Pointwise evaluation of Bochner integrals in Marcinkiewicz spaces

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ABSTRACT

Let $\mathcal{M}_p$, $1 \leq p < \infty$, be the Marcinkiewicz Banach space. The elements of $\mathcal{M}_p$ are equivalence classes of Borel measurable scalar-valued functions on the reals. Let $F$ be an $\mathcal{M}_p$-valued function on a set $\Lambda$ which is Bochner integrable with respect to a $\sigma$-finite measure $\mu$ over $\Lambda$. In Theorem I it is shown that there exists a function-valued function $G$ on $\Lambda$ such that for $\mu$ almost all $\lambda$, $G(\lambda)$ is a representative of $F(\lambda)$, and the function of $t$ obtained by integrating, for each $t$, the scalar-valued function $(G(\cdot))(t)$ with respect to $\mu$, is a representative of the Bochner integral of $F$. In Theorem II we find a large class of functions $G$ for which the above holds. We thereby extend the work of Dunford and Schwartz on $L_p$ spaces to the $\mathcal{M}_p$ spaces. Some applications of these theorems in the study of infinitesimal generators and convolutions are indicated.

1. INTRODUCTION

Let $\mathcal{A}$ be a $\sigma$-algebra over a set $\Gamma$, and let $X$ be a vector space over $\mathbb{F}$ ( = $\mathbb{R}$ or $\mathbb{C}$) whose elements are $\mathcal{A}$-Borel measurable $\mathbb{F}$-valued functions on $\Gamma$.

Suppose that this $X$ is a Banach space and that $F(\cdot)$ is a Bochner integrable function on a measure space $(\Lambda, \mathcal{A}, \mu)$ to $X$. Then its Bochner integral

$$\int_{\Lambda} F(\lambda) \mu(d\lambda)$$

is in $X$, and hence is itself a $\mathcal{A}$-borel measurable function on $\Gamma$ to $\mathbb{F}$. The
question naturally arises as to whether this function can be evaluated at a point \( y \in \mathcal{F} \) by simply integrating the \( \mathcal{F} \)-valued function \( F(\cdot)(y) \) on \( A \), i.e. whether

\[
\int_A F(\lambda)(y) \mu(d\lambda) = \int_A \int \lambda F(\mu)(d\lambda)(y).
\]

In many important situations our vector space \( X \) is merely a complete semi-normed space and only the associated space \( \mathcal{X} \), whose elements are equivalence classes \( f \) of functions \( f \in \mathcal{F} \), is complete normed, i.e. Banach. For such Banach spaces (of equivalence classes) the corresponding question breaks into two questions:

**Question I.** Given \( F(\cdot) \in L^1(A, \mathcal{A}, \mu; \mathcal{X}) \), cf. Def. 1.2c), does there exist a function \( G(\cdot) \) on \( A \) to \( X \) such that

\[
\begin{align*}
G(\lambda) &\in \mathcal{F}(\lambda) \text{ for } \lambda \in A, \\
G(\cdot)(-) &\text{ is } \mathcal{A} \otimes \mathcal{B}(\mathcal{F}) \text{ measurable,}^1 \\
\forall y \in \mathcal{F}, G(\cdot)(y) &\in L_1(A, \mathcal{A}, \mu; \mathcal{F}),
\end{align*}
\]

and

\[
\int_A G(\lambda)(-\mu(d\lambda)) \in \int_A \mathcal{F}(\lambda) \mu(d\lambda)?
\]

An affirmative answer to Question I then raises a second question.

**Question II.** Given \( F(\cdot) \in L^1(A, \mathcal{A}, \mu; \mathcal{X}) \), is there a nice condition (*) on functions \( G(\cdot) \) satisfying (1.1) such that (1.1) together with (*) entails (1.2)?

The aim of this paper is to answer these questions affirmatively, when \( (A, \mathcal{A}, \mu) \) is a \( \sigma \)-finite nonnegative measure space and \( \mathcal{X} \) is the Marcinkiewcz Banach space \( \mathcal{M}_p(\mathbb{R}) \), \( 1 \leq p < \infty \), cf. Def. 2.1, which has been the object of recent study [6] [7], cf. also [1]. In [4, p. 198, Thm. 17] Dunford and Schwartz have shown that when \( (A, \mathcal{A}, \mu) \) and \( (\mathcal{F}, \mathcal{F}, \nu) \) are both finite or \( \sigma \)-finite nonnegative measure spaces, and \( \mathcal{X} = L_1(\mathcal{F}, \mathcal{B}, \nu; \mathcal{F}) \), \( 1 \leq p \leq \infty \), there exists a function \( G(\cdot) \) on \( A \) to \( \mathcal{X} \) such that (1.1) and (1.2) hold; moreover if \( G_0(\cdot) \) is any function on \( A \) to \( \mathcal{X} \) satisfying (1.1), then (1.2) holds for \( G_0 \). We shall use their result and some of their techniques to obtain first an analogous result, Theorem 4.2, for a Banach space \( \mathcal{M}_p(\mathbb{R}) \) closely related to \( \mathcal{M}_p(\mathbb{R}) \). Then by use of the theory of continuous selections due to E. Michael [9], we will prove two main theorems for \( \mathcal{M}_p(\mathbb{R}) \). To state these theorems we must specify our terminology:

### 1.2 Definitions

Let \( \mathcal{A} \) be a \( \sigma \)-algebra over a set \( A \) and \( \mu \) be a countably additive measure on \( \mathcal{A} \) to \([0, \infty]\). Let \( \mathcal{Y} \) be a Banach space over \( \mathcal{F} \).

a) We say that a function \( F(\cdot) \) on \( A \) to \( \mathcal{Y} \) is *ball measurable* iff \( F(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathcal{Y})) \), \( 2 \) where \( \mathcal{B}(\mathcal{Y}) \) is the \( \sigma \)-algebra generated by the family \( \mathcal{N} \) of all open balls in \( \mathcal{Y} \).

b) \( L_1(A, \mathcal{A}, \mu; \mathcal{Y}) \) will denote the set of all functions \( F(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathcal{Y})) \) for which \( \int_A |F(\lambda)| \mu(d\lambda) < + \infty \).

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1 For arbitrary \( \sigma \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \), \( \mathcal{A} \otimes \mathcal{B} \) denotes the \( \sigma \)-algebra generated by their product. For an arbitrary topological space \( Y \), \( \mathcal{B}(Y) \) denotes the \( \sigma \)-algebra of all Borel sets in \( Y \).
2 For \( \sigma \)-algebras \( \mathcal{B}_1, \mathcal{B}_2 \) of subsets of \( \mathcal{F}_1, \mathcal{F}_2 \) respectively, \( M(\mathcal{B}_1, \mathcal{B}_2) \) denotes the set of all measurable functions on \( (\mathcal{F}_1, \mathcal{B}_1) \) to \( (\mathcal{F}_2, \mathcal{B}_2) \).
c) $L^{\text{sep}}_1(\lambda,\mathcal{A},\mu; Y)$ will denote the set of all functions in $L_1(\lambda,\mathcal{A},\mu; Y)$ which have separable range. A function $F(\cdot)$ on $\Lambda$ to $Y$ is said to be Bochner integrable w.r.t. $\mu$ iff $F(\cdot) \in L^{\text{sep}}_1(\lambda,\mathcal{A},\mu; Y)$.

Obviously we have

\[(1.3) \quad F(\cdot) \in M(\mathcal{A}, \mathcal{B}, (Y)) \Leftrightarrow \forall y \in Y, |F(\cdot) - y| \in M(\mathcal{A}, \mathcal{B}(\mathbb{R})).\]

Also, cf. [9, p. 97, Thm. 13],

\[(1.4) \quad \{ \text{when the range of } F(\cdot) \text{ is separable} \}
\quad F(\cdot) \in M(\mathcal{A}, \mathcal{B}, (Y)) \Leftrightarrow F(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathbb{R})).\]

In the following theorem we affirmatively answer Question I for $\mathcal{M}_p(\mathbb{R})$, $1 \leq p < \infty$.

1.5 THEOREM I. Let $(\Lambda, \mathcal{A}, \mu)$ be a $\sigma$-finite nonnegative measure space, and $F(\cdot) \in L^{\text{sep}}_1(\lambda,\mathcal{A},\mu; \mathcal{M}_p(\mathbb{R}))$, $1 \leq p < \infty$. Then there exists a function $G(\cdot)$ on $\Lambda$ to $\mathcal{M}_p(\mathbb{R})$, cf. Def. 2.1c, satisfying the following conditions:

\[a) \quad G(\cdot)(\cdot) \in L^p(\mathcal{A}, \mathcal{B}(\mathbb{R})), \mathcal{M}(\mathcal{F}), \quad \forall t \in \mathbb{R}, G(\cdot)(t) \in L_1(\lambda, \mathcal{A}, \mu; \mathbb{R}).\]

\[b) \quad \int_{\Lambda} G(\cdot)(\cdot) \mu(d\lambda) \in \int_{\Lambda} F(\cdot) \mu(d\lambda)\]

where the last integral is Bochner in the Banach space $\mathcal{M}_p(\mathbb{R})$.

c) \[\sup_{T > 0} \left( \frac{1}{2T} \int_{-T}^{T} |G(\cdot)(t)|^p dt \right)^{1/p} \in L_1(\lambda, \mathcal{A}, \mu; \mathbb{R}).\]

The following theorem provides an affirmative answer to Question II for $\mathcal{M}_p(\mathbb{R})$, $1 \leq p < \infty$.

1.6 THEOREM II. Let $\Lambda, \mathcal{A}, \mu, p, \tilde{F}$ be as in Theorem I. Let $G(\cdot)$ be any function on $\Lambda$ to $\mathcal{M}_p(\mathbb{R})$ such that

\[a) \quad G(\cdot)(\cdot) \in L^p(\mathcal{A}, \mathcal{B}(\mathbb{R})), \mathcal{M}(\mathcal{F}), \quad \forall t \in \mathbb{R}, G(\cdot)(t) \in L_1(\lambda, \mathcal{A}, \mu; \mathbb{R}),\]

\[b) \quad \int_{\Lambda} G(\cdot)(\cdot) \mu(d\lambda) \in \int_{\Lambda} \tilde{F}(\cdot) \mu(d\lambda)\]

and such that for some $T_0 > 0$,

\[\sup_{T > T_0} \left( \frac{1}{2T} \int_{-T}^{T} |G(\cdot)(t)|^p dt \right)^{1/p} \in L_1(\lambda, \mathcal{A}, \mu; \mathbb{R}).\]

Then

\[\int_{\Lambda} G(\cdot)(\cdot) \mu(d\lambda) \in \int_{\Lambda} \tilde{F}(\cdot) \mu(d\lambda)\]

where the last integral is Bochner in the Banach space $\mathcal{M}_p(\mathbb{R})$. 367
2. THE BANACH SPACES $\mathcal{F}_p(\mathbb{R})$ AND $\mathcal{M}_p(\mathbb{R})$

Let $1 \leq p < \infty$. For each $B \in \mathcal{B}(\mathbb{R})$, let $L_p(B)$ be the space of Borel measurable functions $h$ on $B$ to $\mathbb{F}$ for which $\|h\|_p := \left(\int_B |h(s)|^p ds\right)^{1/p} < \infty$. Let $L_p^{\text{loc}}(\mathbb{R})$ be the vector space of all Borel measurable functions $f$ on $\mathbb{R}$ to $\mathbb{F}$ such that $\chi_B(\cdot)f(\cdot) \in L_p(\mathbb{R})$ for all bounded Borel sets $B$ in $\mathbb{R}$.

### 2.1 DEFINITIONS.

Let $1 \leq p < \infty$, $B \in \mathcal{B}(\mathbb{R})$ and $f \in L_p^{\text{loc}}(\mathbb{R})$.

a) For all $h \in L_p(B)$

$$h = \{g : g \in L_p(B) \& \|g - h\|_p = 0\}$$

and

$$L_p(B) = \{h : h \in L_p(B)\}.$$

b) For each $T > 0$

$$\|f\|_{p,T} := \left(\frac{1}{2T} \int_{-T}^T |f(t)|^p dt\right)^{1/p} < \infty.$$

c) For all $f \in L_p^{\text{loc}}(\mathbb{R})$

$$\|f\|_p = \lim_{T \to \infty} \|f\|_{p,T}$$

and

$$M_p(\mathbb{R}) = \{g : g \in L_p^{\text{loc}}(\mathbb{R}) \& \|g\|_p < \infty\}.$$ 

d) For all $f \in M_p(\mathbb{R})$

$$f = \{g : g \in M_p(\mathbb{R}) \& \|g - f\|_p = 0\}$$

and

$$\mathcal{M}_p(\mathbb{R}) = \{g : g \in M_p(\mathbb{R})\}.$$

$\mathcal{M}_p(\mathbb{R})$ given the norm $\|\cdot\|_p$ is called the $p$-th Marcinkiewicz space.

### 2.2. DEFINITIONS.

Let $1 \leq p < \infty$.

a) For all $f \in L_p^{\text{loc}}(\mathbb{R})$

$$|f|_{S_p} = \sup_{T > 0} |f|_{p,T}$$

and

$$S_p(\mathbb{R}) = \{g : g \in L_p^{\text{loc}}(\mathbb{R}) \& \|g\|_{S_p} < \infty\}.$$ 

b) For all $f \in S_p(\mathbb{R})$

$$f = \{g : g \in S_p(\mathbb{R}) \& \|g - f\|_{S_p} = 0\}$$

and

$$\mathcal{S}_p(\mathbb{R}) = \{g : g \in S_p(\mathbb{R})\}.$$
Note that

\[(2.3) \quad \begin{cases} \mathcal{S}_p(\mathbb{R}) \subseteq M_p(\mathbb{R}), \\
\|f\|_{S_p} = 0 \text{ iff } f(\cdot) = 0 \text{ a.e. Leb. iff } |f|_p = 0,
\end{cases}\]

but \(|f|_p = 0\) for all \(f \in L_p(\mathbb{R})\). The last fact shows that equivalence classes \(f\) in \(M_p(\mathbb{R})\) are quite large and a careful distinction must be made between functions \(f\) in \(M_p(\mathbb{R})\) and equivalence classes \(\tilde{f}\) in \(\mathcal{S}_p(\mathbb{R})\).

The following theorem is known. The part for \(\mathcal{S}_p(\mathbb{R})\) is due to Marcinkiewicz [8] and independently Bohr and Følner [3]. The part for \(\mathcal{S}'(\mathbb{R})\) is due to K. Lau [7, Prop. 2.2(i)].

2.4 THEOREM. For \(1 \leq p < \infty\), \(\mathcal{S}_p(\mathbb{R})\) and \(\mathcal{S}'(\mathbb{R})\) are Banach spaces under norms \(\|\cdot\|_{S_p}\) and \(\|\cdot\|_p\) respectively.

The Banach spaces \(\mathcal{S}_p(\mathbb{R})\) and \(\mathcal{S}'(\mathbb{R})\) are closely related, as shown in the following result due to K. Lau, for the proof of which we refer the reader to Lau and Lee [7, Prop. 2.2(ii)].

2.5 PROPOSITION. Let \(1 \leq p < \infty\). Then

a) the correspondence \(J\) defined for all \(f \in \mathcal{S}_p(\mathbb{R})\) by \(J(f) = \tilde{f}\), is a linear contraction on \(\mathcal{S}_p(\mathbb{R})\) onto \(\mathcal{S}'(\mathbb{R})\).

b) For all \(f \in \mathcal{S}_p(\mathbb{R})\), \(\tilde{f} \subseteq J(f)\).

c) For all \(f \in \mathcal{S}'(\mathbb{R})\),

\[
\|\tilde{f}\|_p = \inf \{ \|g\|_{S_p} : g \in J^{-1}(\{\tilde{f}\}) \}.
\]

d) The quotient space \(\mathcal{S}'_p(\mathbb{R})/J^{-1}(\{0\})\) is isometrically isomorphic to \(\mathcal{S}'(\mathbb{R})\).

2.6 TRIVIALITY. Let \(f \in L^\text{loc}_p(\mathbb{R})\) and \(\phi(T) = \|f\|_{p,T}\) for all \(T > 0\). Then

a) \(\phi(\cdot)\) is a continuous function on \((0, \infty)\) to \([0, \infty)\).

b) \n
\[\|f\|_{S_p} = \sup_{T \in \mathbb{Q}_+} \|f\|_{p,T}.\]

\[c) \|f\|_p = \inf_{T_0 \in \mathbb{Q}_+} \sup_{T \geq T_0} \|f\|_{p,T}.\]

PROOF. a) Let \(T_0 > 0\). Set

\[
\psi(T) = \frac{1}{T} \int_{-T}^T |f(s)|^p ds, \quad VT > 0.
\]

Since \(\text{Leb} \{[-T,T] \Delta [-T_0,T_0]\} \to 0\) as \(T \to T_0\), \(\Delta\) denoting the symmetric difference,

\[
|\psi(T) - \psi(T_0)| \leq \int_{[-T,T] \Delta [-T_0,T_0]} |f(s)|^p ds \to 0 \text{ as } T \to T_0.
\]

\[\mathbb{Q}_+ \text{ denotes the set of positive rational numbers.}\]
Therefore $\psi(\cdot)$ is continuous at $T_0$ and hence $\phi(T) = ((1/2T)\psi(T))^{1/\rho}$ is continuous at $T_0$.

b) and c) follow from a) and Def. 2.2a) and 2.1c). □

3. PRODUCT MEASURABLE REPRESENTATIVES

Throughout this section we make the following:

3.1 ASSUMPTION. $\mathcal{A}$ is a $\sigma$-algebra of subsets of a set $\Lambda$ and $\mu$ is a $\sigma$-finite nonnegative measure on $\mathcal{A}$.

$\text{Leb}(\cdot)$ will denote Lebesgue measure on $\mathcal{M}(\mathbb{R})$, and $(\mu \times \text{Leb})(\cdot)$ will denote the product of $\mu$ and $\text{Leb}$ on the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{M}(\mathbb{R})$. The following result, cf. Halmos [5, p. 141, 147], will be used in the proofs of our main theorems.

3.2 TRIVIALITY. Let $P \in \mathcal{A} \otimes \mathcal{M}(\mathbb{R})$. For each $\lambda_0 \in \Lambda$ and $t_0 \in \mathbb{R}$ let

$$P_{\lambda_0,t_0} = \{t \in \mathbb{R} : (\lambda_0, t) \in P\},$$

$$P_{\lambda_0} = \{\lambda \in \Lambda : (\lambda, t_0) \in P\}.$$

Then a) for all $\lambda \in \Lambda$, $P_{\lambda} \in \mathcal{M}(\mathbb{R})$ and for all $t \in \mathbb{R}$, $P_{t} \in \mathcal{A}$.

b) The following are equivalent:

1) $(\mu \times \text{Leb})(P) = 0$.

2) There exists an $N \in \mathcal{A}$ such that $\mu(N) = 0$, and $\text{Leb}(P_{\lambda}) = 0$ for all $\lambda \in \Lambda \setminus N$.

3) There exists an $E \in \mathcal{M}(\mathbb{R})$ such that $\text{Leb}(E) = 0$, and $\mu(P_{t}) = 0$ for all $t \in \mathbb{R} \setminus E$.

3.3 DEFINITION. Let $X$ be a space with seminorm $\| \cdot \|_X$, such that

$$X \subseteq M(\mathcal{M}(\mathbb{R}), \mathcal{M}(\mathbb{R})),$$

let $\hat{X}$ be the normed space of all equivalence classes $f$ of functions $f \in X$, where

$$f = \{g : g \in X \& \| g - f \|_X = 0\}$$

and let $F(\cdot)$ be any function on $\Lambda$ to $\hat{X}$. We say that a function $G(\cdot)$ on $\Lambda$ to $X$ is a representative of $F(\cdot)$, iff $G(\lambda) \in \hat{F}(\lambda)$ for all $\lambda \in \Lambda$.

It follows easily from Triv. 3.2a) that

$$(3.4) \quad G(\cdot)(\cdot) \in M(\mathcal{A} \otimes \mathcal{M}(\mathbb{R}), \mathcal{M}(\mathbb{R})) \Rightarrow \forall t \in \mathbb{R}, \ G(\cdot)(t) \in M(\mathcal{A}, \mathcal{M}(\mathbb{R})).$$

By use of (3.4), it can be easily shown that for the spaces $X = L_\rho(\mathbb{R})$, $X = \mathcal{C}_p(\mathbb{R})$, and $X = \mathcal{C}_b(X)$ a function $F(\cdot) \in M(\mathcal{A}, \mathcal{M}(\mathbb{R}))$ can have representatives $G(\cdot)$ which are not product measurable.

The following useful lemma asserts that the measurability of the function $F(\cdot)(\cdot)$ on $\Lambda \times \mathbb{R}$ entails the ball measurability of the function $\hat{F}(\cdot)$ on $\Lambda$.

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3.5 LEMLMA. Let $X = L_p(\mathbb{R})$ or $S_p(\mathbb{R})$ or $M_p(\mathbb{R})$, $1 \leq p < \infty$. Then under the assumption 3.1,

$$F(\cdot) \in X^A \& F(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F})) = \hat{F}(\cdot) \in M(\mathcal{A}, \mathcal{B} (X)).$$

PROOF. Let $F(\cdot) \in X^A$ and $F(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F}))$.

**Case I.** Let $X = L_p(\mathbb{R})$. Let $g \in L_p(\mathbb{R})$ and define $G(\cdot)$ on $\Lambda$ to $X$ by $G(\cdot)(-) = g(-)$. Then obviously $G(\cdot)(-)$ on $\Lambda \times \mathbb{R}$ to $\mathbb{F}$ is in $M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F}))$ and

$$|F(\cdot)(-) - g(-)|^p \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F})).$$

Also for each $\lambda \in \Lambda$, since $F(\lambda)$ and $g$ are in $L_p(\mathbb{R})$,

$$\int_{\mathbb{R}} |F(\lambda)(t) - g(t)|^p dt \in \mathbb{R}.$$

By (1), (2), and Tonelli’s Theorem [4, p. 194, Thm. 14],

$$\int_{\mathbb{R}} |F(\cdot)(-) - g(\cdot)|^p dt \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R})).$$

Hence $|F(\cdot) - g|^p \in M(\mathcal{A}, \mathcal{B}(\mathbb{R}))$ and by (1.3) $\hat{F}(\cdot)$ is ball measurable.

Next assume that $F(\cdot)$ is $L_p^{loc}(\mathbb{R})$-valued and let $T > 0$. Then

$$\forall \lambda \in \Lambda, \chi_{[-T, T]}(-)F(\lambda)(-) \in L_p(\mathbb{R})$$

and

$$\chi_{[-T, T]}(-)F(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F})).$$

It follows from Case I (cf. (3) et seq.) that for each $g \in L_p^{loc}(\mathbb{R})$,

$$|F(\cdot) - g|_{p, T} = (2T)^{-1/p} \chi_{[-T, T]} F(\cdot) - \chi_{[-T, T]} g |_{p} \in M(\mathcal{A}, \mathcal{B}(\mathbb{R})).$$

**Case II.** Let $X = S_p(\mathbb{R})$. Then by Triv. 2.6b) and (4) for $g \in S_p(\mathbb{R})$

$$|F(\cdot) - g|_{S_p} = \sup_{T \in \mathbb{Q}_{+}} |F(\cdot) - g|_{p, T} \in M(\mathcal{A}, \mathcal{B}(\mathbb{R})).$$

Therefore by (1.3) $\hat{F}(\cdot)$ is ball measurable.

**Case III.** Let $X = M_p(\mathbb{R})$. Then by Triv. 2.6c) and (4) for each $g \in M_p(\mathbb{R})$,

$$|F(\cdot) - g|_{p} = \inf_{T \in \mathbb{Q}_{+}} \left\{ \sup_{T \in \mathbb{Q}_{+}} |F(\cdot) - g|_{p, T} \right\} \in M(\mathcal{A}, \mathcal{B}(\mathbb{R})).$$

Therefore by (1.3) $\hat{F}(\cdot)$ is ball measurable.

The next proposition is useful in applications of Theorem II.

3.6 PROPOSITION. Let $\mathcal{A}$ and $\mu$ be as in 3.1, $1 \leq p < \infty$, and $F(\cdot)$ be a function on $\Lambda$ to $M_p(\mathbb{R})$ such that

(i) $F(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F}))$, 371
(ii) the range of $F(\cdot)$ is a separable subset of $\mathcal{M}_p(\mathbb{R})$, and

(iii) $\sup_{T \geq T_0} |F(\cdot)|_{p, T} \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R})$ for some $T_0 > 0$.

Then

a) $\hat{F}(\cdot) \in L^1_{\text{sep}}(\Lambda, \mathcal{A}, \mu; \mathcal{M}_p(\mathbb{R}))$.

b) $F(\cdot)(t) \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R})$ for almost all (Leb.) $t \in \mathbb{R}$.

c) Letting

$$B = \{ t : t \in \mathbb{R} \& F(\cdot)(t) \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R}) \}$$

and

$$G(\lambda)(t) = \chi_B(t)F(\lambda)(t), \quad \lambda \in \Lambda, t \in \mathbb{R},$$

the function $G(\cdot)$ on $\Lambda$ to $M_p(\mathbb{R})$ satisfies 1.6(i), (ii).

**Proof.** a) By Lemma 3.5, $\hat{F}(\cdot)$ is a ball measurable $\mathcal{M}_p(\mathbb{R})$-valued function. By hypothesis the range of $\hat{F}(\cdot)$ is separable. Obviously from the hypothesis $|F(\cdot)|_{p, T} \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R})$. Therefore $\hat{F}(\cdot) \in L^1_{\text{sep}}(\Lambda, \mathcal{A}, \mu; \mathcal{M}_p(\mathbb{R}))$.

b) Let $n \geq T_0$. Then by (i) and Tonelli’s Theorem, by Hölder’s Inequality, and by (ii), respectively,

$$\frac{1}{2^n} \int_{-n}^{n} \left\{ \int_{\Lambda} |F(\lambda)(t)| \mu(d\lambda) \right\} dt = \int_{\Lambda} \left( \frac{1}{2^n} \int_{-n}^{n} |F(\lambda)(t)| \cdot 1 \right) \mu(d\lambda)$$

$$\leq \int_{\Lambda} \left( \frac{1}{2^n} \int_{-n}^{n} |F(\lambda)(t)|^p dt \right)^{1/p} \left( \frac{1}{2^n} \int_{-n}^{n} dt \right)^{1/p} \mu(d\lambda)$$

$$\leq \int_{\Lambda} \{ \sup_{T \geq T_0} |F(\lambda)(t)|_{p, T} \mu(d\lambda) < + \infty.$$

Hence $F(\cdot)(t) \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R})$ for almost all $t \in [-n, n]$, i.e. the Borel set

$$E_n = \{ t : -n \leq t \leq n \& \int_{\Lambda} |F(\lambda)(t)| \mu(d\lambda) = \infty \}$$

is Lebesgue negligible. Let $E = \bigcup_{-\infty}^{\infty} E_n$. Then $E$ is Lebesgue negligible and

$$\forall t \in \mathbb{R} \setminus E, \quad F(\cdot)(t) \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R}).$$

c) Let $G(\cdot)(-) = \chi_B(\cdot)F(\cdot)(-)$. Then

$$\forall \lambda \in \Lambda, \quad G(\lambda)(-) = F(\lambda)(-) \text{ a.e. Leb.},$$

$$G(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})),$$

$$\forall t \in \mathbb{R}, \quad G(\cdot)(t) \in L_1(\Lambda, \mathcal{A}, \mu; \mathbb{R}).$$

Thus the function $G(\cdot)$ on $\Lambda$ to $M_p(\mathbb{R})$ satisfies 1.6(i), and by (iii) also satisfies 1.6(ii). □

4. QUESTIONS I AND II IN $\mathcal{S}_p(\mathbb{R})$

Questions I and II will now be answered affirmatively for functions

$$\tilde{F}(\cdot) \in L^1_{\text{sep}}(\Lambda, \mathcal{A}, \mu; \mathcal{S}_p(\mathbb{R})).$$
Our proof depends on transforming this problem to one in \( L_p(B) \), cf. Def. 2.1a), where \( B \) is a bounded Borel set in \( \mathbb{R} \). The proof requires the following triviality.

### 4.1 TRIVIALITY

Let \( B \) be a bounded Borel subset of \( \mathbb{R} \) and \( 1 \leq p < \infty \). Then

a) for all \( f \in \mathcal{S}_p(\mathbb{R}) \), \( R \text{str}_p f \in L_p(B) \);

b) defining \( R_B \) on \( \mathcal{S}_p(\mathbb{R}) \) to \( L_p(B) \) by

\[
R_B(f) = \{ \text{str}_p f(\cdot) \} \cdot \forall f \in \mathcal{S}_p(\mathbb{R}),
\]

we have that \( R_B \) is a continuous linear operator on \( \mathcal{S}_p(\mathbb{R}) \) to \( L_p(B) \) with operator norm

\[
| R_B | \leq [2 \sup \{ |t| : t \in B \}]^{1/p}.
\]

**Proof.**

a) follows directly from Def. 2.2a).

b) Obviously \( R_B \) is linear. By (2.3) \( R_B \) is single valued. Since \( B \) is bounded, we have \( B \subseteq [-T_0, T_0] \), for some \( T_0 > 0 \). Thus by (2.3) for all \( f \in \mathcal{S}_p(\mathbb{R}) \),

\[
| R_B(f) |_p = (\int_{B} |f(t)|^p dt)^{1/p} \leq (2T_0)^{1/p} \int_{[-T_0, T_0]} |f|_p \leq (2T_0)^{1/p} \int_{\mathcal{S}_p}.
\]

where \( f \in \mathcal{F} \). Hence \( | R_B | \leq (2T_0)^{1/p} \). \( \square \)

The following theorem is the analogue for \( \mathcal{S}_p(\mathbb{R}) \) of the combined Theorems I and II, and will be needed for their proofs.

### 4.2 THEOREM

Let \( (A, \mathcal{A}, \mu) \) be a \( \sigma \)-finite measure space and

\[
\mathcal{F}(\cdot) \in \mathcal{L}^\mathcal{D}(A, \mathcal{A}; \mathcal{S}_p(\mathbb{R})), \ 1 \leq p < \infty.
\]

Then

a) there exists a function \( G(\cdot) \) on \( A \) to \( \mathcal{S}_p(\mathbb{R}) \) such that

(i) \( G((\cdot))(-) \in M(\mathcal{A} \otimes \mathcal{M}(\mathbb{R}), \mathcal{M}(\mathbb{F})) \),

(ii) \( G(\lambda) \in \mathcal{F}(\lambda) \) for \( \mu \) almost all \( \lambda \in A \),

(iii) \( \forall t \in \mathbb{R}, \ G(\cdot)(t) \in L_1(A, \mathcal{A}, \mu; \mathbb{F}) \),

and

(iv) \( \int_A G(\cdot)(\cdot)(\cdot) \mu(d\lambda) \in \int_A \mathcal{F}(\lambda) \mu(d\lambda) \)

where the last integral is Bochner in \( \mathcal{S}_p(\mathbb{R}) \).

b) If \( G_0(\cdot) \) is any function on \( A \) to \( \mathcal{S}_p(\mathbb{R}) \) such that \( a)(i)-(iii) \) hold for \( G_0 \) then \( G_0((\cdot)(\cdot)(\cdot)) = G((\cdot)(\cdot)(\cdot)) \) a.e. \( \mu \times \text{Leb} \) on \( A \times \mathbb{R} \) and

\[
\int_A G_0(\lambda)(\cdot)(\cdot)(\cdot) \mu(d\lambda) \in \int_A \mathcal{F}(\lambda) \mu(d\lambda).
\]

**Proof.**

a) Let \( n \in \mathbb{N} \) and \( B_n = (n, n + 1] \). By Triv. 4.1,

\[
(R_B \circ \mathcal{F})(\cdot) \in \mathcal{L}^\mathcal{D}(A, \mathcal{A}, \mu; L_p(B_n)),
\]

\( \text{If } f(\cdot) \text{ is a function on a set } A \text{ and } B \subseteq A, \text{ then } R \text{str}_B f(\cdot) \text{ denotes the restriction of } f(\cdot) \text{ to } B. \)
and by [4, p. 113, Thm. 19(c)],

\[(1) \quad R_{B_n}\{\int_A \tilde{F}(\lambda)\mu(d\lambda)\} = \int_A R_{B_n}\{\tilde{F}(\lambda)\}\mu(d\lambda)\]

where the first integral is in \(\mathcal{L}_p(\mathbb{R})\) and the second in \(L_p(B_n)\). By [4, p. 198, Thm. 17] for each \(n \in \mathbb{N}\), there exists a function \(h_n\) and a set \(N_n\) such that

\[(2) \quad h_n(\cdot, t) \in M(\mathcal{A} \otimes \mathcal{B}(B_n), \mathcal{B}(\mathbb{F}))\],

\[(3) \quad N_n \in \mathcal{A} \text{ is } \mu \text{ negligible},\]

\[(4) \quad \forall \lambda \in A \setminus N_n, \quad h_n(\lambda, -) \in R_{B_n}\{\tilde{F}(\lambda)\},\]

\[(5) \quad \forall t \in B_n, \quad h_n(\cdot, t) \in L_1(A, \mathcal{A}, \mu; \mathcal{F}),\]

and

\[(6) \quad \int_A h_n(\lambda, -)\mu(d\lambda) \in \int_A R_{B_n}\{\tilde{F}(\lambda)\}\mu(d\lambda).\]

Now let \(F(\cdot)\) be any representative of \(\tilde{F}(\cdot)\). Then since for each \(\lambda \in A \setminus N_n,\)
\[R_{B_n}F(\lambda)(-) \in R_{B_n}\{\tilde{F}(\lambda)\},\]

it follows by (4) that there exists a Lebesgue negligible Borel subset \(E_n\) of \(B_n\) such that

\[(7) \quad \forall t \in B_n \setminus E_n, \quad h_n(\lambda, t) = F(\lambda)(t).\]

Let \(f(\cdot)\) be a representative of \(\int_A \tilde{F}(\lambda)\mu(d\lambda)\). Then
\[R_{B_n}f(\cdot) \in R_{B_n}\{\int_A \tilde{F}(\lambda)\mu(d\lambda)\} = \int_A R_{B_n}\{\tilde{F}(\lambda)\}\mu(d\lambda),\]

and comparison with (6) shows that there exists a Lebesgue negligible Borel subset \(E_n\) of \(B_n\) such that

\[(8) \quad \forall t \in B_n \setminus E_n, \quad \int_A h_n(\lambda, t)\mu(d\lambda) = f(t).\]

Define

\[(9) \quad G(\lambda)(t) = \sum_{-\infty}^{\infty} \chi_{B_n}(t)h_n(\lambda, t), \quad \lambda \in A, \quad t \in \mathbb{R}.\]

For each \(n, \mathcal{B}(B_n) \subset \mathcal{B}(\mathbb{R})\); therefore

\[\mathcal{A} \otimes \mathcal{B}(B_n) = \sigma\text{-ring} (\mathcal{A} \times \mathcal{B}(B_n)) \subset \sigma\text{-ring} (\mathcal{A} \times \mathcal{B}(\mathbb{R})) = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}).\]

Now for each \(B \in \mathcal{B}(\mathbb{F}),\)

\[\{(\lambda, t) : G(\lambda)(t) \in B\} = \bigcup_{-\infty}^{\infty} h_n^{-1}(B) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R});\]

hence by (2) \(G(\cdot)(-) \in M(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F})).\) Let

\[
N = \bigcup_{-\infty}^{\infty} N_n, \quad E = \bigcup_{-\infty}^{\infty} E_n, \quad \text{and} \quad \forall \lambda \in A \setminus N, \quad E_{n, \lambda} = \bigcup_{-\infty}^{\infty} E_{n, \lambda}.\]

5 If \(\mathcal{F} \subset 2^\mathcal{F}\) then \(\sigma\text{-ring} (\mathcal{F})\) denotes the smallest \(\sigma\text{-ring} \) containing \(\mathcal{F}.)
Then $N$ is $\mu$ negligible, and $E$ and $E^\lambda$ are Lebesgue negligible. By (9) and (7) for each $\lambda \in A \setminus N$, and for all $t \in \mathbb{R} \setminus E^\lambda$,

$$G(\lambda)(t) = \sum_{n=-\infty}^{\infty} \chi_{B_n}(t) F(\lambda)(t) = F(\lambda)(t),$$

and thus for all $\lambda \in A \setminus N$, $G(\lambda) \in \bar{F}(\lambda)$.

Now let $t_0 \in \mathbb{R}$. Then $t_0 \in B_{n_0}$ for some $n_0$, and therefore by (9) and (5)

$$G(\cdot)(t_0) = h_{n_0}(\cdot, t_0) \in I_1(A, \mathcal{S}, \mu, \mathcal{F}).$$

Finally let $t_0 \in \mathbb{R} \setminus E$. Then $t_0 \in B_{n_0} \setminus E_{n_0}$ for some $n_0$ and by (8)

$$\int_A G(\lambda)(t_0)\mu(d\lambda) = \int_A h_{n_0}(\lambda, t_0)\mu(d\lambda) = f(t_0).$$

Since $E$ is Lebesgue negligible, it follows that

$$\int_A G(\lambda)(-\lambda)\mu(d\lambda) \in \mathcal{F}(\lambda)\mu(d\lambda).$$

b) Let $G_0(\cdot)$ be a function on $A$ to $S_p(\mathbb{R})$ such that a) (i)–(iii) (with $G_0$ replacing $G$) hold. Let $N_0$ be a $\mu$ negligible set such that $G_0(\lambda) \in \bar{F}(\lambda)$ for all $\lambda \in A \setminus N_0$. Then by (a) and (2.3) for each $\lambda \in A \setminus (N \cup N_0)$, $G_0(\lambda)(-\lambda) = G(\lambda)(-\lambda)$ a.e. $\mathbb{R}$. Hence by Triv. 3.2b)

$$(\mu \times \text{Leb}) \{(\lambda, t) : G_0(\lambda)(t) \neq G(\lambda)(t)\} = 0,$$

and there exists a Lebesgue negligible Borel set $E_0$ such that for each $t \in \mathbb{R} \setminus E_0$, $G_0(\cdot)(t) = G(\cdot)(t)$ a.e. $\mu$. Hence for each $t \in \mathbb{R} \setminus E_0$,

$$\int_A G_0(\lambda)(t)\mu(d\lambda) = \int_A G(\lambda)(t)\mu(d\lambda).$$

Therefore by (2.3) and (iv) of part a),

$$\int_A G_0(\lambda)(-\lambda)\mu(d\lambda) \in \mathcal{F}(\lambda)\mu(d\lambda).$$

5. PROOFS FOR THEOREMS I AND II

We proceed to deduce Theorems I and II from Theorem 4.2. Starting with the linear contraction $J$ on $\mathcal{H}_p(\mathbb{R})$ onto $\mathcal{M}_p(\mathbb{R})$ defined in 2.5, we first appeal to the following theorem on continuous selections due to E. Michael [10, p. 375, Prop. 7.2].

5.1 THEOREM. Let $X$ and $Y$ be Banach spaces over $\mathbb{F}$ and $J$ be a continuous linear operator on $X$ onto $Y$. Then for each $\alpha > 1$, there exists a continuous (in general non-linear) function $S_\alpha$ on $Y$ into $X$ such that for all $y \in Y$,

a) $S_\alpha(y) \in J^{-1}(\{y\})$

b) $|S_\alpha(y)|_X \leq \alpha \inf \{ |x|_X : x \in J^{-1}(\{y\}) \}$

c) $S_\alpha(ay) = aS_\alpha(y) \forall a \in \mathbb{F}$. 375
From Proposition 2.5 and Theorem 5.1 we at once get the following lemma:

5.2 **Lemma.** Let $1 \leq p < \infty$, $J$ be as in 2.5, and $\alpha > 1$. Then there exists a continuous function $S_\alpha$ on $\mathcal{M}_p(\mathbb{R})$ into $\mathcal{S}_p(\mathbb{R})$ such that for all $f \in \mathcal{M}_p(\mathbb{R})$,

- $S_\alpha(f) \lesssim f$ and $J(S_\alpha(f)) = f$
- $|S_\alpha(f)|_{S_p} \leq \alpha |f|_p$
- $S_\alpha(\alpha f) = a S_\alpha(f)$ for all $a \in \mathbb{R}$.

With the aid of the $S_\alpha$ (with $\alpha = 2$) just obtained, we shall now deduce Theorem I from Theorem 4.2.

5.3 **Proof of Theorem I.** By Lemma 5.2 there exists a continuous function $S_2$ on $\mathcal{M}_p(\mathbb{R})$ to $\mathcal{S}_p(\mathbb{R})$ such that $S_2(f) \lesssim f$ and $|S_2(f)|_{S_p} \leq 2|f|_p$ for all $f \in \mathcal{M}_p(\mathbb{R})$. Since $\hat{F}(\cdot) \in L^1(\mathcal{M}_p(\mathbb{R}))$, it follows that the range of $\hat{F}(\cdot)$ is separable and $\hat{F}(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathcal{S}_p(\mathbb{R})))$, cf. (1.4). Thus, $S_2$ being continuous, the range of $(S_2 \circ \hat{F})(\cdot)$ is separable in $\mathcal{S}_p(\mathbb{R})$ and $(S_2 \circ \hat{F})(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathcal{S}_p(\mathbb{R})))$. Hence by (1.4) $(S_2 \circ \hat{F})(\cdot) \in M(\mathcal{A}, \mathcal{B}(\mathcal{S}_p(\mathbb{R})))$. Also $|(S_2 \circ \hat{F})(\cdot)|_{S_p} \leq 2 \| \hat{F}(\cdot) \|_p$ so that

\begin{equation}
(S_2 \circ \hat{F})(\cdot) \in L^1(\mathcal{M}_p(\mathbb{R}), \mathcal{A}(\mathbb{R})).
\end{equation}

It follows from Theorem 4.2 that there exists a function $G(\cdot)$ on $\mathcal{A}$ to $\mathcal{S}_p(\mathbb{R})$ such that

\begin{align}
G(\lambda) &\in (S_2 \circ \hat{F})(\lambda) \subseteq \hat{F}(\lambda) \text{ for } \mu \text{ almost all } \lambda \in \mathcal{A}, \text{ cf. } 5.2a),
G(\cdot)(-) \in M(\mathcal{A}, \mathcal{B}(\mathcal{S}_p(\mathbb{R}))),
\forall t \in \mathbb{R}, G(\cdot)(t) \in L^1(\mathcal{A}, \mathcal{A}; \mathbb{F}),
\end{align}

and

\begin{equation}
\int_{\mathcal{A}} G(\lambda)(-) \mu(\lambda) = \int_{\mathcal{A}} (S_2 \circ \hat{F})(\lambda) \mu\mu(\lambda).
\end{equation}

Since, cf. (2.3), for all $\lambda \in \mathcal{A}$, $G(\lambda) \in \mathcal{M}_p(\mathbb{R})$, by (2) we have established the conditions 1.5a).

Let $J$ be defined as in Prop. 2.5. Then by 2.5b),

\begin{equation}
\int_{\mathcal{A}} (S_2 \circ \hat{F})(\lambda) \mu(\lambda) \lesssim J\{\int_{\mathcal{A}} (S_2 \circ \hat{F})(\lambda) \mu(\lambda)\}.
\end{equation}

By [4, p. 113, Thm. 19(c)] and Lemma 5.2a)

\begin{equation}
J\{\int_{\mathcal{A}} (S_2 \circ \hat{F})(\lambda) \mu(\lambda)\} = \int_{\mathcal{A}} (J \circ S_2 \circ \hat{F})(\lambda) \mu(\lambda) = \int_{\mathcal{A}} \hat{F}(\lambda) \mu(\lambda).
\end{equation}

Combining (3), (4), and (5) we have that

\begin{align}
\int_{\mathcal{A}} G(\lambda)(-) \mu(\lambda) = \int_{\mathcal{A}} \hat{F}(\lambda) \mu(\lambda),
\end{align}

i.e. we have 1.5b).
Finally, it follows from (2), Def. 2.2a), and (1) that
\[ \sup_{T > 0} \left( \frac{1}{2T} \int_{-T}^{T} |G(\cdot)(t)|^p dt \right)^{1/p} = |(S_2 \circ \hat{F})(\cdot)|_{S_p} \in L_1(A, \mathcal{A}, \mu; \mathbb{R}), \]
i.e. 1.5c) holds. \( \square \)

The following result is needed for the proof of Theorem II. Its proof consists of an easy application of Fatou’s Lemma to an appropriate sequence \((T_n)\) tending to \(+ \infty\).

5.4 TRIVIALITY. Let \( \{\psi_T : T > 0\} \subset M(\mathcal{A}, \mathcal{M}(\mathbb{R})) \) be such that \( 0 \leq \psi_T(\cdot) \leq \psi(\cdot), \) a.e. \( \mu \), where \( \psi(\cdot) \in L_1(A, \mathcal{A}, \mu; \mathbb{R}) \) and
\[ \lim_{T \to \infty} \psi_T(\cdot) \in M(\mathcal{A}, \mathcal{M}(\mathbb{R})). \]

Then
\[ \lim_{T \to \infty} \int_A \psi_T(\lambda) \mu(d\lambda) \leq \int_A \left\{ \lim_{T \to \infty} \psi_T(\lambda) \right\} \mu(d\lambda). \]

5.5 PROOF OF THEOREM II. Since \( \hat{F}(\cdot) \in L_1^{sp}(A, \mathcal{A}, \mu; \mathcal{M}(\mathbb{R})) \), by Theorem I there exists an \( H(\cdot) \) on \( A \) to \( M_p(\mathbb{R}) \) satisfying the following conditions:
\[ \begin{cases} 
H(\lambda) \in \hat{F}(\lambda) \text{ for } \mu \text{ a.a. } \lambda \in A, \\
H(\cdot)(-) \in M(\mathcal{A}, \mathcal{M}(\mathbb{R}), \mathcal{M}(\mathbb{R})), \\
\forall t \in \mathbb{R}, \ H(\cdot)(t) \in L_1(A, \mathcal{A}, \mu; \mathbb{R}), \\
(1) \\
(2) \end{cases} \]

and
\[ (3) \quad \hat{h} = \int_A \hat{F}(\lambda) \mu(d\lambda) \text{ where } h(t) = \int_A H(\lambda)(t) \mu(d\lambda). \]

By hypothesis, \( G(\cdot) \) is a function on \( A \) to \( M_p(\mathbb{R}) \) satisfying condition (1) (with \( G \) replacing \( H \)) and
\[ (4) \quad \sup_{T \geq T_0} |G(\cdot)|_{p,T} \in L_1(A, \mathcal{A}, \mu; \mathbb{R}) \text{ for some } T_0 > 0. \]

Obviously \( \|H(\cdot) - G(\cdot)\|_p = 0, \) a.e. \( \mu. \) It follows from (2) and (4) that for all \( T \geq T_0 \)
\[ (5) \quad |H(\cdot) - G(\cdot)|_{p,T} \leq \sup_{T > 0} |H(\cdot)|_{p,T} + \sup_{T \geq T_0} |G(\cdot)|_{p,T} \in L_1(A, \mathcal{A}, \mu; \mathbb{R}). \]

Define
\[ g(t) = \int_A G(\lambda)(t) \mu(d\lambda), \quad t \in \mathbb{R}. \]

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Now by the generalized Minkowski inequality, cf. [4, p. 530, Ex. 11.13], we have for each $T > 0$ that

$$\left| h - g \right|_{p, T} \leq \left\{ \frac{1}{2T} \int_{-T}^{T} \left[ H(\lambda)(t) - G(\lambda)(t) \right]^p dt \right\}^{1/p}$$

(6)

$$\leq \int_{A} \left\{ \frac{1}{2T} \int_{-T}^{T} \left| H(\lambda)(t) - G(\lambda)(t) \right|^p dt \right\}^{1/p} \mu(d\lambda)$$

$$\leq \int_{A} \left| H(\lambda) - G(\lambda) \right|_{p, T} \mu(d\lambda).$$

Also by Lemma 3.5, $H - G \in M(\mathcal{A}, \mathcal{B}(\mathcal{M}(\mathbb{R})))$, and so by (1.3)

$$\lim_{T \to \infty} \left| H(\cdot) - G(\cdot) \right| \left( p, T \right)_{\mathcal{M}(\mathcal{A})} \mathcal{H}(\mathbb{R}) = 0.$$

Thus by (6), (5), and Triviality 5.4,

$$\left| h - g \right| \leq \lim_{T \to \infty} \int_{A} \left| H(\lambda) - G(\lambda) \right|_{p, T} \mu(d\lambda)$$

$$\leq \int_{A} \left\{ \lim_{T \to \infty} \left| H(\lambda) - G(\lambda) \right|_{p, T} \right\} \mu(d\lambda) = 0.$$

Therefore by (3)

$$g = h = \int_{A} \hat{f}(\lambda) \mu(d\lambda). \quad \square$$

6. SOME APPLICATIONS

We shall briefly indicate some uses of Theorem II. Consider the group \{ $V_s : s \in \mathbb{R}$ \} defined by

$$V_s(f) = \{ f(\cdot + s) \}^{\wedge}, f \in M_{b}(\mathbb{R}).$$

To show that its infinitesimal generator is differentiation, we must evaluate integrals such as

$$\int_{a}^{b} V_s(\hat{g}) ds \text{ and } \int_{0}^{\infty} V_s(\hat{h}) e^{-s} ds,$$

for which Theorem II is useful.

For the next application we first note that Theorem II holds for real and complex measures. Now in [1, p. 19] Bertrandias defined the convolution of an $f$ in $\mathcal{M}_{b}(\mathbb{R})$ with a (bounded) complex measure $\nu$ on $\mathcal{B}(\mathbb{R})$ by

$$f \ast \nu = \lim_{A \to \infty, B \to -\infty} \left\{ \int_{-A}^{B} f(\cdot - s) \nu(ds) \right\}^{\wedge}$$

by first showing that the limit in $\mathcal{M}_{b}(\mathbb{R})$ exists. Moreover, he showed that when $\nu(\cdot)(f)$ is continuous on $\mathbb{R}$,

$$f \ast \nu = \int_{\mathbb{R}} V_{-A}(f) \nu(ds),$$

where the integral is Bochner in $\mathcal{M}_{b}(\mathbb{R})$, cf. [1, p. 21]. This result emerges as a corollary of our Prop. 3.6, and Theorem II extended to complex measures.
REFERENCES