T)) = \Lambda(\Gamma([T_0, T_1])) \dot{+} \Lambda(\Gamma(T \backslash [T_0, T_1[)) \cong [T_0, T_1] \dot{+} T \backslash [T_0, T_1[\cong T]$.

Let now $|T_1| = p > 1$ and $T_1 = \{t_1, \dots, t_p\}$. Set $R^i = \{x \in T : x \geqslant t_i\}$ and $S^i = [T_0, t_i] \dot{+} R^i$. Then we get $T \cong \bigodot_{i=1}^{p} S^i$. Thus, by Propositions 57, 58 and 47, it follows $\Lambda(\Gamma(T)) \cong \bigodot_{i=1}^{p} \Lambda(\Gamma(S^i))$. Since $|\{x \in S^i : l(x) = 1\}| = 1$, $\Lambda(\Gamma(T)) \cong \bigodot_{i=1}^{p} S^i \cong T$. □

Theorem 60. *Let $A \in \mathbf{FBL}$. Then $\Gamma(\Lambda(A)) \cong A$.*

Proof. We will prove the theorem by induction on $n = l(\mathrm{Irr}(\mathbb{I}(A)))$. Assume $l(\mathrm{Irr}(\mathbb{I}(A))) = 1$ and set $I_1 = \{\alpha \in \mathrm{Irr}(\mathbb{I}(A)) : l(\alpha) = 1\}$.

Let us consider two cases:

Case 1: Let $|I_1| = 1$. Then $\mathrm{Irr}(\mathbb{I}(A))$ is a chain, and, by Theorem 48, A is a BL-chain. Hence A is an ordinal sum of MV-chains. From that $\Gamma(\mathrm{Irr}(\mathbb{I}(A))) = \Gamma(\Lambda(A)) \cong A$.

Case 2: Let $|I_1| = p > 1$. Applying Corollary 28 and Propositions 54, $\mathrm{Irr}(\mathbb{I}(A) \cong \mathrm{Irr}(\mathbb{I}(A_1) \odot \cdots \odot \mathrm{Irr}(\mathbb{I}(A_r)$, where for each $i = 1, \dots, r, A_i$ is a BL-comet. By assumption $l(\mathrm{Irr}(\mathbb{I}(A))) = 1$ and by Proposition 43, it follows $l(\mathrm{Irr}(\mathbb{I}(A_i))) = 1$. Thus A_i is a BL-chain, for $i = 1, \dots, r$. Then (see Case 1) $\Gamma(\mathrm{Irr}(\mathbb{I}(A_i))) = \Gamma(\Lambda(A_i)) \cong A_i$. Using Proposition 49, 57 and 58, we have: $\Gamma(\Lambda(A)) = \Gamma(\Lambda(A_1 \times \cdots \times A_r)) = \Gamma(\mathrm{Irr}(\mathbb{I}(A_1)) \odot \cdots \odot \mathrm{Irr}(\mathbb{I}(A_r)) = \Gamma(\Lambda(A_1)) \times \cdots \times \Gamma(\Lambda(A_r)) \cong A_1 \times \cdots \times A_r = A$.

Assume now $l(\mathrm{Irr}(\mathbb{I}(A)))=n>1$.

Suppose first that A is a BL-comet. Let $\mathrm{pivot}(A)<1$. By Corollary 30 and Proposition 49, $\mathrm{Irr}(\mathbb{I}(A))\cong\mathrm{Irr}(\mathbb{I}(A_1))\dot{+}\mathrm{Irr}(\mathbb{I}(A_2))$. Thus by Proposition 57, $\Gamma(\Lambda(A))\cong\Gamma(\Lambda(A_1))\uplus\Gamma(\Lambda(A_2))$. We recall that A_1 is a BL-chain and that, by Proposition 43, $l(\mathrm{Irr}(\mathbb{I}(A_2)))=n-1$. Thus by induction hypothesis $\Gamma(\Lambda(A))\cong A_1\uplus A_2\cong A$.

Finally let $A\in\mathbf{FBL}$. By Corollary 28, $A=A_1\times\cdots\times A_r$, with $A_1,\ldots,A_r$ BL-comets. By Proposition 54, $\mathrm{Irr}(\mathbb{I}(A))\cong\mathrm{Irr}(\mathbb{I}(A_1))\odot\cdots\odot\mathrm{Irr}(\mathbb{I}(A_r))$. Therefore, by Propositions 57 and 58, $\Gamma(\mathrm{Irr}(\mathbb{I}(A)))\cong\Gamma(\mathrm{Irr}(\mathbb{I}(A_1))\odot\cdots\odot\mathrm{Irr}(\mathbb{I}(A_r)))$, that is $\Gamma(\Lambda(A))\cong\Gamma(\Lambda(A_1))\times\cdots\times\Gamma(\Lambda(A_r))\cong A_1\times\cdots\times A_r=A$. □

Acknowledgements

The authors wish to thank the reviewer for his helpful suggestions.

References

[1] P. Agliano, F. Montagna, Varieties of BL algebras I: general properties, Technical Report, University of Siena.
[2] G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, RI, 1984.
[3] R.L.O. Cignoli, I.M.L. D'ottaviano, D. Mundici, Algebraic Foundations of Many-valued Reasoning (Trends in Logic, Studia Logica Library), Kluwer, Dordrecht, 2000.
[4] P. Hájek, Metamathematics of Fuzzy Logic (Trends in Logic, Studia Logica Library), Kluwer, Dordrecht, 1998.
[5] E. Turunen, BL-algebras of Basic Fuzzy logic, Mathware Soft Comput. 6 (1999) 49–61.