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A note on the multiplicity of determinantal ideals

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Abstract

Herzog, Huneke, and Srinivasan have conjectured that for any homogeneous k-algebra, the multiplicity is bounded above by a function of the maximal degrees of the syzygies and below by a function of the minimal degrees of the syzygies. The goal of this paper is to establish the multiplicity conjecture of Herzog, Huneke, and Srinivasan about the multiplicity of graded Cohen–Macaulay algebras over a field k for k-algebras $k[x_1, \dots, x_n]/I$ when I is a determinantal ideal of arbitrary codimension.

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field k, let $\deg(x_i) = 1$ and let $I \subset R$ be a graded ideal of arbitrary codimension. Consider the minimal graded free R-resolution of R/I:

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

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where we denote $\beta_{i,j}(R/I) = \dim \operatorname{Tor}_i^R(R/I,k)_j$ the (i,j)th graded Betti number of R/I. Many important numerical invariants of I and the associated scheme can be read off from the minimal graded free R-resolution of R/I. For instance, the Hilbert polynomial, and hence the multiplicity $e(R/I)$ of I, can be written down in terms of the shifts j such that $\beta_{i,j}(R/I) \neq 0$ for some i, $1 \leqslant i \leqslant p$.

Let c denote the codimension of R/I. Then $c \leqslant p$ and equality holds if and only if R/I is Cohen–Macaulay. We define $m_i(I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$ to be the minimum degree shift at the ith step and $M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$ to be the maximum degree shift at the ith step. We will simply write m_i and M_i when there is no confusion. If R/I is Cohen–Macaulay and has a pure resolution, i.e., $m_i = M_i$ for all i, $1 \leqslant i \leqslant c$, then Huneke and Miller showed in [9] that

$$e(R/I) = \frac{\prod_{i=1}^{c} m_i}{c!}.$$

Generalizing their result Herzog, Huneke, and Srinivasan made the following multiplicity conjecture:

Conjecture 1.1. *If R/I is Cohen–Macaulay then*

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leqslant e(R/I) \leqslant \frac{\prod_{i=1}^{c} M_i}{c!}.$$

Conjecture 1.1 has been extensively studied, and partial results have been obtained. It turns out to be true for the following type of ideals:

- complete intersections [8];
- powers of complete intersection ideals [7];
- perfect ideals with a pure resolution [9];
- perfect ideals with a quasi-pure resolution (i.e., $m_i \geqslant M_{i-1}$) [8];
- perfect ideals of codimension 2 [8];
- Gorenstein ideals of codimension 3 [12];
- perfect stable monomial ideals [8];
- perfect square free strongly stable monomial ideals [8];
- zero-dimensional subschemes Y that are residual to a zero-scheme Z of certain type [6].

The goal of this paper is to prove Conjecture 1.1 for determinantal ideals of arbitrary codimension c, i.e., ideals generated by the maximal minors of a $t \times (t+c-1)$ homogeneous polynomial matrix. Determinantal ideals are a central topic in both commutative algebra and algebraic geometry. Due to their important role, their study has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results about determinantal ideals are due to J.A. Eagon and M. Hochster in [3], and to J.A. Eagon and D.G. Northcott in [4]. J.A. Eagon and M. Hochster proved that generic determinantal ideals are perfect. J.A. Eagon and D.G. Northcott constructed a finite graded free resolution for any determinantal ideal, and as a corollary, they showed that determinantal ideals are perfect. Since then many authors have made important contributions to the study of determinantal ideals, and the reader can look at [1,2,5,11] for background, history and a list of important papers.

In this short note we verify that determinantal ideals I satisfy the Herzog–Huneke–Srinivasan Conjecture which relates the multiplicity $e(R/I)$ to the minimal and maximal shifts in the graded minimal R-resolution of R/I.

Next we outline the structure of the paper. In Section 2, we first recall the basic facts on determinantal ideals I of codimension c defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$ and the associated complexes needed later on. We determine the minimal and maximal shifts in the graded minimal free R-resolution of R/I in terms of the degree matrix $\mathcal{U}$ of $\mathcal{A}$ and we state some technical lemmas used in the inductive process of the proof of our main theorem (cf. Theorem 3.1).

Section 3 is completely devoted to proving Conjecture 1.1 for determinantal ideals I of arbitrary codimension. To prove it we use induction on the codimension c of I and for any c induction on the size t of the homogeneous $t \times (t+c-1)$ matrix whose maximal minors generate I by successively deleting columns and rows of the largest possible degree when we prove the lower bound, and columns and rows of the smallest possible degree when we prove the upper bound. We end the paper with an example which illustrates that the upper and lower bounds for the multiplicity $e(R/I)$ of a determinantal ideal I given in Theorem 3.1 are sharp.

2. Determinantal ideals

In the first part of this section, we provide the background and basic results on determinantal ideals needed in the sequel, and we refer to [2,5] for more details.

Let $\mathcal{A}$ be a homogeneous matrix, i.e., a matrix representing a degree 0 morphism $\phi : F \to G$ of free graded R-modules. In this case, we denote by $I(\mathcal{A})$ the ideal of R generated by the maximal minors of $\mathcal{A}$.

Definition 2.1. A homogeneous ideal $I \subset R$ of codimension c is called a *determinantal* ideal if $I = I(\mathcal{A})$ for some $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$.

Let $I \subset R$ be a determinantal ideal of codimension c generated by the maximal minors of a $t \times (t+c-1)$ matrix $\mathcal{A} = (f_{ij})_{i=1,\dots,t}^{j=1,\dots,t+c-1}$ where $f_{ij} \in k[x_1, \dots, x_n]$ are homogeneous polynomials of degree $a_j - b_i$. The matrix $\mathcal{A}$ defines a degree 0 map

$$F = \bigoplus_{i=1}^{t} R(b_i) \xrightarrow{\mathcal{A}} G = \bigoplus_{j=1}^{t+c-1} R(a_j)$$
$$v \mapsto v \cdot \mathcal{A}$$

where $v = (f_1, \dots, f_t) \in F$ and we assume without loss of generality that $\mathcal{A}$ is minimal; i.e., $f_{ij} = 0$ for all i, j with $b_i = a_j$. If we let $u_{i,j} = a_j - b_i$ for all $j = 1, \dots, t+c-1$

and $i = 1, \dots, t$, the matrix $\mathcal{U} = (u_{i,j})_{i=1,\dots,t}^{j=1,\dots,t+c-1}$ is called the *degree matrix* associated to I. By re-ordering degrees, if necessary, we may also assume that $b_1 \geqslant \cdots \geqslant b_t$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{t+c-1}$. In particular, we have:

$$u_{i,j} \leqslant u_{i+1,j} \quad \text{and} \quad u_{i,j} \leqslant u_{i,j+1} \quad \text{for all } i, j. \tag{1}$$

Note that the degree matrix $\mathcal{U}$ is completely determined by $u_{1,1}, u_{1,2}, \dots, u_{1,c}, u_{2,2}, u_{2,3}, \dots, u_{2,c+1}, \dots, u_{t,t}, u_{t,t+1}, \dots, u_{t,c+t-1}$ because of the identity $u_{i,j} + u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} = 0$ for all i, j. Moreover, the graded Betti numbers in the minimal free R-resolution of $R/I(\mathcal{A})$ depend only upon the integers

$$\{u_{i,j}\}_{1 \leqslant i \leqslant t}^{i \leqslant j \leqslant c+i-1} \subset \{u_{i,j}\}_{i=1,\dots,t}^{j=1,\dots,t+c-1}$$

as described below.

Proposition 2.2. *Let $I \subset R$ be a determinantal ideal of codimension c with degree matrix $\mathcal{U} = (u_{i,j})_{i=1,\dots,t}^{j=1,\dots,t+c-1}$ as above. Then we have*:

(1) $m_i = u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}$ *for* $1 \leqslant i \leqslant c$,
(2) $M_i = u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t,t+c-i} + u_{t,t+c-i+1} + u_{t,t+c-i+2} + \cdots + u_{t,t+c-1}$ *for* $1 \leqslant i \leqslant c$.

Proof. We denote by $\varphi : F \to G$ the morphism of free graded R-modules of rank t and $t + c - 1$, defined by the homogeneous matrix $\mathcal{A}$ associated to I. The Eagon–Northcott complex $\mathcal{D}_0(\varphi^*)$:

$$0 \to \bigwedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \bigwedge^{t} F \to \bigwedge^{t+c-2} G^* \otimes S_{c-2}(F) \otimes \bigwedge^{t} F \to \cdots$$
$$\to \bigwedge^{t} G^* \otimes S_0(F) \otimes \bigwedge^{t} F \to R \to R/I \to 0$$

gives us a graded minimal free R-resolution of R/I (see, for instance, [2, Theorem 2.20]; and [5, Corollaries A2.12 and A2.13]). Now the result follows after a straightforward computation. $\square$

We will now fix the notation and prove the technical lemmas needed in the induction process we will use in next section for proving the multiplicity conjecture for determinantal ideals of arbitrary codimension.

Let $I \subset R$ be a homogeneous ideal of codimension c. Assume that I is determinantal and let $\mathcal{A}$ (respectively $\mathcal{U}$) be the $t \times (t + c - 1)$ homogeneous matrix (respectively degree matrix) associated to I. Let $\mathcal{A}'$ (respectively $\mathcal{U}'$) be the $(t - 1) \times (t + c - 2)$ homogeneous matrix (respectively degree matrix) obtained by deleting the last column and the last row of $\mathcal{A}$, and denote by I' the codimension c determinantal ideal generated by the maximal minors of $\mathcal{A}'$. Since the multiplicity of R/I and R/I' are completely determined by the corresponding degree matrices, it is enough to consider an example of an ideal for any degree matrix. So, from now on, we take

$$\mathcal{A} := \begin{pmatrix} x_1^{u_{1,1}} & x_2^{u_{1,2}} & \cdots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \cdots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \cdots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \cdots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \cdots & x_{c-1}^{u_{t,t+c-2}} & x_c^{u_{t,t+c-1}} \end{pmatrix}$$

and

$$\mathcal{A}' := \begin{pmatrix} x_1^{u_{1,1}} & x_2^{u_{1,2}} & \cdots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \cdots & 0 \\ 0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \cdots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \cdots & 0 \\ 0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \cdots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \cdots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} \end{pmatrix}.$$

Let $J \subset R$ be the codimension $c-1$ determinantal ideal generated by the maximal minors of the $t \times (t+c-2)$ homogeneous matrix

$$\mathcal{B} := \begin{pmatrix} x_1^{u_{1,1}} & x_2^{u_{1,2}} & \cdots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \cdots & 0 \\ 0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \cdots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \cdots & 0 \\ 0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \cdots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \cdots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} \\ 0 & 0 & 0 & \cdots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \cdots & x_{c-1}^{u_{t,t+c-2}} \end{pmatrix}$$

obtained by deleting the last column of $\mathcal{A}$. Analogously, let $\mathcal{A}''$ (respectively $\mathcal{U}''$) be the $(t-1)\times(t+c-2)$ homogeneous matrix (respectively degree matrix) obtained by deleting the first column and the first row of $\mathcal{A}$, and we denote by I'' the codimension c determinantal ideal generated by the maximal minors of $\mathcal{A}''$. Let $\mathcal{C}$ be the $t \times (t+c-2)$ homogeneous matrix obtained by deleting the first column of $\mathcal{A}$ and let $K \subset R$ be the codimension $c-1$ determinantal ideal generated by the maximal minors of $\mathcal{C}$.

The ideal I is obtained from I' by a basic double G-link, as well as from I'' by a basic double G-link. Indeed, we have

Lemma 2.3. *With the above notation,*

(1) $I = J + x_c^{u_{t,t+c-1}} I'$ *and* $I = K + x_1^{u_{1,1}} I''$;
(2) $e(R/I) = e(R/I') + u_{t,t+c-1} \cdot e(R/J)$ *and* $e(R/I) = e(R/I'') + u_{1,1} \cdot e(R/K)$.

Proof. (1) The equalities of ideals are immediate.
(2) It follows from [10, Lemma 4.8]. □

Lemma 2.4. *With the above notation, we have*

(1) $m_i = m_i(I) = m_i(I') + u_{t,t+i-1} = m_i' + u_{t,t+i-1}$ *for all* $1 \leqslant i \leqslant c$,
(2) $M_i = M_i(I) = M_i(I'') + u_{1,c-i+1} = M_i'' + u_{1,c-i+1}$ *for all* $1 \leqslant i \leqslant c$,
(3) $m_i(J) = m_i(I) = m_i$ *for all* $1 \leqslant i \leqslant c-1$, *and*
(4) $M_i(K) = M_i(I) = M_i$ *for all* $1 \leqslant i \leqslant c-1$.

Proof. This follows from Proposition 2.2. □

3. The multiplicity conjecture

Using the fact that the ideal I is obtained from the ideal I' (respectively I'') by a basic double G-link, we can now show that Conjecture 1.1 is true for determinantal ideals of arbitrary codimension.

Theorem 3.1. *Let $I \subset R$ be a determinantal ideal of codimension c. Then the following lower and upper bounds hold*:

(1) $e(R/I) \geqslant \frac{\prod_{i=1}^{c} m_i}{c!}$, *and*
(2) $e(R/I) \leqslant \frac{\prod_{i=1}^{c} M_i}{c!}$.

Proof. As we explained in Section 2, it is enough to prove the result for the ideal I generated by the maximal minors of the $t \times (t+c-1)$ matrix

$$\mathcal{A} := \begin{pmatrix} x_1^{u_{1,1}} & x_2^{u_{1,2}} & \cdots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \cdots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \cdots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \cdots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \cdots & x_{c-1}^{u_{t,t+c-2}} & x_c^{u_{t,t+c-1}} \end{pmatrix}.$$

(1) We proceed by induction on the codimension c of I. If $c = 1$ then I is a principal ideal and the result is trivial. For $c = 2$ the result was proved by Herzog and Srinivasan in [8]. Assume $c \geqslant 3$. We will now induct on t. If $t = 1$ then I is a complete intersection ideal and hence the result is well known. Assume $t > 1$. Let $\mathcal{A}'$ (respectively $\mathcal{B}$) be the matrix obtained by deleting the last column and the last row (respectively the last column)

of the matrix $\mathcal{A}$ and let I' (respectively J) be the ideal generated by the maximal minors of $\mathcal{A}'$ (respectively $\mathcal{B}$). Let m_i, m'_i and $m_i(J)$ be the minimal shifts in the graded minimal free R-resolution of R/I, R/I' and R/J, respectively (see Proposition 2.2 and Lemma 2.4).

By Lemma 2.3(2),

$$e(R/I) = e\big(R/I'\big) + u_{t,t+c-1} \cdot e(R/J),$$

by the induction hypothesis on c and Lemma 2.4(3), we have

$$e(R/J) \geqslant \frac{\prod_{i=1}^{c-1} m_i(J)}{(c-1)!} = \frac{\prod_{i=1}^{c-1} m_i}{(c-1)!},$$

and by the induction hypothesis on t we have

$$e\big(R/I'\big) \geqslant \frac{\prod_{i=1}^{c} m'_i}{(c)!}.$$

Therefore, since $m_i = m'_i + u_{t,t+i-1}$ (Lemma 2.4(1)), we have

$$c!e(R/I) \geqslant \prod_{i=1}^{c} m_i$$

if and only if

$$\begin{aligned}
&\prod_{i=1}^{c} m'_i + cu_{t,t+c-1} \prod_{i=1}^{c-1} m_i \\
&\quad \geqslant \prod_{i=1}^{c} m_i = \prod_{i=1}^{c} \big(m'_i + u_{t,t+i-1}\big) \\
&\quad = u_{t,t+c-1} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + m'_c \sum_{r=0}^{c-2} \big(u_{t,t+r} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1}\big) \\
&\quad = u_{t,t+c-1} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + \sum_{r=0}^{c-2} \big(u_{t,t+r} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} m'_c\big)
\end{aligned}$$

if and only if

$$(c-1)u_{t,t+c-1} \prod_{i=1}^{c-1} m_i \geqslant \sum_{r=0}^{c-2} \big(u_{t,t+r} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} m'_c\big).$$

Since, for all integers i, $1 \leqslant i \leqslant c-1$, and for all integers r, $0 \leqslant r \leqslant c-2$, we have the inequalities

$$\begin{aligned}m_i - m'_{i+1} &= (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}) \\ &\quad - (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{1,i+1} + u_{2,i+2} + u_{3,i+3} + \cdots + u_{t-1,t+i-1}) \\ &= (u_{2,i+1} - u_{1,i+1}) + (u_{3,i+2} - u_{2,i+2}) + \cdots + (u_{t,t+i-1} - u_{t-1,t+i-1}) \geqslant 0,\end{aligned}$$

and

$$u_{t,t+c-1} \geqslant u_{t,t+r},$$

we obtain

$$u_{t,t+c-1} \prod_{i=1}^{c-1} m_i \geqslant \bigl(u_{t,t+r} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} m'_c\bigr)$$

for all r, $0 \leqslant r \leqslant c-2$, and the lower bound follows.

(2) The upper bound is proved similarly. We again proceed by induction on the codimension c of I. If $1 \leqslant c \leqslant 2$ then the result holds. So, let us assume $c \geqslant 3$. We will now induct on t. If $t = 1$ then I is a complete intersection ideal and the result is true. Assume $t > 1$. Let $\mathcal{A}''$ (respectively $\mathcal{C}$) be the matrix obtained by deleting the first column and the first row (respectively the first column) of the matrix $\mathcal{A}$ and let I'' (respectively K) be the ideal generated by the maximal minors of $\mathcal{A}''$ (respectively $\mathcal{C}$). Let M_i, M''_i and $M_i(K)$ be the maximal shifts in the graded minimal free R-resolution of R/I, R/I'' and R/K, respectively.

By Lemma 2.3(2),

$$e(R/I) = e\bigl(R/I''\bigr) + u_{1,1} \cdot e(R/K),$$

by the induction hypothesis on c and Lemma 2.4(4), we have

$$e(R/K) \leqslant \frac{\prod_{i=1}^{c-1} M_i(K)}{(c-1)!} = \frac{\prod_{i=1}^{c-1} M_i}{(c-1)!},$$

and by the induction hypothesis on t we have

$$e\bigl(R/I''\bigr) \leqslant \frac{\prod_{i=1}^{c} M''_i}{(c)!}.$$

By Lemma 2.4(2), $M_i = M_i(I) = M_i(I'') + u_{1,c-i+1} = M''_i + u_{1,c-i+1}$ for all $1 \leqslant i \leqslant c$. Therefore, we have

$$c! e(R/I) \leqslant \prod_{i=1}^{c} M_i$$

if and only if

$$
\begin{aligned}
&\prod_{i=1}^{c} M_i'' + cu_{1,1}\prod_{i=1}^{c-1} M_i\\
&\quad \leqslant \prod_{i=1}^{c} M_i = \prod_{i=1}^{c}\bigl(M_i'' + u_{1,c-i+1}\bigr)\\
&\quad = u_{1,1}\prod_{i=1}^{c-1} M_i + \prod_{i=1}^{c} M_i'' + M_c''\sum_{r=0}^{c-2}\bigl(u_{1,c-r}M_1\cdots M_r M_{r+2}''\cdots M_{c-1}''\bigr)\\
&\quad = u_{1,1}\prod_{i=1}^{c-1} M_i + \prod_{i=1}^{c} M_i'' + \sum_{r=0}^{c-2}\bigl(u_{1,c-r}M_1\cdots M_r M_{r+2}'\cdots M_{c-1}''M_c''\bigr)
\end{aligned}
$$

if and only if

$$
(c-1)u_{1,1}\prod_{i=1}^{c-1} M_i \leqslant \sum_{r=0}^{c-2}\bigl(u_{1,c-r}M_1\cdots M_r M_{r+2}''\cdots M_{c-1}''M_c''\bigr).
$$

Because, for all integers i, $1 \leqslant i \leqslant c-1$, and all integers r, $0 \leqslant r \leqslant c-2$, we have

$$
\begin{aligned}
&M_i - M_{i+1}''\\
&\quad = (u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t-1,t+c-i-1} + u_{t,t+c-i} + u_{t,t+c-i+1} + \cdots + u_{t,t+c-1})\\
&\qquad - (u_{2,c-i+1} + u_{3,c-i+2} + \cdots + u_{t,t+c-i-1} + u_{t,t+c-i} + u_{t,t+c-i+1} + \cdots + u_{t,t+c-1})\\
&\quad = (u_{1,c-i+1} - u_{2,c-i+1}) + (u_{2,c-i+2} - u_{3,c-i+2}) + \cdots + (u_{t-1,t+c-i-1} - u_{t,t+c-i-1})\\
&\quad \leqslant 0,
\end{aligned}
$$

and

$$
u_{1,1} \leqslant u_{1,c-r},
$$

we deduce

$$
u_{1,1}\prod_{i=1}^{c-1} M_i \leqslant \bigl(u_{1,c-r}M_1\cdots M_r M_{r+2}''\cdots M_{c-1}''M_c''\bigr)
$$

for all r, $0 \leqslant r \leqslant c-2$. This completes the proof of the upper bound, and hence the proof of the theorem. $\square$

Since the power I^s of a complete intersection ideal $I \subset R$ is an example of determinantal ideal, we recover Guardo and Van Tuyl's result (see [7, Proposition 3.2]) as a corollary of Theorem 3.1. In fact, we have

Corollary 3.2. *Let $I \subset R$ be a complete intersection ideal and let s be any positive integer. Then Conjecture* 1.1 *is true for R/I^s.*

We will end this note with an example which illustrate that the bounds given in Theorem 3.1 are optimal.

Example 3.3. Let $I \subset R$ be a codimension c determinantal ideal generated by the maximal minors of a $t \times (t+c-1)$ matrix all whose entries are homogeneous polynomials of fixed degree $1 \leqslant d \in \mathbb{Z}$. Thus, we have

$$m_i(I) = M_i(I) = td + (i-1)d \quad \text{for all } i,\ 1 \leqslant i \leqslant c.$$

Therefore, we conclude that

$$e(R/I) = \frac{\prod_{i=1}^{c} m_i(I)}{c!} = \frac{\prod_{i=1}^{c} M_i(I)}{c!} = \frac{\prod_{i=1}^{c}(td + (i-1)d)}{c!} = d^c \binom{t+c-1}{c}.$$

Note added in proof

While the paper was in press, I received a preprint by Herzog and Zheng and a preprint by Migliore, Nagel and Römer where they give another proof of Theorem 3.1.

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