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ORIGINAL ARTICLE

New Concepts of Feasibility and Efficiency of Solutions in Fuzzy Mathematical Programming Problems

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Abstract Some new concepts in regards to $\bar{\alpha}$-feasibility and $\bar{\alpha}$-efficiency of solutions in fuzzy mathematical programming problems are introduced in this paper, where $\bar{\alpha}$ is a vector of distinct satisfaction degrees. Based on the defined concepts, a new method is suggested to solve fuzzy mathematical programming problems. In this sense, the proposed approach enables decision makers to take into account more flexible solutions by allowing desired distinct satisfactions in constraints. In the case of linear problems with fuzzy constraints, multi-parametric programming is employed to obtain the optimal solution as an affine function of distinct satisfaction degrees. In particular, it proves that the obtained solution is convex and continuous. Therefore, the different optimal solutions can be obtained by a simple substituting the new values of satisfaction parameters into the parametric profiles without any further optimization calculations, which is desirable for online optimization and sensitivity analysis of the profit to satisfaction parameters.

Keywords Feasibility and efficiency of solutions · Fuzzy linear programming · Multi-parametric programming · Sensitivity analysis

1. Introduction

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Decision making is possibly the most important and inevitable aspect of application of mathematical methods in various fields of human activity. In the real-world situations, decisions are fuzzy, at least partly. The first step of attempting a practical decision making problem consists of formulating a suitable mathematical model of a system or a situation to be analyzed. Moreover, if we intend to make reasonably adequate mathematical models of such situation, we should be able to introduce fuzziness into our models and to suggest means of processing fuzzy information [1, 2]. Fuzzy logic theory and applications have a vast literature. With regards to documented literature, the development in fuzzy theory and applications can be classified to three phases [3]; phase 1 (1965-1977) can be referred to as academic phase in which the concept of fuzzy theory has been discussed in depth and accepted as a useful tool for decision making. The phase 2 (1978-1988) can be called as transformation phase whereby significant advances in fuzzy set theory and a few applications were developed. The period from 1989 onwards can be the phase 3, the fuzzy boom period, in which tremendous application problems in industrial and business are being tackled with remarkable success.

In the real world situations, the decision maker might not really want to actually maximize or minimize the objective function. Rather, he or she might want to reach some aspiration levels that might not even be definable crisply. Thus he or she might want to improve the present cost situation considerably, and so on. Also, the role of the constraints can be different from that in the classical one, where the violation of any single constraint by any amount renders the solution infeasible. The decision maker might accept small violations of constraints but might also attach different (crisp or fuzzy) degrees of importance to violations of different constraints. Fuzzy mathematical programming offers a number of ways to allow for all these types of imprecisions.

It is necessary to distinguish between flexibility in constraints and goals and uncertainty of the data. Flexibility is modeled by fuzzy sets and may reflect the fact that constraints or goals are linguistically formulated, their satisfaction is a matter of tolerance and degrees or fuzziness [4]. On the other hand, there is uncertainty, corresponding to an objective variability in the model parameters (randomness), or a lack of knowledge of the parameter values (epistemic uncertainty). Randomness comes from the random nature of events and deals with uncertainty regarding to the membership or non-membership of an element in a set. Epistemic uncertainty is concerned with ill-known parameters modeled by fuzzy intervals in the setting of possibility theory [5, 6].

The rest of this paper is organized as follows. In Section 2, we provide some preliminary results in multi-parametric programming that will be used in our methodology. In Section 3, we introduce the concepts of $\bar{\alpha}$-feasible and $\bar{\alpha}$-efficient solutions and then based on the theoretical discussions we provide a computational method for obtaining aforementioned solutions. In Section 4, an application of the method is described in fuzzy linear programming problems. Finally, we present some concluding remarks in Section 5, and state some additional technical results in the Appendixes A and B.

2. Multi-parametric Programming

In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters vary between lower and upper bounds, multi-parametric programming (MPP) is a technique for obtaining (i) the objective function and the optimization parameters as a function of these parameters and (ii) the regions in the space of the parameters where these functions are valid.

Consider the general parametric nonlinear programming problem:

$$\begin{aligned} &\min_x f(x,\theta) \\ &\text{s.t.}\ \ g_i(x,\theta) \le 0,\ \forall i = 1,\cdots,p, \\ &\qquad x \in X \subseteq \mathbb{R}^n, \\ &\qquad \theta \in \Theta \subseteq \mathbb{R}^m, \end{aligned} \tag{1}$$

where f and g_i, for all $i = 1,\cdots,p$, are twice continuously differentiable in x and θ. The first order Karush-Kuhn-Tucker (KKT) optimality conditions for (1) are given as follows:

$$\begin{aligned} &\nabla\mathcal{L} = 0, \\ &\lambda_i g_i(x,\theta) = 0,\ \lambda_i \ge 0,\ \forall i = 1,\cdots,p, \\ &\mathcal{L} = f(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta). \end{aligned} \tag{2}$$

The main sensitivity result for (1) derives directly from system (2), as shown in Theorem 1 [7].

Theorem 1 *Let θ_0 be a vector of parameter values and (x_0,λ_0) a KKT pair corresponding to* (2)*, where λ_0 is nonnegative and x_0 is feasible in* (1)*. Also assume that* (i) *strict complementary slackness (SCS) holds,* (ii) *the binding constraint gradients are linearly independent (LICQ: linear independence constraint qualification), and* (iii) *the second-order sufficiency conditions (SOSC) hold. Then, in the neighborhood of θ_0, there exists a unique, once continuously differentiable function, $z(\theta) = [x(\theta),\lambda(\theta)]$, satisfying* (2) *with $z(\theta_0) = [x(\theta_0),\lambda(\theta_0)]$, where $x(\theta)$ is a unique isolated minimizer for* (1)*, and*

$$\begin{pmatrix} \dfrac{dx(\theta_0)}{d\theta} \\[2ex] \dfrac{d\lambda(\theta_0)}{d\theta} \end{pmatrix} = -(M_0)^{-1}N_0, \tag{3}$$

where

$$M_0 = \left(\begin{array}{c|ccc} \nabla_x^2 \mathcal{L} & \nabla_x g_1 & \cdots & \nabla_x g_p \\ \hline -\lambda_1 \nabla_x^T g_1 & -g_1 & & 0 \\ \vdots & & \ddots & \\ -\lambda_p \nabla_x^T g_p & 0 & & -g_p \end{array}\right),$$

$$N_0 = \left(\nabla_{\theta x}^2 \mathcal{L}, -\lambda_1 \nabla_\theta^T g_1, \cdots, -\lambda_p \nabla_\theta^T g_p\right)^T.$$

Note that the assumptions which are stated in the above theorem ensure the existence of the inverse of M_0 [8].

Corollary 1 *First order estimation of $x(\theta)$ and $\lambda(\theta)$, near $\theta = \theta_0$* [9]*: Under the assumptions of Theorem* 1*, a first order approximation of* $[x(\theta), \lambda(\theta)]$ *in a neighborhood of θ_0 is*

$$\begin{pmatrix} x(\theta) \\ \lambda(\theta) \end{pmatrix} = \begin{pmatrix} x_0 \\ \lambda_0 \end{pmatrix} + (M_0)^{-1} N_0 \, \theta + o\left(||\theta||\right), \tag{4}$$

where $(x_0, \lambda_0) = [x(\theta_0), \lambda(\theta_0)]$*,* $M_0 = M(\theta_0)$*,* $N_0 = N(\theta_0)$ *and* $\phi(\theta) = o\left(||\theta||\right)$ *means that* $\dfrac{\phi(\theta)}{||\theta||} \to 0$ *as* $\theta \to \theta_0$*.*

Despite being a simple and linear expression, Eq.(4) may lead to complex computational problems, since in the general nonlinear case, the Jacobians of system (2) are in the most cases computationally complex. Fortunately, it simplifies when (1) has a linear objective function, linear constraints, and the parameters appear on the right-hand side of the constraints:

$$\begin{aligned} \min\ & z(\theta) = c^T x \\ \text{s.t.}\ & Ax \leq b + F\theta, \\ & x \in X \subseteq \mathbb{R}^n, \\ & \theta \in \Theta \subseteq \mathbb{R}^m, \end{aligned} \tag{5}$$

where c is a constant vector of dimension n, A is a $p \times n$ constant matrix, F is a $p \times m$ constant matrix, b is a constant vector of dimension p, and X and Θ are compact polyhedral convex sets of dimensions n and m, respectively.

An application of Theorem 1 to (5) at $[x(\theta_0), \theta_0]$ gives the following result:

$$\begin{pmatrix} \dfrac{dx(\theta_0)}{d\theta} \\ \dfrac{d\lambda(\theta_0)}{d\theta} \end{pmatrix} = -(M_0)^{-1} N_0, \tag{6}$$

where

$$M_0 = \left(\begin{array}{c|ccc} 0 & A_1^T & \cdots & A_p^T \\ \hline -\lambda_1 A_1 & -V_1 & & 0 \\ \vdots & & \ddots & \\ -\lambda_p A_p & 0 & & -V_p \end{array}\right), \tag{7}$$

$$N_0 = (Y, \lambda_1 F_1, \cdots, \lambda_p F_p)^T,$$

$$V_i = A_i x(\theta_0) - b_i - F_i \theta_0,$$

and Y is a null matrix of dimension $n \times m$. Thus, in the linear optimization problem, the Jacobians reduce to a mere algebraic manipulation of the matrices declared in (5). In the neighborhood of the KKT point, $[x(\theta_0), \theta_0]$, Corollary 1 writes as follows:

$$\begin{pmatrix} x_0(\theta) \\ \lambda_0(\theta) \end{pmatrix} = -(M_0)^{-1}\ N_0\ (\theta - \theta_0) + \begin{pmatrix} x(\theta_0) \\ \lambda(\theta_0) \end{pmatrix}. \tag{8}$$

This is where parametric programming detaches from the sensitivity analysis theory. MPP is based on the sensitivity analysis theory, distinguishing from the latter in targets. Sensitivity analysis provides solutions in the neighborhood of the nominal value of the varying parameters, whereas MPP provides a complete map of the optimal solution in the space of the varying parameters.

The space of θ where this solution (8) remains optimal is defined as the critical region, CR^0 (where CR is an abbreviation referring to the critical region), and can be obtained by using feasibility and optimality conditions. Note that for convenience and simplicity in presentation, we use the notation CR to denote the set of points in the space of θ that lie in CR as well as to denote the set of inequalities which define CR. Feasibility is ensured by substituting $x_0(\theta)$ into the inactive inequalities given in (5), whereas the optimality condition is given by $\bar{\lambda}_0(\theta) \geq 0$, where $\bar{\lambda}_0(\theta)$ corresponds to the vector of active inequalities, resulting in a set of parametric constraints. Let this set be represented by

$$\mathrm{CR}^{\mathrm{R}} = \left\{\hat{A} x_0(\theta) \leq \hat{b} + \hat{F}\theta, \bar{\lambda}_0(\theta) \geq 0, \mathrm{CR}^{\mathrm{IG}}\right\}, \tag{9}$$

where $\hat{A}$, $\hat{b}$ and $\hat{F}$ correspond to the inactive inequalities and $\mathrm{CR}^{\mathrm{IG}}$ represents a set of linear inequalities defining an initial given region. From the parametric inequalities thus obtained, the redundant inequalities are removed and a compact representation of CR^0 is obtained as follows:

$$\mathrm{CR}^0 = \Delta\{\mathrm{CR}^{\mathrm{R}}\}, \tag{10}$$

where Δ is an operator which removes redundant constraints (for a procedure to identify redundant constraints see [10] and see Appendix A for a summary). Note that a CR^0 is a polyhedral region. Once CR^0 has been defined for a solution, $[x(\theta_0), \theta_0]$, the next step is to define the rest of the region, $\mathrm{CR}^{\mathrm{rest}}$, as proposed in [11] (see Appendix B for a summary):

$$\mathrm{CR}^{\mathrm{rest}} = \mathrm{CR}^{\mathrm{IG}} - \mathrm{CR}^{0}. \tag{11}$$

Another set of parametric solutions in each of these regions is then obtained and corresponding CRs are obtained. The algorithm terminates when there are no more regions to be explored. In other words, the algorithm terminates when the solution of the differential equation (6) has been fully approximated by first order expansions. The main steps of the algorithm are outlined in Algorithm 1. Note that while defining the rest of the regions, some of the regions are split and hence the same optimal solution may be obtained in more than one region. Therefore, the regions with the same optimal solution are united and a compact representation of the final solution is obtained.

Algorithm 1 MPP algorithm.

Step 1: In a given region solve (5) by treating θ as a free variable to obtain a feasible point $[\theta_0]$;

Step 2: Fix $\theta = \theta_0$ and solve (5) to obtain $[x(\theta_0), \lambda(\theta_0)]$;

Step 3: Compute $[(M_0)^{-1}N_0]$ from (6);

Step 4: Obtain $[x_0(\theta), \lambda_0(\theta)]$ from (8);

Step 5: Form a set of inequalities, CR^{R}, as described in (9);

Step 6: Remove redundant inequalities from this set of inequalities and define the corresponding CR^{0} as given in (10);

Step 7: Define the rest of the region, $\mathrm{CR}^{\mathrm{rest}}$ as given in (11);

Step 8: If no more regions to explore, go to next step, otherwise go to Step 1;

Step 9: Collect all the solutions and unify the regions having the same solution to obtain a compact representation.

When θ is present on the right-hand side of the constraints, the solution space of (1) is convex and continuous [12]. Since (5) is a special case of (1), its solution has these properties as well. Due to its importance, we state these properties specifically for (5) in the next theorem.

Theorem 2 *Consider* (5) *and let* Θ *be convex. Then the set of feasible parameters* $\Theta_f \subseteq \Theta$ *is convex, the optimizer* $x(\theta) : \Theta_f \to \mathbb{R}^n$ *is continuous and piecewise affine, and the optimal solution* $z(\theta) : \Theta_f \to \mathbb{R}$ *is continuous, convex, and piecewise linear.*

Proof We first prove the convexity of Θ_f and $z(\theta)$. Take generic $\theta_1, \theta_2 \in \Theta_f$ and let $z(\theta_1)$, $z(\theta_2)$ and x_1, x_2 be the corresponding optimal values and minimizers. Let $\alpha \in [0, 1]$ and define $x_\alpha = \alpha x_1 + (1 - \alpha)x_2$, $\theta_\alpha = \alpha\theta_1 + (1 - \alpha)\theta_2$. By feasibility, x_1, x_2 satisfy the constraints $Ax_1 \leq b + F\theta_1$, $Ax_2 \leq b + F\theta_2$. These inequalities can be linearly combined to obtain $Ax_\alpha \leq b + F\theta_\alpha$ and therefore x_α is feasible for the optimization problem (5). Since a feasible solution, $x(\theta_\alpha)$, exists at θ_α, an optimal solution exists at θ_α and hence Θ_f is convex.

The optimal solution at θ_α will be less than or equal to the feasible solution:

$$z(\theta_\alpha) \leq c^T x_\alpha$$

and hence,

$$z(\theta_\alpha) - [\alpha c^T x_1 + (1-\alpha)c^T x_2] \leq c^T x_\alpha - [\alpha c^T x_1 + (1-\alpha)c^T x_2] = 0,$$

or

$$z(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha z(\theta_1) + (1-\alpha)z(\theta_2),\ \forall\theta_1,\theta_2 \in \Theta,\ \forall\alpha \in [0,1],$$

proving the convexity of $z(\theta)$ on Θ_f. Within the closed polyhedral regions, CR^0, in Θ_f the solution $x(\theta)$ is affine (Corollary 1). The boundary between two regions belongs to both closed regions. Because the optimum is unique, the solution must be continuous across its boundary. The fact that $z(\theta)$ is continuous and piecewise linear follows trivially.

Remark 1 Note that when (5) is a multi-parametric linear program the solution procedure described above and the algorithm remains valid. The results presented in Theorem 1 continue to hold true and SOSC is valid as discussed on page 71 in [9]. For multi-parametric linear programming models x is an affine function of θ and λ remains constant in a CR as shown in Chapter 4 in [10] and therefore Corollary 1 can be used.

Hence, at the end of the algorithm, the obtained solution is a conditional piecewise function of the parameters, and Theorem 2 implies that the computed optimal function, $z(\theta)$, is continuous and convex.

Computational Complexity

Under the assumptions of Theorem 1, at the most n constraints can be active at a point in Θ. Thus, given a set of p constraints, all the possible combinations of active constraints are less than or equal to

$$\eta = \sum_{i=0}^{n} \binom{p}{i},$$

where

$$\binom{p}{i} = \frac{p!}{(p-i)!i!}.$$

In the worst case, an estimate of η_r, the number of regions, CR, generated can be obtained as follows. The following analysis does not take into account (i) the reduction of redundant constraints, and (ii) possible empty sets are not further partitioned.

The first critical region, CR^0 is defined by the constraints given in (9). For simplicity assume that CR^{IG} is unbounded. Thus, first CR^0 is defined by p constraints. From Appendix B, CR^{rest} consists of p convex polyhedra CR_l defined by at most p inequalities. For each CR_l, a new CR is determined consisting of $2p$ inequalities (the additional p inequalities come from the condition $CR \subseteq CR_l$), and therefore the corresponding CR^{rest} partition includes $2p$ sets defined by $2p$ inequalities. This way of generating regions can be associated with a search tree. By induction, it is easy to prove that at the tree level $k+1$ there are $k!p^k$ regions defined by $(k+1)p$ constraints. As observed earlier, each CR is the largest set corresponding to a certain combination of active constraints. Therefore, the search tree has a maximum depth of η, as at each level there is one admissible combination less. In conclusion, the number of regions is $\eta_r \leq \sum_{k=0}^{\eta-1} k!p^k$, each one defined by at most ηp linear inequalities.

The algorithm has been well developed and fully automated in MPT Toolbox 3.0 [13] (a free toolbox for MATLAB) that enables us to apply MPP to large-scale multiparametric programming problems.

3. Fuzzy Mathematical Programming

Let us consider the following fuzzy mathematical programming problem,

$$\begin{aligned} &\widetilde{\max}\ f(x,c) \\ &\text{s.t.}\quad g_i(x, a_{i,.}) \preceq 0, \\ &\qquad\ x \geq 0, \\ &\qquad\ i = 1, 2, \cdots, m, \end{aligned} \tag{12}$$

where $x = (x_1, x_2, \cdots, x_n)^T$ is an n-dimensional real decision vector, $c = (c_1, c_2, \cdots, c_n)$ is an n-dimensional fuzzy vector of fuzzy parameters involved in the objective function f. Here the fuzzy parameters are assumed to be characterized by fuzzy numbers as introduced in [14]. For these fuzzy parameters, membership function μ_{c_j}, $j = 1, 2, \cdots, n$, is defined for a fuzzy number c_j, where c_j is a convex continuous fuzzy subset on the real line. $a_{i,.}$ is i-th row of $A = (a_{i,j})$, $(i = 1, \cdots, m; j = 1, \cdots, n)$, where A is a real $m \times n$-dimensional matrix of technical coefficients. The functions f, g_i, $i = 1, 2, \cdots, m$ possess continuous first and second derivatives, i.e., $f, g_i \in C^2$, $i = 1, 2, \cdots, m$. Also, "$\widetilde{\max}$" maximize the objective in a fuzzy sense and "$\preceq$" denotes a fuzzy extension of "$\leq$" on $\mathbb{R}$ which is used to compare the left side of fuzzy constraints with the right side [14].

Unfortunately, the model (12) is not well-defined because:

i. we cannot maximize the fuzzy quantity $f(x,c)$;

ii. the constraints $g_i(x, a_{i,.}) \preceq 0$, $i = 1, 2, \cdots, m$, do not produce a crisp feasible set.

One convenient approach to express a crisp preference of alternatives is comparing fuzzy quantities by use of ranking function $R : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ that maps each fuzzy quantity to real line in which there exist a natural order-ship (See for more detail in [15]). Also, in order to define a deterministic feasible set, an idea is to provide confidence levels α_i at which it is desired that the corresponding i-th fuzzy constraint hold. Therefore, in order to obviate those mentioned restrictions, we introduce the following problem,

$$\begin{aligned} &\max R(f(x,c)) \\ &\text{s.t.}\ \ \mu_i\{g_i(x,a_{i,.})\le 0\}\ge\alpha_i, \\ &\qquad x\ge 0, \\ &\qquad 0<\alpha_i\le 1, \\ &\qquad i=1,2,\cdots,m. \end{aligned} \tag{13}$$

To motivate for a meaningful choice of membership function for each fuzzy constraints, it is argued that if $g_i(x,a_{i,.})\le 0$, then the i-th constraint is absolutely satisfied, where as if $g_i(x,a_{i,.})\ge p_i$, where p_i is the predefined maximum tolerance from zero, as determined by the decision maker, then the i-th constraint is absolutely violated. For $g_i(x,a_{i,.})\in(0,p_i)$, the membership function is monotonically decreasing. If this decrease is along a linear function, then it makes sense to choose the membership function of the i-th constraint ($i=1,2,\cdots,m$) as

$$\mu\{g_i(x,a_{i,.})\le 0\}=\begin{cases}1, & g_i(x,a_{i,.})\le 0,\\ 1-\dfrac{g_i(x,a_{i,.})}{p_i}, & 0\le g_i(x,a_{i,.})\le p_i,\\ 0, & g_i(x,a_{i,.})\ge p_i.\end{cases} \tag{14}$$

Let us begin with the concept of feasible solution to the fuzzy programming problem (12).

Definition 1 *Let $\bar{\alpha}=(\alpha_1,\cdots,\alpha_m)\in(0,1]^m$ be a vector, and*

$$X_{\bar{\alpha}}=\{x\in\mathbb{R}^n\mid x\ge 0,\mu_i\{g_i(x,a_{i,.})\le 0\}\ge\alpha_i,\ i=1,\cdots,m\}. \tag{15}$$

A vector $x\in X_{\bar{\alpha}}$ is called the $\bar{\alpha}$-feasible solution to problem (12).

Following proposition enables us to define feasible set of (12) as an intersection of all α-cuts corresponding to fuzzy constraints.

Proposition 1 *Let $\bar{\alpha}=(\alpha_1,\cdots,\alpha_m)\in(0,1]^m$. Then $X_{\bar{\alpha}}=\bigcap_{i=1}^{m}X^i_{\alpha_i}$, where*

$$X^i_{\alpha_i}=\{x\in\mathbb{R}^n\mid x\ge 0,\mu_i\{g_i(x,a_{i,.})\le 0\}\ge\alpha_i\} \tag{16}$$

for $i=1,2,\cdots,m$ (Namely, X^i_α is the α-cut of the i-th fuzzy constraint).

Proof $\forall\bar{\alpha}=(\alpha_1,\cdots,\alpha_m)\in(0,1]^m$, let $x\in X_{\bar{\alpha}}$. Therefore $\mu_i\{g_i(x,a_{i,.})\le 0\}\ge\alpha_i$, and from (16), we have $x\in X^i_{\alpha_i}$, $i=1,2,\cdots,m$. Therefore $x\in\bigcap_{i=1}^{m}X^i_{\alpha_i}$.

On the other hand, if $x\in\bigcap_{i=1}^{m}X^i_{\alpha_i}$, we have $x\in X^i_{\alpha_i}$ for all $i=1,2,\cdots,m$, therefore $\mu_i\{g_i(x,a_{i,.})\le 0\}\ge\alpha_i$, and hence $x\in X_{\bar{\alpha}}$. This completes the proof.

Proposition 2 *Let $\bar{\alpha}' = (\alpha'_1, \cdots, \alpha'_m)$, $\bar{\alpha}'' = (\alpha''_1, \cdots, \alpha''_m)$, where $\alpha'_i \leq \alpha''_i$ for all i. Then $\bar{\alpha}''$-feasibility of x implies the $\bar{\alpha}'$-feasibility of it.*

Proof The proof is straightforward.

For a given $\alpha \in (0, 1]$, let $x \in \mathbb{R}^n$ be a usual α-feasible solution to (12) (a solution with the same degrees of satisfaction in all of constraints). It means that $\mu_i\{g_i(x, a_{i,.}) \leq 0\} \geq \alpha$, or $x \in X^i_\alpha$ for all $i = 1, 2, \cdots, m$. If $\bar{\alpha} = (\alpha, \cdots, \alpha) \in (0, 1]^m$, then $x \in X_{\bar{\alpha}}$, which implies that the α-feasibility of (12) can be understood as a special case of the $\bar{\alpha}$-feasibility. Therefore we have the following result.

Remark 2 If the problem (12) is not infeasible, then $X_{\bar{\alpha}}$ is not empty.

Proof The proof is straightforward.

Definition 2 *Let $\preceq$ be a fuzzy extension of binary relation $\leq$, and let $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ be an $\bar{\alpha}$-feasible solution to* (12)*, where $\bar{\alpha} = (\alpha_1, \cdots, \alpha_m) \in (0, 1]^m$, and let $f(x, c)$ be a fuzzy objective. The vector $x \in \mathbb{R}^n$ is an $\bar{\alpha}$-efficient solution to* (12) *with maximization of the objective function if there is no $x' \in X_{\bar{\alpha}}$ such that $R(f(x, c)) < R(f(x', c))$.*

Similarly, an $\bar{\alpha}$-efficient solution with minimization of the objective function can be defined.

Notice that any $\bar{\alpha}$-efficient solution to the fuzzy mathematical programming problem is an $\bar{\alpha}$-feasible solution to the fuzzy mathematical programming problem with some additional property.

In the following theorem, we show the necessary and sufficient conditions for an $\bar{\alpha}$-efficient solution to (12).

Theorem 3 *Let $\bar{\alpha} = (\alpha_1, \cdots, \alpha_m) \in (0, 1]^m$, and let $x^* = (x^*_1, \cdots, x^*_n)^T$, $x^*_j \geq 0$, $j = 1, 2, \cdots, n$, be an $\bar{\alpha}$-feasible solution to* (12)*. Then the vector $x^* \in \mathbb{R}^n$ is an $\bar{\alpha}$-efficient solution to* (12) *with maximization of the objective function if and only if x^* is an optimal solution to the following problem,*

$$\begin{aligned} &\max R(f(x, c)) \\ &\text{s.t.} \;\; g_i(x, a_{i,.}) \leq (1 - \alpha_i)p_i, \; i = 1, 2, \cdots, m, \\ &\qquad x_j \geq 0, \; j = 1, 2, \cdots, n, \end{aligned} \tag{17}$$

where p_i is the predefined maximum tolerance.

Proof Let $\bar{\alpha} = (\alpha_1, \cdots, \alpha_m) \in (0, 1]^m$, and let $x^* = (x^*_1, \cdots, x^*_n)^T$, $x^*_j \geq 0$, $j = 1, 2, \cdots, n$, be an $\bar{\alpha}$-efficient solution to (12) with maximization of the objective function. By Definition 1 and Eq.(14), we have $\mu_i(g_i(x^*, a_{i,.}) \preceq 0) \geq \alpha_i$, or $g_i(x^*, a_{i,.}) \leq (1 - \alpha_i)p_i$ for $i = 1, 2, \cdots, m$. Therefore x^* is a feasible solution to (17). Also, by Definition 2, there is no $x' \in X_{\bar{\alpha}}$ such that $R(f(x^*, c)) < R(f(x', c))$, it means that x^* is an optimal solution to (17). On the other hand, if x^* is an optimal solution to the problem (17), clearly, x^* is an $\bar{\alpha}$-feasible solution to the problem (12) and therefore by Definition 2, optimality of x^* implies the $\bar{\alpha}$-efficiency of x^*.

Theorem 3 provides a computational method to solve fuzzy mathematical problem (12). Thus, by assigning a specific $\bar{\alpha}$ by a decision maker, we may substitute the α_j in corresponding constraint of (17), and solve the obtained problem to compute the $\bar{\alpha}$-efficient solution to the problem (12). An $\bar{\alpha}$-efficient solution to (12) has two properties: (i) the solution has different satisfaction degrees corresponding to each constraint, (ii) the acquired solution is optimal. This solution enables decision maker, by assigning desired preferences, to obtain a more flexible and more compatible solution with respect to environmental conditions. This compatibility, particularly, in online optimization is more noticeable.

In Theorem 3, we have discussed a method to fuzzy mathematical programming problems to obtain an $\bar{\alpha}$-efficient solution. If the resulting problem (17) has only one optimal solution, then we have proved that this solution is an $\bar{\alpha}$-efficient solution to the given fuzzy problem. In the case of which the problem (17) has some multiple optimal solutions, in order to find a maximum efficient solution, i.e., an $\bar{\alpha}'$-efficient solution with $\alpha' \geq \alpha$, $i = 1, \cdots, m$, we perform the following two phase approach.

In the two phase approach, Eq.(17) is solved in phase 1, while in phase 2, a solution is obtained which has higher satisfaction degrees than the previous solution. Thus by using this two phase approach, we achieve a better utilization of available resources. Further the solution resulting by this two phase approach is always an $\bar{\alpha}$-efficient solution.

Let us call the problem (17) as phase 1 problem. Let $\bar{\alpha}^0 = (\alpha_1^0, \cdots, \alpha_m^0)$, and $(x^*, R(f(x^*, c)))$ be the optimal solution of phase 1 with $\bar{\alpha}^0$ degree of efficiency. Set $\alpha_i^* = \mu_i\{g_i(x^*, a_{i,.}) \leq 0\} \geq \alpha_i^0$, $i = 1, 2, \cdots, m$. In phase 2, we solve the following problem,

$$\begin{aligned} \max\ & \sum_{i=1}^{m} \alpha_i \\ \text{s.t.}\ & R(f(x, c)) \geq R(f(x^*, c)), \\ & g_i(x, a_{i,.}) \leq (1 - \alpha_i) p_i, \\ & \alpha_i^* \leq \alpha_i \leq 1, \quad i = 1, 2, \cdots, m, \\ & x \geq 0. \end{aligned} \tag{18}$$

Let $(x^{**}, \alpha_1^{**}, \cdots, \alpha_m^{**})$ be an optimal solution to the problem (18) (phase 2). Then we have the following theorem.

Theorem 4 *The optimal solution x^{**} to the problem* (18) *(phase* 2*) is a maximum $\bar{\alpha}$-efficient solution to the problem* (12).

Proof By (18), $\alpha_i^* \geq \alpha_i^0$ and Proposition 2, x^{**} is an $\bar{\alpha}^0$-feasible solution to (12) and this implies that it is feasible in (17). By optimality of x^* in (17) and $R(f(x^{**}, c)) \geq R(f(x^*, c))$, we have optimality of x^{**} in (17) and $R(f(x^{**}, c)) = R(f(x^*, c))$. Therefore, x^{**} is also an $\bar{\alpha}^0$-efficient solution to (12). Also, since $(x^{**}, \alpha_1^{**}, \cdots, \alpha_m^{**})$ is optimal and the coefficients in the objective function of (18) are positive, we have $\alpha_i^{**} = \mu_i\{g_i(x^*, a_{i,.}) \leq 0\}$, $i = 1, \cdots, m$. Now, if possible, let x^{**} be not a maximum $\bar{\alpha}^0$-efficient solution to (12). Then there exist an $\bar{\alpha}^0$-efficient solution x' to (12), such that

$$\alpha_i' \geq \alpha_i^{**},\ i = 1, 2, \cdots, m$$

and for some k,

$$\alpha_k' > \alpha_k^{**},$$

where, $\alpha_i' = \mu_i\{g_i(x^*, a_{i,.}) \leq 0\}$, $i = 1, 2, \cdots, m$, and $R(f(x', c)) \geq R(f(x^*, c))$. We have $(x', \alpha_1', \cdots, \alpha_m')$ feasible for (18) and

$$\sum_{i=1}^{m} \alpha_i^{**} = \sum_{i=1,\ i\neq k}^{m} \alpha_i^{**} + \alpha_k^{**} < \sum_{i=1,\ i\neq k}^{m} \alpha_i' + \alpha_k' = \sum_{i=1}^{m} \alpha_i'.$$

But this implies that $(x^{**}, \alpha_1^{**}, \cdots, \alpha_m^{**})$ is not an optimal solution to (18), and it is clearly a contradiction.

4. Numerical Implementation of the Method in Linear Programming

Here we consider a fuzzy linear programming problem with three fuzzy constraints for numerical presentation.

Example 1 Consider the following problem,

$$\begin{aligned} \max\ & 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ \text{s.t.}\ & x_1 + x_2 + x_3 + x_4 \leq 15, \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 80, \\ & 3x_1 + 4.4x_2 + 10x_3 + 15x_4 \leq 100, \\ & x_1, x_2, x_3, x_4 \geq 0, \end{aligned} \tag{19}$$

with the following membership functions,

$$\mu_i(A_i x, b_i) = \begin{cases} 1, & A_i x < b_i, \\ 1 - \dfrac{A_i x - b_i}{p_i}, & b_i \leq A_i x \leq b_i + p_i, \quad i = 1, 2, 3, \\ 0, & A_i x > b_i + p_i, \end{cases}$$

where $p_1 = 5$, $p_2 = 40$ and $p_3 = 30$ are predefined maximum tolerance from b_i, $i = 1, 2, 3$ ([9]).

By Theorem 3, we can rewrite (19) as follows:

$$\begin{aligned} \max\ & 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ \text{s.t.}\ & x_1 + x_2 + x_3 + x_4 \leq 15 + 5(1 - \alpha_1), \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 80 + 40(1 - \alpha_2), \\ & 3x_1 + 4.4x_2 + 10x_3 + 15x_4 \leq 100 + 30(1 - \alpha_3), \\ & 0 < \alpha_i \leq 1, \quad i = 1, 2, 3, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{20}$$

Some $\bar{\alpha}$-efficient solutions with satisfaction degrees which decision maker's desire can be found in the following table (Table 1).

Table 1: Some optimal solutions to (19) with different satisfactions.

a	b	c	d	e	f
$\bar{\alpha}$	$(0.5, 0.5, 0.2)$	$(0.5, 0.5, 0.8)$	$(0.5, 0.1, 0.5)$	$(0.5, 0.9, 0.5)$	$(0.5, 0.5, 0.5)$
$c^T x$	121.071	108.214	114.643	114.643	114.643
x_1	7.28571	9.85714	8.57143	6.71429	8.57143
x_2	0.0	0.0	0.0	2.32143	0.0
x_3	10.2143	7.64286	8.92857	8.46429	8.92857
x_4	0.0	0.0	0.0	0.0	0.0

If all of the satisfaction degrees are equal, then the $\bar{\alpha}$-feasibility and $\bar{\alpha}$-efficiency reduce to classic α-feasibility and α-optimality (see Table 1, column *f*).

Let x^* be a $(0.5, 0.9, 0.5)$-efficient solution with $c^T x^* = 114.643$ as an optimal objective value (see Table 1, column *e*). In phase 2, we need to solve the following linear programming problem,

$$\begin{aligned} \max\ & \alpha_1 + \alpha_2 + \alpha_3 \\ \text{s.t.}\ & 4x_1 + 5x_2 + 9x_3 + 11x_4 \geq 114.643, \\ & x_1 + x_2 + x_3 + x_4 \leq 15 + 5(1 - \alpha_1), \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 80 + 40(1 - \alpha_2), \\ & 3x_1 + 4.4x_2 + 10x_3 + 15x_4 \leq 100 + 30(1 - \alpha_3), \\ & 0.5 \leq \alpha_1 \leq 1, \\ & 0.9 \leq \alpha_2 \leq 1, \\ & 0.5 \leq \alpha_3 \leq 1, \\ & x_1, x_2, x_3, x_4 \geq 0, \end{aligned}$$

An optimal solution to the above problem is $x^{**} = (4.04762, 5.65476, 7.79762, 0)$. Also $c^T x^{**} = c^T x^* = 114.643$. We have $\mu_1(A_1 x^{**}, b_1) = \mu_3(A_3 x^{**}, b_3) = 0.5$ and $\mu_2(A_2 x^{**}, b_2) = 1$.

Thus, using the two phase approach, we can get an optimal solution x^{**} which not only achieves the optimal objective value but also gives a higher membership value in μ_2.

Sensitivity Analysis

In order to have a comprehensive sensitivity analysis of the solution to the resources, we can employ the MPP algorithm to obtain an optimal solution as an affine function of satisfaction degrees. This representation helps us to understand the relation between results and satisfaction degrees. Following example illustrates the state of the method. (We have implemented MPP by MATLAB's MPT Toolbox [13]).

Example 2 Implementing the MPP algorithm to (20), discloses the following results (Fig. 1, Table 2).

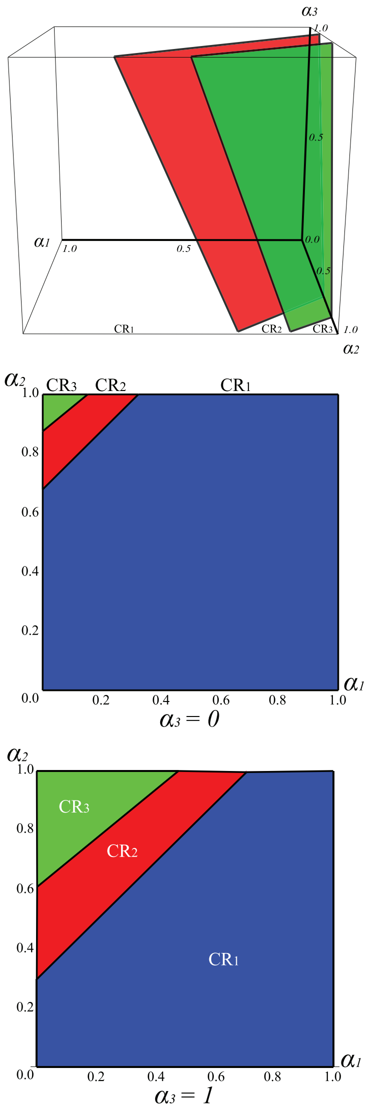

Fig. 1 Critical regions of problem (20)

Table 2: Critical regions and its corresponding optimal solutions.

i	CR^i	Optimal solution
1	$-0.46427 - 0.68070\alpha_1 + 0.68419\alpha_2 + 0.26181\alpha_3 \le 0$ $0 < \alpha_1 \le 1$ $0 < \alpha_2 \le 1$ $0 < \alpha_3 \le 1$	$z^* = 130 - 9.28571\alpha_1 - 21.4286\alpha_3$ $x_1^* = 5.2381 - 4.62585\alpha_1 + 3.06122\alpha_3$ $x_2^* = 5.95238 - 3.14626\alpha_1 + 1.53061\alpha_3$ $x_3^* = 8.80952 + 2.77211\alpha_1 - 4.59184\alpha_3$ $x_4^* = 0$
2	$0.46427 + 0.68070\alpha_1 - 0.68419\alpha_2 - 0.26181\alpha_3 \le 0$ $-0.66210 - 0.62156\alpha_1 + 0.75669\alpha_2 + 0.20268\alpha_3 \le 0$ $0 < \alpha_1$ $\alpha_2 \le 1$ $0 < \alpha_3 \le 1$	$z^* = 130 - 9.28571\alpha_1 - 21.4286\alpha_3$ $x_1^* = 23.3333 + 21.9048\alpha_1 - 26.6667\alpha_2 - 7.14286\alpha_3$ $x_2^* = -16.66667 - 36.30952\alpha_1 + 33.33333\alpha_2 + 14.28571\alpha_3$ $x_3^* = 13.33333 + 9.40476\alpha_1 - 6.66667\alpha_2 - 7.14286\alpha_3$ $x_4^* = 0$
3	$0.66210 + 0.62156\alpha_1 - 0.75669\alpha_2 - 0.20268\alpha_3 \le 0$ $0 < \alpha_1$ $\alpha_2 \le 1$ $0 < \alpha_3 \le 1$	$z^* = 139.891 - 11.3043\alpha_2 - 24.4565\alpha_3$ $x_1^* = 0$ $x_2^* = 22.0109 - 10.8696\alpha_2 + 2.44565\alpha_3$ $x_3^* = 3.31522 + 4.78261\alpha_2 - 4.07609\alpha_3$ $x_4^* = 0$

We can obviously conclude the following items:

1) A complete map is available in all optimal solutions, objective and alternatives as a function of α_1, α_2 and α_3.

2) The space of α_1, α_2 and α_3 has been divided into three regions, CR_1, CR_2 and CR_3, where the profiles of objective and alternatives remain optimal and hence:
(a) One does not have to exhaustively enumerate the complete space of α_1, α_2 and α_3;
(b) The optimal solution can be obtained by a simple substituting the value of α_1, α_2 and α_3 into the parametric profiles without any further optimization calculations.

3) Sensitivity of the objective to the parameters can be identified. For example, in CR_1, the objective is more sensitive to α_3, whereas it is not sensitive to α_2 at all. Thus, for any value of $\bar{\alpha}$ that lies in CR_1, any changes in b_2 will not affect the objective.

The solution to the multi-parametric linear programming, provides a complete map of the optimal decision as a function of the satisfaction degrees and the characteristic partitions of the satisfaction degrees space where this solution is feasible. In this way the solution to the multi-parametric linear programming problem is obtained as piecewise affine function. The online computational effort is small since the online optimization problem is solved offline and no optimizer is ever called online. In contrast, the online optimization problem is reduced to a mere function evaluation problem. This is known as the online optimization via off-line parametric optimization concept.

Here, the sub-model might also be related to interval linear programming where inexact parameters (i.e., degrees of satisfaction) fluctuate between the lower and upper bounds [16-18], but the importance of our method is as follows:

1) The optimal solution is known as a function of satisfaction degrees which are determined by decision maker;

2) Substituting the value of parameters into the parametric profiles we know directly the optimal objective;

3) The sensitivity of the objective to the satisfaction parameters (satisfaction of each constraint) is identified. The decision maker foresees more sensitive operating regions, making the management more efficient and more efficient decision making.

This type of information is very useful for solving reactive or online optimization problems. Such problems usually require a repetitive solution to optimization problems; due to the varying conditions of most processes, the optimal decision/action changes with time. The key advantage of parametric programming is to obtain the optimal solution as a function of the varying parameters without exhaustively enumerating the entire parametric space. The online optimization problem thus reduces to a simple map-reading and function evaluation problem. The corresponding computational effort required by this kind of implementation is very small, as no optimization is done online.

Remark 3 Here, for the sake of simplicity in sensitivity analysis of fuzzy linear programming problems, we have supposed that all degrees of satisfactions are absolutely independent. But in the case of which there are some dependencies between satisfaction degrees in the constraints, we need to reduce the mentioned MPP algorithm to handle these dependencies. As a simple example, if the satisfaction degree of the i-th constraint can be written as a linear combination of some other satisfaction degrees, e.g., $\alpha_i = a_j\alpha_j + a_k\alpha_k,\ j,k \neq i,\ a_j, a_k \in \mathbb{R}$, we have $A_i x \leq b_i + (1 - a_j\alpha_j - a_k\alpha_k)p_i$, therefore, by some simple arithmetic operations, we obtain $A_i x \leq b'_i + p_i a_j(1-\alpha_j) + p_i a_k(1-\alpha_k)$, where $b'_i = b_i - p_i(a_j + a_k)$. Hence, we need to set $F_{i,j} = p_i a_j$ and $F_{i,k} = p_i a_k$ in Eq.(5). Therefore in this case our approach remains valid. Also see Table 1, column *f*, for a special case in which all satisfaction degrees are equal.

5. Concluding Remarks

In this paper, we have been introduced the new concepts of $\bar{\alpha}$-feasibility and $\bar{\alpha}$-efficiency, where $\bar{\alpha}$ is a vector of satisfaction degrees which are determined by decision maker. This new concepts help us to obtain more flexible solutions to fuzzy mathematical programming problems. In addition, in the case of linear problems, we have proved that the desired solution can be obtained by solving a corresponding multi-parametric linear programming and in order to solve the later problem, we have utilized MPP which provides a complete map of the optimal solution as a conditional piecewise linear function of the satisfaction parameters. Benchmark examples have been presented to show the applicability and to describe the proposed procedure.

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Appendix A: Redundancy check for a set of linear constraints

Consider a system of linear constraints:

$$\sum_{j=1}^{N} g_{i,j}\theta_j \leq b_i,\ i = 1, \cdots, k, \cdots, m.$$

Constraint k is redundant if there is a solution to the following problem:

$$\begin{aligned} &\min_{\theta,\epsilon} \epsilon_k \\ &\text{s.t.}\ \sum_{j=1}^{N} g_{i,j}\theta_j + \epsilon_i = b_i,\ i = 1, \cdots, m, \\ &\qquad \epsilon_i \in \mathbb{R}, \end{aligned}$$

such that $\epsilon_k \geq 0$. If $\{\min \epsilon_k\} > 0$, the constraint is said to be strongly redundant; if $\{\min \epsilon_k\} = 0$, simultaneously with another ϵ_i, one of them is said to be weakly redundant.

Appendix B: Definition of rest of the region

Given an initial region, $\mathrm{CR}^{\mathrm{IG}}$ and a region of optimality, CR^0 such that $\mathrm{CR}^0 \subseteq \mathrm{CR}^{\mathrm{IG}}$, a procedure is described in this section to define the rest of the region, $\mathrm{CR}^{\mathrm{rest}} = \mathrm{CR}^{\mathrm{IG}} - \mathrm{CR}^0$. For the sake of simplifying the explanation of the procedure, consider the case when only two parameters, θ_1 and θ_2, are present (see Fig. 2), where $\mathrm{CR}^{\mathrm{IG}}$ is defined by the inequalities: $\{\theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2 \leq \theta_2^U\}$ and CR^0 is defined by the inequalities: $\{C1 \leq 0, C2 \leq 0, C3 \leq 0\}$ where $C1$,$C2$ and $C3$ are linear in θ. The procedure consists of considering one by one of the inequalities which define CR^0. Considering, for example, the inequality $C1 \leq 0$, the rest of the region is given by, $\mathrm{CR}_1^{\mathrm{rest}} : \{C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U\}$, which is obtained by reversing the sign of inequality $C1 \leq 0$ and removing redundant constraints in $\mathrm{CR}^{\mathrm{IG}}$ (see Fig. 3). Thus, by considering the rest of the inequalities, the complete rest of the region is given by: $\mathrm{CR}^{\mathrm{rest}} = \{\mathrm{CR}_1^{\mathrm{rest}} \cup \mathrm{CR}_2^{\mathrm{rest}} \cup \mathrm{CR}_3^{\mathrm{rest}}\}$, where $\mathrm{CR}_1^{\mathrm{rest}}$, $\mathrm{CR}_2^{\mathrm{rest}}$ and $\mathrm{CR}_3^{\mathrm{rest}}$ are given in Table 3 and are graphically depicted in Fig. 4. Note that for the case when $\mathrm{CR}^{\mathrm{IG}}$ is unbounded, simply suppress the inequalities involving $\mathrm{CR}^{\mathrm{IG}}$ in Table 3.

Table 3: Definition of rest of the regions.

Region	Inequalities
$\mathrm{CR}_1^{\mathrm{rest}}$	$C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U$
$\mathrm{CR}_2^{\mathrm{rest}}$	$C1 \leq 0, C2 \geq 0, \theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U$
$\mathrm{CR}_3^{\mathrm{rest}}$	$C1 \leq 0, C2 \leq 0, C3 \geq 0, \theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2$

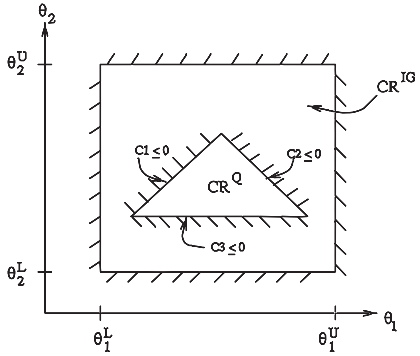

Fig. 2 Critical regions, CR^{IG} and CR^{0}

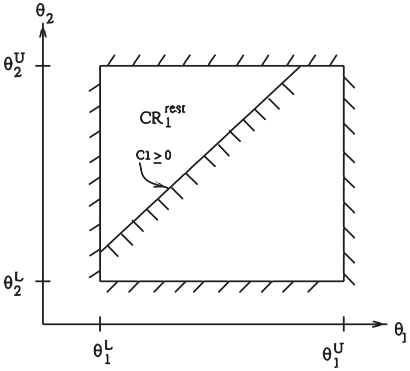

Fig. 3 Division of critical regions: Step 1

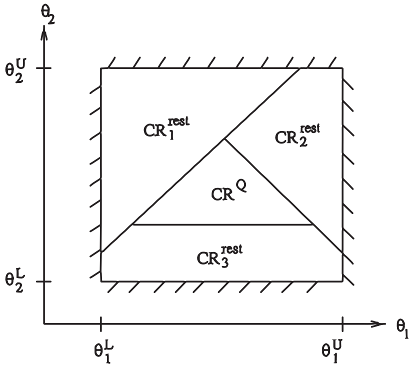

Fig. 4 Division of critical regions: rest of the regions