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Note

Fulkerson's Conjecture and Circuit Covers

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It was conjectured by Fulkerson that the edge-set of any bridgeless graph can be covered by six cycles (union of circuits) such that each edge is in exactly four cycles. We prove that if Fulkerson's conjecture is true, then the edge-set of every bridgeless graph G can be covered by three cycles whose total length is at most $\frac{22}{15}|E(G)|$. We also prove that there are infinitely many bridgeless graphs G whose edge-set cannot be covered by three cycles of total length less than $\frac{22}{15}|E(G)|$

1. INTRODUCTION

Let G be a graph. A *circuit cover* of G is a collection of circuits of G such that each edge of G is in at least one of the circuits. The *length* of a circuit cover is the sum of the lengths of the circuits in the cover. We define a *cycle* to be the union of disjoint circuits. It is clear that a circuit cover can be represented by a collection of cycles. We call a circuit cover a *k-cycle cover* if it can be represented by k cycles. (The empty set $\emptyset$ is regarded as a cycle of any graph.) In other words, a k-cycle cover of G is a collection of k cycles such that each edge of G is in at least one of the cycles.

The shortest circuit cover problem (SCCP) is to find a circuit cover of shortest length. In this paper, we investigate connections between the SCCP and Fulkerson's conjecture. It was conjectured by Fulkerson [4] (or see Seymour [9]) that every bridgeless cubic graph has six perfect matchings such that each edge is in exactly two of the matchings. This conjecture can be reformulated (see Jaeger [6]) as

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Fulkerson's Conjecture. Every bridgeless graph has a 6-cycle cover such that each edge is in exactly four cycles.

Bermond, Jackson, and Jaeger [1] have proved that every bridgeless graph G has a 3-cycle cover of length at most $\frac{5}{3}|E(G)|$. We shall prove that

THEOREM 1.1. *If Fulkerson's conjecture is true, then every bridgeless graph G has a 3-cycle cover of length at most $\frac{22}{15}|E(G)|$.*

THEOREM 1.2. *There are infinitely many bridgeless graphs G that have no 3-cycle cover of length less than $\frac{22}{15}|E(G)|$.*

2. 3-CYCLE COVERS

The *symmetric difference* of two sets A and B is denoted by $A \oplus B = (A \cup B) \setminus (A \cap B)$. Let $\mathscr{C}$ be a cycle cover of a graph G; $l(\mathscr{C})$ denotes the length of $\mathscr{C}$ and $S(\mathscr{C})$ is the set of edges in $\mathscr{C}$ which are contained in precisely one cycle of $\mathscr{C}$.

LEMMA 2.1. *Let $\mathscr{C}$ be a 3-cycle cover of a graph G; then G has a 3-cycle cover of length at most $2|E(G)| + \frac{2}{3}|S(\mathscr{C})| - \frac{1}{3}l(\mathscr{C})$.*

Proof. Let $\mathscr{C} = \{C_1, C_2, C_3\}$ and set $S_i = S(\mathscr{C}) \cap C_i$, $1 \leqslant i \leqslant 3$. We have that

$$l(\mathscr{C}) = |C_1| + |C_2| + |C_3| \qquad \text{and} \qquad |S(\mathscr{C})| = |S_1| + |S_2| + |S_3|. \quad (2.1)$$

Let $\mathscr{C}' = \{C_2 \oplus C_1, C_3 \oplus C_1, C_2 \oplus C_3 \oplus C_1\}$. Then $\mathscr{C}'$ covers each element in S_1 exactly three times, each element in $C_1 \setminus S_1$ exactly once, and each element of the rest exactly twice. It follows that the length of $\mathscr{C}'$ is

$$l(\mathscr{C}') = 3|S_1| + |C_1 \setminus S_1| + 2|E(G) \setminus C_1| = 2|E(G)| + 2|S_1| - |C_1|.$$

Similarly, there are 3-cycle covers $\mathscr{C}''$, $\mathscr{C}'''$ such that

$$l(\mathscr{C}'') = 2|E(G)| + 2|S_2| - |C_2|, \qquad l(\mathscr{C}''') = 2|E(G)| + 2|S_3| - |C_3|.$$

Using these equalities and taking (2.1) into account, we obtain that

$$l(\mathscr{C}') + l(\mathscr{C}'') + l(\mathscr{C}''') = 6|E(G)| + 2|S(\mathscr{C})| - l(\mathscr{C}).$$

Hence, one of the three 3-cycle covers $\mathscr{C}'$, $\mathscr{C}''$, $\mathscr{C}'''$ must be of length at most $2|E(G)| + \frac{2}{3}|S(\mathscr{C})| - \frac{1}{3}l(\mathscr{C})$, as required by the lemma. ∎

The central part of the proof of Jaeger's 8-flow Theorem [5] is that every bridgeless graph has a 3-cycle cover. A traditional approach to the shortest circuit cover problem is to find directly a cycle cover of length as short as possible. As an alternative, we try to find a 3-cycle cover of length as long as possible and then relate the length of the long 3-cycle cover to upper bounds of short cycle covers. More precisely, we shall prove that

THEOREM 2.2. *If a graph G has a 3-cycle cover of length at least m, then G has a 3-cycle cover of length at most $\frac{3}{8}|E(G)| - \frac{5}{9}m$.*

Proof. Choose a 3-cycle cover $\mathscr{C}$ of G such that $l(\mathscr{C})$ is maximum; then $l(\mathscr{C}) \geqslant m$. Let l denote the minimum length of a 3-cycle cover of G. By Lemma 2.1,

$$l \leqslant 2\,|E(G)| + \tfrac{2}{3}\,|S(\mathscr{C})| - \tfrac{1}{3} l(\mathscr{C}). \tag{2.2}$$

Let T be the set of elements which are in all of the three cycles of $\mathscr{C}$; then

$$l(\mathscr{C}) = 2\,|E(G)| + |T| - |S(\mathscr{C})|. \tag{2.3}$$

Set $\mathscr{C} = \{C_1, C_2, C_3\}$. Since $S(\mathscr{C}) \cup T = C_1 \oplus C_2 \oplus C_3$, we see that $\{C_1, C_2, C_3, S(\mathscr{C}) \cup T\}$ is a 4-cycle cover of G which covers each element of T exactly four times and each element of $E(G)\backslash T$ exactly twice. It follows that any three cycles of $\{C_1, C_2, C_3, S(\mathscr{C}) \cup T\}$ form a 3-cycle cover of G, and by the maximality of $l(\mathscr{C})$, $|S(\mathscr{C}) \cup T| \leqslant |C_i|$, $1 \leqslant i \leqslant 3$. Thus, $|S(\mathscr{C})| + |T| \leqslant \frac{1}{3}(|C_1| + |C_2| + |C_3|) = \frac{1}{3} l(\mathscr{C})$. Combining this with (2.3) yields that $S(\mathscr{C}) \leqslant |E(G)| - \frac{1}{3} l(\mathscr{C})$, which together with (2.2) gives that $l \leqslant \frac{8}{3}|E(G)| - \frac{5}{9} l(\mathscr{C}) \leqslant \frac{8}{3}|E(G)| - \frac{5}{9}m$, as required. ∎

3. PROOF OF MAIN THEOREMS

Proof of Theorem 1.1. If Fulkerson's conjecture is true, then there is a 6-cycle cover F such that each edge of G is contained in exactly four cycles of F. Let $\mathscr{C}$ be any subset of three cycles of F. (Note that F is formed by six cycles.) We see that each edge of G must appear in at least one cycle of $\mathscr{C}$; that is, $\mathscr{C}$ is a 3-cycle cover of G. Let l denote the minimum length of a 3-cycle cover of G. By Lemma 2.1,

$$l \leqslant 2\,|E(G)| + \tfrac{2}{3}\,|S(\mathscr{C})| - \tfrac{1}{3} l(\mathscr{C}).$$

There are totally $\binom{6}{3} = 20$ choices for $\mathscr{C}$. List these twenty 3-cycle covers as $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{20}$. Then we have that

$$l \leqslant 2\,|E(G)| + \tfrac{2}{3}|S(\mathscr{C}_i)| - \tfrac{1}{3} l(\mathscr{C}_i), \qquad 1 \leqslant i \leqslant 20.$$

Summing over all i, $1 \leqslant i \leqslant 20$, yields that

$$20l \leqslant 40\,|E(G)| + \tfrac{2}{3} \sum_{i=1}^{20} |S(\mathscr{C}_i)| - \tfrac{1}{3} \sum_{i=1}^{20} l(\mathscr{C}_i),$$

and so

$$l \leqslant 2\,|E(G)| + \tfrac{1}{30} \sum_{i=1}^{20} |S(\mathscr{C}_i)| - \tfrac{1}{60} \sum_{i=1}^{20} l(\mathscr{C}_i). \tag{3.1}$$

Now, for each edge $e \in E(G)$, there are two and only two cycles of F, say C', C'', which do not contain e. So, $e \in S(\mathscr{C}_i)$ if and only if $\{C', C''\} \subseteq \mathscr{C}_i$. There are precisely four such $\mathscr{C}_i$'s (precisely four choices for the third cycle). Thus e is counted precisely four times in the sum $\sum_{i=1}^{20} |S(\mathscr{C}_i)|$. Since this is true for every $e \in E(G)$,

$$\sum_{i=1}^{20} |S(\mathscr{C}_i)| = 4\,|E(G)|. \tag{3.2}$$

We note that each edge of G appears in precisely four cycles of F and each cycle of F appears in precisely $\binom{5}{2} = 10$ of the twenty 3-cycle cover $\mathscr{C}_i$'s. Therefore each edge of G is counted precisely 40 times in the sum $\sum_{i=1}^{20} l(\mathscr{C}_i)$. It follows that

$$\sum_{i=1}^{20} l(\mathscr{C}_i = 40\,|E(G)|. \tag{3.3}$$

Substituting (3.3) and (3.2) into (3.1) gives that $l \leqslant \frac{22}{15}\,|E(G)|$ and completes the proof of Theorem 1.1. ∎

DEFINITION. Denote by uv the edge with ends u and v. Let v be a vertex in a cubic graph and let vx, vy, vz be the three edges incident with v. We say that v is *split* into three vertices (of degree one) if v is replaced by three new vertices v_1, v_2, v_3 and vx, vy, vz are replaced by three new edges v_1x, v_2y, v_3z.

Proof of Theorem 1.2. Let G be any bridgeless cubic graph. We associate with each vertex of G a copy of the Petersen graph, denoted by P_v. Let P'_v be the graph obtained by splitting a vertex of P_v into three vertices, say v'_1, v'_2, v'_3. Now, for each vertex v of G, split v into three vertices, say v_1, v_2, v_3, and identify v_i with v'_i, $1 \leqslant i \leqslant 3$. Denote the resulting graph by G_p. We have that $|E(G_p)| = 15\,|V(G)| + |E(G)|$. Now let J be the graph obtained from G_p by contracting all the edges of G. (These are the graphs used by Jamshy and Tarsi in [8].) Then $|E(J)| = 15\,|V(G)|$. Since G is bridgeless, G_p is bridgeless, and so is J. We claim that

J has no 3-cycle cover of length less than $\frac{22}{15}|E(J)|$. If this is false, let $\mathscr{C}=\{C_1, C_2, C_3\}$ be a 3-cycle cover of J such that $l(\mathscr{C})<\frac{22}{15}|E(J)$ $(=22\,|V(G)|)$. For each $v \in V(G)$, set $\mathscr{C}_v=\{C_1 \cap P'_v, C_2 \cap P'_v, C_3 \cap P'_v\}$. Then $\mathscr{C}_v$ is a 3-cycle cover of the Petersen graph. From the fact that $l(\mathscr{C})=\sum_{v \in V(G)} l(\mathscr{C}_v)$, it follows that the Petersen graph has a 3-cycle cover of length less than 22, but it is easy to check that this is impossible. This contradiction proves the claim. Since J is constructed from an arbitrary bridgeless cubic graph, the theorem follows. ∎

Remark. In the proofs of Lemma 2.1, Theorem 2.2, and Theorem 1.1, the key property we need for a graph is that the symmetric difference of two cycles is a cycle. Since a binary matroid has such property, all the three results could be directly generalized to binary matroids. For example, a generalization of Theorem 1.1 could be as follows. Let M be a binary matroid on a set E. If E can be covered by at most six cycles such that each element of E is in exactly four cycles, then E can be covered by at most three cycles whose total length is at most $\frac{22}{15}|E|$.

4. Open Problems

In Theorem 2.2, if $m \geqslant 2\,|E(G)|$, then G has a 3-cycle cover of length at most $\frac{14}{9}|E(G)|$, a significant improvement on $\frac{5}{3}|E(G)|$ (even better than the bound $\frac{8}{5}|E(G)|$ obtained by assuming that G has a nowhere-zero 5-flow [7]). This leads us to the problem of finding a longest 3-cycle cover of a graph. We propose the following two conjectures.

Conjecture 4.1. Every bridgeless graph G has a 3-cycle cover of length at least $2\,|E(G)|$.

Remark. Using the arguments in [3], one may show that every bridgeless graph G has a 3-cycle cover of length at least $\frac{16}{9}|E(G)|$.

Conjecture 4.2. Every bridgeless cubic graph has three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3=\varnothing$.

Remark. The matching polytope theorem of Edmonds [2] implies that every bridgeless cubic graph has three distinct perfect matchings; it is known (see Jaeger [6]) that every 3-edge-connected cubic graph has three spanning trees T_1, T_2, T_3 such that $T_1 \cap T_2 \cap T_3=\varnothing$.

If $\mathscr{C}$ is a 3-cycle cover, of length at least $2\,|E(G)|$, of a cubic graph G, then each cycle of $\mathscr{C}$ must be a 2-factor of G, and so the complement is a perfect matching. It follows that Conjectures 4.1 and 4.2 are equivalent for cubic graphs, and they are weaker than Fulkerson's conjecture.

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