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A Legendre spectral element method for eigenvalues in hydrodynamic stability

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Abstract

A Legendre polynomial-based spectral technique is developed to be applicable to solving eigenvalue problems which arise in linear and nonlinear stability questions in porous media, and other areas of Continuum Mechanics. The matrices produced in the corresponding generalised eigenvalue problem are sparse, reducing the computational and storage costs, where the superimposition of boundary conditions is not needed due to the structure of the method. Several eigenvalue problems are solved using both the Legendre polynomial-based and Chebyshev tau techniques. In each example, the Legendre polynomial-based spectral technique converges to the required accuracy utilising less polynomials than the Chebyshev tau method, and with much greater computational efficiency.

Keywords: Spectral methods; Porous media; Sparse matrices; Hydrodynamic stability

1. Introduction

Convection in porous media is a widely explored subject due to the immense variety of applications such as bio-remediation, geothermal reservoir systems, contaminant movement in soil, solid matrix heat exchangers, solar power converters and oil extraction. These and many other examples are described in [16], and specific references may be found in [19, pp. 238, 239]. Another recent use of porous media is in heat transfer mechanisms through the use of porous foams and heat pipes, see, e.g. Amili and Yortsos [1].

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The study of the stability of such systems is a key aspect in their physical interpretation (see e.g. [16,19]), and continues to be one of the most pursued topics in fluid mechanics. Analysis of these stability problems usually requires the derivation of eigenvalues and eigenfunctions, where very few of these systems can be solved analytically. There is, therefore, substantial motivation behind the development of accurate and efficient techniques to quantify the stability of fluid motion in porous media. To illustrate the nature of the porous convection problems which may be solved we refer to Straughan [19, Chapter 19] where many such examples are given, and the compound matrix and Chebyshev tau methods are described. Such second order systems of equations with complex coefficients occur naturally in many porous convection studies and we describe a method which permits one to solve very efficiently with high accuracy for eigenvalues and eigenfunctions in such a class of problem.

Two powerful existing techniques for finding eigenvalues and eigenfunctions are the compound matrix (see e.g. [2,3,5,6,8–10,12,20]) and the Chebyshev tau method (see, e.g. [4]). The compound matrix method, which belongs to the family of shooting techniques, performs competently for stiff differential equations, with the specific purpose of reducing rounding error, as explored in [10,20], see also the references therein. The Chebyshev tau technique is a spectral method. This method calculates as many eigenvalues as required as opposed to just one at a time as is done in the compound matrix method.

In this paper a Legendre polynomial-based spectral method is analysed. This generates sparse matrices, where the standard boundary conditions for porous media problems are contained within the method, negating the need for their superimposition onto the matrices as is necessary with the Chebyshev tau method, see Haidvogel and Zang [11], for example. Several different examples of its application to porous media are presented to demonstrate its adaptability, accuracy and relevant ease in implementation. In each example this method is compared to the Chebyshev tau technique to assess accuracy and speed of convergence. The results clearly demonstrate that the Legendre method coupled with the Arnoldi technique of finding matrix eigenvalues leads to substantial computational advantages. This lends the technique to a considerable number of extremely useful applications. In analysing a hydrodynamic stability problem one usually has to determine a neutral curve, and this often involves hundreds of eigenvalue calculations to accommodate different parameter values within the model, making the sparsity of the Legendre polynomial-based spectral method a crucial advantage. While the exponentially fast convergence of a spectral method usually means that traditional techniques which yield full matrices do not present an issue because very few polynomials are required, some practical eigenvalue problems do need many polynomials. Such cases are parallel flow situations, see, e.g. Dongarra et al. [4], where one may need upward of 200 polynomials and high-precision arithmetic. Another motivation for the spectral technique employed here is its ability to extend in a natural way to two- and three-dimensional stability problems for which the matrices are large and then their sparse structure is a major advantage. If one tries to use a technique such as the Chebyshev tau method in higher dimensions it is not so clear how one incorporates the boundary conditions, in addition to the matrices being full. The spectral technique advocated here extends naturally to two and three space dimensions by using tensor products of the basis elements in x, y and z.

The Legendre idea employed here was introduced for the solution of differential equations by Shen [17]. No eigenvalue calculations were considered. Kirchner [13] developed a method for solving eigenvalues for the Orr–Sommerfeld equation which essentially uses the technique of Shen [17]. The Orr–Sommerfeld equation is fourth order and not characteristic of the equations governing flow in porous media. The main contribution here is to show how to adapt the Shen method naturally to eigenvalue problems for porous convection. Since the Orr–Sommerfeld equation is fourth order the version of the Shen method adopted by

Kirchner [13] is different from that given here where we concentrate on writing the equations as a system of coupled second order equations. For completeness, we show how other problems in fluid mechanics which involve coupled second and fourth order equations may be solved by a combination of the ideas described in [13] and those given here.

The layout of the paper now follows. In Section 2 we illustrate the technique by application to a simple problem, the simple harmonic motion equation. Then in Sections 3 and 4 we illustrate the method by application to two different convection problems in porous media. Section 3 treats the problem of Hadley flow where convection is driven by vertical and horizontal temperature gradients. Section 4 deals with multi-component convection in a porous medium where oscillatory instabilities may arise due to competition between a temperature field and two different salt fields. From the mathematical viewpoint Section 3 effectively treats a fourth order equation with complex coefficients, written as two second order equations, whereas Section 4 analyses an eighth order system, expressed as four interconnected second order equations. To illustrate the versatility of the Legendre polynomial–Galerkin method we show how Bénard convection in a fluid may be treated in Section 5. This is different from convection in a porous medium because it involves a fourth order equation coupled with a second order one. In this way one sees how the Legendre polynomial method may be applied to a variety of problems in hydrodynamic stability. The paper is completed in Section 6 by analysing the benefits of the technique described here as compared with competing methods.

2. Structure of the technique

Consider the domain $\Omega = (-1, 1)$, with the Hilbert space

$$H_0^1(\Omega) = \{v : v, v' \in L_2(\Omega), v(-1) = v(1) = 0\},$$

where

$$L_2(\Omega) = \{v : v(\Omega) \to \mathbb{C}, \int_\Omega |v|^2 \, dx < \infty\}.$$

Let $(\cdot, \cdot)$ be the inner product on $L^2(\Omega)$, e.g. $(f, g) = \int_\Omega f \bar{g} \, dx$, $\bar{g}$ being complex conjugate and $\| \cdot \|$ the associated norm. If the setting is real, the space L^2 employed will involve real, functions rather than complex ones. To motivate the Legendre polynomial-based spectral technique we begin with the equation

$$u'' + \lambda u = 0, \tag{1}$$

where $u \in H_0^1(\Omega)$, and $u = 0$ at $z = \pm 1$.

Eq. (1) can be solve numerically by replacing the infinite dimensional space $H_0^1(\Omega)$ by a finite dimensional space $S_N \subset H_0^1(\Omega)$ of dimension $N \in \mathbb{N}$. Assuming that a basis $\phi_1, \dots, \phi_N$ of S_N can be constructed, the solution u to (1) may be approximated by $u = \sum_{k=1}^N u_k \phi_k$ and then (1) replaced by

$$\sum_{k=1}^N u_k \phi_k'' + \lambda \sum_{k=1}^N u_k \phi_k = 0, \tag{2}$$

where the u_k are the Fourier coefficients.

Let $L_i,\ i \in \mathbb{N}$, be the ith Legendre polynomial on $(-1, 1)$ with $S_N = \mathscr{P}^{N+1}(\Omega) \cap H_0^1(\Omega)$, where $\mathscr{P}^p(\Omega)$ denotes the polynomials of degree p on Ω. Using the identity

$$(2i+1)L_i(z) = L'_{i+1}(z) - L'_{i-1}(z), \tag{3}$$

cf. Sneddon [18, p. 69], for $p \geqslant 2$, we define the basis function

$$\phi_i(z) = \int_{-1}^{z} L_i(s)\,\mathrm{d}s = \frac{L_{i+1} - L_{i-1}}{2i+1}, \quad i = 1, \ldots, p-1, \tag{4}$$

cf. Shen [17, p. 1492]. By the definition of Legendre polynomials the basis functions ϕ_i are linearly independent, such that $S_N = \mathrm{span}\{\phi_i\}$ $i = 1, \ldots, N$ with $N = \dim(S_N)$. A crucial aspect of these basis functions is their inclusion in the space $H_0^1(\Omega)$ or, more specifically in this context, that $\phi_i(-1) = \phi_i(1) = 0$. This follows when utilising the relation $L_i(\pm 1) = (\pm 1)^i$. This inherent structure clearly avoids the need for the superimposition of the boundary conditions in the resulting matrix—a fact frequently needed with, e.g. the Chebyshev tau analysis. This inherent structure is highly significant in the method's applicability to two- and three-dimensional porous problems as discussed in Section 1.

To solve (2) we multiply by ϕ_i and integrate over Ω to find

$$\left(\sum_{k=1}^{N} u_k \phi''_k, \phi_i\right) + \lambda \left(\sum_{k=1}^{N} u_k \phi_k, \phi_i\right) = 0, \quad i = 1, \ldots, N. \tag{5}$$

By making use of the divergence theorem and utilising (4) we can observe that

$$\left(\sum_{k=1}^{N} u_k \phi''_k, \phi_i\right) = -\sum_{k=1}^{N} u_k (\phi'_k, \phi'_i) = -\left(\sum_{k=1}^{N} u_k L_k, L_i\right). \tag{6}$$

System (5), with the rearrangement of (6), may be solved by utilising the inherent orthogonality of Legendre polynomials within the specified inner product, where

$$(L_i, L_j) = \int_{\Omega} L_i(z) L_j(z)\,\mathrm{d}z = \begin{cases} \dfrac{2}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7}$$

This procedure leads to a generalised eigenvalue problem of the form

$$A\mathbf{u} = \lambda B \mathbf{u}, \tag{8}$$

where $\mathbf{u} = (u_1, \ldots, u_N)^{\mathrm{T}}$. By using the orthogonal behaviour shown in (7) the matrix A can be derived from (6) yielding diagonal elements $A_{i,i} = 2/(2i+1)$, where $i = 1, \ldots, N$. This desirable feature that the matrix A is diagonal is due to the fact that ϕ_i is selected so that $\phi'_i = L_i$.

Similarly,

$$\left(\sum_{k=1}^{N} u_k \phi_k, \phi_i\right) = \left(\sum_{k=1}^{N} u_k \left(\frac{L_{k+1} - L_{k-1}}{2k+1}\right), \frac{L_{i+1} - L_{i-1}}{2i+1}\right),$$

which, utilising (7), yields the symmetric banded matrix B with elements

$$B_{i,j} = \begin{cases} \dfrac{4}{(2j-1)(2j+1)(2j+3)}, & j=i, \qquad i=1,\dots,N, \\ \dfrac{-2}{(2j-3)(2j-1)(2j+1)}, & j=i+2, \quad i=1,\dots,N-2, \end{cases}$$

which is of bandwidth 4. The equivalent procedure with the Chebyshev tau approach yields (full) matrices A and B which are not of banded structure as they are here.

System (8) is a sparse eigenvalue problem making it ideal for specific sparse iterative solvers such as the implicitly restarted Arnoldi method (IRAM) as presented in the ARPACK package (see [14]). This reduces computational and storage requirements needed by the QZ algorithm (see e.g. [7]), which is necessary for a technique like the Chebyshev tau method, since A and B are full with B frequently singular. The speed up achieved with the Arnoldi technique is a notable feature presented here.

We now begin with application to porous convection and convection in a fluid. While we study three distinct but representative problems we stress that the techniques are easily adaptable to many other hydrodynamic stability problems, and even stability problems in other areas of Continuum Mechanics. For example, we have investigated stability problems in some viscoelastic flows and also a stability problem for a thermoelastic plate.

3. Hadley flow

Hadley flow refers to convection in a layer of porous medium where the basic temperature field varies in the vertical (i.e. z direction) as well as along one of the horizontal directions, which we will define as the x direction. This system is presented in more detail by Nield [15] and is also used as a test case for the Chebyshev tau technique in [20]. It is a useful example as the equations have complex coefficients dependent on the z variable, and can be very sensitive to small variations in the parameters, making it beneficial as a test of the method's accuracy.

Defining the porous medium to be contained in the layer $z \in (-H/2, H/2)$ we adopt the temperature field boundary conditions

$$T = T_0 \mp \tfrac{1}{2}\triangle T - \beta_T x, \quad z = \pm\tfrac{1}{2}H,$$

where $\triangle T$ is the temperature differential in the z direction and β_T is some constant of proportionality. Employing a non-dimensionalised form of the temperature field boundary conditions, the steady-state solution has the form

$$\begin{aligned} \overline{U} &= R_H z, \\ \overline{T} &= -R_V z + \tfrac{1}{24}R_H^2(z-4z^3) - R_H x, \end{aligned} \tag{9}$$

where $z \in (-\frac{1}{2}, \frac{1}{2})$, R_H and R_V are the vertical and horizontal Rayleigh numbers, respectively, and $\bar{U}(z)$ is the x-component of velocity. Defining $a^2 = k^2 + m^2$ with k and m being the x and y wavenumber, the non-dimensionalised perturbation equations from (9) are

$$\begin{aligned} &(D^2 - a^2)W + a^2 S = 0, \\ &(D^2 - a^2 - \mathrm{i}\sigma - \mathrm{i}k\overline{U}(z))S + \mathrm{i}ka^{-2}R_H DW - (D\overline{T})W = 0, \end{aligned} \tag{10}$$

where $D = \mathrm{d}/\mathrm{d}z$. System (10) is subject to the boundary conditions

$$W = S = 0, \quad z = \pm\tfrac{1}{2}.$$

In (10), $W(z)$ and $S(z)$ are the third component of velocity and temperature field perturbation, respectively. Adopting the Legendre-based spectral technique, (10) reduces to the generalised matrix eigenvalue problem,

$$A\mathbf{x} = \sigma B\mathbf{x},$$

where here $\mathbf{x} = (w_1, \ldots, w_N, s_1, \ldots, s_N)$, and the matrices are given by

$$A = \begin{pmatrix} D_2 - a^2 P & a^2 P \\ \dfrac{\mathrm{i}k}{a^2} R_H D + \left(R_v - \dfrac{R_H^2}{24} \right) P + \dfrac{R_H^2}{8} Q^2 & D_2 - a^2 P - \dfrac{\mathrm{i}k}{2} R_H Q^1 \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 \\ 0 & iP \end{pmatrix},$$

recalling that in deriving A and B we switch $(-\frac{1}{2}, \frac{1}{2})$ to the domain $(-1, 1)$ of the Legendre polynomials. To place the matrices A and B in context, if we define a generic function $g_f = (\sum_{k=1}^{N} g_k \phi_k,\ \phi_i)$, with $\mathbf{g} = (g_1, \ldots, g_N)^{\mathrm{T}}$, then the ith rows of the $N \times N$ matrices $D_2,\ D,\ P$ and Q^i multiplied by $\mathbf{g}$ correspond to $(g_f'', \phi_i),\ (g_f', \phi_i),\ (g_f, \phi_i)$ and $(z^i g_f, \phi_i)$, respectively. The matrix representations D_2 and P were derived in Section 2 when demonstrating the simplest case, whereas the remaining matrices are given in Appendices A.1– A.3. All the matrix representations are banded in structure.

An important aspect of this method is its behaviour when the coefficients of the porous equations are functions of z. Using the recurrence formula (see e.g. [18, p. 68]),

$$zL_n = \frac{n+1}{2n+1} L_{n+1} + \frac{n}{2n+1} L_{n-1} \tag{11}$$

each $z^n \phi_k,\ n \in \mathbb{N}$ term can be expressed as a combination of Legendre polynomials. This in turn allows the relevant inner product to be evaluated using the orthogonality conditions (7), as shown in A.2 and A.3 for z and z^2. Due to the inherent nature of the recurrence formula, as the powers of z become larger the bandwidth of the corresponding matrix also grows.

Proposition 1. *If $U(z) \in \mathscr{P}^k(\Omega)$ for some $k \in \mathbb{N}$ then $(Ug_f, \phi_i),\ i = 1, \ldots, N$ has bandwidth $2k+2$.*

Proof. Assuming that $m \geqslant k+1$, by repeated application of recurrence relation (11) it clearly follows that

$$U(z)L_{m+1} = a_1 L_{m+k+1} + a_2 L_{m+k} + \cdots + a_{2k} L_{m-k+1},$$
$$U(z)L_{m-1} = b_1 L_{m+k-1} + b_2 L_{m+k-2} + \cdots + b_{2k} L_{m-k-1},$$

for some constants $a_i,\ b_i,\ i = 1, \ldots, 2k$. The function $U(z)\phi_m$ can now be represented as

$$\frac{U(z)L_{m+1} - U(z)L_{m-1}}{2m+1} = \sum_{s=0}^{2k+2} c_i L_{m+k+1-s} \tag{12}$$

Table 1
Comparison of the Legendre and Chebyshev tau techniques with the results denoted by σ_L and σ_C, respectively, with N being the number of polynomials

N	σ_L	σ_C
14	−0.2934315592	−0.2912641416
16	−0.2934328110	−0.2934658698
18	−0.2934327663	−0.2934479056
20	−0.2934327661	−0.2934319875
22	−0.2934327661	−0.2934327166
24	−0.2934327661	−0.2934327711
26	−0.2934327661	−0.2934327661

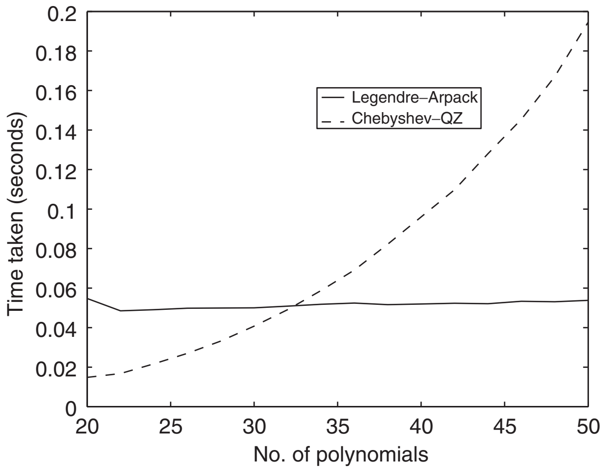

Fig. 1. Number of polynomials used against computational time.

for some constants c_i. Consider row i in the matrix representation of the inner product $(Ug_f,\ \phi_i)$. The only terms that are non-zero in the inner product of $U(z)g_f$ and ϕ_i are the $(i+1)$th or $(i-1)$th Legendre polynomials. If we consider this using (12) we have indices $i+1$ and $i-1$ when $m=i-k+s$ and $m=i-k-2+s$, respectively, for $s=0,\dots,2k+2$, for those m which are greater than or equal to 1. At its maximum this yields $2k+3$ distinct values of m, each of which represents an entry into the ith row of the matrix. Hence, as the diagonal term is included in every row, the matrix has bandwidth $2k+2$. $\square$

Table 1 presents the leading eigenvalue in the spectrum as obtained using both the Legendre-based spectral technique and the Chebyshev tau method with the x-wavenumber $k=0$, and the y-wavenumber $m=10$, $R_H=114.2$ and $R_V=100$ fixed such that the method determines the value of σ. While we only present one eigenvalue, similar behaviour is observed for other eigenvalues.

Convergence of both methods is evident from Table 1, where the Legendre method clearly requires fewer polynomials to converge to the required accuracy. In fact, the better convergence rate of the Legendre polynomial method is striking since the Chebyshev tau method requires approximately 30% more polynomials to achieve the same accuracy. The results are also in accordance with those published by Straughan and Walker [20]. Fig. 1 provides a visual representation of the computational time required to converge to the first eigenvalue of the spectrum as the number of polynomials is increased. The Legendre–Arpack line refers to the solutions obtained via the Legendre polynomial-based spectral method utilising the Arnoldi method obtained from the ARPACK system (see [14]), whilst the Chebyshev-QZ line refers to the Chebyshev tau method coupled with the QZ algorithm (see e.g. [7]).

It is clear from Fig. 1 that as the matrices associated with the generalised eigenvalue problem grow in size the Legendre–Arnoldi method is substantially more computationally efficient than the Chebyshev tau-QZ technique.

4. Multi-component convection–diffusion

Here we study another representative porous convection eigenvalue problem. In this case we solve an eighth order system which models multi-component diffusion in a porous medium as presented in [21].

Consider a porous medium contained in the layer $z \in (0, d)$ with constant boundary temperatures $T = 0\,°\mathrm{C}\ (z = 0)$ and $T = T_U \geqslant 4\,°\mathrm{C}\ (z = d)$, respectively. The fluid saturating the porous medium is water and so if $T_U > 4\,°\mathrm{C}$ the physical picture models a layer of gravitationally unstable water lying beneath a layer which is gravitationally stable (since water has a density maximum at approximately 4 °C). This results in convection in the lower layer which may penetrate into the upper layer.

The fluid is assumed to have two different species dissolved in it, where we will denote $C^\beta,\ \beta = 1, 2$, to be the the concentration of component β. The density is assumed quadratic in the temperature field and linear with respect to these concentrations such that

$$\rho = \rho_0 \left(1 - \alpha(T-4)^2 + \sum_{\beta=1}^{2} \alpha_\beta (C^\beta - C_0^\beta) \right),$$

where ρ_0 and C_0^β are density and salt references, respectively, and α and α_β are the thermal and solute coefficients. In the equations below $p_{,i}$ and $T_{,t}$ denote the partial derivatives with respect to the ith spatial coordinate and time, respectively, e.g. $p_{,i} = \partial p/\partial x_i$ and $T_{,t} = \partial T/\partial t$, and $\triangle$ is the spatial Laplacian.

Employing Darcy's law to model fluid flow along with the incompressibility condition and the equations of conservation of temperature and solute yields the system

$$\begin{aligned} p_{,i} &= -\frac{\mu}{k} v_i - g\rho_0 \left(1 - \alpha(T-4)^2 + \sum_{\beta=1}^{2} \alpha_\beta (C^\beta - C_0^\beta) \right) k_i, \\ T_{,t} + v_i T_{,i} &= \kappa \triangle T, \\ C^\beta_{,t} + v_i C^\beta_{,i} &= \kappa_\beta \triangle C^\beta, \end{aligned} \tag{13}$$

where the variables $p,\ \mu,\ k,\ v_i$ and g represent pressure, dynamic viscosity, permeability, velocity and gravitational acceleration, respectively.

Table 2
Comparison of the Legendre and Chebyshev tau techniques with the results denoted by σ_L and σ_C, respectively, with N being the number of polynomials

N	σ_L	σ_C
6	−5.60913318	−5.61227689
8	−5.60913183	−5.60921498
10	−5.60913183	−5.60913147
12	−5.60913183	−5.60913180
14	−5.60913183	−5.60913183

Defining a^2 to be the wavenumber, the non-dimensionalised linear perturbation equations arising from (13) are

$$\begin{aligned}&(D^2-a^2)W-2(\zeta-z)a^2S-a^2\Psi^1-a^2\Psi^2=0,\\&(D^2-a^2)S-RW=\sigma S,\\&(D^2-a^2)\Psi^1-R_1W=P_1\sigma\Psi^1,\\&(D^2-a^2)\Psi^2-R_2W=P_2\sigma\Psi^2,\end{aligned}\tag{14}$$

where $D=\mathrm{d}/\mathrm{d}z$, $\zeta=4/T_U$, R and R_β are the thermal and solute Rayleigh numbers, respectively, and the P_β are salt Prandtl numbers. Here W, S, Ψ^1, Ψ^2 are the z-dependent parts of the perturbations of velocity, temperature, solute 1 and solute 2. The appropriate boundary conditions are

$$W=S=\Psi^1=\Psi^2=0,\quad z=0,1.$$

The Legendre polynomial scheme advocated here applied to (14) reduces to solving the generalised matrix eigenvalue problem

$$A\mathbf{x}=\sigma B\mathbf{x},$$

where $\mathbf{x}=(w_1,\ldots,w_N,s_1,\ldots,s_N,\psi_1^1,\ldots,\psi_N^1,\psi_1^2,\ldots,\psi_N^2)$, with ψ_i^α being the coefficients in the expansion of Ψ^α, $\alpha=1,2$, in terms of the basis ϕ_i. The matrices A and B given by

$$A=\begin{pmatrix}D_2-a^2P & a^2Q^1-a^2bP & -a^2P & -a^2P\\ -RP & D_2-a^2P & 0 & 0\\ -R_1P & 0 & D_2-a^2P & 0\\ -R_2P & 0 & 0 & D_2-a^2P\end{pmatrix},$$

$$B=\begin{pmatrix}0&0&0&0\\0&P&0&0\\0&0&P_1P&0\\0&0&0&P_2P\end{pmatrix},$$

where $b=2\zeta-1$ with the matrix representations D_2, P and Q^1 as presented in Section 3.

We here present results for the leading eigenvalue of the spectrum. These are shown in Table 2 for fixed variables $a^2=21.344$, $\zeta=0.14286$, $R=228.009$, $R_1=-291.066$, $R_2=261$, $P_1=4.5454$ and $P_2=4.7619$.

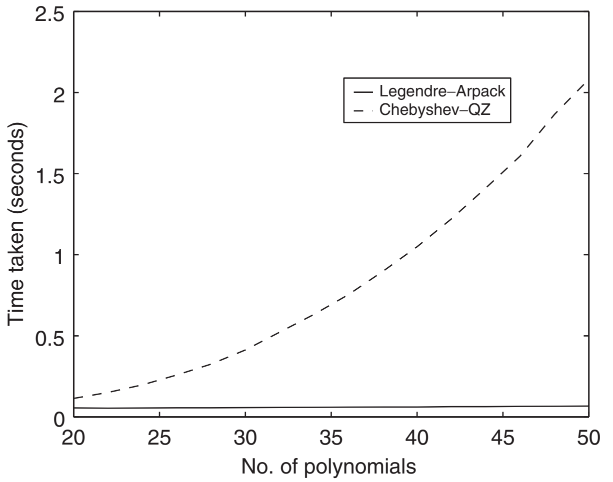

Fig. 2. Number of polynomials used against computational time.

Again, the convergence rate of the Legendre polynomial method is striking and requires fewer polynomials to converge to the required accuracy than the Chebyshev tau method. We observe that the Chebyshev tau method requires approximately 75% more polynomials to achieve the same accuracy. Fig. 2 provides a visual representation of the computational time required to converge to the required eigenvalue as the number of polynomials is increased. Similar to Fig. 1, the Legendre–Arpack line refers to the solutions obtained via the Legendre polynomial-based spectral method coupled with the Arnoldi algorithm, whilst the Chebyshev-QZ line refers to the Chebyshev tau method.

Fig. 2 again demonstrates a high-level computational efficiency of the Legendre–Arnoldi method when compared with the Chebyshev tau-QZ technique.

5. Bénard convection

In hydrodynamic stability calculations the eigenvalue equation which arises from the Navier–Stokes equations is naturally fourth order as opposed to second order from Darcy's law in porous media. When one deals with convection problems in fluid mechanics one is, therefore, usually faced with solving a system comprised of a fourth order equation combined with one or more second order equations. The basis ϕ_i defined in (4) is inadequate to cope with the fourth order equation (unless the fluid layer is subject to artificial stress-free boundary conditions). Therefore, we now combine the Kirchnner [13] technique (which is also used by Shen [17], but not for eigenvalue problems) with the basis in (4). To illustrate the idea we restrict attention to the classical Bénard problem, cf. Straughan [19, p. 49].

If a fluid layer is heated from below, once the gravitational effect has been overcome the fluid rises creating convective motion, which is known as Bénard convection. If we suppose the fluid is contained in the infinite layer $\mathbb{R}^2 \times \{z \in (0,\ 1)\}$, with fixed upper and lower boundary temperatures, the perturbation equations to the steady-state solution are found to be, cf. Straughan [19, p. 50],

$$\begin{aligned} &u_{i,t} + u_j u_{i,j} = -p_{,i} + \Delta u_i + k_i R\theta, \\ &u_{i,i} = 0, \\ &Pr(\theta_{,t} + u_i\theta_{,i}) = Rw + \Delta\theta, \end{aligned} \tag{15}$$

where u_i, p and θ are the non-dimensionalised velocity, pressure and temperature, respectively, Pr and R are the Prandtl and Rayleigh numbers, respectively, and $w = u_3$. The boundary conditions are that $u_i = \theta = 0$ on $z = 0, 1$ and u_i, θ satisfy a plane tiling planform. Note that θ represents a perturbation to the steady-state temperature field so the zero boundary conditions on $z = 0, 1$ are consistent. The plan forms represent the horizontal shape of the convection cells formed at the onset of instability. These cells form a regular horizontal pattern tiling the (x, y) plane, e.g. hexagons, where the wavenumber a (cf. [19, p. 51]) is a measure of the width (to depth) of the convection cell. Defining $a^2 = k^2 + m^2$ with k and m being the x and y wavenumber, the linearised equations governing instability from (15) are

$$\begin{aligned} &(D^2 - a^2)^2 W - a^2 RS = \sigma(D^2 - a^2)W, \\ &(D^2 - a^2)S + RW = \sigma Pr S, \end{aligned} \tag{16}$$

with boundary conditions

$$W = DW = S = 0, \quad z = 0, 1.$$

Here $W(z)$ and $S(z)$ are the vertical component of velocity and temperature field as functions of z.

For the W part we follow the method of Kirchner [13]. Thus, consider the Hilbert space

$$H_0^2(\Omega) = \{v : v, v', v'' \in L_2(\Omega), v(\pm 1) = v'(\pm 1) = 0\}.$$

Here $W \in H_0^2(\Omega)$ and $S \in H_0^1(\Omega)$. The basis functions for the finite dimensional space $S_N \subset H_0^1(\Omega)$ are chosen as in (4) and we turn our attention to building a basis for some finite dimensional space $T_N \subset H_0^2(\Omega)$ of dimension $N \in \mathbb{N}$. Defining $T_N = \mathscr{P}^{N+3} \cap H_0^2(\Omega)$ we introduce the set of basis functions for $i = 1, \ldots, N$ as in [13] (see also [17, p. 1496]), and so define β_i by

$$\begin{aligned} \beta_i(z) &= \int_{-1}^{z}\int_{-1}^{s} L_{i+1}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{-1}^{z}\int_{-1}^{s} \frac{L'_{i+2}(t) - L'_i(t)}{2i+3}\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{(2i+3)}\int_{-1}^{z} [L_{i+2}(s) - L_i(s)]\,\mathrm{d}s \\ &= \frac{L_{i+3} - L_{i+1}}{(2i+3)(2i+5)} - \frac{L_{i+1} - L_{i-1}}{(2i+1)(2i+3)}. \end{aligned} \tag{17}$$

By the definition of Legendre polynomials the basis functions β_i are linearly independent, such that $T_N = \mathrm{span}\{\beta_i\}$ $i = 1, \ldots, N$ with $N = \dim(T_N)$, cf. Kirchner [13].

The system (16) may now be written in terms of the basis functions, such that

$$\sum_{k=1}^{N} w_k(D^4-2a^2D^2+a^4)\beta_k - a^2R\sum_{k=1}^{N} s_k\phi_k = \sigma\sum_{k=1}^{N}(D^2-a^2)w_k\beta_k, \tag{18}$$

$$\sum_{k=1}^{N} s_k(D^2-a^2)\phi_k + R\sum_{k=1}^{N} w_k\beta_k = \sigma Pr\sum_{k=1}^{N} s_k\phi_k. \tag{19}$$

The method is now to take the inner product of (18) with β_i and the inner product of (19) with ϕ_i and derive a finite dimensional generalised eigenvalue problem for σ.

The key to the method is that $\beta_i''=L_{i+1}$ which leads to a diagonal matrix associated with D^4. Since $\beta_i'=(L_{i+2}-L_i)/(2i+3)$ the D^2 operator also leads to a banded matrix. To see this note that

$$\begin{aligned}(D^4W,\ \beta_i) &= -(D^3W,\beta_i')+\beta_i D^3W|_{-1}^{1}\\ &=(D^2W,\beta_i'')-\beta_i'D^2W|_{-1}^{1}\\ &=\sum_{k=1}^{N} w_k(\beta_k'',\beta_i'')\\ &=\sum_{k=1}^{N} w_k(L_{k+1},\ L_{i+1})=\frac{2w_i}{2i+3},\end{aligned}$$

where we have used the forms for β_i, β_i' and the fact that $L_i(\pm1)=(\pm1)^i$.

A similar calculation shows that

$$(D^2W,\ \beta_i)=-\left(\sum_{k=1}^{N} w_k\frac{L_{k+2}-L_k}{2k+3},\ \frac{L_{i+2}-L_i}{2i+3}\right),$$

which is the $(i+1)$th row of the matrix representation $-P$ as presented in Section 3.

After some calculations we can show Eqs. (18), (19) reduce to the generalised eigenvalue problem

$$A\mathbf{x}=\sigma B\mathbf{x},$$

where $\mathbf{x}=(w_1,\ldots,w_N,\ s_1,\ldots,s_N)$, and the matrices A and B are now given by

$$A=\begin{pmatrix} D_4(\beta)-2a^2D_2(\beta)+a^4BB & -Ra^2PB\\ RBP & D_2-a^2P\end{pmatrix},$$

$$B=\begin{pmatrix} D_2(\beta)-a^2B & 0\\ 0 & PrP\end{pmatrix}.$$

Table 3
Comparison of the Legendre and Chebyshev tau techniques with the results denoted by σ_L and σ_C, respectively, with N being the number of polynomials

N	σ_L	σ_C
6	9.978751578	9.770168425
8	9.978787315	9.982167291
10	9.978787485	9.978681120
12	9.978787486	9.97878353
14	9.978787486	9.978787384
16	9.978787486	9.978787484
18	9.978787486	9.978787486

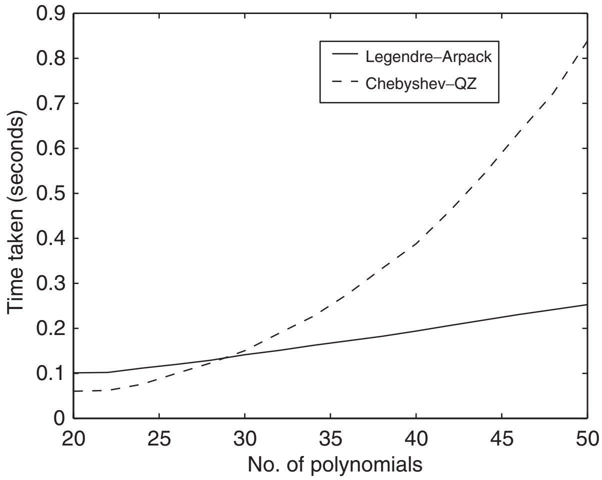

Fig. 3. Number of polynomials used against computational time.

Here D_2 and P are the matrix representations defined for the basis function ϕ_i in Section 3, whereas $D_4(\beta)$ and $D_2(\beta)$ are the matrix versions of the operators D^4 and D^2 with respect to the β_i basis. The matrices BB, PB and BP, corresponding to (W, β_i), (S, β_i), and (W, ϕ_i), respectively, are presented in Appendices A.4–A.6.

Table 3 presents results for the leading eigenvalue of the spectrum for both the Legendre and D^2 Chebyshev tau techniques, with variables fixed at $a^2 = 5$, $R = 111.3$ and $Pr = 6$. The D^2 Chebyshev technique is used as it is desirable to reduce the order of the differential equations whenever possible when using this spectral technique (see [4]).

Again both methods converge to the required accuracy, with the Legendre spectral method requiring less polynomials. In fact, the Chebyshev tau method requires approximately 50% more polynomials to achieve the same accuracy. Fig. 3 provides a visual representation of the computational time required to locate the full spectrum of eigenvalues as the number of polynomials is increased. Similar to Fig. 1 the Legendre–Arpack line refers to the solutions obtained via the Legendre polynomial-based spectral method coupled with the Arnoldi algorithm, whilst the Chebyshev-QZ line refers to the Chebyshev tau method coupled with the QZ algorithm.

Fig. 3 again demonstrates a high-level computational efficiency even when the full eigenvalue spectrum is calculated.

It is worth noting that the generalised eigenvalue problem for the Legendre method employs matrices of order $2N$ compared to matrices of order $3N$ for the D^2 Chebyshev tau technique. There is, therefore, an advantage in using the Legendre method in that smaller matrices are employed. This is a major consideration when employing the basis functions β_i. In a similar manner to the basis functions ϕ_i, $z^n\beta_i$ can be expressed as Legendre polynomials using identity (11), which inherently increases the bandwidth of the matrix to as the powers of z increase. In fact, we may now show

Proposition 2. *Let* $g_b = \sum_{k=1}^{N} g_k\beta_k$. *If* $U(z) \in \mathscr{P}^k(\Omega)$ *for some* $k \in \mathbb{N}$ *then* (Ug_b, β_i), $i = 1, \ldots, N$ *has bandwidth* $2k + 8$.

The proof of Proposition 2 is analogous to the proof of Proposition 1.

6. Conclusions

A Legendre polynomial-based spectral method is presented for solving stability problems associated with the analysis of porous media. The specific choice of basis functions leads to sparse matrices, with banded sub-matrices of size $N \times N$, where N is the number of Legendre polynomials used. To capitalise on this inherent structure we make use of a parallel sparse matrix iterative solver. In this paper we use the implicitly restarted Arnoldi method (IRAM) as presented in the ARPACK package (see [14]). This is seen to substantially reduce the computational and storage requirements as opposed to those needed by the QZ algorithm (see e.g. [7]). Thus, the sparsity of the matrices in the Legendre technique described here is a significant advantage. In Figs. 1 and 3 the Chebyshev tau-QZ technique is seen to be approximately as fast as the Legendre–Arnoldi technique when the number of polynomials is less than 30 or so. In Fig. 2 when the number of equations is greater, and consequently, the matrices are larger, the Legendre–Arnoldi technique is faster even for a small number of polynomials. Thus, for 2 or 3-D stability problems when a large number of polynomials are required we expect the Legendre–Arnoldi method to be worthy of employment.

The current method is particularly advantageous in that it extends naturally to two- and three-dimensional eigenvalue problems. This is easily achieved by using tensor products of basis elements in x, y and z.

Sections 3–5 analyse different examples of hydrodynamic systems, which are convection in a porous medium with an inclined temperature gradient (Hadley flow), multi-component convection–diffusion in a porous medium, and Bénard convection in a fluid. The resulting eigenvalue problems are solved using both the Legendre polynomial-based and Chebyshev tau spectral techniques. In each of these cases the Legendre polynomial-based spectral technique converges to the required eigenvalue utilising less polynomials than the Chebyshev tau method, and with substantially greater computational efficiency, especially since the Legendre technique allows us to employ the Arnoldi algorithm.

While we have here concentrated on convection problems in porous media with the equations governed by Darcy's law, we can adapt the ideas here to many other classes of stability problem in Continuum Mechanics. For example, stability in porous media with a different governing law such as that of Brinkman, viscoelastic flows, and stability problems in elasticity or thermoelasticity.

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Appendix A.

This appendix presents the matrix form of the inner products relevant to Sections 3–5. The generic functions g_p and g_b are used such that $g_p = \sum_{k=1}^{N} g_k\phi_k$ and $g_b = \sum_{k=1}^{N} g_k\beta_k$, where $\mathbf{g} = (g_1, \ldots, g_N)^{\mathrm{T}}$. The notation $A_{i,*}$ refers to the ith row of matrix A.

A.1. *Calculation of* (g'_p, ϕ_i)

The sum can be written as combination of Legendre polynomials using (4) such that

$$\left(\sum_{k=1}^{N} g_k\phi'_k, \phi_i\right) = \left(\sum_{k=1}^{N} g_k L_k, \frac{L_{i+1} - L_{i-1}}{2i+1}\right).$$

By applying the orthogonality relationship in (7), the skew-symmetric matrix representation D can be derived, where $(g'_p, \phi_i) = D_{i,*}\mathbf{g}$, with

$$D_{i,\,i+1} = \frac{2}{(2i+1)(2i+3)}, \quad i = 1, \ldots, N-1,$$

where D is of bandwidth 2.

A.2. *Calculation of* (zg_p, ϕ_i)

Using the basis functions (4) and the recurrence relation (11) we have

$$\left(\sum_{k=1}^{N} g_k z\phi_k, \phi_i\right) = \left(\sum_{k=1}^{N} \frac{g_k}{2k+1}\left(\frac{(k+2)L_{k+2}}{2k+3} - \frac{(2k+1)L_k}{(2k-1)(2k+3)} - \frac{(k-1)L_{k-2}}{2k-1}\right), \frac{L_{i+1} - L_{i-1}}{2i+1}\right).$$

By applying the orthogonality relationship in (7), the symmetric matrix representation Q^1 can be derived, where $(zg_p, \phi_i) = Q^1_{i,*}\mathbf{g}$, with

$$Q^1_{i,j} = \begin{cases} \dfrac{2(i+1)}{(2i-1)(2i+1)(2i+3)(2i+5)}, & j = i+1, \\[2ex] \dfrac{-2(i+2)}{(2i+1)(2i+3)(2i+5)(2i+7)}, & j = i+3, \end{cases}$$

where Q^1 is of bandwidth 6.

A.3. Calculation of $(z^2 g_p, \phi_i)$

Using the basis functions (4) and repeatedly applying the recurrence relation (11) we have

$$\left(\sum_{k=1}^{N} g_k z^2 \phi_k, \phi_i\right) = \left(\sum_{k=1}^{N} \frac{g_k}{2k+1}\left(\frac{(k+2)(k+3)L_{k+3}}{(2k+3)(2k+5)} + \frac{(k^2+k-3)L_{k+1}}{(2k-1)(2k+5)} - \frac{(k^2+k-3)L_{k-1}}{(2k-3)(2k+3)} - \frac{((k-1)(k-2))L_{k-3}}{(2k-1)(2k-3)}\right), \frac{L_{i+1}-L_{i-1}}{2i+1}\right).$$

By applying the orthogonality relationship in (7), the symmetric matrix representation Q^2 can be derived, where $(z^2 g_p, \phi_i) = Q^2_{i,*}\mathbf{g}$, with

$$Q^2_{i,j} = \begin{cases} \dfrac{4(i^2+i-3)}{(2i-3)(2i-1)(2i+1)(2i+3)(2i+5)}, & j = i, \\[2ex] \dfrac{6}{(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)}, & j = i+2, \\[2ex] \dfrac{-2(i+3)(i+2)}{(2i+1)(2i+3)(2i+5)(2i+7)(2i+9)}, & j = i+4, \end{cases}$$

where Q^2 is of bandwidth 8.

A.4. Calculation of (g_b, β_i)

Using the basis functions (17) we have

$$\left(\sum_{k=1}^{N} g_k \beta_k, \beta_i\right) = \left(\sum_{k=1}^{N} \frac{g_k(L_{k+3}-L_{k+1})}{(2k+3)(2k+5)} - \frac{g_k(L_{k+1}-L_{k-1})}{(2k+1)(2k+3)}, \frac{L_{i+3}-L_{i+1}}{(2i+3)(2i+5)} - \frac{L_{i+1}-L_{i-1}}{(2i+1)(2i+3)}\right).$$

By applying the orthogonality relationship in (7), the symmetric matrix representation *BB* can be derived, where $(g_b, \beta_i) = (BB)_{i,*}\mathbf{g}$, with

$$(BB)_{i,j} = \begin{cases} \dfrac{12}{(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)}, & j=i, \\ \dfrac{-8}{(2i+1)(2i+3)(2i+5)(2i+7)(2i+9)}, & j=i+2, \\ \dfrac{2}{(2i+3)(2i+5)(2i+7)(2i+9)(2i+11)}, & j=i+4, \end{cases}$$

where *BB* is of bandwidth 8.

A.5. Calculation of (g_p, β_i)

Using the basis functions (4) and (17) we have

$$\left(\sum_{k=1}^{N} g_k \phi_k, \beta_i\right) = \left(\sum_{k=1}^{N} \frac{g_k(L_{k+1} - L_{k-1})}{(2k+1)}, \right.$$
$$\left. \frac{L_{i+3} - L_{i+1}}{(2i+3)(2i+5)} - \frac{L_{i+1} - L_{i-1}}{(2i+1)(2i+3)}\right).$$

By applying the orthogonality relationship in (7), the matrix representation *PB* can be derived, where $(g_p, \beta_i) = (PB)_{i,*}\mathbf{g}$, with

$$(PB)_{i,j} = \begin{cases} \dfrac{-6}{(2i-1)(2i+1)(2i+3)(2i+5)}, & j=i, \\ \dfrac{6}{(2i+1)(2i+3)(2i+5)(2i+7)}, & j=i+2, \\ \dfrac{-2}{(2i+3)(2i+5)(2i+7)(2i+9)}, & j=i+4, \\ \dfrac{2}{(2i-3)(2i-1)(2i+1)(2i+3)}, & j=i-2, \end{cases}$$

where *PB* is of bandwidth 8.

A.6. Calculation of (g_b, ϕ_i)

Using the basis functions (4) and (17) we have

$$\left(\sum_{k=1}^{N} g_k\beta_k, \phi_i\right) = \left(\sum_{k=1}^{N} \frac{g_k(L_{k+3}-L_{k+1})}{(2k+3)(2k+5)} - \frac{g_k(L_{k+1}-L_{k-1})}{(2k+1)(2k+3)}, \frac{L_{i+1}-L_{i-1}}{(2i+1)}\right).$$

By applying the orthogonality relationship in (7), the matrix representation BP can be derived, where $(g_b, \phi_i) = (BP)_{i,*}\mathbf{g}$, with

$$(BP)_{i,j} = \begin{cases} \dfrac{-6}{(2i-1)(2i+1)(2i+3)(2i+5)}, & j = i, \\ \dfrac{2}{(2i+1)(2i+3)(2i+5)(2i+7)}, & j = i+4, \\ \dfrac{6}{(2i-3)(2i-1)(2i+1)(2i+3)}, & j = i-2, \\ \dfrac{-2}{(2i-5)(2i-3)(2i-1)(2i+1)}, & j = i-4, \end{cases}$$

where PB is of bandwidth 8.

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