On the Non-existence of Graphs with Transitive Generalized Dicyclic Groups

LEWIS A. NOWITZ

Computer Applications, Inc., and New York University, New York, New York

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ABSTRACT

In this paper it is shown that no finite (undirected) graph can have a faithful, transitive representation of a generalized dicyclic group as the full group of automorphisms on its vertices.

By a graph $X$ we mean a finite set $V$ with a set $E$ of unordered pairs of distinct elements of $V$. Unordered pairs will be indicated by brackets. The elements of $V$ are called vertices of $X$, denoted $V(X)$; the elements of $E$ are the edges of $X$, denoted $E(X)$. $G(X)$ denotes the automorphism group of $X$, considered as a permutation group on $V(X)$. Although $X$ has little apparent structure, the possible permutation groups $G(X)$ are severely limited. For example, most transitive Abelian groups cannot be automorphism groups of graphs. If a doubly transitive group of degree $n$ is the automorphism group of a graph, then it must be isomorphic to the symmetric group [1–5].

A generalized dicyclic group $\mathcal{G}$ is generated by an Abelian group $A$ and an element $b \notin A$, such that $b^4 = 1$, $b^2 \in A$, $b^2 \neq 1$, and $b^{-1}ab = a^{-1}$, for $a \in A$.

THEOREM. A transitive generalized dicyclic group $\mathcal{G}$ is not the group of any graph $X$.

PROOF: We will make use of the following two results of Sabidussi [6]:

1. If $G(X)$ is a regular group and if $G(X)$ is not isomorphic to the cyclic group of order 2, then $X$ is connected.

2. Let $G$ be a group, and let $H$ be a subset of $G$ which does not contain the identity. Define $X_{G,H}$ by:

$$V(X_{G,H}) = \{G\}, \quad E(X_{G,H}) = \{[g, gh], g \in G, h \in H\}.$$ 

Then $X_{G,H}$ is connected if and only if $H$ is a set of generators of $G$. 

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LEMMA. The only faithful representation of a generalized dicyclic group \( \bar{G} \) as a group of permutations is the regular representation.

PROOF: Let \( K \) be a proper subgroup of \( \bar{G} \), i.e., \( \{1\} \subset K \). Let \( k \in A \cap K \), \( ab \in Ab \). Then \((ab)^{-1} kab = b^{-1}a^{-1}kab = b^{-1}kb = k^{-1}\). Therefore, the group generated by \( k \) is normal in \( \bar{G} \). Now let \( kb \in Ab \cap K \). Then \((kb)^2 = kbb = bk^{-1}kb = b^2 \in A \), so the group generated by \( b^2 \) is normal in \( \bar{G} \).

Since every proper subgroup \( K \) of \( G \) itself contains a proper subgroup which is normal in \( \bar{G} \), no representation of \( \bar{G} \) on the cosets of such a group \( K \) is faithful [7]. Therefore the only transitive faithful representation of \( \bar{G} \) is the regular representation.

We now prove the main result. Assume the contrary: There exists a graph \( X \) with transitive group \( \bar{G}(X) \). Then \( \bar{G} \) is regular, and \( X \) is connected. Therefore we may identify \( V(X) \) with elements of \( \bar{G} \). Moreover, \( E(X) = \{[g, hh]\} \) where \( g \in \bar{G} \), \( h \in H \), and \( H \) is a set of generators of \( \bar{G} \).

Let \( A \) be the Abelian subgroup of \( \bar{G} \) indicated above and let \( b \notin A \). Then

\[
V(X_{\bar{G}, H}) = \{a_0 = 1, a_1, \ldots, a_{n-1}, a_0b, a_1b, \ldots, a_{n-1}b\},
\]

where the order of \( G \) is \( 2n \) and where the order of \( A \) is \( n \).

Now let \( \rho \) be a permutation of \( V(X_{\bar{G}(X), H}) \) defined by

\[
\rho: g \rightarrow bg, \quad g \in A,
\]

\[
\rho: g \rightarrow b^{-1}g, \quad g \notin A.
\]

Then \( \rho \) is an automorphism of \( X \).

Let

\[
a \in A, \quad b \in Ab, \quad a \in A \cap H, \quad a^*b \in Ab \cap H.
\]

We define \([g_1, g_2] \cong [g_1', g_2']\) if and only if both are in \( E(X) \) or both are not in \( E(X) \).

CASE 1a: Both vertices of \( V(X) \) are in \( \{A\} \):

\[
[p(a), p(a\bar{a})] \cong [ba, ba\bar{a}] \cong [a, a\bar{a}].
\]

CASE 1b: Both vertices of \( V(X) \) are in \( \{Ab\} \):

\[
[ab, a\bar{a}b] = [ab, ab(b^{-1}\bar{a}b)] = [ab, ab\bar{a}^{-1}].
\]

Furthermore

\[
[p(ab), p(ab\bar{a}^{-1})] \cong [b^{-1}ab, b^{-1}ab\bar{a}^{-1}]
\]

\[
\cong [a^{-1}, a^{-1}\bar{a}^{-1}] \cong [ab, ab\bar{a}^{-1}].
\]
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Case 2: One vertex in \( \{A\} \), one vertex in \( \{Ab\} \):

\[
[p(a), p(aa*b)] \cong [ba, b^{-1}aa*b] \cong [ba, a^{-1}a*^{-1}] \cong \\
\cong [ba, ba((a^{-1}b^{-1}) a^{-1}a*^{-1})] \cong \\
\cong [ba, ba(b^{-1}a) a^{-1}a*^{-1}] \cong \\
\cong [ba, ba(b^{-1}a*^{-1})] \cong [ba, ba(a*b)^{-1}] \cong \\
\cong [ba(a*b)^{-1}, ba(a*b)^{-1} a*b] \cong [a, aa*b].
\]

Cases 1 and 2 exhaust all possibilities.

So \( \rho \) is an automorphism of \( X \) and \( \rho \notin \bar{G} \), which proves the theorem.

Corollary. Let \( \bar{K} \) be a generalized dicyclic group with exactly one element of order two. Let \( G(X) \) be a primitive graph group containing \( \bar{K} \) as a transitive subgroup. Then \( G(X) \cong \) symmetric group on \( n \) symbols (\( \Sigma_n \)) where \( n \) is the cardinal number of \( V(X) \).

Proof: We have shown that \( \bar{K} \) must be regular. But such groups \( \bar{K} \) are \( B \)-groups, i.e., every primitive group containing the regular representation of \( \bar{K} \) as a subgroup is doubly transitive [7, 8]. Therefore \( G(X) \cong \Sigma_n \).

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Reference