Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 279 (2004) 566–612

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Symmetric group blocks of small defect

Gordon James [a], Andrew Mathas [b,*]

[a] Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK
[b] School of Mathematics F07, University of Sydney, Sydney NSW 2006, Australia

Received 13 November 2003

Available online 1 June 2004

Communicated by Georgia Benkart

1. Introduction

This paper is concerned with the modular representation theory of the symmetric groups. Throughout, we fix a positive integer n and a prime $p > 0$ and we consider representations of the symmetric group $\mathfrak{S}_n$ of degree n over a field of characteristic p. We adopt the standard notation for the representation theory of the symmetric groups from [8].

It is well known that a p-block of a symmetric group $\mathfrak{S}_n$ is determined by its p-core and its weight, and that the weight of a block is equal to the defect of the block if p exceeds the weight [8]. In this paper we shall be concerned mainly with blocks of small defect.

Let λ and μ be partitions of n with μ being p-regular. As usual, the symmetric group $\mathfrak{S}_n$ has a Specht module $S(\lambda)$ and a p-modular irreducible module $D(\mu)$. The decomposition number $[S(\lambda):D(\mu)]$ is defined to be the composition multiplicity of $D(\mu)$ in $S(\lambda)$. The following facts are known about blocks of weight w:

(a) If $w = 0$ or 1 then all the decomposition numbers of the block are 0 or 1.
(b) If $w = 2$ and $p > 2$ then all the decomposition numbers of the block are 0 or 1 [21].
(c) If $w = 4$ then some decomposition numbers of a block can be greater than 1, even if $p > w$.

* Corresponding author.
E-mail address: a.mathas@maths.usyd.edu.au (A. Mathas).

doi:10.1016/j.jalgebra.2004.04.006

Moreover, if $w = 0, 1$, or 2 then there is a known method for determining the decomposition numbers [18]. The situation for the case $w = 3$ is still not properly understood. In particular, the decomposition number

$$\left[S(2p-2, 2p-2, p-1, 1) : D(3p-3, 2p-1)\right]$$

is yet to be determined for $p > 5$. This is just one of a collection of decomposition numbers for weight 3 which we are unable to evaluate.

Our investigation of blocks of weight 3 grew out of an attempt to improve upon the earlier results of Martin and Russell [17] by explicitly calculating the decomposition numbers. This led us to discover various errors and omissions in [17] which place in doubt the claim made there that when $p > w$ all the decomposition numbers are 0 or 1. Note, incidentally, that if the decomposition numbers for a given block are known to be 0 or 1 then the decomposition numbers can, in principle, be determined by applying Schaper's theorem [9,19].

In this paper we concentrate on understanding blocks of *small defect*. By definition, if k is a field of characteristic p then a block B of $k\mathfrak{S}_n$ has *small defect* if $p > w$, where w is the p-weight of B. These terms will be introduced below.

2. Basic results

By using a result of Brundan and Kleshchev, we are able to improve upon the presentation of several of the basic techniques used in [17,20,21] for estimating decomposition numbers. In order to state these results recall that the *diagram* of a partition λ is the set of *nodes*

$$[\lambda] = \left\{(i, j) \mid 1 \leqslant j \leqslant \lambda_i\right\}.$$

We think of $[\lambda]$ as being an array of crosses in the plane and we will refer to the rows and columns of $[\lambda]$ which should be interpreted in the obvious way.

A node $x \in [\lambda]$ is *removable* if $[\lambda] \setminus \{x\}$ is the diagram of a partition. Similarly, a node $y \notin [\lambda]$ is *addable* if $[\lambda] \cup \{y\}$ is the diagram of a partition. The node $x = (i, j)$ is called an r-node if $r \equiv j - i \pmod p$. A removable r-node $x \in [\lambda]$ is *normal* if whenever y is an addable r-node in $[\lambda]$ which is in an earlier row than x then there are more removable r-nodes between x and y than there are addable r-nodes [11].

Finally, recall that a partition μ is *p-regular* if no p non-zero parts of μ are equal. Then $D(\mu) \neq 0$ if and only if μ is p-regular.

Proposition 2.1. *Assume that λ and μ are partitions of n with μ being p-regular, and that k is a positive integer such that*

(1) *λ has at most k removable r-nodes*; *and*
(2) *μ has a least k normal r-nodes.*

Then $[S(\lambda):D(\mu)]$ is either zero or is equal to an explicit decomposition number of $\mathfrak{S}_{n-k}$. More precisely, if λ has fewer than k removable r-nodes then $[S(\lambda):D(\mu)]=0$; and if λ has exactly k removable r-nodes then $[S(\lambda):D(\mu)]=[S(\bar{\lambda}):D(\overline{\mu})]$, where $\bar{\lambda}$ is the partition obtained from λ by removing its k r-nodes, and $\overline{\mu}$ is the partition obtained from μ by removing its lowest k normal r-nodes.

Proof. By r-restricting $D(\mu)$ k times, we obtain an $\mathfrak{S}_{n-k}$-module which contains $D(\overline{\mu})$ as a submodule. If λ has fewer than k removable r-nodes then by r-restricting $S(\lambda)$ k times we obtain the zero module, so $[S(\lambda):D(\mu)]=0$. If λ has exactly k removable r-nodes then $[S(\lambda):D(\mu)]=[S(\bar{\lambda}):D(\overline{\mu})]$ by [3, Lemma 2.13]. □

We now recall the notion of an *abacus* from [6]. A p-abacus has p runners, which we label as runner 1 to runner p, reading from left to right. The bead positions on the abacus are labelled $1,2,3,\ldots$, reading from left to right and then top to bottom. Thus, the beads on runner r have labels $r+pk$, for some $k\geqslant 0$.

Recall that if $\lambda=(\lambda_1,\lambda_2,\ldots)$ with $\lambda_i=0$, whenever $i>k$, then λ has an abacus configuration with k beads at positions $\{\lambda_i+k-i+1 \mid 1\leqslant i\leqslant k\}$. Note that if λ has k non-zero parts then λ can be represented on an abacus with k' beads whenever $k'\geqslant k$. For example, the partition $(15,13,6,4^2,2^2)$ can be represented as an abacus with 10 and 11 beads, respectively, as follows:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \bullet \\
\bullet & \cdot & \cdot & \bullet & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bullet & \cdot & \cdot & \bullet \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad\text{and}\quad
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \bullet & \bullet & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \bullet \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot \\
\bullet & \cdot & \cdot & \cdot & \cdot
\end{array}.$$

An abacus representation with k beads can be converted into one with $k+1$ beads by shifting all beads one position to the right and then adding a new bead at position 1.

It is convenient to say that a bead on runner r is an r-node. This changes the definition of r-node above by a constant and causes no harm. With this convention, removing an r-node from a partition corresponds to moving a bead on runner r one space to the left (with an obvious modification if $r=1$), and adding an r-node corresponds to moving a bead on runner $r-1$ one space to the right (with an obvious modification if $r=1$).

By definition, a *p-core* is partition which has an abacus configuration in which all of the beads are positioned as high as possible on each runner. A partition has *p-weight w* if its abacus configuration can be obtained by starting with the abacus configuration of a p-core and sliding w (not necessarily distinct) beads down one position on their runner. In this way, we attach a p-core to each partition of weight w.

Finally, recall that all of the irreducible constituents of a Specht module $S(\lambda)$ belong to the same block and, further, that $S(\lambda)$ and $S(\mu)$ belong to the same block if and only if λ and μ have the same p-core [8]. Consequently, $S(\lambda)$ and $S(\mu)$ belong to the same block if and only if they are of the same weight and they have abacus configurations which have the same number of beads on each runner. We will say that two partitions λ and μ belong to a block B if $S(\lambda)$ and $S(\mu)$ are both contained in B.

We can now present some corollaries of Proposition 2.1.

Corollary 2.2. *Suppose that the partition λ of n has exactly k removable r-nodes and no addable r-nodes. Let μ be a p-regular partition of n. Then $[S(\lambda) : D(\mu)]$ is equal to an explicit decomposition number of $\mathfrak{S}_{n-k}$ which is in a block of the same weight as λ.*

Proof. We may assume that μ is in the same block as λ. Hence μ has exactly k more removable r-nodes than addable r-nodes and so has at least k normal r-nodes. The corollary now follows immediately from Proposition 2.1. (The remark that the block of $\mathfrak{S}_{n-k}$ has the same weight as λ follows from the fact that the abacus configuration of $\bar{\lambda}$ can be obtained from that of λ by swapping runners $r-1$ and r.) $\square$

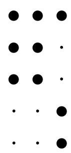
Corollary 2.3. *Suppose that B is a block of $\mathfrak{S}_n$ with the property that for every partition in B there exists an r such that the partition has a removable r-node but no addable r-node. Then we can equate each decomposition number of B with an explicit decomposition number for a smaller symmetric group.*

From now on, we assume that we are dealing with a block of weight w.

Corollary 2.4. *Suppose that $w \leqslant 3$. Then every decomposition number for the principal block of $\mathfrak{S}_{wp}$ is either zero or can be equated with an explicit decomposition number of $\mathfrak{S}_{wp-1}$.*

Proof. The p-core of the principal block of $\mathfrak{S}_{wp}$ is empty, and so it can be represented on an abacus with w beads on each runner, with all the beads pushed as far up as possible. Suppose that $S(\lambda)$ belongs to the principal block of $\mathfrak{S}_{wp}$, so that the abacus configuration for λ is obtained from the p-core configuration by moving w beads, not necessarily distinct, down one position on their runners. Since $w \leqslant 3$, we see that in the abacus configuration for λ, for each r, we can move at most one bead from runner r to runner $r-1$. In other words, λ has at most one removable r-node. Now suppose that μ is p-regular and choose a normal r-node of μ, for some r. Proposition 2.1 now allows us to deduce that $[S(\lambda) : D(\mu)]$ is either zero or equal to a decomposition number of $\mathfrak{S}_{wp-1}$. $\square$

Note that the weight of a partition of $\mathfrak{S}_{wp-1}$ must be less than w, and all the decomposition numbers for blocks of weight $0, 1$, or 2 are known [8,18]. Therefore, Corollary 2.4 determines the decomposition numbers of $\mathfrak{S}_{3p}$. Note, too, that the proof fails when $w = 4$ because λ may have more than one removable r-node in this case. For example, suppose that $p = 3$ and consider the partition $\lambda = (6, 4, 1^2)$, which has the abacus configuration:



Corollary 2.5 (Scopes [20]). *Suppose that B is a block of $\mathfrak{S}_n$ such that the abacus configuration of every partition in B has the property that runner i contains at least w more beads than runner $i-1$, for some i. Then each decomposition number for B can be equated with an explicit decomposition number for some smaller symmetric group.*

Proof. Making w slides from the p-core, no position which we reach allows us to move a bead on runner $i-1$ one space to the right. Therefore, we may apply Corollary 2.3, with $r = i$, to obtain the desired result. □

We remark that Scopes proved the stronger result that the block B is Morita equivalent to the block whose abacus configuration is obtained by interchanging runners i and $i-1$.

3. Methods for estimating decomposition numbers

We now present a collection of techniques for gathering information about decomposition numbers. These ideas determine the decomposition numbers for blocks of weight 0, 1, or 2, and go some way in dealing with blocks of higher weight. Many examples will appear later in this paper.

Suppose we are given a partition λ and that we are trying to find $[S(\lambda) : D(\mu)]$, for all μ. We may assume that λ and μ are in the same block and that $\mu \trianglerighteq \lambda$, since otherwise $[S(\lambda) : D(\mu)] = 0$. (Recall that $\mu \trianglerighteq \lambda$ if $\sum_{i=1}^{k} \mu_i \geqslant \sum_{i=1}^{k} \lambda_i$, for all $k \geqslant 1$. We say that μ *dominates* λ.) In particular, the number of (non-zero) parts of μ cannot exceed the number of parts of λ. Hence, whatever abacus we use to represent λ we can also use to represent μ. This follows because the number of parts of a partition can be read off its abacus configuration by counting the number of beads after the first gap.

Also, recalling the definition of normal node from the last section, observe that the normal r-nodes for μ can also be read off an abacus configuration for μ by considering the beads on runners $r-1$ and r in the abacus.

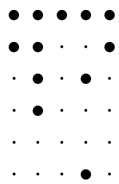
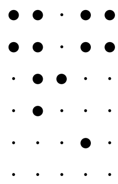
Our first rule is the abacus version of Corollary 2.2.

Rule 1. Suppose that λ has an abacus configuration such that exactly k beads on runner r can be moved one space to the left and that none of the beads on runner $r-1$ can be moved one space to the right, for some r. Then $[S(\lambda) : D(\mu)] = [S(\bar{\lambda}) : D(\overline{\mu})]$, where the abacus configuration for $\bar{\lambda}$ is obtained from that for λ by moving to the left the k possible beads on runner r, and the abacus configuration for $\overline{\mu}$ is obtained by moving to the left the k beads on runner r corresponding to the lowest k normal nodes in μ.

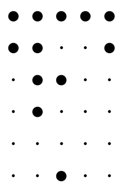
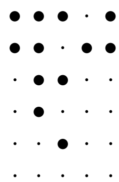
Notes.

(a) In practice, μ frequently has no addable r-nodes, so to obtain $\overline{\mu}$ one simply moves to the left the k possible beads on runner r.
(b) If $k \geqslant 1$ then Rule 1 equates $[S(\lambda) : D(\mu)]$ with a decomposition number of the same weight in a smaller symmetric group.

Example. If λ and μ correspond to

 and 

and $r = 4$, then $\bar{\lambda}$ and $\overline{\mu}$ correspond to

 and .

(Here, we could also apply Rule 1 with $r = 2$, but not with $r = 5$.)

Rule 2. Given a partition λ, Schaper's theorem [9,19] gives us a linear combination of Specht modules $S(\nu)$, where $\nu \rhd \lambda$ and ν belongs to the same block as λ. If we know (for example, by induction) all of the decomposition numbers for the Specht modules $S(\nu)$ appearing in this sum then, in the Grothendieck group of $\mathfrak{S}_n$, we can rewrite this sum as a linear combination of irreducible modules $D(\mu)$ with non-negative integer coefficients. Schaper's theorem then tells us that:

(a) if $D(\mu)$ appears in this linear combination with multiplicity $m > 1$, then $m \geqslant [S(\lambda) : D(\mu)] \geqslant 1$; and
(b) if $D(\mu)$ appears in this linear combination with multiplicity $m \leqslant 1$, then $[S(\lambda) : D(\mu)] = m$.

Note that Rule 2 gives us both upper and lower bounds on $[S(\lambda) : D(\mu)]$. Our next rule will provide another upper bound.

Suppose that μ has exactly k normal r-nodes and let $\overline{\mu}$ be the partition obtained from μ by removing these nodes. Also, let Ω denote the set of partitions of $n-k$ which are obtained from λ by removing k r-nodes. Then Kleshchev's Branching Theorem shows that

$$\big[S(\lambda):D(\mu)\big] \leqslant \sum_{\omega\in\Omega}\big[S(\omega):D(\overline{\mu})\big].$$

(Here, we interpret the right-hand side to be zero when Ω is empty.)

Rule 3. We may iterate the process just defined, first removing all of the k_1 normal r_1-nodes from μ, then taking all the k_2 normal r_2-nodes from the partition $\overline{\mu}$, and so on, until we reach a stage where we can evaluate the decomposition numbers on the right-hand side of the inequality.

Note. We do not increase the weight of the partitions involved when we apply Rule 3. By this we mean that the weight of $\overline{\mu}$ is at most the weight of μ. To see this, first observe that, in general, if the number of beads on runner $r-1$ is a and the number of beads on runner r is b, then moving a bead left from runner r to runner $r-1$ decreases the weight by $a-b+1$ (of course, a negative decrease corresponds to an increase). Hence, by induction, moving k beads left from runner r to runner $r-1$ decreases the weight by $k(a-b+k)$. Now suppose that μ has exactly k normal r-nodes. Then the definition of normal implies that $k \geqslant b-a$; thus, $k(a-b+k) \geqslant 0$, so removing the k normal r-nodes does not increase the weight.

Observe that we can always apply Rule 3 to get an upper bound on $[S(\lambda):D(\mu)]$ because at some point we will be able to evaluate the right-hand side of the inequality, if need be by persevering until we reach the empty partition. If in applying Rule 3 we remove k_1 normal r_1-nodes, k_2 normal r_2-nodes, and so on, then we refer to $r_1^{k_1}r_2^{k_2}\dots$ as a Kleshchev sequence for μ.

The next rule is due to the first author [5].

Rule 4. Assume that λ and μ are partitions of n with μ being p-regular, and $\lambda_1=\mu_1$. Let

$$\bar{\lambda}=(\lambda_2,\lambda_3,\dots) \quad \text{and} \quad \overline{\mu}=(\mu_2,\mu_3,\dots).$$

Then $[S(\lambda):D(\mu)]=[S(\bar{\lambda}):D(\overline{\mu})]$.

This rule says that we can remove the first rows of λ and μ without changing the decomposition multiplicity $[S(\lambda):D(\mu)]$. Analogously, we have the following rule (see [2,5]).

Rule 5. Assume that λ and μ are partitions of n with μ being p-regular, and that λ and μ have the same first column. Let

$$\lambda^{(1)}=(\lambda_1-1,\lambda_2-1,\dots), \qquad \mu^{(1)}=(\mu_1-1,\mu_2-1,\dots).$$

Then $[S(\lambda):D(\mu)]=[S(\lambda^{(1)}):D(\mu^{(1)})]$.

Notes.

(a) Removing the first column from a partition corresponds to putting a bead in the first gap.
(b) Removing the first row from a partition corresponds to removing the last bead.

Rule 6. Assume that λ is p-regular and we know the decomposition numbers for every $S(\nu)$ with $\nu \trianglerighteq \lambda$. Then we can express $D(\lambda)$ as a linear combination of the Specht modules $S(\nu)$ with $\nu \trianglerighteq \lambda$. For all r, the r-restriction of this linear combination of Specht modules is a module for $\mathfrak{S}_{n-1}$.

We can apply Rule 6 to give an upper bound on a decomposition number $[S(\lambda) : D(\mu)]$ whenever we know the decomposition numbers for every $S(\nu)$ with $\nu \triangleright \lambda$; see the example at the end of Section 5. Moreover, we can iterate this process and perform a sequence $r_1^{k_1} r_2^{k_2} \ldots$ of restrictions, rather than just a single r-restriction.

Rule 6, which involves restricting an irreducible module, again gives us an upper bound on decomposition numbers. We do not list the corresponding result involving inducing an irreducible module, which would give us a lower bound, for the following reason. If $[S(\lambda) : D(\mu)] \geqslant 1$, then inducing simple modules would *perhaps* give this information, but Rule 1 would *certainly* give it.

Rule 7. Assume that λ and μ are partitions of n with μ being p-regular. Then $[S(\lambda) : D(\mu)] = [S(\lambda') : D(\mu^*)]$, where λ' is the conjugate of λ and μ^* is the image of μ under the Mullineux map [1,4].

We reiterate that the rules we have stated deal very well with many decomposition numbers of blocks of small defect. Moreover, as we shall see, Rules 1–6 add credibility to the conjecture which we discuss next.

Our conjecture relates certain decomposition numbers for different primes.

Let λ and μ be partitions, μ being p-regular, and suppose that λ and μ have the same p-core and the same weight. Represent λ on some abacus with p runners. We shall discuss the decomposition number $[S(\lambda) : D(\mu)]$ so we may assume that $\mu \trianglerighteq \lambda$; hence μ and λ can be represented on abacuses which have the same number of beads. Suppose that p' is a prime greater than p. Our conjecture equates certain p'-modular decomposition numbers with p-modular decomposition numbers. Let λ^+ denote the partition obtained from λ by adding $p' - p$ empty runners to the abacus (in any places) and let μ^+ denote the partition obtained from μ by adding $p' - p$ empty runners to the abacus configuration for μ (in the same places). We now put forward the following conjecture.

Conjecture 3.1. *Suppose that $p > w$. Then $[S(\lambda^+) : D(\mu^+)] = [S(\lambda) : D(\mu)]$.*

Let B be a block of $\mathfrak{S}_n$ of weight w. Then B is a *block of small defect* if $p > w$.

Note that by Rule 1 the decomposition number $[S(\lambda^+) : D(\mu^+)]$ is independent of where the $p' - p$ empty runners are inserted into the abacuses of λ and μ (the empty runners do, however, need to be in the same places). Unless stated otherwise we will assume that the abacuses for λ^+ and μ^+ are obtained from those for λ and μ, respectively, by adding $p' - p$ empty runners at the end.

Rather than working with the symmetric group in characteristic p if, instead, we work with the Hecke algebra of type A at a complex pth root of unity then our conjecture is true, without any restriction on p. This is part of the main result of our paper [10].

We remark that the assumption that $p > w$ in Conjecture 3.1 is necessary. To see this let $\lambda = (3, 1^2)$ and $\mu = (5)$ and take $p = 2$. Then λ and μ are partitions of 2-weight 2 and $[S(3, 1^2) : D(5)] = 2$. These partitions have the following abacus configurations:

$$\lambda = \begin{array}{cc} \cdot & \bullet \\ \bullet & \cdot \\ \cdot & \bullet \\ \cdot & \cdot \end{array}, \qquad \mu = \begin{array}{cc} \bullet & \bullet \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \bullet \end{array}.$$

So we may take $\lambda^+ = (5, 2, 1)$ and $\mu = (8)$ with $p' = 3$ by adding an empty right hand runner. However, $[S(5, 2, 1) : D(8)] = 1$ when $p' = 3$. So $[S(\lambda) : D(\mu)] \neq [S(\lambda^+) : D(\mu^+)]$ in this case.

We give further evidence in support of our conjecture after the examples below.

Example. Suppose that $p = 5$ and $\lambda = (8, 8, 4, 1)$ and $\mu = (12, 9)$. Then we can represent λ and μ on an abacus as follows:

$$\lambda = \begin{array}{ccccc} \cdot & \bullet & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot \\ \bullet & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \mu = \begin{array}{ccccc} \bullet & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}.$$

Now let $p' = 7$. Then Rule 1 (applied 3 times) ensures that $[S(\lambda^+) : D(\mu^+)]$ is the same if

$$\lambda^+ = \begin{array}{ccccccc} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \mu^+ = \begin{array}{ccccccc} \cdot & \bullet & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

or if

$$\lambda^+ = \begin{array}{ccccccc} \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \mu^+ = \begin{array}{ccccccc} \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}.$$

Conjecture 3.1 says that $[S(\lambda^+) : D(\mu^+)] = [S(\lambda) : D(\mu)]$. Thus, in this example, our conjecture says that the decomposition multiplicity

$$\big[S(2p-2, 2p-2, p-1, 1) : D(3p-3, 2p-1)\big]$$

is the same for $p = 7$ as for $p = 5$. We know of no way to compute this multiplicity in general; however, using extensive computer calculations Lübeck and Müller [13,14] have shown that if $p = 5$ then $[S(2p-2, 2p-2, p-1, 1) : D(3p-3, 2p-1)] = 1$.

If our conjecture is correct then it follows, as in the example above, that

$$\big[S(2p-2, 2p-2, p-1, 1) : D(3p-3, 2p-1)\big] = 1, \quad \text{for all } p > 3.$$

On the other hand, if $[S(2p-2, 2p-2, p-1, 1) : D(3p-3, 2p-1)] \neq 1$, for any $p > 3$, then this provides a counterexample to the "$pe > n$ conjecture" of [7, Section 4].

We find it remarkable that if Conjecture 3.1 is correct then we can produce a computer-free proof of Lübeck and Müller's result above. That is, we can deduce that $[S(8^2, 4, 1) : D(12, 9)] = 1$ when $p = 5$. Here is how this comes about. First, using Rules 1–7,

$$1 = \big[S(8, 5, 4, 2) : D(19)\big], \quad \text{when } p = 5.$$

Indeed, the decomposition matrices of $\mathfrak{S}_n$ and $p = 5$ can be calculated by hand for $n \leqslant 20$. Next,

$$\begin{aligned}
&\big[S(8, 5, 4, 2) : D(19)\big] \\
&\quad = \big[S\big(4^2, 3^2, 2, 1^3\big) : D\big(5^3, 4\big)\big], \quad \text{when } p = 5, \text{ by Rule 7}, \\
&\quad = \big[S\big(8^2, 5^2, 4, 1^3\big) : D\big(9^3, 6\big)\big], \quad \text{when } p = 7, \text{ if Conjecture 3.1 is correct}, \\
&\quad = \big[S\big(8, 5^3, 4, 2^3\big) : D\big(11^3\big)\big], \quad \text{when } p = 7, \text{ by Rule 7}.
\end{aligned}$$

Now, we are unable to evaluate the last decomposition number when $p = 7$ using Rules 1–7; however, using Rule 2, we can show that $[S(8, 5^3, 4, 2^3) : D(11^3)]$ is either 1 or 2. By investigating these two possibilities, we can show that $[S(8, 5^3, 4, 2^3) : D(11^3)] = [S(10^2, 6, 5, 1^2) : D(11^3)]$, when $p = 7$, irrespective of the actual value of $[S(8, 5^3, 4, 2^3) : D(11^3)]$. In turn,

$$\begin{aligned}
&\big[S\big(10^2, 6, 5, 1^2\big) : D\big(11^3\big)\big] \\
&\quad = \big[S\big(6^2, 4, 3, 1^2\big) : D\big(7^3\big)\big], \quad \text{when } p = 5, \text{ if Conjecture 3.1 is correct}, \\
&\quad = \big[S\big(6, 4^2, 3, 2^2\big) : D(12, 9)\big], \quad \text{when } p = 5, \text{ by Rule 7}.
\end{aligned}$$

Again, using Rules 1–7, we are only able to show that this last decomposition number is either 1 or 2; however, by pursuing these two possibilities in turn it can be shown that

$$\big[S\big(6, 4^2, 3, 2^2\big) : D(12, 9)\big] = \big[S\big(8^2, 4, 1\big) : D(12, 9)\big], \quad \text{when } p = 5.$$

Thus, by a very circuitous route, we have shown that if our conjecture is true then one can deduce by hand that $[S(8^2,4,1):D(12,9)]=1$ when $p=5$.

The argument for showing that $[S(8^2,4,1):D(12,9)]=1$ when $p=5$, consists of alternating applications of Conjecture 3.1 and Rule 7 (conjugation). In the absence of a proof of our conjecture, similar arguments suggest that there are at least $p-3$ projective indecomposable modules in blocks of weight 3 which cannot, as yet, be determined. These are the blocks corresponding to the p-cores

$$(2^{p-2}),\ (3^{p-3}),\ (4^{p-4}),\ (5^{p-5}),\dots,(p-2)^2.$$

Notice that these cores occur in conjugate pairs, so there at least $(p-3)/2$ independent decomposition numbers of weight 3 in characteristic p which current theory is unable to determine.

As evidence in support of our conjectures we present the following propositions which show that our conjectures are compatible with Rules 1–5. We remark that in the proofs of Propositions 3.2–3.6, the hypothesis that $p>w$ has immediate effect only in the proof of Proposition 3.3.

Proposition 3.2. *Assume that $p>w$ and that λ and μ are partitions of n of weight w and with the same p-core and that μ is p-regular. Assume that $[S(\alpha^+):D(\beta^+)]=[S(\alpha):D(\beta)]$ whenever α and β are partitions of an integer less than n, with weight w.*

Suppose that λ has an abacus configuration such that, for some r, exactly $k>0$ beads on runner r can be moved one space to the left and that none of the beads on runner $r-1$ can be moved one space to the right, as in Rule 1. Then $[S(\lambda^+):D(\mu^+)]=[S(\lambda):D(\mu)]$.

Proof. Adopt the notation of Rule 1. Note that $\bar\lambda$ and $\overline{\mu}$ have weight w. We may assume that the abacuses for λ^+ and μ^+ are obtained by inserting $p'-p$ empty runners between runners r and $r+1$ of the abacuses for λ and μ, respectively; consequently, $(\bar\lambda)^+=\overline{\lambda^+}$ and $(\overline{\mu})^+=\overline{\mu^+}$. Therefore,

$$\begin{aligned}\big[S\big(\lambda^+\big):D\big(\mu^+\big)\big]&=\big[S\big(\overline{\lambda^+}\big):D\big(\overline{\mu^+}\big)\big],\quad\text{by Rule 1,}\\&=\big[S\big((\bar\lambda)^+\big):D\big((\overline{\mu})^+\big)\big]\\&=\big[S(\bar\lambda):D(\overline{\mu})\big],\quad\text{by our induction hypothesis,}\\&=\big[S(\lambda):D(\mu)\big],\quad\text{by Rule 1.}\qquad\square\end{aligned}$$

Proposition 3.3. *Assume that $p>w$ and that λ and μ are partitions of n of weight w and with the same p-core and that μ is p-regular. Assume that $[S(\nu^+):S(\mu^+)]=[S(\nu):D(\mu)]$ whenever $\nu\rhd\lambda$ and that by applying Rule 2 we can deduce that $[S(\lambda):D(\mu)]\leqslant m$. Then $[S(\lambda^+):D(\mu^+)]\leqslant m$.*

Proof. Applying Schaper's theorem to $S(\lambda)$ gives a linear combination $\sum_\nu a_\nu S(\nu)$ of Specht modules $S(\nu)$, where $a_\nu\neq 0$ only if λ and ν belong to the same block and $\nu\rhd\lambda$.

Therefore, if $a_\nu \neq 0$ then λ and ν can both be represented on an abacus with the same number of beads and the partitions ν that arise are determined by sliding beads up and down the runners of an abacus for λ in a specific way. Moreover, the coefficient a_ν of a $S(\nu)$ in this linear combination depends on the p-adic evaluation of the hook lengths involved. Since $p > w$, no hook length in either of the partitions λ or λ^+ is divisible by p^2. Hence, Schaper's theorem applied to $S(\lambda^+)$ produces the linear combination $\sum_\nu a_\nu S(\nu^+)$ of Specht modules. By assumption, the decomposition numbers $[S(\nu^+) : D(\mu^+)]$ have already been proved to be equal to $[S(\nu) : D(\mu)]$. Therefore, the information provided for $[S(\lambda) : D(\mu)]$ by Rule 2 gives the same information for $[S(\lambda^+) : D(\mu^+)]$. □

Proposition 3.4. *Assume that $p > w$ and that λ and μ are partitions of n of weight w and with the same p-core and that μ is p-regular. Suppose that $[S(\alpha^+) : D(\beta^+)] = [S(\alpha) : D(\beta)]$ whenever α and β are partitions of an integer less than n, with weight at most w. If Rule* 3 *gives $[S(\lambda) : D(\mu)] \leqslant \sum_{\omega\in\Omega}[S(\omega) : D(\overline{\mu})]$ then $[S(\lambda^+) : D(\mu^+)] \leqslant \sum_{\omega\in\Omega}[S(\omega) : D(\overline{\mu})]$.*

Proof. Once again, we assume that the abacuses for λ^+ and μ^+ are obtained by inserting $p' - p$ empty runners between runners r and $r + 1$ of the abacuses for λ and μ, respectively. Then $\overline{\mu^+} = (\overline{\mu})^+$ and hence

$$\begin{aligned}
\bigl[S(\lambda^+) : D(\mu^+)\bigr] &\leqslant \sum_{\omega\in\Omega}\bigl[S(\omega^+) : D\bigl((\overline{\mu})^+\bigr)\bigr], \quad \text{by Rule 3,}\\
&= \sum_{\omega\in\Omega}\bigl[S(\omega) : D(\overline{\mu})\bigr], \quad \text{by our induction hypothesis.}
\end{aligned}$$

Note that we are justified in applying our induction hypothesis, in the light of the note to Rule 3. □

Proposition 3.5. *Assume that $p > w$ and that λ and μ are partitions of n with the same p-core and of weight w, with μ p-regular. Assume, too, that λ and μ have the same first row, as in Rule* 4. *Suppose that $[S(\alpha^+) : D(\beta^+)] = [S(\alpha) : D(\beta)]$ whenever α and β are partitions of an integer less than n, with weight at most w. Then $[S(\lambda^+) : D(\mu^+)] = [S(\lambda) : D(\mu)]$.*

Proof. Since λ and μ have the same p-core, we can represent them on abacuses with the same number of beads. As these partitions also have the same first row, the last bead on the abacus for λ is in the same position as the last bead for μ. Removing this bead does not increase the weight.

Adopt the notation of Rule 4. Note that $(\lambda^{(1)})^+ = (\lambda^+)^{(1)}$ and $(\mu^{(1)})^+ = (\mu^+)^{(1)}$. Therefore,

$$\begin{aligned}
\bigl[S(\lambda^+) : D(\mu^+)\bigr] &= \bigl[S\bigl((\lambda^+)^{(1)}\bigr) : D\bigl((\mu^+)^{(1)}\bigr)\bigr], \quad \text{by Rule 4,}\\
&= \bigl[S\bigl((\lambda^{(1)})^+\bigr) : D\bigl((\mu^{(1)})^+\bigr)\bigr]
\end{aligned}$$

$$= \big[S\big(\lambda^{(1)}\big) : D\big(\mu^{(1)}\big)\big], \quad \text{by our induction hypothesis,}$$
$$= \big[S(\lambda) : D(\mu)\big], \quad \text{by Rule 4.} \qquad \square$$

There is a more general version of row removal [2] which says that if $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$, for some s, then

$$\big[S(\lambda) : D(\mu)\big] = \big[S(\lambda_1, \ldots, \lambda_s) : D(\mu_1, \ldots, \mu_s)\big] \times \big[S(\lambda_{s+1}, \lambda_{s+2}, \ldots) : D(\mu_{s+1}, \mu_{s+2}, \ldots)\big].$$

However, this result is not obviously compatible with Conjecture 3.1 when $s > 1$ because it is easy to find examples where $\lambda_1^+ + \cdots + \lambda_s^+ \neq \mu_1^+ + \cdots + \mu_s^+$.

The previous remark also applies for the general version of column removal (Rule 5). Even so, we do have the following result.

Proposition 3.6. *Assume that $p > w$ and that λ and μ are partitions of n with the same p-core and of weight w, with μ p-regular. Assume, too, that λ and μ have the same first column, as in Rule 5. Suppose that $[S(\alpha^+) : D(\beta^+)] = [S(\alpha) : D(\beta)]$ whenever α and β are partitions of an integer less than n, with weight at most w. Then $[S(\lambda^+) : D(\mu^+)] = [S(\lambda) : D(\mu)]$.*

Proof. Since λ and μ have the same first column, the first gap in the abacus for λ is in the same position as the first gap in the abacus for μ; say this is position i. Suppose that position i on the p-abacus is position i^+ on the p'-abacus. Then, by repeated applications of Rule 5, $[S(\lambda^+) : D(\mu^+)] = [S(\alpha) : D(\beta)]$, where α is obtained from λ^+ by filling the gaps up to and including the gap at position i^+ and β is obtained from μ^+ in the same way. Similarly, $[S((\lambda^{(1)})^+) : D((\mu^{(1)})^+)] = [S(\alpha) : D(\beta)]$, where we adopt the notation of Rule 5. Therefore,

$$\big[S\big(\lambda^+\big) : D\big(\mu^+\big)\big] = \big[S\big(\big(\lambda^{(1)}\big)^+\big) : D\big(\big(\mu^{(1)}\big)^+\big)\big]$$
$$= \big[S\big(\lambda^{(1)}\big) : D\big(\mu^{(1)}\big)\big], \quad \text{by our induction hypothesis,}$$
$$= \big[S(\lambda) : D(\mu)\big], \quad \text{by Rule 5.} \qquad \square$$

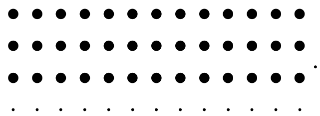
Roughly speaking, Propositions 3.2–3.6 say that if all decomposition numbers were determined by Rules 1–5, then Conjecture 3.1 would be true by induction.

Finally, we remark that Conjecture 3.1 is not obviously compatible with Rule 6 because the restriction of a block of weight w can have arbitrarily large weight (in particular, the weight can be larger than p).

4. Blocks of weight 3

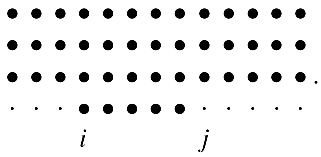
Now let $w = 3$ and assume that $p > 3$. By repeatedly applying Corollary 2.3 we can reduce the calculation of *all* decomposition numbers for blocks of weight 3 down to considering only certain blocks or, equivalently, p-cores. We now describe the abacuses for this minimal collection of p-cores. We assume, without loss of generality, that each of our abacuses has exactly 3 beads on runner 1 and at least 3 beads on every other runner.

Case 1. All of the runners contain exactly 3 beads:



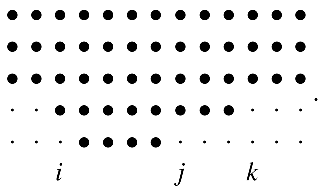
Note that in this case the p-core is empty. In the light of Corollary 2.4 and the remark which follows it, we do not need to pursue Case 1 further. (Case 1 is the only case which has to be considered if $w = 1$.)

Case 2. The first $i - 1$ runners contain exactly 3 beads; runners i up to $j - 1$ contain 4 beads; and runners j to p contain 3 beads. Here, $1 < i < j \leqslant p + 1$, so there are $\binom{p}{2}$ such p-cores:



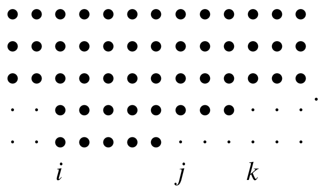
Note that in this case the p-core is $(i - 1)^{j-i}$, a partition of $ij - i^2 + i - j$. (Cases 1 and 2 are the only cases which have to be considered if $w = 2$.)

Case 3. The first $i - 1$ runners contain exactly 3 beads; runner i contains 4 beads; runners $i + 1$ to $j - 1$ contain 5 beads; runners j to $k - 1$ contain 4 beads; and runners k to p contain 3 beads. Here, we allow $2 < i + 1 < j \leqslant k \leqslant p + 1$, so there are $\binom{p-1}{3} + \binom{p-1}{2} = \binom{p}{3}$ such p-cores:



Note that in this case the p-core is $((p - k + 2i)^{j-i-1}, (i - 1)^{k-i})$.

Case 4. The first $i-1$ runners contain exactly 3 beads; runners i to $j-1$ contain 5 beads; runners j to $k-1$ contain 4 beads; and runners k to p contain 3 beads. Here, we allow $1 < i < j \leqslant k \leqslant p+1$, so there are $\binom{p}{3} + \binom{p}{2} = \binom{p+1}{3}$ such p-cores:



Note that in this case the p-core is $((p-k+2i-1)^{j-i}, (i-1)^{k-i})$. (The reader should have no difficulty working out the additional cases which arise for $w=4$, and for higher weights.)

Thus, in general there are $\binom{p+1}{3} + \binom{p}{3} + \binom{p}{2} + 1 = 2\binom{p+1}{3} + 1$ different p-cores which need to be considered. In each case, the decomposition numbers for the different cores are often very similar (depending on the parameters i, j, k); however, the explosion of delicate subcases makes it very difficult to write down a convincing argument for general p.

Of the four cases that need to be considered when $w=3$, Case 3 was overlooked in [17], thereby further jeopardizing their claim that the decomposition numbers for $w=3$ can be determined and that they all have value 0 or 1. We now apply our methods to obtain certain decomposition numbers for $w=3$ in Cases 2–4. Some of these results already appear in [17].

Suppose that we have fixed a p-core ρ *and* an abacus configuration for ρ as above. Then, as in [17,21], we use the following notation for the partitions of weight 3 with p-core ρ:

(1) Let $\langle r\rangle$ be the partition whose abacus is obtained by moving the last bead on runner r (of the abacus for ρ), down 3 places.
(2) Let $\langle r^2\rangle$ be the partition whose abacus is obtained by moving the last bead on runner r down 2 places and the second last bead on runner r, down one place.
(3) Let $\langle r^3\rangle$ be the partition whose abacus is obtained by moving the last 3 beads on runner r, down one place each.
(4) For $r \neq s$, let $\langle r, s\rangle$ be the partition whose abacus is obtained by moving the last bead on runner r down 2 places and the last bead on runner s down one place.
(5) For $r \neq s$, let $\langle r^2, s\rangle$ be the partition whose abacus is obtained by moving the last 2 beads on runner r down one place each, and the last bead on runner s down one place.
(6) For r, s, t distinct, let $\langle r, s, t\rangle$ be the partition whose abacus is obtained by moving the last bead on runners r, s and t down one place each.

4.1. Some decomposition numbers in Case 2

Assume that the p-core belongs to Case 2. That is, runners up to runner $i-1$ contain 3 beads; runner i contains 4 beads; after this, there are some or no runners with 4 beads;

any remaining runners contain 3 beads. Let runner j be the first runner after runner i with 3 beads.

By Rule 1, we can equate the decomposition number $[S(\lambda) : D(\mu)]$ with a decomposition number of weight 3 in a smaller symmetric group for all partitions λ and μ in this block except for when λ is one of the partitions α^*, β^*, γ^*, $\alpha^\natural$, $\beta^\natural$, $\gamma^\natural$, $\alpha^{(u)}$, $\beta^{(u)}$, or $\gamma^{(u)}$ where

$$\alpha^* = \langle i^2 \rangle, \qquad \beta^* = \langle i, i-1 \rangle, \qquad \gamma^* = \langle i-1 \rangle$$

and

$$\alpha^\natural = \langle i^3 \rangle, \qquad \beta^\natural = \langle (i-1)^2, i \rangle, \qquad \gamma^\natural = \langle (i-1)^2 \rangle$$

and

$$\alpha^{(u)} = \langle i^2, u \rangle, \quad \text{for } 1 \leqslant u \leqslant p \text{ and } u \neq i-i, i$$

(Rule 1 deals with the case $u = i-1$, and $u = i$ gives $\alpha^\natural$),

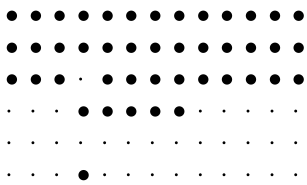
$$\beta^{(u)} = \langle i-1, i, u \rangle, \quad \text{for } 1 \leqslant u \leqslant p \text{ and } u \neq i-1, i$$

(Rule 1 deals with the case $u = i-1$, and $u = i$ gives β^*), and

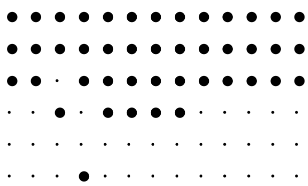
$$\gamma^{(u)} = \langle i-1, u \rangle, \quad \text{for } 1 \leqslant u \leqslant p \text{ and } u \neq i-1, i.$$

(Rule 1 deals with the case $u = i$, and $u = i-1$ gives γ^*).

We call α^*, β^*, γ^*, $\alpha^\natural$, $\beta^\natural$, $\gamma^\natural$, $\alpha^{(u)}$, $\beta^{(u)}$, $\gamma^{(u)}$ the *exceptional partitions* for Case 2. The abacuses for the exceptional partitions in Case 2 are as follows:

$\alpha^* =$ 

so $\alpha^* = (2p - j + 2i, i^{j-i}, 1^{p-i})$;

$\beta^* =$ 

so $\beta^* = (2p - j + 2i, i^{j-i-1}, i-1, 1^{p-i+1})$;

$$\gamma^* = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\gamma^* = (2p - j + 2i - 1, i^{j-i}, 1^{p-i+1})$;

$$\alpha^\natural = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\alpha^\natural = (p - j + 2i, i^{j-i}, 1^{2p-i})$;

$$\beta^\natural = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\beta^\natural = (p - j + 2i, i^{j-i-1}, i-1, 1^{2p-i+1})$;

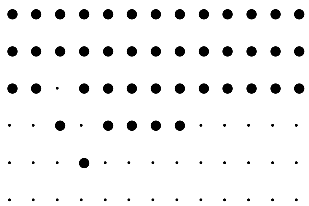
$$\gamma^\natural = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\gamma^\natural = (p - j + 2i - 1, i^{j-i}, 1^{2p-i+1})$;

$$\alpha^{(u)} = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} + \text{one move on runner } u,$$

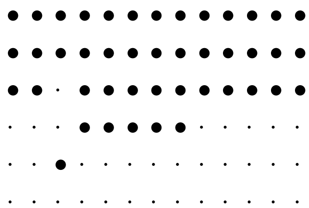
so

$$\alpha^{(u)} = \begin{cases} \big(p-j+2i, u-j+i+1, (i+1)^{j-i}, 2^{p-u}, 1^{u-i-1}\big), & \text{if } j \leqslant u \leqslant p, \\ \big(p-j+i+u, p-j+2i+1, (i+1)^{j-u-1}, i^{u-i}, 1^{p-1}\big), & \text{if } i < u < j, \\ \big(p-j+2i, i^{j-i}, u+1, 2^{p-i}, 1^{i-u-1}\big), & \\ \quad \text{if } 1 \leqslant u < i-1; & \end{cases}$$

$\beta^{(u)} =$  + one move on runner u,

so

$$\beta^{(u)} = \begin{cases} \big(p-j+2i, u-j+i+1, (i+1)^{j-i-1}, i, 2^{p-u}, 1^{u-i}\big), \\ \quad \text{if } j \leqslant u \leqslant p, \\ \big(p-j+i+u, p-j+2i+1, (i+1)^{j-u-1}, i^{u-i-1}, i-1, 1^{p-i+1}\big), \\ \quad \text{if } i < u < j, \\ \big(p-j+2i, i^{j-i-1}, i-1, u+1, 2^{p-i+1}, 1^{i-u-2}\big), \\ \quad \text{if } 1 \leqslant u < i-1; \end{cases}$$

$\gamma^{(u)} =$  + one move on runner u,

so

$$\gamma^{(u)} = \begin{cases} \big(p-j+2i-1, u-j+i+1, (i+1)^{j-i}, 2^{p-u}, 1^{u-i}\big), & \text{if } j \leqslant u \leqslant p, \\ \big(p-j+u+i, p-j+2i, (i+1)^{j-u-1}, i^{u-i}, 1^{p-i+1}\big), & \text{if } i < u < j, \\ \big(p-j+2i-1, i^{j-i}, u+1, 2^{p-i+1}, 1^{i-u-2}\big), & \text{if } 1 \leqslant u < i-1. \end{cases}$$

Note. The three partitions $\alpha^\natural, \beta^\natural, \gamma^\natural$ are p-singular and, moreover, $\alpha^* \rhd \beta^* \rhd \gamma^*$ and $\alpha^\natural \rhd \beta^\natural \rhd \gamma^\natural$ and $\alpha^{(u)} \rhd \beta^{(u)} \rhd \gamma^{(u)}$ for all u with $1 \leqslant u \leqslant p$ and $u \neq i-1, i$. Also, if $\lambda \in \{\alpha, \beta, \gamma\}$ then

$$\lambda^* \rhd \lambda^{(j-1)} \rhd \lambda^{(j-2)} \rhd \cdots \rhd \lambda^{(i+1)} \rhd \lambda^{(p)} \rhd \lambda^{(p-1)} \rhd \cdots \rhd \lambda^{(j)} \rhd \lambda^{(i-2)} \rhd \lambda^{(i-3)} \cdots \\ \rhd \lambda^{(1)} \rhd \lambda^\natural.$$

Proposition 4.1. *Assume that* $1 \leqslant v \leqslant p$ *and* $v \neq i-1, i$.

(1) *For* λ *arbitrary and* $\mu \in \{\alpha^*, \alpha^{(v)}\}$ *we can compute* $[S(\lambda) : D(\mu)]$.
(2) *For* λ *an exceptional partition and* $\mu \in \{\beta^*, \beta^{(v)}\}$ *we can compute* $[S(\lambda) : D(\mu)]$.
(3) *For* λ *an exceptional partition and* $\mu \in \{\gamma^*, \gamma^{(v)}\}$ *we can compute* $[S(\lambda) : D(\mu)]$.

Proof. Recall that the decomposition numbers are known for blocks of weight $0, 1$, and 2. We prove that we can reduce the calculation of the decomposition numbers in the proposition to one of these cases:

(1) Suppose that $\mu \in \{\alpha^*, \alpha^{(v)}\}$. Then μ has 2 normal i-nodes and every λ in the same block as μ has at most 2 removable i-nodes. Therefore, we can apply Proposition 2.1 to compute $[S(\lambda) : D(\mu)]$.

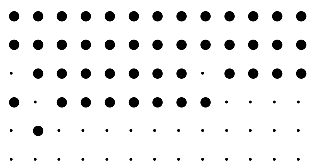
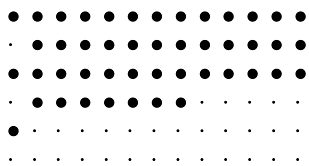
(2) Suppose that $\mu \in \{\beta^*, \beta^{(v)}\}$.

If $i \neq 2$ and $\mu \neq \beta^{(i-2)}$ then μ has exactly one normal $(i-1)$-node, and λ has at most 1 removable $(i-1)$-node, so we can apply Proposition 2.1 again.

Assume that $\mu = \beta^{(i-2)}$ and $i \neq 2, 3$. Then μ has exactly one normal $(i-2)$-node, and λ has at most 1 removable $(i-2)$-node, so we can apply Proposition 2.1 again. Note that if $\mu = \beta^{(i-2)}$ and $i = 3$, then μ is p-singular.

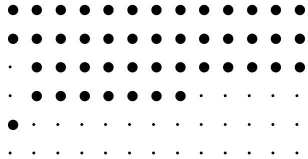
Assume that $i = 2$ and μ is p-regular. Then $\mu = \beta^{(v)}$ for some v with $j \leqslant v \leqslant p$. We need only consider those partitions λ for which the first part of μ is larger than the first part of λ (since, otherwise, either $\mu \ntrianglerighteq \lambda$ or we can apply row removal). Therefore, $\lambda \in \{\gamma^\natural, \gamma^{(j)}, \ldots, \gamma^{(p)}\}$. But the first columns of $\beta^{(v)}$ and $\gamma^{(u)}$ have the same length and we can apply Rule 5. Also, unless $v = j$, we see that $\beta^{(v)}$ has a normal v-node while $\gamma^\natural$ has no removable v-node; so, $[S(\lambda) : D(\mu)] = 0$ by Proposition 2.1.

We are now left with one final case, namely, $i = 2$, $\mu = \beta^{(j)}$ (with $j \leqslant p$, since otherwise μ is p-singular) and $\lambda = \gamma^\natural$.

$\beta^{(j)} =$  , $\gamma^\natural =$  .

It is possible to prove that $[S(\gamma^\natural) : D(\beta^{(j)})] = 1$ by applying Rules 2 and 3, but it is tricky to apply Schaper's theorem without making a mistake. We therefore prove that $[S(\gamma^\natural) : D(\beta^{(j)})] = 1$ as follows (recall that $i = 2$).

Let ϕ be the abacus

$\phi =$  .

Using the Littlewood–Richardson rule, we now add a skew p-hook to ϕ in all possible ways to see, in the Grothendieck group, that:

$$-S(\langle 1^2\rangle) + S(\langle 1, j\rangle) - S(\langle 1, j+1\rangle) + \cdots \pm S(\langle 1, p\rangle) + (-1)^j \big(S(\langle 1, 2\rangle) - S(\langle 1, 3\rangle) + \cdots \pm S(\langle 1, j-1\rangle)\big) + S(\langle 1\rangle) = 0.$$

Now, $\gamma^\natural = \langle 1^2\rangle$. Therefore, $[S(\gamma^\natural) : D(\beta^{(j)})]$ is equal to the multiplicity of $D(\beta^{(j)})$ in

$$S(\langle 1, j\rangle) - S(\langle 1, j+1\rangle) + \cdots \pm S(\langle 1, p\rangle) + (-1)^j \big(S(\langle 1, 2\rangle) - S(\langle 1, 3\rangle) + \cdots \pm S(\langle 1, j-1\rangle)\big) + S(\langle 1\rangle).$$

This is equal to the multiplicity of $D(\beta^{(j)})$ in $S(\langle 1, j\rangle) - S(\langle 1, j+1\rangle) + \cdots \pm S(\langle 1, p\rangle)$ because $\beta^{(j)}$ does not dominate the other terms (consider the first two parts). In turn, this multiplicity is equal to $[S(\langle 1, j\rangle) : D(\beta^{(j)})]$ since $\beta^{(j)}$ does not dominate the other terms ($\beta^{(j)}$ and all the other terms have the same first column, $\beta^{(j)}$ ends in $j-2$ ones, while $\langle 1, k\rangle$ ends in $k-2$ ones, for $j \leqslant k \leqslant p$). Finally, $[S(\langle 1, j\rangle) : D(\beta^{(j)})] = 1$ by two applications of Rule 5 followed by the defect 1 result.

(3) Suppose that $\mu \in \{\gamma^*, \gamma^{(v)}\}$. Note that μ has exactly one normal $(i-1)$-node (except if $i = 2$, $j = p+1$ and $\mu = \gamma^{(v)}$), but every exceptional partition λ has at most 1 removable $(i-1)$-node, so we can apply Proposition 2.1 again. Suppose that $i = 2$, $j = p+1$ and $\mu = \gamma^{(v)}$ (here, $i < v < p$). We need only consider those λ where the first part of μ exceeds the first part of λ, and λ has a removable v-node. It is easy to check that there are no such partitions, so we have finished. □

We remark that a more detailed analysis shows that the part of the decomposition matrix with the rows and columns indexed by the exceptional partitions has the following block diagonal form:

$$\begin{matrix} * & 0 & \cdots & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ 0 & * & * & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & * & * \\ 0 & \cdots & \cdots & 0 & * \end{matrix},$$

where the blocks are certain 3×3 matrices (with singular columns omitted) which are labelled by triples $\{\alpha^?, \beta^?, \gamma^?\}$. The ordering of 3×3 blocks is compatible with the ordering of the partitions given before the statement of Proposition 4.1. See Appendix A for the case $p = 5$.

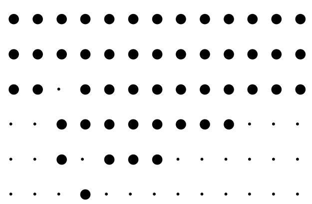
4.2. Some decomposition numbers in Case 3

Assume that the p-core belongs to Case 3. That is, runners up to runner $i-1$ contain 3 beads; runner i contains 4 beads; runner $i+1$ contains 5 beads; after this, there are some or no runners with 5 beads; after this, there are some or no runners with 4 beads; any remaining runners contain 3 beads. Let runner j be the first runner with 4 beads; let runner k be the first runner after runner i with 3 beads.

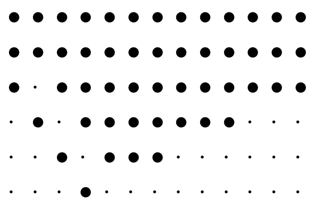
By Rule 1, we can equate the decomposition number $[S(\lambda):D(\mu)]$ with a decomposition number of weight 3 in a smaller symmetric group for all partitions λ and μ in the block, except for when λ is one of the partitions α, β, γ, or δ where

$$\alpha=\langle i^2, i+1\rangle, \qquad \beta=\langle i-1, i, i+1\rangle, \qquad \gamma=\langle i^2\rangle, \quad \text{and} \quad \delta=\langle i, i-1\rangle.$$

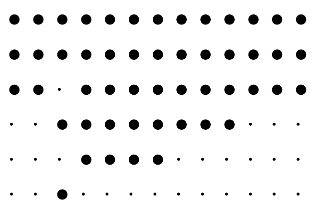
We call $\alpha, \beta, \gamma, \delta$ the *exceptional partitions* for Case 3. The abacus configurations for the exceptional partitions in Case 3 are as follows:

$\alpha =$ 

so $\alpha=(2p-k-j+3i+2, (p-k+2i+1)^{j-i-2}, p-k+2i, i^{k-i}, 1^{p-i})$;

$\beta =$ 

so $\beta=(2p-k-j+3i+2, (p-k+2i+1)^{j-i-2}, p-k+2i, i^{k-i-1}, i-1, 1^{p-i+1})$;

$\gamma =$ 

so $\gamma = (2p-k-j+3i+1, (p-k+2i+1)^{j-i-1}, i^{k-i}, 1^{p-i})$;

$$\delta = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\delta = (2p-k-j+3i+1, (p-k+2i+1)^{j-i-1}, i^{k-i-1}, i-1, 1^{p-i+1})$.

Proposition 4.2. *The part of the decomposition matrix whose rows and columns are labelled by $\alpha, \beta, \gamma, \delta$ is*

	α	β	γ	δ
α	1	·	·	·
β	1	1	·	·
γ	1	·	1	·
δ	1	1	1	1

(*omitted entries are zero*).

Proof. Note that $\alpha \rhd \beta \rhd \delta$ and $\alpha \rhd \gamma \rhd \delta$ but $\beta \not\trianglerighteq \gamma$ and $\gamma \not\trianglerighteq \beta$. Let

$$\alpha_{(1)} = \beta_{(1)} = \big(2p-k-j+3i+2, (p-k+2i+1)^{j-i-2}, p-k+2i\big),$$
$$\gamma_{(1)} = \delta_{(1)} = \big(2p-k-j+3i+1, (p-k+2i+1)^{j-i-1}\big)$$

and $\alpha_{(2)} = \gamma_{(2)} = (i^{k-i}, 1^{p-i})$ and $\beta_{(2)} = \delta_{(2)} = (i^{k-i-1}, i-1, 1^{p-i+1})$. Note that

$$\big[S(\gamma_{(1)}) : D(\alpha_{(1)})\big] = \big[S\big(p-j+i+1, 1^{j-i-1}\big) : D\big(p-j+i+2, 1^{j-i-2}\big)\big] = 1,$$

by Rule 5 (column removal), and that

$$\big[S(\beta_{(2)}) : D(\alpha_{(2)})\big] = \big[S\big(i-1, 1^{p-i+1}\big) : D\big(i, 1^{p-i}\big)\big] = 1,$$

by Rule 4 (row removal). Next,

$$\big[S(\lambda) : D(\mu)\big] = \big[S(\lambda_{(1)}) : D(\mu_{(1)})\big]\big[S(\lambda_{(2)}) : D(\mu_{(2)})\big]$$

for all $\lambda, \mu \in \{\alpha, \beta, \gamma, \delta\}$ by Rule 4, again. From this the proposition follows. $\square$

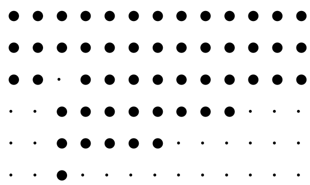
4.3. *Some decomposition numbers in Case 4*

Assume that the p-core belongs to Case 4. That is, runners up to runner $i-1$ contain 3 beads; runner i contains 5 beads; after this, there are some or no runners with 5 beads; after this, there are some or no runners with 4 beads; any remaining runners contain 3 beads. Let runner j be the first runner with 4 beads; let runner k be the first runner after runner i with 3 beads.

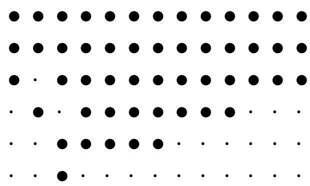
By Rule 1, we can equate the decomposition number $[S(\lambda) : D(\mu)]$ with a decomposition number of weight 3 in a smaller symmetric group for all λ and μ in the block except for when λ is one of the partitions α, β, γ or δ where

$$\alpha = \langle i^3\rangle, \qquad \beta = \langle i^2, i-1\rangle, \qquad \gamma = \langle i-1, i\rangle, \qquad \delta = \langle i-1\rangle.$$

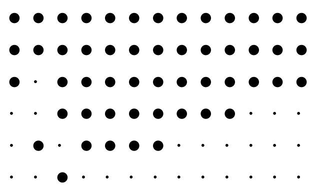
We call $\alpha, \beta, \gamma, \delta$ the *exceptional partitions* for Case 4. The abacus configurations for the exceptional partitions in Case 4 are as follows:

$\alpha =$ 

so $\alpha = (2p-k-j+3i, (p-k+2i)^{j-i}, i^{k-i}, 1^{p-i})$;

$\beta =$ 

so $\beta = (2p-k-j+3i, (p-k+2i)^{j-i}, i^{k-i-1}, i-1, 1^{p-i+1})$;

$\gamma =$ 

so $\gamma = (2p-k-j+3i, (p-k+2i)^{j-i-1}, p-k+2i-1, i^{k-i}, 1^{p-i+1})$;

$$\delta = \begin{array}{ccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

so $\delta = (2p-k-j+3i-1, (p-k+2i)^{j-i}, i^{k-i}, 1^{p-i+1})$.

Proposition 4.3. *The part of the decomposition matrix whose rows and columns are labelled by $\alpha, \beta, \gamma, \delta$ is*

	α	β	γ	δ
α	1	·	·	·
β	1	1	·	·
γ	1	1	1	·
δ	1	1	1	1

(*omitted entries are zero*).

Proof. Note that $\alpha \rhd \beta \rhd \gamma \rhd \delta$. First, each of α, β, γ and δ has exactly 3 removable i-nodes. Also, the 3 removable i-nodes in α are normal. Therefore, for $\lambda \in \{\alpha, \beta, \gamma, \delta\}$ we have $[S(\lambda) : D(\alpha)] = [S(\bar{\lambda}) : D(\bar{\alpha})]$ as in Proposition 2.1. However, here $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = \bar{\delta}$, so $[S(\lambda) : D(\alpha)] = 1$.

Next, remove the first column from β, γ, δ; see Note (a) which follows Rule 5. We can now apply a similar argument to the above, using 2 removable i-nodes, to deduce that $[S(\lambda) : D(\beta)] = 1$ for $\lambda \in \{\beta, \gamma, \delta\}$.

Finally, remove the first i columns from γ, δ, using Rule 5, and use the one remaining removable i-node to deduce that $[S(\lambda) : D(\gamma)] = 1$, for $\lambda \in \{\gamma, \delta\}$. The proof of the proposition is now complete. □

Of course, Propositions 4.1–4.3 hardly scratch the surface of the problem of calculating the decomposition numbers, since they evaluate $[S(\lambda) : D(\mu)]$ only when λ and μ are both exceptional. It is still necessary to calculate $[S(\lambda) : D(\mu)]$ when λ is exceptional and μ is arbitrary. In Case 4, for example, these answers depend upon the values of i, j, and k, and there are a very large number of separate cases that have to be considered.

5. The case $p = 5$

We have written a computer program, using the GAP package SPECHT [15], to apply Rules 1–7 to find all the decomposition numbers when $p = 5$ and $w = 3$. All the decomposition numbers were determined once we had assumed Lübeck and Müller's result that $[S(8^2, 4, 1) : D(12, 9)] = 1$. Consequently, at least in principle, all of the decomposition numbers of the symmetric groups for blocks of weight 3 in characteristic 5 are now known. The decomposition matrices for Cases 2–4 when $p = 5$ are given in the appendix.

In outline, the program first finds all the partitions ν which dominate the last of the exceptional partitions and then uses Rules 1 and 2 to find the decomposition numbers $[S(\nu) : D(\mu)]$ whenever ν is not exceptional. For the exceptional partitions λ, the program applies Schaper's theorem (Rule 2); this often determines the decomposition numbers $[S(\lambda) : D(\mu)]$. If this decomposition number is not determined then Rule 2 gives us an integer $m > 1$ such that $m \geqslant [S(\lambda) : D(\mu)] \geqslant 1$. The program next checks to see whether the answer is given by one of Rules 3–6. Finally, as a last resort, the program tries to apply Rule 7 in order to show that $[S(\lambda) : D(\mu)] = 1$. The program also does parallel computations with two different primes which it uses, along with Rule 1, to check the consistency of its calculations (compare Conjecture 3.1).

Finally, in order to check our calculations we compared the matrices that we computed with the decomposition matrices of the corresponding Hecke algebra of type A [16] at a complex pth root of unity—which are known by the LLT algorithm [12]. Since we were able to compute these decomposition numbers using only Rules 1–7 (and Lübeck and Müller's result), these two sets of decomposition matrices should agree because $p > w$ (this affects only Rule 2). In all cases the symmetric group and Hecke algebra decomposition multiplicities were the same.

Here is a small example of the technique in action.

Example. Suppose that we are in Case 4, with $i = p$, $j = k = p + 1$. Thus the p-core has the following abacus:

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \end{array}.$$

Then the exceptional partitions for this core are: $\alpha = \langle p^3\rangle$, $\beta = \langle p^2, p-1\rangle$, $\gamma = \langle p-1, p\rangle$, and $\delta = \langle p-1\rangle$. We will show that the non-zero decomposition numbers for these exceptional partitions are as follows:

		$\langle p\rangle$	$\langle p^2\rangle$	$\langle p, p-1\rangle$	α	β	γ	δ
	$\langle p\rangle$	1						
	$\langle p^2\rangle$	·	1					
	$\langle p, p-1\rangle$	1	1	1				
$\alpha =$	$\langle p^3\rangle$	·	·	·	1			
$\beta =$	$\langle p^2, p-1\rangle$	·	1	·	1	1		
$\gamma =$	$\langle p-1, p\rangle$	·	1	1	1	1	1	
$\delta =$	$\langle p-1\rangle$	·	·	·	1	1	1	1

First, one readily checks that the partitions which index the rows of this matrix are precisely the partitions ν such that ν has the same p-core as δ and $\nu \trianglerighteq \delta$.

Suppose that $\nu, \mu \in \{\langle p\rangle, \langle p^2\rangle, \langle p, p-1\rangle\}$. Then $p-1$ applications of Rule 1 allow us to equate $[S(\nu) : D(\mu)]$ with a decomposition number in Case 2 with $i = p,\ j = k = p+1$. In practice, though, it is much easier to apply Rule 2 (Schaper's theorem) to evaluate $[S(\nu) : D(\mu)]$.

Now consider the exceptional partitions $\alpha, \beta, \gamma, \delta$.

Rule 2 immediately implies that $S(\alpha)$ is irreducible.

Next, applying Schaper's theorem to β gives the following linear combination of Specht modules: $S(\langle p^2\rangle) + S(\alpha)$. From what we have already deduced, this is equal to $D(\langle p^2\rangle) + D(\alpha)$. Rule 2 now gives us the row of the matrix which is labelled by β.

Similarly, applying Schaper's theorem to γ gives

$$-S\big(\langle p\rangle\big) + S\big(\langle p, p-1\rangle\big) + S(\alpha) + S(\beta) = 2D\big(\langle p^2\rangle\big) + D\big(\langle p, p-1\rangle\big) + 2D(\alpha) + D(\beta).$$

By Rule 2, $[S(\gamma) : D(\langle p, p-1\rangle)] = [S(\gamma) : D(\beta)] = 1$ and we also know that

$$2 \geqslant \big[S(\gamma) : D\big(\langle p^2\rangle\big)\big] \geqslant 1 \quad \text{and} \quad 2 \geqslant \big[S(\gamma) : D(\alpha)\big] \geqslant 1.$$

Now, γ and $\langle p^2\rangle$ have the same first part so $[S(\gamma) : D(\langle p^2\rangle)] = [S(\gamma_{(2)}) : D(\langle p^2\rangle_{(2)})]$ by Rule 4. But $\gamma_{(2)}$ and $\langle p^2\rangle_{(2)}$ belong to a block of weight 2, so $[S(\gamma_{(2)}) : D(\langle p^2\rangle_{(2)})] \leqslant 1$. Hence $[S(\gamma) : D(\langle p^2\rangle)] = 1$.

If $[S(\gamma) : D(\alpha)] = 2$, then

$$D(\gamma) = S\big(\langle p-1, p\rangle\big) - S\big(\langle p^2, p-1\rangle\big) - S\big(\langle p, p-1\rangle\big) + S\big(\langle p\rangle\big).$$

If we $p^2(p-1)^2 \cdots 2^2 1^2$ restrict this, as in Rule 6, we do not obtain a module (an irreducible module occurs with negative multiplicity). This contradiction implies that $[S(\gamma) : D(\alpha)] = 1$, and all the decomposition numbers for $S(\gamma)$ are now known.

Finally, we apply Schaper's theorem to δ. This gives

$$S\big(\langle p\rangle\big) - S\big(\langle p^2\rangle\big) - S\big(\langle p, p-1\rangle\big) + S(\alpha) + S(\beta) + S(\gamma) = 3D(\alpha) + 2D(\beta) + D(\gamma).$$

Thus, the only decomposition numbers for $S(\delta)$ which are still in doubt are $[S(\delta) : D(\alpha)]$ and $[S(\delta) : D(\beta)]$.

We apply Rule 3, with the Kleshchev sequence p^3 (which leads to a block of weight 0) to conclude that $[S(\delta) : D(\alpha)] \leqslant 1$. Hence, $[S(\delta) : D(\alpha)] = 1$.

Now, δ and β have the same first column, so by Rule 5,

$$\big[S(\delta) : D(\beta)\big] = \big[S\big(\delta^{(1)}\big) : D\big(\beta^{(1)}\big)\big].$$

But $\delta^{(1)}$ and $\beta^{(1)}$ belong to a block of weight 2, so $[S(\delta^{(1)}) : D(\beta^{(1)})] \leqslant 1$. Hence $[S(\delta) : D(\gamma)] = 1$.

We have now completed the example.

Notes.

(a) Some of the decomposition numbers in the last example were computed in different ways in Proposition 4.3. The method in the example uses only Rules 1–7 (in fact we used all the rules except Rule 7).
(b) The arguments used in the example apply equally well for any $p > 3$. As a consequence of many other instances of this phenomenon, we were led to formulate Conjecture 3.1.

Acknowledgment

This research was supported, in part, by EPSRC grant GR/S06639/01.

Appendix A. Decomposition numbers of weight 3 in characteristic 5

In this appendix we list the non-zero entries in the rows indexed by exceptional partitions for all of the decomposition matrices in Cases 2–4 when $p = 5$. These matrices, combined with the results of this paper (specifically Rule 1), determine the decomposition matrices for all blocks of weight 3 for all symmetric groups when $p = 5$.

We remark that we have also used our program to calculate the decomposition numbers in Cases 1–4 when $p = 7$. This calculation took over one month to complete, on a reasonably fast computer. We were unable to determine whether the following two decomposition numbers are equal 1 or 2:

Case 2: $(i, j) = (p-1, p+1)$,
$\big[S\big\langle (p-1)^2 : p\big\rangle : D\big(\langle p-1, p\rangle\big)\big]$,
Core: $\big(5^2\big) = \big((p-2)^2\big)$,

Case 2: $(i, j) = (p-2, p+1)$,
$\big[S\big\langle (p-1)^2, p-1\big\rangle : D\big(\langle p, p-1, p-2\rangle\big)\big]$,
Core: $\big(4^3\big) = \big((p-3)^3\big)$.

Conjecture 3.1 and our calculations for $p = 5$ imply that these decomposition numbers should both be equal to 1. *Assuming this*, we were able to compute all of the remaining decomposition numbers for Cases 2–4 when $p = 7$. We again found that $[S(\lambda) : D(\mu)] \leqslant 1$ in all cases.

A.1. *Case 2:* $(i, j) = (p, p+1)$

	$\langle p\rangle$	$\langle p-1, p\rangle$	$\langle p^2, p-1\rangle$	$\langle p, p-2\rangle$	$\langle p-2\rangle$	$\langle p-2, p\rangle$	$\langle p, p-3\rangle$	$\langle p-3\rangle$	$\langle p-3, p\rangle$	$\langle p-4\rangle$	$\langle p-4, p\rangle$	$\langle p^2\rangle$	$\langle p, p-1\rangle$	$\langle p-1\rangle$	$\langle p^2, p-2\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p-1, p-2\rangle$	$\langle p^2, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p-1, p-3\rangle$	$\langle p^2, p-4\rangle$	$\langle p, p-1, p-4\rangle$	$\langle p-1, p-4\rangle$
$\langle p^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\langle p, p-1\rangle$	1	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
$\langle p-1\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
$\langle p^2, p-2\rangle$	1	1	1	1	1	1	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.
$\langle p, p-1, p-2\rangle$	1	.	.	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.
$\langle p-1, p-2\rangle$	1	.	1	1	.	1	.	.	.	.	.	1	1	.	1	1	1	.	.	.	.	.	.
$\langle p^2, p-3\rangle$	1	.	.	1	1	1	1	1	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.
$\langle p, p-1, p-3\rangle$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.	.	.	.
$\langle p-1, p-3\rangle$	1	1	1	1	.	1	1	.	1	.	.	.	.	.	1	1	1	1	1	1	.	.	.
$\langle p^2, p-4\rangle$	.	.	.	.	.	.	1	1	1	1	1	.	.	.	.	.	.	1	.	.	1	.	.
$\langle p, p-1, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.
$\langle p-1, p-4\rangle$	.	.	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	1	1	1	1	1	1
$\langle p^3\rangle$	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.
$\langle (p-1)^2, p\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
$\langle (p-1)^2\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	1

A.2. *Case 2:* $(i, j) = (p-1, p)$

	$\langle p-1\rangle$	$\langle p\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1\rangle$	$\langle p, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p, p-4\rangle$	$\langle p, p-1, p-4\rangle$	$\langle (p-1)^2\rangle$	$\langle p-1, p-2\rangle$	$\langle p-2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p-2, p\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-2, p-3\rangle$	$\langle (p-1)^2, p-4\rangle$	$\langle p-1, p-2, p-4\rangle$	$\langle p-2, p-4\rangle$
$\langle (p-1)^2\rangle$	1	1	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\langle p-1, p-2\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
$\langle p-2\rangle$	1	.	1	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
$\langle (p-1)^2, p\rangle$	.	.	1	1	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
$\langle p, p-1, p-2\rangle$	1	1	1	1	1	1	.	.	.	.	.	1	1	1	1	1	.	.	.	.	.	.	.
$\langle p-2, p\rangle$	1	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	1	.	.	.	.	.	.
$\langle (p-1)^2, p-3\rangle$	.	.	.	1	1	.	1	1	1	.	.	.	.	.	1	1	.	1	.	.	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.
$\langle p-2, p-3\rangle$	.	.	.	1	.	1	1	.	1	.	.	.	.	.	1	1	1	1	1	1	.	.	.
$\langle (p-1)^2, p-4\rangle$	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.	1	.	.	1	.	.
$\langle p-1, p-2, p-4\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.
$\langle p-2, p-4\rangle$	.	.	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	1	1	1	1	1	1
$\langle (p-1)^3\rangle$	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.
$\langle (p-2)^2, p-1\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
$\langle (p-2)^2\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	1

A.3. *Case 2:* $(i, j) = (p-2, p-1)$

	$\langle p-2\rangle$	$\langle p\rangle$	$\langle p, p-2\rangle$	$\langle p-1\rangle$	$\langle p-2, p-1\rangle$	$\langle p, p-1\rangle$	$\langle p-1, p-2\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-3\rangle$	$\langle p-1, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-1, p-4\rangle$	$\langle p-1, p-2, p-4\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$	$\langle p-3\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^2, p-4\rangle$	$\langle p-3, p-4\rangle$
$\langle (p-2)^2\rangle$	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\langle p-2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
$\langle p-3\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.
$\langle (p-2)^2, p\rangle$	1	1	.	1	1	1	1	1	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
$\langle p, p-2, p-3\rangle$	1	.	1	.	1	1	1	.	.	1	.	1	.	.	.	1	1	1	1	1	.	.	.	.	.	.
$\langle p-3, p\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	1	1	1	.	.	.	.	.
$\langle (p-2)^2, p-1\rangle$	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	1	.	.	1	1	1	1	1	1	.	.	.	.	.	.	.	1	1	.	1	1	.	.	.
$\langle p-3, p-1\rangle$	1	.	.	.	.	.	1	.	1	.	.	1	.	.	.	.	.	.	1	1	1	1	1	1	.	.
$\langle (p-2)^2, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	1	.	1	1	1	.	.	.	.	.	.	1	1	.	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	1	1	1	1	1
$\langle (p-2)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2, p-2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	1

A.4. *Case 2:* $(i, j) = (p-3, p-2)$

	$\langle p, p-3\rangle$	$\langle p-1, p-3\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p-2\rangle$	$\langle p-3, p-2\rangle$	$\langle p, p-2\rangle$	$\langle p-1, p-2\rangle$	$\langle p-2, p-3\rangle$	$\langle p-2, p\rangle$	$\langle p-2, p-1\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p, p-4\rangle$	$\langle p-1, p-4\rangle$	$\langle p-2, p-4\rangle$	$\langle p-4, p-3\rangle$	$\langle (p-3)^2\rangle$	$\langle p-4\rangle$	$\langle (p-3)^2, p\rangle$	$\langle p, p-3, p-4\rangle$	$\langle p-4, p\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle p-1, p-3, p-4\rangle$	$\langle p-4, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle p-2, p-3, p-4\rangle$	$\langle p-4, p-2\rangle$
$\langle (p-3)^2\rangle$	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\langle p-4\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
$\langle (p-3)^2, p\rangle$	1	.	.	.	1	1	1	.	1	1	.	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.
$\langle p, p-3, p-4\rangle$	.	.	.	.	.	1	1	.	1	.	.	.	.	1	.	.	1	1	1	1	1	.	.	.	.	.	.	.
$\langle p-4, p\rangle$	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	1	1	.	.	.	.	.	.
$\langle (p-3)^2, p-1\rangle$	1	1	1	1	.	.	1	1	1	1	1	1	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.
$\langle p-1, p-3, p-4\rangle$	.	.	.	.	.	.	1	1	.	.	.	.	.	1	1	.	.	.	.	1	1	.	1	1	.	.	.	.
$\langle p-4, p-1\rangle$	1	1	.	1	.	.	.	.	1	.	.	1	1	.	.	.	1	.	.	1	1	1	1	1	1	.	.	.
$\langle (p-3)^2, p-2\rangle$	.	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	.	1	.	.	1	.	1	.	1	1	.	.	.	.	.	.	1	1	.	1	1	.
$\langle p-4, p-2\rangle$	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1
$\langle (p-3)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	.
$\langle (p-4)^2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
$\langle (p-4)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1

A.5. *Case 2:* $(i, j) = (p-1, p+1)$

	$\langle p\rangle$	$\langle p-1\rangle$	$\langle p, p-1\rangle$	$\langle p-1, p\rangle$	$\langle p^2\rangle$	$\langle p^2, p-1\rangle$	$\langle p, p-2\rangle$	$\langle p^2, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p-3, p\rangle$	$\langle p^2, p-3\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle p-4, p\rangle$	$\langle p^2, p-4\rangle$	$\langle p, p-2, p-4\rangle$	$\langle p-4, p-1\rangle$	$\langle (p-1)^2\rangle$	$\langle p-1, p-2\rangle$	$\langle p-2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p-2, p\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-2, p-3\rangle$	$\langle (p-1)^2, p-4\rangle$	$\langle p-1, p-2, p-4\rangle$	$\langle p-2, p-4\rangle$
$\langle (p-1)^2\rangle$	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
$\langle p-1, p-2\rangle$	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
$\langle p-2\rangle$	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
$\langle (p-1)^2, p\rangle$	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
$\langle p, p-1, p-2\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.	.
$\langle p-2, p\rangle$	1	1	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	.	.	.	.	.	.
$\langle (p-1)^2, p-3\rangle$	.	.	.	1	.	1	.	1	1	1	1	1	1	.	1	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	1	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.
$\langle p-2, p-3\rangle$	.	.	.	1	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.	1	1	.	1	1	1	.	.	.
$\langle (p-1)^2, p-4\rangle$	.	.	.	1	.	.	.	.	.	.	1	1	1	.	1	1	1	.	1	.	.	.	.	.	.	1	.	.	1	.	.
$\langle p-1, p-2, p-4\rangle$	.	.	.	.	.	.	.	1	.	1	.	.	1	1	.	.	1	1	.	.	.	.	.	.	.	1	1	.	1	1	.
$\langle p-2, p-4\rangle$	.	.	.	1	.	.	.	.	1	1	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	1	1	1	1	1	1
$\langle (p-1)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.
$\langle (p-2)^2, p-1\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	1	1	.
$\langle (p-2)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	1

A.6. *Case 2*: $(i, j) = (p-2, p)$

	$\langle p-1\rangle$	$\langle p-2, p-1\rangle$	$\langle p-1, p\rangle$	$\langle p-2, p\rangle$	$\langle p, p-1\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p-1, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p, p-1, p-4\rangle$	$\langle (p-1)^2, p-4\rangle$	$\langle p, p-2, p-4\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$	$\langle p-3\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle (p-2)^2, p-4\rangle$	$\langle p-3, p-4\rangle$
$\langle (p-2)^2\rangle$	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\langle p-2, p-3\rangle$	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
$\langle p-3\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.
$\langle (p-2)^2, p-1\rangle$	1	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.
$\langle p-3, p-1\rangle$	.	1	.	1	.	.	1	1	.	1	.	.	.	.	.	.	.	1	1	1	1	1	1	.	.	.	.	.
$\langle (p-2)^2, p\rangle$	1	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
$\langle p, p-2, p-3\rangle$	.	1	.	.	1	1	.	1	1	.	1	.	1	.	.	.	.	.	.	.	1	1	1	1	1	.	.	.
$\langle p-3, p\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	1	.	.
$\langle (p-2)^2, p-4\rangle$	.	.	.	.	.	1	.	.	1	.	1	1	.	1	1	1	1	.	.	.	.	.	.	1	1	.	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	1	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	1	1	1	1	1
$\langle (p-2)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2, p-2\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	1	1

A.7. *Case 2:* $(i, j) = (p-3, p-1)$

	$\langle p-3, p-2\rangle$	$\langle p, p-2\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-2, p-1\rangle$	$\langle p-3, p-1\rangle$	$\langle p-1, p-2\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-2)^2, p\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p, p-2, p-4\rangle$	$\langle p-1, p-2, p-4\rangle$	$\langle p-4, p-3\rangle$	$\langle (p-3)^2\rangle$	$\langle p-4\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-3)^2, p\rangle$	$\langle p, p-3, p-4\rangle$	$\langle p-4, p\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle p-1, p-3, p-4\rangle$	$\langle p-4, p-1\rangle$
$\langle (p-3)^2\rangle$	.	.	.	1	1	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
$\langle p-4\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
$\langle (p-3)^2, p-2\rangle$	.	.	.	1	1	1	.	.	.	1	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
$\langle p-4, p-2\rangle$	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.
$\langle (p-3)^2, p\rangle$	1	1	.	.	.	1	1	1	1	.	1	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.
$\langle p, p-3, p-4\rangle$	1	.	1	.	.	.	1	.	.	.	1	1	.	1	.	1	.	.	1	1	1	1	.	.	.	.
$\langle p-4, p\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	1	1	1	.	.	.
$\langle (p-3)^2, p-1\rangle$	1	1	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	1	.	.	1	.	.
$\langle p-1, p-3, p-4\rangle$	.	.	1	.	.	.	1	1	1	.	.	1	1	1	1	.	.	.	.	.	1	1	.	1	1	.
$\langle p-4, p-1\rangle$	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	1	.	.	.	.	1	1	1	1	1	1
$\langle (p-3)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	1	1	.
$\langle (p-4)^2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.
$\langle (p-4)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1

A.8. *Case 2*: $(i, j) = (p-2, p+1)$

	$\langle p-1, p\rangle$	$\langle p-1, p-2\rangle$	$\langle p-2, p\rangle$	$\langle p-2, p-1\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p^2\rangle$	$\langle p^2, p-1\rangle$	$\langle p^2, p-2\rangle$	$\langle (p-1)^2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p, p-3\rangle$	$\langle p-1, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-4, p-1\rangle$	$\langle (p-1)^2, p-4\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$	$\langle p-3\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^2, p-4\rangle$	$\langle p-3, p-4\rangle$
$\langle (p-2)^2\rangle$	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
$\langle p-2, p-3\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
$\langle p-3\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.
$\langle (p-2)^2, p\rangle$	1	1	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.
$\langle p, p-2, p-3\rangle$	.	.	.	.	.	1	.	.	1	1	.	1	1	1	.	.	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.
$\langle p-3, p\rangle$	.	.	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	.	.	.	.	.
$\langle (p-2)^2, p-1\rangle$	.	.	1	.	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	1	.	.	.
$\langle p-3, p-1\rangle$	1	.	1	.	1	.	.	1	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	.	.
$\langle (p-2)^2, p-4\rangle$	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	1	1	1	1	1	.	.	.	.	.	.	1	1	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	1	.
$\langle p-3, p-4\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1	1	.	1	1
$\langle (p-2)^3\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2, p-2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	1	.
$\langle (p-3)^2\rangle$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	1	1

A.9. *Case 2:* $(i, j) = (p-3, p)$

	$\langle p-2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-2, p\rangle$	$\langle p-3, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-1)^2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-4\rangle$	$\langle p-4, p-3\rangle$	$\langle (p-3)^2\rangle$	$\langle p-4\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle p-4, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-3)^2, p\rangle$	$\langle p, p-3, p-4\rangle$
$\langle (p-3)^2\rangle$	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
$\langle p-4\rangle$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
$\langle (p-3)^2, p-1\rangle$	.	.	1	1	.	.	1	1	1	1	1	.	.	.	.	1	.	1	.	.	.	.	.
$\langle p-1, p-3, p-4\rangle$	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	1	.	1	.	.	.	.	.
$\langle p-4, p-1\rangle$	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	1	1	1	1	.	.	.	.
$\langle (p-3)^2, p-2\rangle$	1	1	.	.	1	.	1	1	1	.	1	1	.	.	.	.	.	1	.	1	.	.	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.
$\langle p-4, p-2\rangle$	.	1	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	1	1	1	1	.	.
$\langle (p-3)^2, p\rangle$	1	.	.	.	.	1	.	.	.	.	.	1	1	.	.	.	.	.	.	1	.	1	.
$\langle p, p-3, p-4\rangle$	.	1	.	.	1	1	.	.	.	.	.	1	1	1	1	.	.	.	.	1	1	1	1
$\langle p-4, p\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	1
$\langle (p-3)^3\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	1	1
$\langle (p-4)^2, p-3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.
$\langle (p-4)^2\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1

A.10. Case 2: $(i, j) = (p-3, p+1)$

	$\langle p-2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p^2, p-2\rangle$	$\langle p^2, p-3\rangle$	$\langle (p-1)^2, p\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^2\rangle$	$\langle p-4\rangle$	$\langle (p-3)^2, p\rangle$	$\langle p-4, p\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle p-4, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle p-4, p-2\rangle$
$\langle (p-3)^2\rangle$	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.
$\langle p-4\rangle$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.
$\langle (p-3)^2, p\rangle$	1	1	.	.	1	.	1	1	.	.	.	1	1	.	.	1	.	1	.	.	.	.	.
$\langle p, p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	1	.	.	.	.	.
$\langle p-4, p\rangle$	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.
$\langle (p-3)^2, p-1\rangle$	.	.	1	1	1	1	1	1	1	1	1	.	1	1	.	.	.	1	.	1	.	.	.
$\langle p-1, p-3, p-4\rangle$	.	.	.	.	.	.	.	.	1	.	.	.	1	1	.	.	.	1	.	1	.	.	.
$\langle p-4, p-1\rangle$	.	.	.	1	1	1	.	1	.	.	1	.	.	.	.	.	.	1	1	1	1	.	.
$\langle (p-3)^2, p-2\rangle$	.	.	.	1	.	1	.	.	1	1	1	.	.	1	1	.	.	.	.	1	.	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.
$\langle p-4, p-2\rangle$	.	.	1	1	.	1	.	.	.	.	1	.	.	1	1	.	.	.	.	1	1	1	1
$\langle (p-3)^3\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1
$\langle (p-4)^2, p-3\rangle$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.
$\langle (p-4)^2\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.

A.11. Case 3: $(i, j, k) = (p-1, p+1, p+1)$

	$\langle p\rangle$	$\langle p^2\rangle$	$\langle p, p-1\rangle$	$\langle p-1, p\rangle$	$\langle p^2, p-1\rangle$	$\langle p^3\rangle$	$\langle p, p-2\rangle$	$\langle p^2, p-2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^2\rangle$	$\langle p-1, p-2\rangle$
$\langle (p-1)^2, p\rangle$	1	1	1	.	.	1	.	.	1	.	.	.
$\langle p, p-1, p-2\rangle$	.	.	1	.	1	1	1	1	1	1	.	.
$\langle (p-1)^2\rangle$	.	1	1	.	1	1	.	.	1	.	1	.
$\langle p-1, p-2\rangle$	.	.	.	1	1	1	.	1	1	1	1	1

A.12. Case 3: $(i, j, k) = (p-2, p, p)$

	$\langle p-1\rangle$	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p-2, p\rangle$	$\langle (p-1)^3\rangle$	$\langle p-1, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$
$\langle (p-2)^2, p-1\rangle$	.	.	1	1	.	1	1	.	1	.	.	1	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	1	.	.	.	.	1	1	1	1	1	.	.
$\langle (p-2)^2\rangle$	1	1	1	1	1	1	1	1	1	.	.	1	.	1	.
$\langle p-2, p-3\rangle$	.	.	.	.	.	.	.	.	1	.	1	1	1	1	1

A.13. Case 3: $(i, j, k) = (p-3, p-1, p-1)$

	$\langle p-2, p-1\rangle$	$\langle p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	1	.	1	1	.	1	1	.
$\langle p-2, p-3, p-4\rangle$	1	.	.	.	.	1	1	.
$\langle (p-3)^2\rangle$	1	1	1	1	1	1	1	1
$\langle p-3, p-4\rangle$	.	.	.	.	.	1	1	1

A.14. Case 3: $(i, j, k) = (p-2, p, p+1)$

	$\langle p-1, p-2\rangle$	$\langle p, p-1\rangle$	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-2, p\rangle$	$\langle p^2, p-1\rangle$	$\langle (p-1)^3\rangle$	$\langle p-1, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$
$\langle (p-2)^2, p-1\rangle$	.	.	.	1	.	.	.	.	1	.	.	.	1	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	.	.	1	.	.	1	.	.	1	1	1	.	.
$\langle (p-2)^2\rangle$	1	1	1	1	1	1	1	1	1	.	.	.	1	.	1	.
$\langle p-2, p-3\rangle$	1	.	.	.	.	1	.	1	1	1	1	1	1	1	1	1

A.15. Case 3: $(i, j, k) = (p-3, p-1, p)$

	$\langle (p-2)^2, p-3\rangle$	$\langle p-2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	1	.	.	1	1	.	.	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	.	.	.	1	1	.
$\langle (p-3)^2\rangle$	1	1	1	1	1	1	1	1	1	1
$\langle p-3, p-4\rangle$	.	1	.	.	.	.	1	1	1	1

A.16. Case 3: $(i, j, k) = (p-3, p-1, p+1)$

	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	1	.	.	.	.	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	1	.	.	1	1	.
$\langle (p-3)^2\rangle$	1	1	1	1	1	1	1	1
$\langle p-3, p-4\rangle$	.	.	1	.	1	1	1	1

A.17. Case 3: $(i, j, k) = (p-2, p+1, p+1)$

	$\langle p-1, p\rangle$	$\langle p^2, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p^2, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p^3\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p^2, p-3\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2\rangle$	$\langle p-2, p-3\rangle$
$\langle (p-2)^2, p-1\rangle$	1	1	1	1	.	.	.	1	1	1	.	.	.	.	1	.	.	.
$\langle p-1, p-2, p-3\rangle$	.	.	.	1	1	.	1	1	1	1	1	1	1	1	1	1	.	.
$\langle (p-2)^2\rangle$	.	.	1	1	.	.	1	.	1	.	.	.	.	.	1	.	1	.
$\langle p-2, p-3\rangle$	.	.	.	.	.	1	1	.	1	.	.	.	.	1	1	1	1	1

A.18. Case 3: $(i, j, k) = (p-3, p, p)$

	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	.	1	.	1	1	1	1	.	1	1	.	1	1	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	.	1	.	.	.	.	.	.	1	1	1	1	.
$\langle (p-3)^2\rangle$	1	.	1	1	1	.	.	1	1	1	1	.	1	.	1	1
$\langle p-3, p-4\rangle$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	1

A.19. Case 3: $(i, j, k) = (p-3, p, p+1)$

	$\langle p, p-1, p-2\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-1)^2, p\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p\rangle$	$\langle p^2, p-2\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	.	.	1	.	.	1	.	.	.	.	1	1	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	1	.	.	.	1	.	.	1	1	1	1	.
$\langle (p-3)^2\rangle$	1	1	.	.	1	1	1	1	1	1	.	1	.	1	1
$\langle p-3, p-4\rangle$	.	1	.	.	.	.	.	1	.	1	.	1	.	1	1

A.20. Case 3: $(i, j, k) = (p-3, p+1, p+1)$

	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^2\rangle$
$\langle (p-3)^2, p-2\rangle$	1	1	1	1	.	.	.	1	1	1	1	.
$\langle p-2, p-3, p-4\rangle$	.	.	.	1	1	.	1	1	1	1	1	.
$\langle (p-3)^2\rangle$	.	.	1	1	.	.	1	.	1	.	1	1
$\langle p-3, p-4\rangle$	.	.	.	.	.	1	1	.	1	.	1	1

A.21. Case 4: $(i, j, k) = (p, p+1, p+1)$

	$\langle p^2\rangle$	$\langle p, p-1\rangle$	$\langle p^3\rangle$	$\langle p^2, p-1\rangle$	$\langle p-1, p\rangle$	$\langle p-1\rangle$
$\langle p^3\rangle$	.	.	1	.	.	.
$\langle p^2, p-1\rangle$	1	.	1	1	.	.
$\langle p-1, p\rangle$	1	1	1	1	1	.
$\langle p-1\rangle$	.	.	1	1	1	1

A.22. Case 4: $(i, j, k) = (p-1, p, p)$

	$\langle p-1\rangle$	$\langle (p-1)^2\rangle$	$\langle p-1, p\rangle$	$\langle p\rangle$	$\langle p, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p-1, p-2\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle p-2\rangle$
$\langle (p-1)^3\rangle$	.	1	1	1	1	1	.	1	.	.	.
$\langle (p-1)^2, p-2\rangle$	.	.	.	.	.	.	.	1	1	.	.
$\langle p-2, p-1\rangle$	1	1	1	.	.	1	1	1	1	1	.
$\langle p-2\rangle$	.	1	.	.	1	1	.	1	1	1	1

A.23. Case 4: $(i, j, k) = (p-2, p-1, p-1)$

	$\langle p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle p-1\rangle$	$\langle p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	1	1	1	1	1	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	.	.	.	.	.	1	1	.	.
$\langle p-3, p-2\rangle$	.	1	.	.	1	1	1	1	1	.
$\langle p-3\rangle$	1	.	.	1	1	.	1	1	1	1

A.24. Case 4: $(i, j, k) = (p-3, p-2, p-2)$

	$\langle p-3, p-2\rangle$	$\langle p-2\rangle$	$\langle p-2, p-3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	1	1	1	.	.
$\langle (p-3)^2, p-4\rangle$	.	.	.	.	1	.	.
$\langle p-4, p-3\rangle$	1	.	.	1	1	1	.
$\langle p-4\rangle$	.	.	1	1	1	1	1

A.25. Case 4: $(i, j, k) = (p-1, p, p+1)$

	$\langle p-1\rangle$	$\langle p\rangle$	$\langle (p-1)^2\rangle$	$\langle p, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p^2\rangle$	$\langle p^2, p-1\rangle$	$\langle p-1, p-2\rangle$	$\langle p, p-2\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle p-2\rangle$
$\langle (p-1)^3\rangle$	1	1	1	.	.	1	1	.	.	.	1	.	.	.
$\langle (p-1)^2, p-2\rangle$	.	.	1	1	1	1	1	1	1	1	1	1	.	.
$\langle p-2, p-1\rangle$	.	.	.	.	.	.	.	.	.	.	1	1	1	.
$\langle p-2\rangle$	1	.	1	.	1	.	1	1	.	1	1	1	1	1

A.26. Case 4: $(i, j, k) = (p-2, p-1, p)$

	$\langle p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-2, p\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle (p-1)^2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-2, p-3\rangle$	$\langle p-1, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	1	1	1	1	1	1	1	1	.	.	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	.	1	.	.	.	1	1	1	1	1	1	1	.	.
$\langle p-3, p-2\rangle$	.	1	.	.	.	1	.	.	.	.	.	1	1	1	.
$\langle p-3\rangle$	1	1	1	.	1	1	.	1	1	.	1	1	1	1	1

A.27. Case 4: $(i, j, k) = (p-3, p-2, p-1)$

	$\langle p-3, p-1\rangle$	$\langle p-2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	1	1	1	1	1	.	.
$\langle (p-3)^2, p-4\rangle$	1	.	.	.	1	1	1	.	.
$\langle p-4, p-3\rangle$	.	.	.	1	.	.	1	1	.
$\langle p-4\rangle$	1	.	1	1	.	1	1	1	1

A.28. Case 4: $(i, j, k) = (p-2, p-1, p+1)$

	$\langle p-1, p-2\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-1)^2\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-1, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	1	1	1	.	1	1	.	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	.	.	1	1	1	1	1	1	1	.	.
$\langle p-3, p-2\rangle$	.	.	.	.	.	.	.	.	1	1	1	.
$\langle p-3\rangle$	1	.	1	1	.	1	.	1	1	1	1	1

A.29. Case 4: $(i, j, k) = (p-3, p-2, p)$

	$\langle (p-3)^2, p-2\rangle$	$\langle p-2, p\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-3)^2, p\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	1	1	1	1	1	.	.
$\langle (p-3)^2, p-4\rangle$	.	.	.	.	1	1	1	.	.
$\langle p-4, p-3\rangle$	1	.	.	1	.	.	1	1	.
$\langle p-4\rangle$	1	.	1	1	.	1	1	1	1

A.30. Case 4: $(i, j, k) = (p-3, p-2, p+1)$

	$\langle p-2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-2)^2\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	.	1	1	1	.
$\langle (p-3)^2, p-4\rangle$	.	.	1	1	1	1	.
$\langle p-4, p-3\rangle$	.	.	.	.	.	1	.
$\langle p-4\rangle$	.	1	1	.	1	1	1

A.31. Case 4: $(i, j, k) = (p-1, p+1, p+1)$

	$\langle p\rangle$	$\langle p^2\rangle$	$\langle p^2, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p^3\rangle$	$\langle p, p-2\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p^2, p-2\rangle$	$\langle p-2, p\rangle$	$\langle (p-1)^3\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle p-2\rangle$
$\langle (p-1)^3\rangle$	1	1	.	.	1	.	.	.	.	1	.	.	.
$\langle (p-1)^2, p-2\rangle$	.	1	1	1	1	1	.	1	.	1	1	.	.
$\langle p-2, p-1\rangle$	.	.	.	1	1	.	1	1	1	1	1	1	.
$\langle p-2\rangle$	.	.	.	.	.	.	.	.	.	1	1	1	1

A.32. Case 4: $(i, j, k) = (p-2, p, p)$

	$\langle p-2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle p, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle (p-1)^3\rangle$	$\langle p-1, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	.	1	1	1	1	1	1	1	1	1	.	.	.	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	.	.	1	.	.	.	.	.	1	1	.	1	.	1	1	.	.
$\langle p-3, p-2\rangle$	1	1	1	.	1	.	1	.	1	1	.	1	1	1	1	1	1	.
$\langle p-3\rangle$	.	.	1	.	.	.	.	1	1	.	.	.	.	.	1	1	1	1

A.33. Case 4: $(i, j, k) = (p-3, p-1, p-1)$

	$\langle p-3, p-2\rangle$	$\langle p-2, p-1\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p-1, p-3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-2)^3\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	1	1	1	1	1	1	.	1	.	.
$\langle (p-3)^2, p-4\rangle$	.	1	.	.	.	.	.	1	.	1	.	.
$\langle p-4, p-3\rangle$	.	.	1	.	1	.	1	1	1	1	1	.
$\langle p-4\rangle$	1	.	.	.	.	1	1	.	.	1	1	1

A.34. Case 4: $(i, j, k) = (p-2, p, p+1)$

	$\langle p-2, p-1\rangle$	$\langle p, p-1\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-1)^2, p\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p^2, p-1\rangle$	$\langle p^2, p-2\rangle$	$\langle (p-1)^3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	1	1	.	1	.	.	1	1	1	.	.	.	.	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	.	1	1	1	1	1	1	1	1	1	1	.	1	1	1	.	.
$\langle p-3, p-2\rangle$	.	.	.	.	.	.	.	.	1	.	.	1	1	.	1	1	1	.
$\langle p-3\rangle$	1	.	.	1	.	1	.	1	.	1	.	.	.	1	1	1	1	1

A.35. Case 4: $(i, j, k) = (p-3, p-1, p)$

	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-3)^2, p\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	1	1	1	1	1	1	1	1	1	.	1	.	.
$\langle (p-3)^2, p-4\rangle$	.	.	.	1	.	.	.	.	1	1	1	.	1	.	.
$\langle p-4, p-3\rangle$	.	1	1	.	.	1	.	1	.	.	1	1	1	1	.
$\langle p-4\rangle$	1	.	1	1	.	.	1	1	.	1	.	.	1	1	1

A.36. Case 4: $(i, j, k) = (p-3, p-1, p+1)$

	$\langle p-1, p-2, p-3\rangle$	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle p, p-1, p-3\rangle$	$\langle (p-3)^2, p-1\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	.	1	.	1	1	1	1	.
$\langle (p-3)^2, p-4\rangle$	.	.	1	.	1	1	1	1	1	.
$\langle p-4, p-3\rangle$	.	.	.	.	.	.	.	1	1	.
$\langle p-4\rangle$	1	.	.	1	1	.	1	.	1	1

A.37. Case 4: $(i, j, k) = (p-2, p+1, p+1)$

	$\langle p-1, p\rangle$	$\langle (p-1)^2, p\rangle$	$\langle (p-1)^2, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-1)^3\rangle$	$\langle p, p-1, p-3\rangle$	$\langle p-1, p-2, p-3\rangle$	$\langle (p-1)^2, p-3\rangle$	$\langle p-3, p-1\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle p-3, p-2\rangle$	$\langle p-3\rangle$
$\langle (p-2)^3\rangle$	1	1	.	.	1	.	.	.	.	1	.	.	.
$\langle (p-2)^2, p-3\rangle$	.	1	1	1	1	1	.	1	.	1	1	.	.
$\langle p-3, p-2\rangle$	.	.	.	1	1	.	1	1	1	1	1	1	.
$\langle p-3\rangle$	.	.	.	.	.	.	.	.	.	1	1	1	1

A.38. Case 4: $(i, j, k) = (p-3, p, p)$

	$\langle p-1, p-2, p-3\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-2, p-3\rangle$	$\langle p, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle p, p-3\rangle$	$\langle (p-3)^2, p\rangle$	$\langle (p-2)^3\rangle$	$\langle p-4, p-2\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4, p-3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	.	1	1	1	1	1	1	1	1	1	.	1	.	.
$\langle (p-3)^2, p-4\rangle$	.	.	.	1	.	.	.	.	.	1	.	1	.	.
$\langle p-4, p-3\rangle$	1	1	1	.	1	.	1	.	1	1	1	1	1	.
$\langle p-4\rangle$	.	.	1	.	.	.	.	1	1	.	.	1	1	1

A.39. Case 4: $(i, j, k) = (p-3, p, p+1)$

	$\langle p-1, p-2, p-3\rangle$	$\langle p, p-1, p-2\rangle$	$\langle p, p-2, p-3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-2)^2, p\rangle$	$\langle (p-3)^2, p\rangle$	$\langle p^2, p-2\rangle$	$\langle p^2, p-3\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^3\rangle$	$\langle p-4\rangle$
$\langle (p-3)^3\rangle$	1	1	.	1	.	.	1	1	1	1	.
$\langle (p-3)^2, p-4\rangle$	.	.	1	1	1	1	1	1	1	1	.
$\langle p-4, p-3\rangle$	.	.	.	.	.	.	.	.	1	1	.
$\langle p-4\rangle$	1	.	.	1	.	1	.	1	.	1	1

A.40. Case 4: $(i, j, k) = (p-3, p+1, p+1)$

	$\langle p, p-1, p-2\rangle$	$\langle (p-2)^2, p-1\rangle$	$\langle (p-2)^2, p-3\rangle$	$\langle (p-3)^2, p-2\rangle$	$\langle (p-2)^3\rangle$	$\langle (p-3)^3\rangle$
$\langle (p-3)^3\rangle$	1	1	.	.	1	1
$\langle (p-3)^2, p-4\rangle$	.	1	1	1	1	1
$\langle p-4, p-3\rangle$	.	.	.	1	1	1
$\langle p-4\rangle$	.	.	.	.	.	1

References

[1] C. Bessenrodt, J.B. Olsson, On residue symbols and the Mullineux conjecture, J. Algebraic Combin. 7 (1998) 227–251.
[2] S. Donkin, A note on decomposition numbers for general linear groups and symmetric groups, Math. Proc. Cambridge Philos. Soc. 97 (1985) 57–62.
[3] J. Brundan, A. Kleshchev, Representation theory of symmetric groups and their double covers, in: Proceedings of Groups, Combinatorics and Geometry, Durham, 2001, World Scientific, River Edge, NJ, 2003, pp. 31–53.
[4] B. Ford, A. Kleshchev, A proof of the Mullineux conjecture, Math. Z. 226 (1997) 267–308.
[5] G.D. James, On the decomposition numbers of the symmetric groups III, J. Algebra 71 (1981) 115–122.
[6] G.D. James, Some combinatorial results involving Young diagrams, Math. Proc. Cambridge Philos. Soc. 83 (1978) 1–10.
[7] G.D. James, The decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leqslant 10$, Proc. London Math. Soc. 60 (1990) 225–265.
[8] G.D. James, A. Kerber, The Representation Theory of the Symmetric Group, in: Encyclopedia Math. Appl., vol. 16, Cambridge Univ. Press, Cambridge, UK, 1981.
[9] G.D. James, A. Mathas, A q-analogue of the Jantzen–Schaper theorem, Proc. London Math. Soc. (3) 74 (1997) 241–274.
[10] G.D. James, A. Mathas, Equating decomposition numbers for different primes, J. Algebra 258 (2002) 599–614.
[11] A.S. Kleshchev, Branching rules for symmetric groups and applications, in: R.W. Carter, J. Saxl (Eds.), Algebraic Groups and Their Applications, Kluwer Academic, Dordrecht, 1998, pp. 103–130.
[12] A. Lascoux, B. Leclerc, J. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996) 205–263.
[13] F. Lübeck, J. Müller, private communication.
[14] F. Lübeck, M. Neunhöffer, Enumerating large orbits and direct condensation, Experiment. Math. 10 (2001) 197–205.
[15] A. Mathas, Decomposition matrices of Hecke algebras of type **A**, in: M. Schönert, et al. (Eds.), Gap: Groups, Algorithms and Programming, 3.4.4, RWTH, Aachen, 1997.
[16] A. Mathas, Hecke Algebras and Schur Algebras of the Symmetric Group, in: Univ. Lecture Ser., vol. 15, Amer. Math. Soc., Providence, RI, 1999.
[17] S. Martin, L. Russell, Defect 3 blocks of symmetric group algebras, J. Algebra 213 (1999) 304–339.
[18] M. Richards, Some decomposition numbers for Hecke algebras of general linear groups, Math. Proc. Cambridge Philos. Soc. 119 (1996) 383–402.
[19] K.D. Schaper, Charakterformeln für Weyl-Moduln und Specht-Moduln in Primcharacteristik, Diplomarbeit, Universität Bonn, 1981.
[20] J.C. Scopes, Cartan matrices and Morita equivalence for blocks of the symmetric groups, J. Algebra 142 (1991) 441–455.
[21] J.C. Scopes, Symmetric group blocks of defect two, Quart. J. Math. Oxford (2) 46 (1995) 201–234.