Spectrum of some triangulated categories

Umesh V. Dubey$^a$, Vivek M. Mallick$^b$,*

$^a$ The Institute of Mathematical Sciences, C.I.T. campus, Taramani, Chennai 600113, India
$^b$ Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193 Bellaterra (Barcelona), Spain

**ARTICLE INFO**

Article history:
Received 9 November 2011
Available online 26 May 2012
Communicated by Michel Van den Bergh

Keywords:
Tensor triangular geometry
Spectrum
Equivariant sheaves
Superschemes

**ABSTRACT**

In this paper, we compute the triangular spectrum (as defined by P. Balmer) of two classes of tensor triangulated categories which are quite common in algebraic geometry. One of them is the derived category of $G$-equivariant sheaves on a smooth scheme $X$, for a finite group $G$. The other class is the derived category of split superschemes.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

This paper studies the prime spectrum of two tensor triangulated categories. Triangulated categories have been one of the most influential objects in mathematics. Introduced by Grothendieck and Verdier to study Serre duality in a relative setting, this idea was soon developed by Verdier and Illusie who studied the derived category of the abelian category of coherent sheaves, and the triangulated category of perfect complexes respectively. Slowly the abstract homological construction of triangulated categories permeated into other subjects like topology, modular representation theory and even Kasparov’s KK theory. Balmer’s paper [3] gives a nice summary of the elegant history.

In algebraic geometry, triangulated categories mostly appear as the derived category of the abelian category of coherent sheaves on a variety and as the category of perfect complexes on a variety. The latter category, as was observed by Neeman [22], are just the compact objects of the derived category of quasi-coherent sheaves (in case the scheme is quasi-compact and separated). From now on we shall call the derived category of the category of coherent sheaves, the derived category of the variety. Gabriel [11] and Rosenberg [24] proved that the category of quasi-coherent sheaves completely determine the underlying variety. Bondal and Orlov [6] proved that...
a smooth variety can be reconstructed from the derived category of coherent sheaves provided that either the canonical bundle or the anti-canonical bundle is ample. But the ampleness condition here is crucial, as Mukai [19] gave an example of two nonisomorphic varieties whose derived categories are equivalent.

Balmer [3] proved that in addition to the triangulated structure on a derived category, if we also consider the tensor structure induced by the tensor structure in the category of coherent sheaves, we have enough information to reconstruct the variety. He gave a method to reconstruct, by constructing “the Spec” of the tensor triangulated category. The definition of Spec is quite general and applies to any tensor triangulated category. Spectrum has been computed for few other triangulated categories, for example [4]. In his ICM talk [4, Section 4.1], Balmer stressed the importance of computing Spec for more examples. We demonstrate two such examples. In both these examples, the Spec turns out to be a scheme. This reconfirms the already known fact that the Spec is not a good invariant of the tensor triangulated category. This raises the question of whether one can define a finer geometric invariant. The first author is presently working on this.

In Section 2, we recall the definition of Spec. We also recall some facts about $G$ sheaves and prove some lemmas which shall be useful in the next section.

In Section 3 we compute the Spec of the derived category of the abelian category of coherent $G$-equivariant sheaves on some smooth quasi-projective scheme $X$. Since the scheme is quasi-projective there exists an orbit space, see [20], which we denote as $X/G$. As $G$ is a finite group and hence we get a finite map $\pi: X \to X/G$ which is also a perfect morphism. Recall that a $G$ equivariant sheaf is defined as follows

**Definition 1.1.** A $G$-sheaf (or $G$-equivariant sheaf or an equivariant sheaf with respect to the group $G$) on $X$ is a sheaf $\mathcal{F}$ together with isomorphisms $\rho_g: \mathcal{F} \to g^*\mathcal{F}$ for all $g \in G$ such that the following diagram

$$
\begin{array}{cccc}
\mathcal{F} & \xrightarrow{\rho_h} & h^*\mathcal{F} & \xrightarrow{h^*\rho_g} & h^*g^*\mathcal{F} \\
\downarrow \rho_{gh} & & & & \downarrow (gh)^* \mathcal{F} \\
\end{array}
$$

is commutative for any pair $g, h \in G$. A $G$-sheaf is a pair $(\mathcal{F}, \rho)$.

The category of coherent $G$-sheaves is denoted as $\text{Coh}^G(X)$ and for simplicity we denote by $D^G(X)$, the bounded derived category of coherent $G$-sheaves. Consider the affine map $\pi: X \to X/G$. Then $D^G(X)$ admits a functor from the category of perfect complexes (see [27]) $D^{\text{per}}(X/G)$,

$$
\pi^*: D^{\text{per}}(X/G) \to D^G(X).
$$

Since we consider only quasi-projective varieties therefore the perfect complexes are nothing but bounded complexes of vector bundles [27].

We prove the following theorem.

**Theorem 1.2.** Assume that the scheme $X$ is a smooth quasi-projective variety over a field $k$ of characteristic $p$ with an action of a finite group $G$. If $p > 0$, assume that the order of $G$ is coprime to $p$. The induced map

$$
\text{Spec}(\pi^*): \text{Spec}(D^G(X)) \to \text{Spec}(D^{\text{per}}(X/G))
$$

is an isomorphism of locally ringed spaces.
The proof involves some computation using results from representation theory.

Finally, in Section 4, we compute the Spec of the tensor triangulated category of perfect complexes over a split superscheme.

Superschemes, studied by Manin and Deligne (see for example [18]), are also an important object of study in modern algebraic geometry, specially due to applications in physics. The following definition of split superscheme is given in Manin [17, pp. 84–85].

**Definition 1.3.**

1. A ringed space \((X, \mathcal{O}_X)\) is called **superspace** if the ring \(\mathcal{O}_X(U)\) associated to any open subset \(U\) is supercommutative and each stalk is local ring. A superspace is called **superscheme** if in addition the ringed space \((X, \mathcal{O}_{X,0})\) is a scheme and \(\mathcal{O}_{X,1}\) is a coherent sheaf over \(\mathcal{O}_{X,0}\) (where the subscript 0 denotes the even part and the subscript 1 denotes the odd part). We shall denote by \(J_X\) the ideal sheaf generated by \(\mathcal{O}_{X,1}\) inside \(\mathcal{O}_X\).

Manin has also given example of superschemes which are not split superschemes. An important example of a split superscheme is super projective space \(\mathbb{P}^n|_m\). We consider the triangulated category \(D^{per}(X)\) of "perfect complexes" (the definition being modified appropriately in the super setting) on this superscheme.

**Theorem 1.4.** Let \(X\) be a quasi-compact, quasi-separated, split superscheme. Let \(X_0 := (X, \mathcal{O}_{X,0})\) be the 0-th part of this superscheme. Here \(X_0\) is by definition a scheme. Then we have an isomorphism of locally ringed spaces

\[
f : X_0 \rightarrow \text{Spec}(D^{per}(X)).
\]

The proof of homeomorphism adapts the classification of thick tensor ideals due to Thomason [26] as demonstrated by Balmer [3]. Again, following Balmer [3] we use the generalized localization theorem of Neeman [22, Theorem 2.1] to finish the proof.

Finally, we would like to mention that recently we came across a paper [16] which proves a version of Theorem 1.2 for stacks. But we would like to mention that our proof is different and is completely scheme theoretic.

This article contains proofs of the results announced in [10]. We also would like to thank the referee for their suggestions.

2. Preliminaries

In this section we shall recall various basic definitions and facts which are used explicitly or implicitly later.

2.1. Some definitions from category theory

As we are borrowing many definitions and results from Balmer’s papers [2,3] so we shall work only with an essentially small categories i.e. categories equivalent to a small category. We recall first some basic definitions.
Definition 2.1 (Triangulated category). An additive category $\mathcal{D}$ with a functorial isomorphism $T$ (called translation or shift), and a collection of sextuple $(a, b, c, f, g, h)$ with objects $a, b, c$ and morphisms $f, g, h$, called distinguished triangles, satisfying certain axioms (cf. [28,14]), is called triangulated category. Traditionally the image of any object, say $a$, via functor $T^i$ is denoted as $a[i]$ and a distinguished triangle is denoted in a similar way as short exact sequences: $a \rightarrow b \rightarrow c \rightarrow a[1]$.

Example 2.2. Let $\mathcal{A}$ be an abelian category and $K^\bullet(\mathcal{A})$ (resp. $D^\bullet(\mathcal{A})$), for $(\bullet = -, +$ or $b)$, be the homotopy (resp. derived) category of an abelian category $\mathcal{A}$. Then both additive categories are triangulated categories, see [14, pp. 25 and 35] for proof. In particular we are interested in the cases when $\mathcal{A} = \text{Coh}^G(X)$ for some variety $X$ with an action of some finite group $G$; see Subsection 2.3 for more details. When group $G$ is trivial then $\mathcal{A}$ is an abelian category of coherent sheaves on variety $X$. Another class of examples which we shall consider later comes from an abelian categories $\mathcal{A} = \text{Coh}(\mathcal{O}_X)$ for some split superscheme $X$.

Example 2.3. The category of perfect complexes on a scheme is a triangulated category. See [27] for definitions.

2.2. Triangular spectrum

In this section we shall recall some definitions and results from Balmer's papers [2] and [3]. Suppose $\mathcal{D}$ is an essentially small triangulated category.

Definition 2.4. A tensor triangulated category is a triple $(\mathcal{D}, \otimes, 1)$ consisting of a triangulated category with symmetric monoidal bifunctor which is exact in each variable. The unit is denoted by 1 (or Id).

Definition 2.5. A thick tensor ideal $\mathcal{A}$ of $\mathcal{D}$ is a full subcategory containing 0 and satisfying the following conditions:

(a) $\mathcal{A}$ is triangulated: if any two terms of a distinguished triangle are in $\mathcal{A}$ then third term is also in $\mathcal{A}$. In particular direct sum of any two objects of $\mathcal{A}$ is again in $\mathcal{A}$ and this we refer as an additivity. This property applied to the distinguished triangles $a[-1] \rightarrow 0 \rightarrow a \xrightarrow{=} a$ and $a \rightarrow 0 \rightarrow a[1] \xrightarrow{+1} a[1]$ shows that $\mathcal{A}$ is closed under translations.

(b) $\mathcal{A}$ is thick: If $a \oplus b \in \mathcal{A}$ then $a \in \mathcal{A}$.

(c) $\mathcal{A}$ is tensor ideal: if $a$ or $b \in \mathcal{A}$ then $a \otimes b \in \mathcal{A}$.

If $\mathcal{E}$ is any collection of objects of $\mathcal{D}$ then we shall denote by $\langle \mathcal{E} \rangle$ the smallest thick tensor ideal generated by this subset in $\mathcal{D}$.

Now we shall give an explicit description of a thick tensor ideal generated by some collection $\mathcal{E}$ in a tensor triangulated category. This description follows Bondal [5]. Recall $\text{add}(\mathcal{E})$ was defined as the additive category generated by $\mathcal{E}$ and closed under taking shifts inside $\mathcal{D}$. Similarly define $\text{ideal}(\mathcal{E})$ as the full subcategory generated by objects of the form $\bigoplus_i a_i \otimes x_i$ where $a_i \in \mathcal{D}$ and $x_i \in \mathcal{E}$. Since $\mathcal{E}$ is contained in $\text{ideal}(\mathcal{E})$, it is closed under taking finite direct sum, shifts and tensoring with any object of $\mathcal{D}$. Recall that there is an operation on subcategories $\mathcal{A}, \mathcal{B}$, denoted by $\mathcal{A} \star \mathcal{B}$, and defined as the full subcategory generated by objects $x$ which fit in a distinguished triangle of the form

$$a \rightarrow x \rightarrow b \rightarrow a[1]$$

with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

As observed in Section 2.2, Bondal et al. [5], if $\mathcal{A}$ and $\mathcal{B}$ are closed under shifts and direct sums then $\mathcal{A} \star \mathcal{B}$ is also closed under shifts and direct sums. Similarly we can see that if $\mathcal{A}$ and $\mathcal{B}$ are tensor ideal then $\mathcal{A} \star \mathcal{B}$ is also tensor ideal. Take $\text{smd}(\mathcal{A})$ to be the full subcategory generated by all direct
summands of objects of \( A \). Now combining these two operations we can define a new operation on collections of subcategories as follows

\[ A \diamond B := \text{smd}(A \ast B). \]

Using this operation we can define the full subcategories \( \langle E \rangle^n \) for each non-negative integer as

\[ \langle E \rangle^n := \langle E \rangle^{n-1} \diamond \langle E \rangle^0 \quad \text{where} \quad \langle E \rangle^0 := \text{smd}(\text{ideal}(E)). \]

Now we can see the following description of ideal generated by a collection \( E \).

**Lemma 2.6.** \( \langle E \rangle = \bigcup_{n \geq 0} \langle E \rangle^n \).

Proof of the above lemma follows from the fact that right hand side subcategory is a thick tensor ideal and contains every thick tensor ideal containing the collection \( E \).

**Definition 2.7.**

(a) An additive functor, \( F : D_1 \to D_2 \), is called exact (or triangulated) if it commutes with translation functor and takes distinguished triangle to a distinguished triangle.

(b) An exact functor, \( F : D_1 \to D_2 \), is called a tensor functor if there exists a natural isomorphism \( \eta(a, b) : F(a) \otimes F(b) \to F(a \otimes b) \) for objects \( a \) and \( b \) of \( D_1 \).

(c) A tensor functor, \( F : D_1 \to D_2 \), is called dominant if \( \langle F(D_1) \rangle = D_2 \).

Note that every unital tensor functor is a dominant tensor functor.

**Definition 2.8.** A prime ideal of \( D \) is a proper thick tensor ideal \( P \subset D \) such that \( a \otimes b \in P \) implies that either \( a \in P \) or \( b \in P \). And triangular spectrum of \( D \) is defined as set of all prime ideals, i.e.

\[ \text{Spc}(D) = \{ P \mid \text{P is a prime ideal of } D \}. \]

The Zariski topology on this set is defined as follows: closed sets are of the form

\[ Z(S) := \{ P \in \text{Spc}(D) \mid S \cap P = \emptyset \}, \]

where \( S \) is a family of objects of \( D \); or equivalently we can define the open subsets to be of the form

\[ U(S) := \text{Spc}(D) \setminus Z(S). \]

In particular, we shall denote by

\[ \text{supp}(a) := Z(\{a\}) = \{ P \in \text{Spc}(D) \mid a \notin P \}, \]

the basic closed sets and hence \( U(\{a\}) \) are the basic open sets.

A collection of objects \( S \subset D \) is called a tensor multiplicative family of objects if \( 1 \in S \) and for \( a, b \in S \), \( a \otimes b \in S \).

We shall recall here the following lemma (see Balmer [3, Lemma 2.2]) which we shall need later,

**Lemma 2.9.** Let \( D \) be a non-zero tensor triangulated category and \( I \subset D \) be a thick tensor ideal. Suppose \( S \subset D \) is a tensor multiplicative family of objects such that \( S \cap I = \emptyset \) Then there exists a prime ideal \( P \in \text{Spc}(D) \) such that \( I \subset P \) and \( P \cap S = \emptyset \).
Balmer [3] had also proved the functoriality of Spc on all essentially small tensor triangulated category with a morphism given by a unital tensor functors but it is not difficult to see that it is also true for an essentially small tensor triangulated categories with morphism given by a dominant tensor functor i.e. we have the following result:

**Proposition 2.10.** Given $F : \mathcal{D}_1 \to \mathcal{D}_2$ a dominant tensor functor, the map $$\text{Spc}(F) : \text{Spc}(\mathcal{D}_2) \to \text{Spc}(\mathcal{D}_1)$$
defined as $P \mapsto F^{-1}(P)$ is well defined, continuous and for all objects $a \in \mathcal{D}_1$, we have $\text{Spc}(F)^{-1}(\text{supp}(a)) = \text{supp}(F(a))$ in $\text{Spc}(\mathcal{D}_2)$.

This defines a contravariant functor $\text{Spc}(-)$ from the category of essentially small tensor triangulated categories with dominant tensor functors as morphisms to the category of topological spaces. So if $F, G$ are two dominant tensor functors then $\text{Spc}(G \circ F) = \text{Spc}(F) \circ \text{Spc}(G)$.

**Proof.** See Balmer [3, Proposition 3.6]. □

Now we shall recall the definition of a structure sheaf defined on $\text{Spc}(\mathcal{D})$ as in Balmer [3, Section 6].

**Definition 2.11.** For any open set $U \subset \text{Spc}(\mathcal{D})$, let $Z := \text{Spc}(\mathcal{D}) \setminus U$ be the closed complement and let $\mathcal{D}_Z$ be the thick tensor ideal of $\mathcal{D}$ supported on $Z$. We denote by $\mathcal{O}_Z$ the sheafification of the following presheaf of rings: $U \mapsto \text{End}(1_U)$ where $1_U \in \mathcal{D}_Z$ is the image of the unit 1 of $\mathcal{D}$ via the localization map. And the restriction maps are defined using localization maps in the obvious way. The sheaf of commutative ring $\mathcal{O}_Z$ makes the topological space $\text{Spc}(\mathcal{D})$ a ringed space, which we shall denote by $\text{Spec}(\mathcal{D}) := (\text{Spc}(\mathcal{D}), \mathcal{O}_Z)$.

The construction of spectrum given in Balmer [3, Theorem 6.3] was extended by Buan, Krause and Solberg [8, Theorem 8.5] from topologically noetherian schemes to more general quasi-compact, quasi-separated schemes.

**Theorem 2.12.** (See Balmer [4, Theorem 54.]) Let $X$ be a quasi-compact and quasi-separated scheme. Suppose $\mathcal{D}^\text{perf}(X)$ denotes the tensor triangulated category of perfect complexes. Then

$$\text{Spec}(\mathcal{D}^\text{perf}(X)) \simeq X$$

as ringed spaces.

### 2.3. $G$-sheaves

Throughout this section, $k$ is a field and $G$ is a finite group whose order is coprime to the characteristic of $k$. By a variety, we mean an integral separated scheme of finite type over $k$. Let $X$ be a smooth quasi-projective variety over $k$, with an action of a finite group $G$ i.e. there is a group homomorphism from $G$ to the automorphism group of algebraic variety $X$. We say $G$ acts freely on $X$ if $gx \neq x$ for any $x \in X$ and any $g \in G$ with $g \neq e$. Recall following general result proved in Mumford’s book [20, p. 66] for the existence of finite group quotient,

**Theorem 2.13.** Let $X$ be an algebraic variety and $G$ a finite group of automorphisms of $X$. Suppose that for any $x \in X$, the orbit $Gx$ of $x$ is contained in an affine open subset of $X$. Then there is a pair $(Y, \pi)$ where $Y$ is a variety and $\pi : X \to Y$ a morphism, satisfying:

1. as a topological space, $(Y, \pi)$ is the quotient of $X$ for the $G$-action; and
2. if $\pi_*\langle \mathcal{O}_X \rangle^G$ denotes the subsheaf of $G$-invariants of $\pi_*\langle \mathcal{O}_X \rangle$ for the action of $G$ on $\pi_*\langle \mathcal{O}_X \rangle$ deduced from 1, the natural homomorphism $\mathcal{O}_Y \to \pi_*\langle \mathcal{O}_X \rangle^G$ is an isomorphism.
The pair \((Y, \pi)\) is determined up to an isomorphism by these conditions. The morphism \(\pi\) is finite, surjective and separable. \(Y\) is affine if \(X\) is affine.

If further \(G\) acts freely on \(X\), \(\pi\) is an étale morphism.

In the remark after the proof [20, p. 69], Mumford further showed that quasi-projective varieties always satisfy the hypothesis of above theorem. We denote this quotient space (if it exists) as \(X/G\).

Definition 2.14.

1. For a variety \(X\) with a \(G\) action, and \(H \subset G\) a subgroup, let \(X^H\) be the subvariety of fixed points of \(H\).
2. A \(G\)-invariant component is defined to be a minimal \(G\)-invariant subvariety of \(X\) with reduced structure such that its dimension is equal to \(\dim X\).

Proposition 2.15. With the notation in the above paragraph,

1. \(X^H\) is a closed subvariety.
2. If \(H_1 \subseteq H_2\) are subgroups then we have a reverse inclusion \(X^{H_2} \subseteq X^{H_1}\).
3. If \(Z\) is any \(G\)-invariant component of \(X\) then there exists an open subset of \(Z\) with free action of \(G/H\) for unique subgroup \(H\).
4. If \(Z\) is any \(G\)-invariant subvariety of \(X\) then there exists the set of subgroups \(H_i\) for \(i = 1, \ldots, r\) such that \(G/H_i\) acts freely on \(W_i\) for \(i = 1, \ldots, r\). Here \(r\) is the number of \(G\)-invariant components of \(Z\). Also note that the open subsets \(W_i\) are pairwise disjoint, and \(\dim(Z \setminus \bigcup_i W_i) < \dim Z\).

Proof. Proof of 1. Since \(X^h = \bigcap_{h \in H} X^h\) where \(X^h\) is a fixed points of automorphism corresponding to \(h\) under the action. It is enough to prove that the invariant of any automorphism of a variety is a closed subset. Since \(X\) is separated, the diagonal and the graph of any automorphism will be closed subset of \(X \times X\). The intersection of graph of automorphism with the diagonal will be closed subset of the diagonal. Hence the invariant of the automorphism \(h\) will be closed in \(X\).

Proof of 2. It clearly follows from the formulae \(X^{H_1} = \bigcap_{h \in H} X^h\).

Proof of 3. Since for any algebraic subvariety there exists the subgroup \(H\) such that \(G/H\) acts faithfully (or effectively), we can assume that \(G\) acts faithfully on \(Z\). Since for a faithful action, \(Z^H\) is a proper subset of \(Z\) for any nontrivial normal subgroup \(H\) of \(G\), the open subset of \(Z\) defined as

\[ W = Z - \left( \bigcup_{H \in G} Z^H \right), \]

where union on right side is over all nontrivial normal subgroups, is non-empty and it is easy to see that \(G\) acts freely on \(W\).

Proof of 4. Using 3, it is enough to prove that any algebraic subset can be uniquely written as union of \(G\)-invariant components of \(Z\), and an algebraic subset of dimension strictly less than \(\dim Z\). Since \(Z\) is noetherian, it will be finite union of irreducible closed subsets. Take finite set \(S\) of generic points of irreducible subsets of \(Z\), which have the same dimension as \(Z\). Now the action of \(G\) on \(Z\) induces an action on the finite set \(S\); since an automorphism of \(Z\) will take any irreducible subset to another irreducible subset of the same dimension. Thus \(S\) can be uniquely written as a disjoint union of \(G\)-invariant subsets. By taking union of closure of these generic points in each invariant subset, we get the \(G\)-invariant components of \(Z\). Clearly, any non-empty intersection of \(W_i\) and \(W_j\) for \(i \neq j\) will give a proper \(G\)-invariant component, and this will contradict the minimality.

We shall now look at some properties of \(G\)-sheaves (Definition 1.1). The \(G\)-sheaves form a category \(\text{QCoh}^G(X)\) as follows. Given two \(G\)-sheaves \((\mathcal{F}, \rho)\) and \((\mathcal{G}, \psi)\), the group of morphisms
of $\mathcal{O}_X$-modules $\text{Hom}_X(\mathcal{F}, \mathcal{G})$ gets a $G$ action, where $g \in G$ acts on $\theta$ to give $\psi_g^{-1} \circ g^*\theta \circ \rho_g$. $\text{Hom}_{\mathcal{Q}\text{Coh}^G(X)}(\mathcal{F}, \mathcal{G}, \psi)$ is defined to be the group of $G$-invariant morphisms in $\text{Hom}_X(\mathcal{F}, \mathcal{G})$.

$\mathcal{Q}\text{Coh}^G(X)$ is an abelian category. Define $\text{Coh}^G(X)$ to be the abelian subcategory of $\mathcal{Q}\text{Coh}^G(X)$ consisting of objects $(\mathcal{F}, \rho)$, for which $\mathcal{F}$ is coherent. In Tohoku paper of Grothendieck [13] it was proved that $\mathcal{Q}\text{Coh}^G(X)$ has enough injectives. Also, for finite $G$ and quasi-projective $X$, there is an ample invertible $G$-sheaf, allowing $G$-equivariant locally free resolutions (see [13,7]). Therefore derived functors of various functors like $\pi_*, \pi^*$ and $\otimes$ will always exist, in a similar fashion as in the non-equivariant case, and for simplicity we shall write $\pi_*, \pi^*$ and $\otimes$ for $R\pi_*, R\pi^*$ and $\otimes^L$ respectively.

**Definition 2.16.** Let $\mathcal{D}^G(X)$ be the bounded derived category of $\text{Coh}^G(X)$.

**Remark 2.17.** $\text{Coh}^G(X)$ can be given a tensor structure $\otimes: \text{Coh}^G(X) \times \text{Coh}^G(X) \to \text{Coh}^G(X)$ in the obvious way. The tensor structure on the derived category $\mathcal{D}^G(X)$ is given by the derived functor of $\otimes$. For more details on this tensor structure, one can refer to [7]. Also $\mathcal{D}^G(X)$ has a natural structure of a $k$-linear category. We shall use this fact later.

**Remark 2.18.** Note that here and elsewhere (for example Theorem 1.2), we assume $X$ to be smooth to make the definition of $\mathcal{D}^G(X)$ meaningful. It might be possible that with a proper but more general definition of $\mathcal{D}^G(X)$ we can remove the assumption that $X$ is smooth. But we will not consider that question in this paper.

Given an algebraic variety $X$ with an action of a finite group $G$ we have a natural morphism $\pi : X \to Y := X/G$ which further gives a functor $\pi_* : \text{Coh}^G(X) \to \text{Coh}^G(Y)$ and by taking $G$-invariant part of image we can define a functor $\pi_*^G : \text{Coh}^G(X) \to \text{Coh}^G(Y)$ i.e. $\pi_*^G(\mathcal{F}, \rho) = (\pi_*^G(\mathcal{F}, \rho))^G$ for all $(\mathcal{F}, \rho) \in \text{Coh}^G(X)$. We have the following result when $G$ acts freely on $X$ (see Mumford’s book [20] for proof).

**Proposition 2.19.** Let $\pi : X \to Y$ be a natural quotient morphism given by free action of the finite group $G$ on $X$. The map $\pi_*^G : \text{Coh}^G(Y) \to \text{Coh}^G(X)$ is an equivalence of abelian categories with the quasi-inverse $\pi_*^G$. Further locally free sheaves corresponds to locally free sheaves of the same rank.

We can extend above equivalence to get a tensor equivalence $\pi_*^G$ between the categories $\mathcal{D}^b(Y)$ and $\mathcal{D}^G(X)$.

Next we prove that there exists a canonical (or isotypic) decomposition, similar to finite dimensional representation of finite groups. Suppose $X$ is a smooth quasi-projective variety over a field $k$, with the structure morphism $\eta : X \to \text{Spec}(k)$. The category of all coherent sheaves on affine variety $\text{Spec}(k)$ can be identified with category of all finite dimensional vector spaces and the category of all $G$-equivariant sheaves can be identified with finite dimensional $k$-linear $G$ representations. See [7] for details.

Let $(\mathcal{G}, \lambda)$ be an object in $\text{Coh}^G(X)$. We shall denote $\eta^*(V) \otimes \mathcal{G}$ by $V \otimes \mathcal{G}$ for simplicity.

For the trivial action of $G$ on $X$, the association of a $G$-sheaf $\mathcal{G}$ on $X$ to its $G$-invariant subsheaf is functorial. More precisely, the exact functor

$$\langle \_ \rangle^G : \text{Coh}^G(X) \to \text{Coh}^G(X)$$

induces an exact functor

$$\langle \_ \rangle^G : \mathcal{D}^G(X) \to \mathcal{D}^G(X).$$

Note that the action of $G$ on an object in the image of this functor is trivial. Thus the image of $\langle \_ \rangle^G$ lies in $\mathcal{D}^b(X)$, where $\mathcal{D}^b(X)$ is considered as a subcategory of $\mathcal{D}^G(X)$ consisting of objects with trivial
G-action (see first paragraph of Section 4.4 in [7]). For a vector space \( V \) over \( k \) with an action of \( G \), define the exact functor

\[
\text{Hom}_G(V, \_ ) = (\eta^* V \otimes \_)^G : D^G(X) \to D^G(X).
\]

Notice that each object contained in the image of the functor \( \text{Hom}_G(V, \_ ) \) are trivial \( G \)-sheaves. Thus the image of \( \text{Hom}_G(V, \_ ) \) lies in \( D^b(X) \). Let \( V_\lambda \) be an irreducible representation of the group \( G \). We have the evaluation map from \( V_\lambda \otimes V_\lambda^* \) to \( k \). We can pullback the usual evaluation map from the representation category to the bounded derived category of \( G \)-equivariant sheaves. Thus we have the following morphism,

\[
\eta^*(ev_{V_\lambda}) \otimes id : V_\lambda \otimes V_\lambda^* \otimes F \to F.
\]

Now by using the fact that the \( G \)-invariant part of a \( G \)-module \( V \) is a direct summand of \( V^* \otimes V \), and the map \( \eta^*(ev) \otimes id \) we get the following map, which we denote by \( ev_F \),

\[
ev_F : \bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, F) \to F.
\]

We have the following lemma which is used later to prove canonical decomposition.

**Lemma 2.20.** The association sending \( F \) to \( \bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, F) \) gives an exact functor from \( D^G(X) \) to itself. Further, the objectwise morphism \( ev_F \) induces a natural transformation between this functor and the identity functor.

**Proof.** Since the association \( \text{Hom}_G(V, \_ ) \) is a functor, it is easy to see that the association taking \( F \) to \( \bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, F) \) is functorial. Consider a morphism \( f : F_1 \to F_2 \) in \( D^G(X) \). Now the naturality of the morphism \( ev \) follows from the commutativity of following diagrams,

\[
\begin{array}{ccc}
\bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, F_1) & \longrightarrow & \bigoplus\lambda V_\lambda \otimes V_\lambda^* \otimes F_1 \\
\downarrow & & \downarrow f \\
\bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, F_2) & \longrightarrow & \bigoplus\lambda V_\lambda \otimes V_\lambda^* \otimes F_2
\end{array}
\]

Here \( f : F_1 \to F_2 \) is a morphism compatible with the action of the finite group \( G \) and therefore gives commutativity of the left square. \( \Box \)

We recall a general result about \( G \) actions.

**Lemma 2.21.** Suppose \( M \) is a \( k \)-linear \( G \)-representation (need not be finite dimensional) for finite group \( G \). The following canonical evaluation map is an isomorphism

\[
ev : \bigoplus\lambda V_\lambda \otimes \text{Hom}_G(V_\lambda, M) \to M.
\]

**Proof.** See [12, Proposition 4.1.15]. \( \Box \)
Definition 2.22. We define amplitude length to be the integral function

\[
\text{ampl}: D^G(X) \to \mathbb{Z}; F^i \mapsto \left| \left\{ i \in \mathbb{Z} \mid H^i(F) \neq 0 \right\} \right|
\]

that is, it is the number of non-zero hypercohomologies \(H^i\) of a bounded complex.

We prove the canonical decomposition of any object using pullback and reduction to affine case.

Proposition 2.23. Suppose \(X\) is an algebraic set (need not be a smooth variety) over a field \(K\) with trivial action of a finite group \(G\) whose order is coprime to \(\text{char}(K)\). Let \(F^i\) be a bounded complex of \(G\)-equivariant coherent sheaves i.e. \(\text{ampl}(F^i) < \infty\). There exists a direct sum decomposition of \(F^i\) as follows

\[
F^i = \bigoplus_{\lambda} V_\lambda \otimes F^i_{\lambda}
\]

where \(V_\lambda\) are finite dimensional irreducible representations of \(G\) and \(F^i_{\lambda} = (V^*_\lambda \otimes F^i)^G =: \text{Hom}_G(V_\lambda, F^i)\). Here the complexes \(F^i_{\lambda}\) are trivial \(G\)-equivariant sheaves or usual sheaves.

Proof. We shall divide proof into two steps. In the first step we prove the case of coherent sheaf concentrated in degree zero (which we refer as a pure sheaf). In the second step we prove isomorphism of the map \(ev\) using the first step.

Step 1. Let \(F^i\) be a complex with a coherent sheaf concentrated at zero, say \(F\). We can assume that the variety \(X\) is affine as it is enough to prove isomorphism on any affine cover. Hence we can assume that \(F = M\). Thus we reduce the problem to proving that the following map is a bijection.

\[
ev: \bigoplus_{\lambda} V_\lambda \otimes \text{Hom}_G(V_\lambda, M) \to M.
\]

This map is an equivariant morphism, see [12, p. 184] for more discussions on this. It is enough to prove that the map \(ev\) is bijection as a \(k\)-linear morphism but this follows from Lemma 2.21.

Step 2. Since \(ev\) is a natural transformation, the full subcategory of \(D^G(X)\), on which \(ev\) is a natural isomorphism, is thick. By Step 1, it contains shifts of sheaves and hence must be the whole of \(D^G(X)\).

Hence using these two steps we have the canonical decomposition as stated and further it is easy to observe that \(F^i_{\lambda}\) are trivial as \(G\)-sheaves i.e. all \(\rho_g\) are identity, see Definition 1.1. \(\square\)

We shall use Proposition 2.23 in the following form.

Corollary 2.24. Let \(X\) be a smooth algebraic variety defined over \(k\) with a \(G\) action. Let \(U \subset X\) be a (possibly singular) \(G\)-invariant, locally closed subset, with \(i_U: U \to X\) being the inclusion. Suppose \(H\) is a subgroup of \(G\) with the property that it acts trivially on \(U\). Then for any object \((G, \rho) \in D^G(X)\) we have the canonical decomposition,

\[
(F, \rho) = \bigoplus_{\lambda} W_\lambda \otimes (F, \rho)_\lambda
\]

where \(F = i^*_U G\) and \((F, \rho)_\lambda = (W^*_\lambda \otimes (F, \rho))^H\) and \(W_\lambda\) is a finite dimensional irreducible representation of the subgroup \(H\), and sum is over all finite dimensional irreducible representation of \(H\). The subgroup \(H\) acts trivially on \((F, \rho)_\lambda\), and this will induce the natural action of the group \(G/H\) on \((F, \rho)_\lambda\).
Proof. Note that \( G \) has finite amplitude length, and hence so does \( \mathcal{F} \). Thus the above proposition applies to \( U \). \( \square \)

Definition 2.25. For an object \( \mathcal{F} \) of \( \mathcal{D}^G(X) \), we define

\[
\text{supph} \mathcal{F} = \bigcup_{i \in \mathbb{Z}} \text{supp} H^i(\mathcal{F}).
\]

Now we shall give a distinguished triangle for any complex of \( G \)-equivariant coherent sheaf \( \mathcal{F} \) over \( X \). We have the following result:

Proposition 2.26. Let \( G, k, X \) and \( Y \) be as above.

1. Suppose \( V \) is a \( G \)-invariant open subset of \( Y \) with the induced action of \( G \) on \( V \), which is trivial. Then for \( G \) in \( \mathcal{D}^G(Y) \),

\[
i^*_V(g^G) = (i^*_V g)^G.
\]

2. Suppose \( G \) acts faithfully on \( X \). If \( \mathcal{F} \in \mathcal{D}^G(X) \) with \( \text{supph}(\mathcal{F}) = X \) then we have a distinguished triangle

\[
\pi^* \pi^*_+ \mathcal{F} \to \mathcal{F} \to \mathcal{F}_1
\]

with \( \text{supph}(\mathcal{F}_1) \subset \text{supph}(\mathcal{F}) \). Same is true if we have faithful action of \( G \) on \( \text{supph}(\mathcal{F}) \subset X \).

Proof. Proof of 1. It follows from the definition of \( G \)-equivariant functions.

Proof of 2. Since \( G \) acts faithfully on \( X \) we can use Proposition 2.15 to get an open subset \( U \subseteq X \) with free action of the group \( G \). We shall use induction on amplitude length, \( \text{ampl}(\mathcal{F}) \). When \( \text{ampl}(\mathcal{F}) = 1 \) then \( \mathcal{F} \) is a shift of a coherent sheaf so enough to prove for coherent sheaf. Now using the fact that \( \text{supph}(\mathcal{F}) = X \) we have \( i_U^*(\mathcal{F}) \neq 0 \). There is a natural morphism coming from adjunction and inclusion of \( G \)-invariant part, say \( \eta : \pi^* \pi^+_G(\mathcal{F}) \to \mathcal{F} \). Using flat base change and part 1 of 2.26 we get an isomorphism \( i^*_U \pi^* \pi^+_G(\mathcal{F}) \simeq \pi^* \pi^+_G(i^*_U \mathcal{F}) \). Now this will give an isomorphism, as \( G \) act freely on \( U \), i.e. \( i^*_U(\eta) : i^*_U \pi^* \pi^+_G(\mathcal{F}) \to i^*_U \mathcal{F} \) is an isomorphism. Hence cone of the map \( \eta \) will have support outside an open set \( U \). This completes the first step of induction.

Now assume that for all \( \mathcal{F} \) with \( \text{ampl}(\mathcal{G}) \leq (n-1) \) we have such a distinguished triangle. Now consider \( \mathcal{F} \) with \( \text{ampl}(\mathcal{F}) = n \) with highest cohomology in degree \( n \). We have usual truncation distinguished triangle \( \tau^{(n-1)}(\mathcal{F}) \to \mathcal{F} \to \mathcal{H}^n(\mathcal{F})[-n] \). Using exactness of \( i^*_U \) and argument similar to first step of induction we have the following commutative diagram (we have used the same notation \( \eta \) for different sheaves),

\[
\begin{array}{ccc}
i^*_U \pi^* \pi^+_G \tau^{(n-1)}(\mathcal{F}) & \to & i^*_U \pi^* \pi^+_G \mathcal{F} \\
\downarrow i^*_U(\eta) & & \downarrow i^*_U(\eta) \\
i^*_U \tau^{(n-1)}(\mathcal{F}) & \to & i^*_U \mathcal{F}
\end{array}
\]

Since both the extreme vertical arrows are isomorphism using induction hypothesis, we have isomorphism of the middle \( i^*_U(\eta) \). Therefore cone of the map \( \eta \) will have proper support. \( \square \)

Lemma 2.27. Let \( \pi : X \to Y \) be the quotient map as before.

1. Given \( \mathcal{F} \in \mathcal{D}^G(X) \) we have \( \text{supph}(\pi_* \mathcal{F}) = \pi(\text{supph} \mathcal{F}) \).
2. There exists a tower of distinguished triangles for each object \( F \) in \( D^G(X) \),

\[
\begin{array}{cccc}
F_0 & \rightarrow & F_1 & \rightarrow & \ldots & \rightarrow & F_{m-1} & \rightarrow & F_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1 & & \ldots & & \ldots & & G_{m-1} & & G_m
\end{array}
\]

where \( G_i = \bigoplus_j W_{\lambda_j} \otimes \pi^* \pi_*^{G/H_j}(F_{\lambda_i}) \) with the sum being over the irreducible representations of the corresponding \( H_j \)'s, \( \text{supp}(F_m) \subseteq \cdots \subseteq \text{supp}(F) \).

Furthermore,

\[
\text{supp}(\pi_*^{G/H_j}(F_{\lambda_i})) \subseteq \text{supp}(\pi_*(F_{\lambda_i})) = \pi(\text{supp}(F_{\lambda_i})).
\]

**Proof.** Proof of 1. Consider \( F \in D^G(X) \) a complex of \( G \)-sheaves. We have the special case of the Grothendieck–Leray spectral sequence \([15, \text{p. } 74, (3.4)]\) as follows

\[
E_2^{p,q} = R^p \pi_*(\mathcal{H}^q(F)) \Rightarrow R^{p+q} \pi_*(F).
\]

Since \( R^p \pi_* = 0 \) for each \( p > 0 \) the above spectral sequence will degenerate and we get that \( \pi_*(\mathcal{H}^i(F)) = \mathcal{H}^i(\pi_*(F)) \). Here as before \( \mathcal{H}^i(F) \) represents the \( i \)-th cohomology sheaf of the complex \( F \). Now this will give the equality,

\[
\text{supp}(\pi_*(F)) = \bigcup_i \text{supp}(\mathcal{H}^i(\pi_*(F))) = \bigcup_i \text{supp}(\pi_* \mathcal{H}^i(F)).
\]

Suppose we prove the assertion for pure sheaves, i.e. complexes of sheaves concentrated on degree 0, then the following observation will complete the proof.

\[
\text{supp}(\pi_*(F)) = \bigcup_i \text{supp}(\pi_* \mathcal{H}^i(F)) = \bigcup_i \pi(\text{supp}(\mathcal{H}^i(F))) = \pi(\text{supp}(F)).
\]

Now it remains to prove the assertion for pure sheaves. We shall denote by \( F_{\mid U} \) the restriction of the sheaf \( F \) on the open set \( U \) of \( X \). Suppose \( V_j \) is an open affine cover of \( Y \) and \( U_j := \pi^{-1}(V_j) \) is the affine open cover of \( X \). We shall denote the restriction of the map \( \pi \) on \( U_j \) with the same notation \( \pi \).

Now using the flat base change we have \( \pi_*(F_{\mid U_j}) = \pi_*(F)_{\mid V_j} \) for any sheaf \( F \) on \( X \). Suppose the above assertion is true for affine case then the following observations will complete the proof.

\[
\pi(\text{supp}(F)) = \pi \left( \bigcup_j \text{supp}(F) \cap U_j \right) = \bigcup_j \pi(\text{supp} F_{\mid U_j}) = \bigcup_j \text{supp}(\pi_*(F)_{\mid V_j}) = \bigcup_j \text{supp}(\pi_* F) \cap V_j = \text{supp}(\pi_* F).
\]

It remains to prove the assertion for pure sheaves on affine varieties. Suppose \( \pi : \text{Spec } B \rightarrow \text{Spec } A \) is a quotient map for the action of \( G \) on \( \text{Spec } B \), and \( N \) is a pure \( G \)-equivariant sheaf on \( \text{Spec}(B) \), corresponding to the \( B \)-module \( N \). Since \( A \) and \( B \) are noetherian rings, this reduces to the following fact:

\[
V(\text{ann}(A)N) = \pi(V(\text{ann}(N))).
\]
Here $\text{ann}(N)$ denotes the annihilator ideal and $V(\text{ann}(N))$ denotes the closed set given by all prime ideal containing the ideal $\text{ann}(N)$. Let $\tilde{\pi} : A \to B$ be the algebra map corresponding to $\pi$.

Now to show $V(\text{ann}(AN)) = \pi(V(\text{ann}(N)))$ it is enough to prove that

$$\tilde{\pi}^{-1}(\text{ann}(N)) = \text{ann}(AN).$$

This follows as $x \in \tilde{\pi}^{-1}(\text{ann}(N))$ iff $\tilde{\pi}(x)N = 0$. This is equivalent to $x(AN) = 0$ which in turn holds iff $x \in \text{ann}(AN)$. This concludes the proof of 1.

**Proof** of 2. To prove the first part we use induction on the dimension of the homological support of $\mathcal{F}$. Note that the homological support is invariant under the action of $G$. If dimension is zero then it will be set of $G$-invariant points and we shall get the direct sums of skyscrapers on these points. If we have free action of $G/H$ for some subgroup $H$ then we have the canonical decomposition by 2.24. This proves that the induction starts.

For the induction step, assume that for all $\tilde{G}$ with $\dim \text{supph}(\tilde{G}) \leq n - 1$, we have a tower as in the statement of the lemma. Now consider $\mathcal{F}$ with $\dim \text{supph}(\mathcal{F}) = n$. Here $\text{supph}(\mathcal{F})$ is a union of $G$-invariant components and using Proposition 2.15 we get subsets $U_i$, open in $\text{supph}(\mathcal{F})$ for $i = 1, \ldots, r$ and subgroups $H_i$ for $i = 1, \ldots, r$. As observed before, these $U_i$ are mutually disjoint and there is a free action of group $G/H_i$ on $U_i$ for $i = 1, \ldots, r$. Consider the open subset $U_1 \subset \text{supph}(\mathcal{F})$. Let $i_{U_1}$ be the inclusion of $U_1$ in $X$. By 2.24, we can decompose $i_{U_1}^*(\mathcal{F})$ as

$$i_{U_1}^*(\mathcal{F}) = \bigoplus_\lambda W_\lambda \otimes \mathcal{F}_\lambda,$$

where each $W_\lambda$ is an irreducible representation of subgroup $H_1$, and the $\mathcal{F}_\lambda$'s are $G/H_1$-sheaves over the open subset $U_1$. Using adjunction and 2.19, we get a canonical isomorphism, $\eta_\lambda : \pi^*\pi_\lambda^*G/H_1(\mathcal{F}_\lambda) \to \mathcal{F}_\lambda$ in $\mathcal{D}^G(U_1)$. Putting these together, we get an isomorphism

$$\bigoplus_\lambda W_\lambda \otimes \pi^*\pi_\lambda^*G/H_1(\mathcal{F}_\lambda) \xrightarrow{\sim} i_{U_1}^*(\mathcal{F}) = \bigoplus_\lambda W_\lambda \otimes \mathcal{F}_\lambda. \tag{1}$$

Let $\mathcal{F}_{\lambda_1} = (i_{U_1})_*\mathcal{F}_\lambda$. Then, $\mathcal{F}_\lambda \cong i_{U_1}^*\mathcal{F}_{\lambda_1}$, since the adjunction map $i_{U_1}^*i_{U_1}^!\mathcal{F}_{\lambda_1} \to \mathcal{F}_\lambda$ induces an isomorphism on stalks, as $U_1$ is open in $\text{supph}(\mathcal{F}_{\lambda_1})$. Also, since $U_1$ is open in $\text{supph}(\mathcal{F})$, there exists an open subset $\tilde{U}_1 \subset X$ such that $\tilde{U}_1 \cap \text{supph}(\mathcal{F}) = U_1$. Let $\tilde{U}_1 = \tilde{U}_1 \cup (X \setminus \text{supph}(\mathcal{F}))$, and $V_1 = \pi(\tilde{U}_1)$.

Now we shall prove that

$$\pi^*\pi_\lambda^*G/H_1^*i_{U_1}^!(\mathcal{F}_{\lambda_1}) \cong i_{U_1}^!\pi^*\pi_\lambda^*G/H_1(\mathcal{F}_\lambda).$$

This follows from flat base change and some functorial properties, by considering the diagram,

$$\begin{array}{ccc}
\tilde{U}_1 & \xrightarrow{i_{U_1}} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
V_1 & \xleftarrow{i_{V_1}} & Y
\end{array}$$

and from the following sequence of canonical isomorphisms:

$$i_{\tilde{U}_1}^!(\pi^*\pi_{\lambda_1}^*G/H_1(\mathcal{F}_{\lambda_1})) \cong \pi^*i_{V_1}^!(\pi_{\lambda_1}^*(\mathcal{F}_{\lambda_1}))^{G/H_1} \cong \pi^*(i_{V_1}^!\pi_{\lambda_1}^*(\mathcal{F}_{\lambda_1}))^{G/H_1} \cong \pi^*(\pi_{\lambda_1}^*i_{U_1}^!(\mathcal{F}_{\lambda_1}))^{G/H_1} \cong \pi^*\pi_{\lambda_1}^*G/H_1^*i_{U_1}^!(\mathcal{F}_{\lambda_1}).$$
Proposition 3.1. The morphism of locally ringed spaces

\[ \text{Spec} \pi^* : \text{Spec} D^G(X) \to \text{Spec} D^\text{perf}(Y) \]

is an isomorphism. Since by Balmer [4, Theorem 54] \( \text{Spec} D^\text{perf}(Y) \cong Y \) as schemes,

\[ \text{Spec} D^G(X) \cong Y \]

as schemes.

We only have to prove the first isomorphism. We know there are two exact functors \( \pi^* : D^\text{perf}(Y) \to D^G(X) \) and \( \pi_* : D^G(X) \to D^\text{perf}(Y) \). We also know that the map \( \pi^* \) is a unital tensor functor and hence it will give the map \( \text{Spec}(\pi^*) : \text{Spec}(D^G(X)) \to \text{Spec}(D^\text{perf}(Y)) \). Note that \( \pi_* \) need not be a tensor functor. We shall prove that \( \text{Spec}(\pi^*) \) is a closed bijection and induces an isomorphism for the structure sheaves.

To simplify the proof we will break it in several steps. The first two steps will prove that \( \text{Spec}(\pi^*) \) gives a bijection of sets on the underlying topological spaces of the two Specs in question. The next step will show that the underlying topological spaces are homeomorphic. Then finally in Step 4 we prove that the Specs of the tensor triangulated categories under consideration, are isomorphic as ringed spaces.

Step 1: \( \text{Spec}(\pi^*) \) is onto

Suppose \( q \in \text{Spec}(D^\text{perf}(Y)) \) is a prime ideal then we want to construct a prime ideal \( p \) in \( \text{Spec}(D^G(X)) \) such that \( q = (\pi^*)^{-1}(p) \). Recall that \( (\pi^*)(q) \) denotes the thick tensor ideal generated by the image of \( q \) via functor \( \pi^* \) in a tensor triangulated category \( D^G(X) \). We have a following lemma which uses the explicit description of thick tensor ideal \( (\pi^*)(q) \).

Lemma 3.2. \( \pi_*(\langle \pi^*(q) \rangle) \subseteq q \).
Proof. To prove this lemma, we use Lemma 2.6 i.e.,

\[ \langle \pi^*(q) \rangle = \bigcup_{n \geq 0} \langle \pi^*(q) \rangle^n \]

where \( \langle \pi^*(q) \rangle^n \) constructed inductively by taking \( \langle \pi^*(q) \rangle^0 \) as the summands of tensor ideal generated by \( \pi^*(q) \) and \( \langle \pi^*(q) \rangle^n \) to be the thick tensor ideal containing cone of morphism between any two objects of \( \langle \pi^*(q) \rangle^{(n-1)} \) and \( \langle \pi^*(q) \rangle^0 \). Here cone of a morphism refers to the third object of any distinguished triangle having this morphism as a base or equivalently we can use \( \circ \) operation. The above equality follows from Lemma 2.6 proved earlier.

We shall use induction on \( n \) in the above explicit description. For \( n = 0 \), given \( \mathcal{F} \in \mathcal{q} \),

\[ \pi_*(\pi^*(\mathcal{F}) \otimes \mathcal{G}) = \mathcal{F} \otimes \pi_*(\mathcal{G}) \in \mathcal{q}, \]

and hence \( \pi_*(\langle \pi^*(q) \rangle^0) \subseteq \mathcal{q} \) using thickness of \( \mathcal{q} \).

Using induction suppose we know that \( \pi_*(\langle \pi^*(q) \rangle^{(n-1)}) \subseteq \mathcal{q} \). Since \( \pi_* \) is an exact functor, it follows that the image under \( \pi_* \) of a cone of any morphism is a cone of \( \pi_* \) of the morphism. Hence using the triangulated ideal property and thickness of \( \mathcal{q} \) it follows that \( \pi_*(\langle \pi^*(q) \rangle^n) \subseteq \mathcal{q} \). Therefore we have \( \pi_*(\langle \pi^*(q) \rangle) = \pi_*(\bigcup_{n \geq 0} \langle \pi^*(q) \rangle^n) \subseteq \mathcal{q}. \]

Lemma 3.3. \( \pi^*(\mathcal{D}^\text{per}(Y) \setminus \mathcal{q}) \cap \langle \pi^*(q) \rangle = \emptyset. \)

Proof. To prove this by contradiction, suppose that there exists an object \( \mathcal{G} \in (\mathcal{D}^\text{per}(Y) \setminus \mathcal{q}) \) such that \( \pi^*(\mathcal{G}) \in \langle \pi^*(q) \rangle \). Then using the above lemma \( \pi_*(\pi^*(\mathcal{G}) \subseteq \mathcal{q} \). On the other hand, the projection formula implies \( \pi_*(\pi^*(\mathcal{G}) = \mathcal{G} \otimes \pi_*(\mathcal{O}_X) \), which we saw is in \( \mathcal{q} \).

Using the primality of \( \mathcal{q} \) it follows that \( \pi_*(\mathcal{O}_X) \in \mathcal{q} \). Now \( (\pi_*(\mathcal{O}_X)) = \mathcal{O}_Y \) is a direct summand of \( \pi_*(\mathcal{O}_X) \) by the canonical decomposition of a \( \mathcal{G} \)-sheaves on \( Y \). Hence \( \mathcal{O}_Y \) is an object of \( \mathcal{q} \); which is absurd. \( \square \)

To complete Step 1, we apply Balmer's result 2.9 to get a prime ideal \( \mathcal{p} \), such that \( \pi^*(\mathcal{D}^\text{per}(Y) \setminus \mathcal{q}) \cap \mathcal{p} = \emptyset \) and \( \langle \pi^*(q) \rangle \subseteq \mathcal{p} \). Hence we shall get \( q = (\pi^*)^{-1}(\mathcal{p}) \) which proves the surjectivity of the map \( \text{Spec}(\pi^*) \).

Step 2: Injectivity of \( \text{Spec}(\pi^*) \)

First we prove a technical lemma.

Lemma 3.4. Let \( \pi : X \to Y \) be the quotient map as before. Let \( \mathcal{p} \) be a prime ideal in \( \mathcal{D}^\mathcal{G}(X) \) and suppose that \( (\pi^*)^{-1}(\mathcal{p}) = \mathcal{q}_Y \). Here, \( y \) is the point in \( Y \) corresponding to \( \mathcal{q}_Y \) in \( \text{Spec}(\mathcal{D}^\text{per}(Y) \cap \mathcal{q}) \).

1. Let \( \mathcal{F} \in \mathcal{D}^\mathcal{G}(X) \) be such that its homological support is contained in \( (X \setminus \pi^{-1}(y)) \). Then, it is an object of \( \mathcal{p} \).
2. Let \( \mathcal{F} \) be an object of \( \mathcal{p} \). Then \( \text{supp}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y)) \).

Proof. Proof of 1. Using 2 of 2.27, there is a tower whose lower terms \( \mathcal{G}_i := \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^*\pi^G_{H^1}(\mathcal{F}_{\lambda_i}) \) have support contained in the subset \( X \setminus \pi^{-1}(y) \). Since \( \text{supp}(W_{\lambda_i} \otimes \mathcal{O}_X) = X \), we have \( \text{supp}(\pi^*\pi^G_{H^1}(\mathcal{F}_{\lambda_i}) \subseteq X \setminus \pi^{-1}(y)) \). Using 1 of 2.27, the support of \( \pi^G_{H^1}(\mathcal{F}_{\lambda_i}) \) will be in \( Y \setminus \{y\} \) and hence \( \pi^G_{H^1}(\mathcal{F}_{\lambda_i}) \in \mathcal{q}_Y \). We know \( \pi^*(\mathcal{q}_Y) \subseteq \mathcal{p} \) where \( \text{Spec}(\pi^*)(\mathcal{p}) = \mathcal{q}_Y \) is given. This will prove \( \pi^*\pi^G_{H^1}(\mathcal{F}_{\lambda_i}) \in \mathcal{p} \) and hence \( \mathcal{G}_i \in \mathcal{p} \). Now using the tower and the definition of a triangulated ideal, \( \mathcal{F} \) is contained in \( \mathcal{p} \).
Proof of 2. Suppose $\text{supph}(\mathcal{F}) \cap \pi^{-1}(y) \neq \emptyset$ and hence we get $\mathcal{F}' = \mathcal{F} \otimes O_{\pi^{-1}(y)} \in p$. Observe that $\text{supph}(\mathcal{F}') = \pi^{-1}(y) = \pi^{-1}(y)$. Now applying the same procedure as in 2 of 2.27, we shall get a distinguished triangle

$$\bigoplus_{\lambda} W_{\lambda} \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_{\lambda}) \to \mathcal{F}' \to \mathcal{F}'' \to$$

with $\text{supph}(\mathcal{F}'') \subseteq \text{supph}(\mathcal{F}')$. Since the $G$-invariant subset $\text{supph}(\mathcal{F}'')$ is a proper subset of $\pi^{-1}(y)$ therefore $\text{supph}(\mathcal{F}'') \cap \pi^{-1}(y) = \emptyset$. Using 1 above, we get that $\mathcal{F}'' \in p$. Hence using triangulated ideal property the third object of distinguished triangle will be in $p$ i.e. $\bigoplus_{\lambda} W_{\lambda} \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_{\lambda}) \in p$. But this gives $\pi^*\pi_*^{G/H}(\mathcal{F}'_{\lambda}) \in p$ with $\text{supph}(\pi_*^{G/H}(\mathcal{F}'_{\lambda})) \subseteq \bar{y}$ as $W_{\lambda} \otimes O_X$ is not in any $p$ because $W_{\lambda} \otimes W_{\lambda} \otimes O_X$ contains the $O_X$ as direct summand, see Proposition 10.30 of [9]. And, at least for one $\lambda$, say $\lambda_0$, we have $\text{supph}(\pi_*^{G/H}(\mathcal{F}'_{\lambda_0})) = \bar{y}$ which gives $\pi^*\pi_*^{G/H}(\mathcal{F}'_{\lambda_0}) \notin q_y$. This is a contradiction as $\pi^*(\mathcal{D}^{\text{per}}(Y)) \cap p = \pi^*(q_Y)$. □

Proposition 3.5. Suppose $X$ is a smooth quasi-projective varieties of dimension $n$. Then the map $\text{Spec}(\pi^*) : \text{Spec}(D^G(X)) \to \text{Spec}(D^{\text{per}}(Y))$ is injective.

Proof. We prove this proposition by contradiction. Let $p_1, p_2$ be two distinct points of $\text{Spec}(D^G(X))$ which maps to the same point $q_Y$ i.e. $(\pi^*)^{-1}(p_1) = (\pi^*)^{-1}(p_2) = q_Y$. Let $\mathcal{F} \in p_1$ be a complex of $G$-equivariant sheaves. Now we use the above lemma.

Using 2, we have $\text{supph}(\mathcal{F}) \subseteq (X - \pi^{-1}(y))$. Therefore using 1, and the fact that $(\pi^*)^{-1}(p_2) = q_Y$, we get that $\mathcal{F} \in p_1 \cap p_2$. Hence $p_1 \subseteq p_2$. Similarly, $p_2 \subseteq p_1$ implying that $p_1 = p_2$. This contradicts the assumption that $p_1 \neq p_2$, and hence proves the proposition. □

Step 3: $\text{Spec}(\pi^*)$ is closed and hence is a homeomorphism

Here we need bijection of the above step to prove closedness of the map $\text{Spec}(\pi^*)$. We shall use the fact that $W \otimes O_X \notin p$ for any finite dimensional representation and any prime ideal $p$. This follows from the fact that the representation on $W^* \otimes W \otimes O_X$, coming from $W \otimes O_X$, has the trivial representation as a direct summand, see Proposition 10.30 of [9]. Since $\text{supph}(\mathcal{F}), \mathcal{F} \in D^G(X)$, are the basic closed sets therefore it is enough to prove that their image under the map $\text{Spec}(\pi^*)$ are closed. Now to prove this we shall use the description given in Lemma 2.27 for any object of $D^G(X)$. Writing $G_{\lambda_j} = \pi_*^{G/H}(\mathcal{F}_{\lambda_j})$ for simplicity, we have the following lemma.

Lemma 3.6. $\text{Spec}(\pi^*)(\text{supph}(\mathcal{F})) = \bigcup_j \bigcup_j \text{supph}(G_{\lambda_j}).$

Proof. Given $\mathcal{F} \in p$ we have $G_{\lambda_j}$’s as in Lemma 2.27. Now,

$$\mathcal{F} \in p \iff W_{\lambda_j} \otimes \pi^*(G_{\lambda_j}) \in p \quad \forall j, \lambda_j$$

$$\iff \pi^*(G_{\lambda_j}) \in p, \quad \text{since} \quad W_{\lambda_j} \otimes O_X \notin p.$$

Therefore

$$\mathcal{F} \notin p \iff \exists \lambda_j \text{ such that } \pi^*(G_{\lambda_j}) \notin p.$$

Let $p \in \text{supph}(\mathcal{F})$ and hence by the definition $\mathcal{F} \notin p$. Now using the above observation there exists a $\lambda_j$ such that $\pi^*(G_{\lambda_j}) \notin p$. Hence $\mathcal{F} \notin (\pi^*)^{-1}(p) = \text{Spec}(\pi^*)(p)$ and hence $\text{Spec}(\pi^*)(p) \in \text{supph}(G_{\lambda_j})$. Therefore $\text{Spec}(\pi^*)(\text{supph}(\mathcal{F})) \subseteq \bigcup_j \bigcup_j \text{supph}(G_{\lambda_j}).$
Conversely suppose \( q \in \bigcup \bigcup \text{supp}(G_{\lambda_j}) \) and hence \( q \in \text{supp}(G_{\lambda_j}) \) for some \( \lambda_j \). Therefore by definition \( G_{\lambda_j} \notin q \) but using the bijection of the map \( \text{Spec}(\pi^*) \) we have \( G_{\lambda_j} \notin (\pi^*)^{-1}(p) = q \) for some \( p \). Now it follows that \( \pi^*(G_{\lambda_j}) \notin p \) and once again using the above observation we have \( \mathcal{F} \notin p \) i.e. \( p \in \text{supp}(\mathcal{F}) \). Hence we have \( \bigcup \bigcup \text{supp}(G_{\lambda_j}) \subseteq \text{Spec}(\pi^*)(\mathcal{F}) \). □

Since union in right hand side of above lemma is finite it follows that the image of \( \text{supp}(\mathcal{F}) \) under the map \( \text{Spec}(\pi^*) \) is closed for all \( \mathcal{F} \in \mathcal{D}^G(X) \). Hence the map \( \text{Spec}(\pi^*) \) is a closed map and therefore it is a homeomorphism.

Remark 3.7.

1. The classification of thick tensor ideals in \( \mathcal{D}^G(X) \) is given by Thomason subsets of \( X/G \). More precisely, the thick tensor ideals in \( \mathcal{D}^G(X) \) are generated by objects, whose images have as their support a Thomason subset of \( Y \). Thus the bijection can be restated as a bijection between thick tensor ideals of \( \mathcal{D}^G(X) \) and \( G \)-invariant Thomason subsets of \( X \).

2. In a fashion, similar to [26, Theorem 4.1], one can use the classification of thick tensor ideals of \( \mathcal{D}^G(X) \) to give a classification of strictly full tensor ideals of \( \mathcal{D}^G(X) \).

Step 4: \( \text{Spec}(\pi^*) \) is an isomorphism

In this step we shall prove that the above homeomorphism \( \text{Spec}(\pi^*) \) is, in fact, an isomorphism. We begin by proving the following lemma which we shall use later.

Lemma 3.8. There exists a natural transformation \( \eta : \pi^* \pi_*^G \to \text{Id} \) (resp. \( \mu : \text{Id} \to \pi_*^G \pi^* \)) such that \( \eta(\mathcal{O}_X) = \text{Id} \) (resp. \( \mu(\mathcal{O}_Y) = \text{Id} \)) where \( \pi^* \pi_*^G(\mathcal{O}_X) = \mathcal{O}_X \) (resp. \( \pi_*^G \pi^*(\mathcal{O}_Y) = \mathcal{O}_Y \)).

Proof. We shall prove the existence of \( \eta \), as \( \mu \) can be found using similar arguments. Since the functor \( \pi^* \) is a left adjoint of the functor \( \pi_* \) we have a natural transformation \( \eta' : \pi^* \pi_* \to \text{Id} \) given by the adjunction property. We also have a natural transformation given by inclusion of \( G \)-invariant part of sheaves on \( Y \), say \( I \). Now composing with the functors \( \pi^* \) and \( \pi_* \) we get another natural transformation which composed with \( \eta' \) gives the \( \eta \) i.e. \( \eta := \eta' \circ (\pi^* \cdot I \cdot \pi_*). \) Now to prove \( \eta(\mathcal{O}_X) = \text{Id} \) we can assume that \( X \) is an affine variety. Suppose \( \tilde{A} \) is a structure sheaf of \( X \) and \( \tilde{B} \) is the structure sheaf of \( Y \). Since \( \pi^* \) is a unital tensor functor, \( \pi^*(\mathcal{O}_Y) = \mathcal{O}_X \). This implies \( R^r \pi^* = 0 \) for \( i > 0 \). Similarly, using the Leray spectral sequence one can deduce \( R^r \pi_* = 0 \) for \( i > 0 \). Thus we get a morphism \( \pi^* \pi_*^G(\tilde{A}) \to \tilde{A} \), in place of its derived functors. Now clearly the multiplication map \( A \otimes_B (\pi A)^G \to A \) is just inverse of the natural identification map of \( A \) with \( A \otimes_B (\pi A)^G \). Hence the map \( \eta(\mathcal{O}_X) : \tilde{A} \to \tilde{A} \) is an identity map. Similarly we can prove that \( \mu(\mathcal{O}_Y) = \text{Id} \). □

Recall the definitions of structure sheaves and associated map of the sheaves given by the unital tensor functor of underlying tensor triangulated categories 2.2. i.e. given a unital functor \( \pi^* : \mathcal{D}^{\mathcal{O}_Y}(Y) \to \mathcal{D}^G(X) \) the morphism \( \text{Spec}(\pi^*) \) induces a map of the structure sheaves, \( \text{Spec}(\pi^*)^\# : \mathcal{O}_Y \to \mathcal{O}_X \). We shall prove that this map is an isomorphism by observing that \( \text{Spec}(\pi^*)^\#(V) \) is an isomorphism for every open set \( V \subseteq \text{Spec}(\mathcal{D}^{\mathcal{O}_Y}(Y)) \). If we take \( U = \pi^{-1}(V) \), \( Z = Y \setminus V \) and \( Z' = X \setminus U \) then we have a functor \( \pi^* : \mathcal{D}^{\mathcal{O}_Y}(Y) \to \mathcal{D}^G(X) \) which will induce a map \( \pi^+ := \text{Spec}(\pi^*)^\#(V) : \text{End}_{\mathcal{D}^{\mathcal{O}_Y}(Y)}(\mathcal{O}_Y) \to \text{End}_{\mathcal{D}^G(X)}(\mathcal{O}_X) \).

Lemma 3.9. The map \( \pi^+ : \text{End}_{\mathcal{D}^{\mathcal{O}_Y}(Y)}(\mathcal{O}_Y) \to \text{End}_{\mathcal{D}^G(X)}(\mathcal{O}_X) \) is surjective.
Proof. Suppose \([\mathcal{O}_Y \leftarrow G \xrightarrow{\pi} \mathcal{O}_Y]\) is an element of \(\text{End}_{D^b_{\mathit{per}}(Y)}(\mathcal{O}_Y)\) then the map \(\pi^*\) will send it to an element \([\mathcal{O}_X \leftarrow \pi^*(G) \xrightarrow{\pi^*} \mathcal{O}_X]\) of \(\text{End}_{D^b_{\mathit{per}}(X)}(\mathcal{O}_X)\). It is now enough to prove that this map is a surjection.

Let \([\mathcal{O}_X \leftarrow \mathcal{F} \xrightarrow{b} \mathcal{O}_X]\) be a given element then using the functor \(\pi_*^G\) we shall get an element \([\mathcal{O}_Y \leftarrow \pi_*^G(\mathcal{F}) \xrightarrow{\pi_*^G(b)} \mathcal{O}_Y]\) in \(\text{End}_{D^b_{\mathit{per}}(Y)}(\mathcal{O}_Y)\) as \(\text{supp}(C(\pi_*^G(t))) \subseteq Z\) using the flat base change and the canonical isomorphism, \(\pi_*^G(\mathcal{F}) \simeq (\pi_*^G(\mathcal{F}))^G \simeq \pi_*^G(\mathcal{F}^G)\).

\[
\begin{array}{ccc}
\pi^*\pi_*^G(\mathcal{F}) & \xrightarrow{\eta(\mathcal{F})} & \mathcal{F} \\
\pi^*\pi_*^G(t) & \downarrow \pi^* & \downarrow \eta(\mathcal{O}_X) \\
\mathcal{O}_X & \xrightarrow{t} & \mathcal{O}_X \\
\end{array}
\]

Now the last assertion \(C(\eta(\mathcal{F})) \in D^G_{\mathit{per}}(X)\) is equivalent to \(i_U^*C(\eta(\mathcal{F})) \simeq 0\) in \(D^G(U)\) but as the functor \(i_U^*\) is exact this assertion is the same as \(C(i_U^*\eta(\mathcal{F})) \simeq 0\). Since a cone of an isomorphism is zero, it is enough to check that the map \(i_U^*\eta(\mathcal{F})\) is an isomorphism. And this follows from the following commutative diagram.

\[
\begin{array}{ccc}
i_U^*\pi^*\pi_*^G(\mathcal{F}) & \xrightarrow{i_U^*\eta(\mathcal{F})} & i_U^*\mathcal{F} \\
\pi^*\pi_*^G(i_U^*\mathcal{F}) & \xrightarrow{\eta(i_U^*\mathcal{F})} & i_U^*\mathcal{F} \\
\pi^*\pi_*^G(\mathcal{O}_U) & \xrightarrow{\eta(\mathcal{O}_U)} & \mathcal{O}_U \\
\end{array}
\]

In above diagram we had used the same notations \(\pi\) and \(\eta\) for its restriction on open subsets. Here the top left vertical isomorphism comes from the flat base change formula and using the following canonical isomorphism.

\[
i_U^*\pi^*\pi_*^G(\mathcal{F}) \simeq \pi^*(\pi_*^G(\mathcal{F}))^G \simeq \pi^*(\pi_*^G(i_U^*\mathcal{F}))^G \simeq \pi^*(\pi_*^G(i_U^*\mathcal{F}))^G = \pi^*\pi_*^G(i_U^*\mathcal{F}).
\]

This proves that \(\pi_*^G\) is surjective. \(\square\)
Lemma 3.10. \( \pi_Y^* \) is injective.

Proof. Let \( \mathcal{O}_Y \xrightarrow{\pi} \mathcal{G} \xrightarrow{\alpha} \mathcal{O}_Y \) in \( \text{End}_{\mathcal{D}^{\text{per}}(Y)}(\mathcal{O}_Y) \) maps to zero in \( \text{End}_{\mathcal{D}^{\text{per}}(X)}(\mathcal{O}_X) \) i.e. \( \mathcal{O}_X \xrightarrow{\pi^*(s)} \mathcal{O}_X \). \( \pi^*(\mathcal{G}) \rightarrow \mathcal{O}_X \) is zero which is equivalent to the existence of \( \mathcal{F} \) and a map \( t : \mathcal{F} \rightarrow \pi^*\mathcal{G} \) with \( \text{supph}(\mathcal{C}(t)) \subseteq Z' \) such that \( \pi^*(a) \circ t = 0 \).

Now the map \( \pi_Y^*(t) : \pi_Y^*(\mathcal{F}) \rightarrow \pi_Y^*(\pi^*(\mathcal{G})) \) gives \( \pi_Y^*(\pi^*(a)) \circ \pi_Y^*(t) = 0 \) and as proved earlier we know that \( \text{supph}(\mathcal{C}(\pi_Y^*(t))) \subseteq Z \) whenever \( \text{supph}(\mathcal{C}(t)) \subseteq Z' \).

Hence the element \([\mathcal{O}_Y \xrightarrow{\pi_Y^*(s)} \pi^*(\pi^*(\mathcal{G})) \xrightarrow{\pi_Y^*(t)} \mathcal{O}_Y]\) = 0 in \( \text{End}_{\mathcal{D}^{\text{per}}(Y)}(\mathcal{O}_Y) \). We shall prove that \([\mathcal{O}_Y \xrightarrow{\pi^*(s)} \pi_Y^*(\pi^*(\mathcal{G})) \mathcal{O}_Y]\) as an elements of \( \text{End}_{\mathcal{D}^{\text{per}}(Y)}(\mathcal{O}_Y) \). Now using Lemma 3.8 we have a map \( \mu(\mathcal{G}) : \mathcal{G} \rightarrow \pi_Y^*(\pi^*(\mathcal{G})) \) which gives the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\mu(\mathcal{G})} & \pi_Y^*(\pi^*(\mathcal{G})) \\
\downarrow{s} & & \downarrow{\pi_Y^*(s)} \\
\mathcal{O}_Y & \xrightarrow{\mu(\mathcal{O}_Y)} & \mathcal{O}_Y
\end{array}
\]

Therefore it remains to prove that \( i_Y^*\mathcal{C}(\mu(\mathcal{G})) = 0 \) but as before this is equivalent to proving \( \mathcal{C}(i_Y^*\mu(\mathcal{G})) = 0 \) since the functor \( i_Y^* \) is an exact functor. Again using the fact that a cone of an isomorphism is zero it is enough to prove that \( i_Y^*\mu(\mathcal{G}) \) is an isomorphism. This clearly follows from the following commutative diagrams:

\[
\begin{array}{ccc}
i_Y^*\mathcal{G} & \xrightarrow{i_Y^*\mu(\mathcal{G})} & i_Y^*(\pi_Y^*\pi^*(\mathcal{G})) \\
\downarrow{i_Y^*(s)} & & \downarrow{\pi_Y^*(s)} \\
i_Y^*(\mathcal{G}) & \xrightarrow{\mu(i_Y^*\mathcal{G})} & \pi_Y^*\pi^*(i_Y^*\mathcal{G})
\end{array}
\]

Here again as earlier the top right vertical isomorphism comes from the flat base change and the following sequence of natural isomorphisms.

\[
i_Y^*(\pi_Y^*\pi^*(\mathcal{G})) \simeq i_Y^*(\pi_Y^*(\pi^*(\mathcal{G}))) \simeq (i_Y^*\pi_Y^*(\pi^*(\mathcal{G})))^G \simeq \pi_Y^*(i_Y^*\pi^*(\mathcal{G})) \simeq \pi_Y^*(\pi^*(i_Y^*\mathcal{G})).
\]

This proves injectivity of the map \( \pi_Y^* \). \( \square \)

From the above two lemmas it follows that \( \pi_Y^* \) is an isomorphism and hence \( \text{Spec}(\pi^*) \) is an isomorphism of the locally ringed spaces \( \text{Spec}(\mathcal{D}^{\text{per}}(Y)) \) and \( \text{Spec}(\mathcal{D}^G(X)) \).

4. Example: Superschemes

In this section, we shall recall the basic definition of superscheme and some properties of it. Then, we shall relate various notions for superschemes with usual schemes.
4.1. Superalgebra

An associative \( \mathbb{Z}/2\mathbb{Z} \)-graded ring is an associative ring \( R \) with a direct sum decomposition \( R = R^0 \oplus R^1 \) as an additive group so that multiplication preserves the grading i.e. \( R^i R^j \subseteq R^{i+j} \) for \( i, j \in \mathbb{Z}/2\mathbb{Z} \). There exists a parity function which takes values in the ring \( \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) for every homogeneous element of \( R \) i.e. if \( r \in R^1 \) then the parity is denoted by \( \bar{r} = i \). Now we restrict to following important class of rings,

**Definition 4.1.** An associative \( \mathbb{Z}/2\mathbb{Z} \) graded ring with unity, \( R = R^0 \oplus R^1 \) is called supercommutative if the supercommutator of a ring \( R \) is zero i.e. \( [r_1, r_2] := r_1 r_2 - (-1)^{r_1 r_2} r_2 r_1 = 0 \) for all \( r_1, r_2 \in R \). Further ring is called \( k \)-superalgebra if \( R \) is supercommutative \( k \)-algebra with \( k \subseteq R^0 \). We shall assume that \( 2 \in R \) is invertible. This will ensure that the elements with parity 1 are nilpotents.

As usual we can define an abelian category of left modules over any \( k \)-superalgebra \( R \), say \( \mathcal{Mod}(R) \). An object of this category is a \( \mathbb{Z}/2\mathbb{Z} \)-graded abelian group with a left \( R \)-module structure which is compatible with the grading i.e. \( R^i M^j \subseteq M^{i+j} \) for all \( i, j = 0, 1 \). Morphism between these objects is a graded morphism compatible with the action of \( R \). Similarly there exists a parity function defined for each homogeneous element of a module \( M \) as above. We can define the parity change functor \( \Pi : \mathcal{M}od(R) \rightarrow \mathcal{M}od(R) ; \ M \mapsto (\Pi M) \) with \( \mathbb{Z}/2\mathbb{Z} \) grading given by \( (\Pi M)^0 = M^1 \) and \( (\Pi M)^1 = M^0 \). There exists an exact faithful functor from \( \mathcal{M}od(R) \) as follows

\[
ff : \mathcal{M}od(R) \rightarrow \mathcal{M}od(R^0) \times \mathcal{M}od(R^0).
\]

A canonical right module structure on left \( R \) modules is given by \( mr := (-1)^{\bar{m} \bar{r}} rm \). Now using this structure we can define tensor product of two left \( R \)-modules \( M_1 \) and \( M_2 \) as the quotient of \( M_1 \otimes_{R^0} M_2 \) by the submodule generated by homogeneous elements \( \{r_1 m_1 \otimes m_2 - (-1)^{\bar{r}_1 \bar{m}_1} m_1 \otimes r_1 m_2 \mid r_1 \in R^1, m_i \in M^i \} \). Here \( M_1 \otimes_{R^0} M_2 \) is defined as a tensor product of two \( \mathbb{Z}/2\mathbb{Z} \) graded modules over a commutative ring \( R^0 \). The tensor product \( M_1 \otimes_{R^0} M_2 \) is then a \( \mathbb{Z}/2\mathbb{Z} \) graded module with \( \bar{m} \otimes \bar{n} = \bar{m} + \bar{n} \). The commutativity constraint is similar to the case of tensor product of supervector spaces. Another important notion in commutative algebra is localization. It is easy to define localization of rings and modules if multiplicative set is contained in the center of a ring. For super commutative ring we can define localization at any homogeneous prime ideal. It is easy to observe that given an \( R \) module \( M \) and a prime ideal \( p \), the localization \( M_p = 0 \) iff \( (R \otimes \mathbb{Z}/2\mathbb{Z})p \subseteq 0 \) where \( J : = R \cdot R^1 \). One can also prove Nakayama’s lemma for superrings by using arguments similar to [1, Proposition 2.6].

**Proposition 4.2** (Nakayama’s lemma). Suppose a finitely generated \( R \) module \( M \) satisfies \( IM = M \) for the homogeneous ideal \( I \) given by the intersection of all maximal homogeneous ideals then \( M = 0 \).

Now using Nakayama’s lemma we get the following result whose proof is similar to the commutative case.

**Corollary 4.3.** Suppose \( (R, \mathfrak{m}) \) is a local superring. Let \( M, M_1 \) and \( M_2 \) be finitely generated \( R \) modules.

1. A finitely generated module \( M = 0 \) if and only if \( M \otimes R/\mathfrak{m} = 0 \).
2. \( M_1 \otimes M_2 = 0 \) if and only if \( M_1 = 0 \) or \( M_2 = 0 \).

4.2. Split superscheme

Given any topological space \( X \) we can define a super ringed space by attaching a sheaf of superrings on \( X \). We shall denote a sheaf of superrings with \( \mathbb{Z}/2\mathbb{Z} \) grading as \( \mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1} \). Similarly we can define sheaf of modules and parity change functor \( \Pi \) over such a ringed space as before. We have the following definition:
Definition 4.4. A ringed space \((X, \mathcal{O}_X)\) is called a superspace if ring \(\mathcal{O}_X(U)\) associated to any open subset \(U\) is supercommutative and each stalk is local ring. A superspace is called superscheme if in addition ringed space \((X, \mathcal{O}_{X,0})\) is a scheme and \(\mathcal{O}_{X,1}\) is a coherent sheaf over \(\mathcal{O}_{X,0}\).

A superscheme \(X\) is called quasi-compact and quasi-separated if \((X, \mathcal{O}_{X,0})\) is quasi-compact and quasi-separated. Similarly a superscheme is (topologically) noetherian if \((X, \mathcal{O}_{X,0})\) is (topologically) noetherian. We shall use these notions later to borrow results developed by Grothendieck. We say that a superscheme is affine if the even part of structure sheaf \((X, \mathcal{O}_{X,0})\) is affine. It is easy to see that any affine superscheme has as its ring of global functions, a super commutative ring. Equivalently an affine superscheme associated to any super commutative ring can be defined in a manner similar to usual affine schemes. Note that in the definition of superscheme the odd part is a coherent sheaf of modules over the even part. Therefore if even part of a superscheme is noetherian then we shall get the left (or two sided) noetherian superscheme. Given a superscheme \((X, \mathcal{O}_X)\) we can define sheaf of ideal \([17, \text{p. 83}] J_X := \mathcal{O}_X \cdot \mathcal{O}_{X,1}. \) Define \(Gr X := \bigoplus_{i \geq 0} J_X^i/J_X^{i+1}\) where \(J_X^i := \mathcal{O}_X\) and we denote the first term of \(Gr X\) as \(Gr_0 X = \mathcal{O}_X/J_X\). Now using these notation we can define structure sheaves of even scheme and reduced scheme associated to the superscheme \(X\) as follows

\[ O_{X\text{red}} := Gr_0 X \quad \text{and} \quad O_{X\text{red}} := \mathcal{O}_X/\sqrt{J_X}. \]

Here \(J_X/J_X^2\) is a locally free sheaf of finite rank \(0/d\) for some \(d\) over \(O_{X\text{red}}\). And \(Gr X\) is a Grassmann algebra over \(O_{X\text{ed}}\) of locally free sheaf \(J_X/J_X^2\). Following particular class of superschemes are defined in Manin \([17, \text{p. 85}]\).

Definition 4.5. A superscheme \((X, \mathcal{O}_X)\) is called split if the graded sheaf \(Gr X\) with mod 2 grading is isomorphic as a locally superringed sheaf to the structure sheaf \(\mathcal{O}_X\).

Manin has also given a way to construct such a split superscheme. If we take purely even scheme \((X, \mathcal{O}_X)\) and a locally free sheaf \(V\) over \(\mathcal{O}_X\) then we can define symmetric algebra of odd locally free sheaf \(\Pi V\), which is denoted \(S(\Pi V)\), then \((X, S(\Pi V))\) is a split superscheme. An important example is given by projective superspace \(\mathbb{P}^{n|m}\) where the locally free sheaf \(V\) is \(\mathcal{O}(-1)^n\). An example of a nonsplit superscheme given in Manin \([17, \text{p. 86}]\) is Grassmann superscheme \(G(1|1, \mathbb{C}^2)\) which is also an example of a superprojective scheme.

We can define an abelian category of sheaf of left modules over \(\mathcal{O}_X\), denoted by \(\mathcal{M}\text{od}^\ell(\mathcal{O}_X)\) or \(\mathcal{M}\text{od}(\mathcal{O}_X)\). As above we have a natural right module structure given by the Koszul sign rule. When \((X, \mathcal{O}_X)\) is affine superscheme given by super ring \(R\) then we can define the sheaf of module associated to any \(R\)-module \(M\) similar to commutative case. Hence we can define quasi-coherent and coherent sheaves over any superscheme. Therefore we shall get two abelian subcategories namely the category of all quasi-coherent sheaves and coherent sheaves. We denote them by \(\mathcal{Q}\text{oh}(\mathcal{O}_X)\) and \(\mathcal{C}\text{oh}(\mathcal{O}_X)\) respectively. Now similar to affine case we have forgetful functor as follows

\[ ff : \mathcal{M}\text{od}(\mathcal{O}_X) \rightarrow \mathcal{M}\text{od}(\mathcal{O}_{X,0}) \times \mathcal{M}\text{od}(\mathcal{O}_{X,0}). \]

It is an exact faithful functor. We can easily see that

\[ \mathcal{Q}\text{oh}(\mathcal{O}_X) = ff^{-1}(\mathcal{Q}\text{oh}(\mathcal{O}_{X,0}) \times \mathcal{Q}\text{oh}(\mathcal{O}_{X,0})), \]

\[ \mathcal{C}\text{oh}(\mathcal{O}_X) = ff^{-1}(\mathcal{C}\text{oh}(\mathcal{O}_{X,0}) \times \mathcal{C}\text{oh}(\mathcal{O}_{X,0})). \]

One can also define locally free sheaves on superscheme.

Definition 4.6. A sheaf \(\mathcal{F}\) on a superscheme \(X\) is said to be locally free of rank \(m|n\) if it is locally isomorphic to \((\mathcal{O}_X)^{\oplus m} \oplus (\Pi \mathcal{O}_X)^{\oplus n}\).
We can define the tensor product of two sheaves of modules over superscheme similar to usual scheme. We shall use the canonical identification of sheaf of left and right modules by Koszul sign rule. Define tensor product of two sheaves of modules \( F_1 \) and \( F_2 \) as the sheaf associated to pre sheaf given by

\[
U \mapsto (F_1 \otimes F_2)(U) := F_1(U) \otimes_{\mathcal{O}_X(U)} F_2(U).
\]

Note that with this definition of tensor structure the commutative constraint is given by sign rule i.e.

\[
F \otimes G \cong G \otimes F \quad \text{where the isomorphism is given by},
\]

\[
f \otimes g \mapsto -g \otimes f \quad \text{if both } F \text{ and } G \text{ are odd},
\]

\[
f \otimes g \mapsto g \otimes f \quad \text{otherwise},
\]

where \( f \) and \( g \) are sections on some open set \( U \).

Now we can prove some easy properties of this tensor product by just reducing to affine case,

**Lemma 4.7.** Suppose \((X, \mathcal{O}_X)\) is a split superscheme and \( F \) and \( G \) are \( \mathcal{O}_X \)-modules. Then we have:

1. \((\Pi F) \otimes G = F \otimes (\Pi G) = \Pi (F \otimes G)\).
2. \( F \otimes \mathcal{O}_{X_{rd}} \) has trivial action of \( J_X \) and hence it is an \( \mathcal{O}_{X_{rd}} \)-module.

Given a split superscheme \((X, \mathcal{O}_X = S(\Pi V) = \Pi \Lambda(V))\) there is one more forgetful functor as follows

\[
ff : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_{X_{rd}}) \times \text{Mod}(\mathcal{O}_{X_{rd}}).
\]

This functor is defined using the obvious inclusion of \( \mathcal{O}_{X_{rd}} \) inside \( \mathcal{O}_X \) which comes from the definition of split superscheme. Note also that the Grassmann algebra constructed from locally free sheaf \( V \) gives a locally free sheaf of \( \mathcal{O}_{X_{rd}} \) module. Therefore structure sheaf \( \mathcal{O}_X \) is locally free sheaf as an \( \mathcal{O}_{X_{rd}} \) module.

Similar to usual scheme we can take \( \mathcal{D}(X) := \mathcal{D} (\text{Mod}(X)) \) the derived category of abelian category \( \text{Mod}(X) \). There are various triangulated subcategories like \( \mathcal{D}^+(\text{qc}/X) := \mathcal{D}^+(\text{QCoh}(X)) \) and \( \mathcal{D}^-(\text{coh}/X) := \mathcal{D}^-(\text{Co}(X)) \) where \( z = +, -, b \) or \( \theta \). For convenience we shall denote by \( \mathcal{D}^B(X^0) := \mathcal{D}^B(\text{Mod}(\mathcal{O}_X^0)) \) (resp. \( \mathcal{D}^B(X_{rd}) := \mathcal{D}^B(\text{Mod}(\mathcal{O}_{X_{rd}})) \)) the derived category of modules over purely even scheme \((X, \mathcal{O}_X^0)\) (resp. \( X_{rd} = (X, Gr_0 X) \)), \( \mathcal{D}_{qc}(X) \) (resp. \( \mathcal{D}_{coh}(X) \)) will denote the full subcategory of \( \mathcal{D}(X) \) containing all complexes of \( \mathcal{O}_X \)-modules with quasi-coherent (resp. coherent) cohomology sheaves.

We need definitions of derived functors and various relations between them for unbounded complexes of modules over superschemes. To extend various functors to unbounded complexes we need notion of K-injective (K-projective) resolutions, see [25]. Following definition was given in [25].

**Definition 4.8.** An unbounded complex \( A \) of an abelian category is called K-injective (resp. K-projective) if for every acyclic complex \( S \), the complex \( \text{Hom} (S, A) \) (resp. \( \text{Hom} (A, S) \)) is acyclic.

It is proved in the same paper, that an abelian category for which inverse (resp. direct) limit exists, and which has enough injectives (resp. projectives) admits a K-injective (resp. K-projective) resolution for any unbounded complex, see Corollary 3.9 (resp. Corollary 3.5) in [25]. Similar to the scheme case the abelian category \( \text{QCoh}(X) \) of all quasi-coherent sheaves over superscheme has arbitrary small co-products. Therefore we can extend various functors to unbounded derived category as demonstrated by Spaltenstein (see [25, Section 6]). Moreover the abelian category \( \text{QCoh}(X) \) will have K-flat resolution for every unbounded complex and hence derived functor of tensor product functor can be
extended to unbounded derived category and various relation among these functors can be extended from bounded derived category case to unbounded derived category, see [25] for more details.

Following criterion using Nakayama’s lemma will be used later.

**Proposition 4.9.** Suppose \((R, m)\) is a local superring. Suppose \(M^*\), \(M_1\) and \(M_2\) are bounded complexes of finitely generated \(R\)-modules.

1. \(M^*\) is acyclic iff \(M^* \otimes R/m\) is acyclic.
2. \(M_1 \otimes M_2\) is acyclic iff \(M_1\) or \(M_2\) is acyclic.

**Proof.** The proof of (i) is similar to the proof of Thomason [26, Lemma 3.3(a)]. Indeed, using spectral sequence mentioned in the proof of Thomason [26, Lemma 3.3(a)] the proof reduces to the case of finitely generated modules which follows from the above result 4.3.

The proof of (ii) follows from the proof of (i) using following natural isomorphism

\[
(M_1 \otimes M_2) \otimes R/m \cong (M_1 \otimes R/m) \otimes_{R/m} (M_2 \otimes R/m).
\]

Next we give some results which we need in computation of spectrum. As in the case of schemes we can define the support of a quasi-coherent sheaf as a subset of \(X\) containing all super prime ideals where the stalk of the sheaf is non-zero. Since nontriviality of the stalk at any point \(p\) is a local property we can check it in an affine open set containing \(p\). Now from the earlier observation \(\mathcal{F}_p = 0\) iff \(\mathcal{F}^0_p = 0 = \mathcal{F}^1_p\) as stalks of a sheaves of \(\mathcal{O}_{Xrd}\) modules \(\mathcal{F}^0\) and \(\mathcal{F}^1\). Therefore for a quasi-coherent sheaf \(\mathcal{F}\) we have \(\text{supp}(\mathcal{F}) = \text{supp}(\mathcal{F}^0) \cup \text{supp}(\mathcal{F}^1)\). Now the assignment of support can be extended to the derived category as follows

\[
\text{supph}(\mathcal{F}) := \bigcup_{i \in \mathbb{Z}} \text{supp}(\mathcal{H}^i(\mathcal{F})).
\]

Consider a split superscheme \((X, \mathcal{O}_X)\). Note that, the inclusion \(i : X_{rd} \to X\) given by the surjection \(i^* : \mathcal{O}_X \to \mathcal{O}_{X_{rd}}\). \(i^*\) splits to give a projection \(p : X \to X_{rd}\) with \(p \circ i = \text{id}_{X_{rd}}\). Let \(i_* : \mathcal{D}_{qc}(X_{rd}) \to \mathcal{D}_{qc}(X)\) and \(i^* : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X_{rd})\) be the induced derived functors.

**Proposition 4.10.** For an ideal \(\mathcal{E}\) in \(\mathcal{D}_{qc}(X_{rd})\), and for an \(\mathcal{O}_{X_{rd}}\) module, \(\mathcal{G}\):

1. \(i_*(i^*\mathcal{G}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}}\). Further \(\text{supph}(i^*\mathcal{G}) = \text{supph}(\mathcal{G}) \subset X\).
2. \(i_*(i^*\mathcal{E})) \subset \mathcal{E}\).
3. \(i_*\) is dominant, that is the thick tensor ideal generated by the image of \(i_*\) is \(\mathcal{D}_{qc}(X)\).

**Proof.** The proof of 1 is clear from the definition.

**Proof of 2.** We shall use the definition of \((\mathcal{E})\) given in Lemma 2.6. Since \(i_*\) is an exact functor, it is enough to prove that \(i_*(\text{ideal}(i^*\mathcal{E})) \subset \mathcal{E}\). Thus, it is enough to see that for \(A \in \mathcal{D}_{qc}(X_{rd})\) and \(\mathcal{J} \in \mathcal{E}\),

\[
i_*(A \otimes_{\mathcal{O}_{X_{rd}}} i^*\mathcal{J}) = i_*(i^*p^*A \otimes_{\mathcal{O}_{X_{rd}}} i^*\mathcal{J}) = i_*i^*(p^*A \otimes_{\mathcal{O}_X} \mathcal{J})
\]

\[
= (p^*A \otimes_{\mathcal{O}_X} \mathcal{J}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}} = \mathcal{J} \otimes_{\mathcal{O}_X} (p^*A \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}})
\]

\(\in (\mathcal{E}).\)

**Proof of 3.** Since \((X, \mathcal{O}_X)\) is a split superscheme, we have identification of \(\mathcal{O}_X\) with \(\text{Gr } X\). The sheaf \(\text{Gr } X\) is an exterior algebra over purely odd locally free sheaf \(\mathcal{II}^i : = J_X / J_X^{i-1}\) and each subquotient \(J_X^i / J_X^{i+1}\) can be identified with \(\mathcal{II}^i A^i\mathcal{V}\). Hence each subquotient is purely odd or purely even locally
free sheaves. The $\mathbb{Z}$-grading on sheaf $\text{Gr} X$ gives a filtration for structure sheaf $\mathcal{O}_X$ and hence we have following tower for structure sheaf $\mathcal{O}_X$:

$$
\begin{array}{ccccccc}
\mathcal{O}_X & \xleftarrow{J_X} & J_X^{n-1} & \cdots & J_X^1 & J_X^0 & \mathcal{O}_{\text{red}} \\
\mathcal{O}_{\text{red}} & \xleftarrow{\Pi^{n-1} \Lambda^{n-1} \mathcal{V}} & \Pi^{n} \Lambda^{n} \mathcal{V}
\end{array}
$$

In above tower, each of the terms in the lower row is complex of either purely odd or purely even sheaves. And using property of tensor proved in 4.7, we have $\Pi^i A^i \mathcal{V} = (\Pi^i \mathcal{O}_{\text{red}}) \otimes A^i \mathcal{V}$. Therefore the ideal generated by the image of the functor $i_*$ contains the all the terms in the lower row of the above tower and hence $i_*$ is a dominant functor. $\square$

We shall denote the functor $i_*$ by $i_{\text{red}}$ from now on.

We now define the another important triangulated subcategory of $\mathcal{D}_{\text{qc}}(X)$.

**Definition 4.11.** A complex $\mathcal{F}$ of sheaves of modules over the superscheme $(X, \mathcal{O}_X)$ is called **strictly perfect** if it is quasi-isomorphic to a bounded complex of locally free coherent sheaves of $\mathcal{O}_X$ modules. A complex $\mathcal{F}$ is called **perfect** if it is locally quasi-isomorphic to a bounded complex of locally free coherent sheaves.

For a perfect complex $\mathcal{F}$, one can define $\text{supph} \mathcal{F}$ in the same was as before, that is $\text{supph} \mathcal{F} = \bigcup \text{supp} H^i \mathcal{F}$.

We shall denote the triangulated subcategory of all perfect complexes by $\mathcal{D}^{\text{per}}(X)$. As in the scheme case, we can extend various functors defined on sheaves of modules over the superscheme to these triangulated categories. Hence we can prove $\mathcal{D}^{\text{per}}(X)$ is a tensor triangulated category with the tensor structure given by the derived functor of the usual tensor product defined earlier.

We need to recall a few more results which might be proved in a way similar to the commutative case. First we need a definition.

**Definition 4.12.** An object $t$ in a triangulated category $\mathcal{T}$, which is closed under the formation of arbitrary small coproducts, is said to be **compact** if $\text{Hom}(t, \_)$ respects coproducts. In a triangulated category $\mathcal{T}$, the full subcategory of all compact objects is denoted as $\mathcal{T}^c$.

Now we shall use the following results.

1. The category of perfect complexes over affine schemes is equivalent to the category of projective modules over the respective superalgebras.
2. (Compare [5, Corollary 3.3.5].) If $X$ is an affine superscheme then the obvious functor $\mathcal{D}(\text{qc}/X) \to \mathcal{D}_{\text{qc}}(X)$ has a quasi-inverse $R \Gamma(X, \_)$.
3. (Compare [5, Eq. (3.4), p. 12].) Suppose $X$ is a superscheme and suppose $X = U_1 \cup U_2$ where $U_1$ and $U_2$ are open and suppose $U_{12} := U_1 \cap U_2$. Let $j_1, j_2$ and $j_{12}$ be the inclusions of $U_1$, $U_2$ and $U_{12}$ in $X$ respectively. Suppose $A$ is a K-injective complex on $X$ and $E$ be another object in $\mathcal{D}(X)$. Then we have a distinguished triangle

$$
\text{Hom}(E, A) \to \text{Hom}(j_1^* E, j_1^* A) \oplus \text{Hom}(j_2^* E, j_2^* A) \to \text{Hom}(j_{12}^* E, j_{12}^* A) \xrightarrow{+} \in \mathcal{D}(X).
$$

in $\mathcal{D}(X)$.
4. (Compare [5, Proposition 3.3.1].) (Reduction principle.) If $P$ is a property satisfied by superschemes, and if
(a) $P$ is true of affine schemes; and
(b) if $P$ holds for $U_1, U_2$ and $U_{12}$, then it is true for $X$.

5. (Compare [5, Lemma 3.3.6].) If $X$ is an affine superscheme, then the category of compact objects in $\mathcal{D}(X)$ is the category of perfect complexes.

We can extend the forgetful functor defined earlier using exactness,

$$ff : \mathcal{D}^\ddagger(\mathcal{X}) \to \mathcal{D}^\ddagger(X^0) \times \mathcal{D}^\ddagger(X^0)$$

and

$$ff : \mathcal{D}^\ddagger_{qc}(\mathcal{X}) \to \mathcal{D}^\ddagger_{qc}(X^0) \times \mathcal{D}^\ddagger_{qc}(X^0).$$

Here $\ddagger \in \{+, -, b, \emptyset\}$. We can have similar forgetful functors in the case of coherent sheaves. If we restrict to split superschemes, we can also define forgetful functors in the case of locally free sheaves (or vector bundles). Hence for a split superscheme, we have the following forgetful functor for the triangulated subcategory of perfect complexes,

$$ff : \mathcal{D}^{per}(\mathcal{X}) \to \mathcal{D}^{per}(X_{\text{rd}}) \times \mathcal{D}^{per}(X_{\text{rd}}).$$

Note that this functor may not be a tensor functor.

### 4.3. Main results

As the forgetful functor is an exact functor we have the following relation between supports as in the case of sheaves:

$$\text{supp}(\mathcal{F}) = \text{supp}(ff(\mathcal{F})) = \text{supp}(\mathcal{F}^0) \cup \text{supp}(\mathcal{F}^1).$$

Above observation gives the following result similar to the result of Thomason [26, Lemma 3.3(c)].

**Lemma 4.13.** Suppose $X$ is a quasi-compact and quasi-separated superscheme and $\mathcal{F} \in \mathcal{D}^{per}(X)$. Then the subset $\text{supp}(\mathcal{F})$ is closed and $X \setminus \text{supp}(\mathcal{F})$ is a quasi-compact subset of $X$.

Using this property of supports we can prove the following result,

**Lemma 4.14.** The pair $(X, \text{supp})$ defined as above gives a support data on the triangulated category $\mathcal{D}^{per}(X)$.

**Proof.** Since the forgetful functor is an exact functor and we have the equality $\text{supp}(\mathcal{F}) = \text{supp}(ff(\mathcal{F}))$ therefore the support data properties (SD 1)–(SD 4) of [3, Definition 3.1] are easy to prove. We shall just prove (SD 5) here, which states that $U(\mathcal{F}_1 \otimes \mathcal{F}_2) = U(\mathcal{F}_1) \cup U(\mathcal{F}_2)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are perfect complexes and $U(\mathcal{F}_1) = X \setminus \text{supp}\mathcal{F}_i$. This is equivalent to the statement that for every $x \in X$, $(\mathcal{F}_1 \otimes \mathcal{F}_2)_x$ is acyclic if and only if either $(\mathcal{F}_1)_x$ or $(\mathcal{F}_2)_x$ is acyclic. Since checking nontriviality of the stalk is a local question, we can assume that $X$ is an affine superscheme. First we observe that any perfect complex $\mathcal{F}$ is a strict perfect complex and hence a bounded complex of finitely generated projective modules. Hence by taking local superring $R = O_{X,x}$, and observing that $(\mathcal{F}_1 \otimes \mathcal{F}_2)_x \cong (\mathcal{F}_1)_x \otimes (\mathcal{F}_2)_x$, the proof follows from the result 4.9(2).

**Definition 4.15.** A subset $Z \subset X$ is said to be Thomason if $Z = \bigcup \alpha Z_\alpha$ where each $Z_\alpha$ is closed and $X \setminus Z_\alpha$ is quasi-compact.

Note that if $X$ is noetherian, the Thomason subsets match with specialization closed subsets.

We shall now prove that above support data is in fact classifying support data as defined in Balmer [3]. We need the following classification (see [3]) of thick tensor subcategories of $\mathcal{D}^{per}(X)$ which we prove by relating it with the case of schemes.
**Proposition 4.16.** Given a quasi-compact and quasi-separated split superscheme \((X, \mathcal{O}_X)\) we have a bijection,

\[
\theta : \{Y \subset X \mid Y \text{ is a Thomason subset}\} \sim \{\mathcal{I} \subset \mathbf{D}^{\text{per}}(X) \mid \mathcal{I} \text{ radical thick } \otimes \text{-ideal}\}
\]

defined by \(Y \mapsto \{\mathcal{F} : \mathcal{F} \in \mathbf{D}^{\text{per}}(X) \mid \text{supph}(\mathcal{F}) \subset Y\}\), with inverse, say \(\eta, \mathcal{I} \mapsto \text{supph}(\mathcal{I}) := \bigcup_{\mathcal{F} \in \mathcal{I}} \text{supph}(\mathcal{F})\).

**Proof.** Using support data properties (SD 1)-(SD 5) (see Balmer [3, Definition 3.1]) we can prove that \(\theta(Y)\) is a radical thick tensor ideal and hence the map \(\theta\) is well defined. To prove that \(\eta(\mathcal{I})\) is a Thomason subset, it is enough to prove that for any \(y \in \eta(\mathcal{I})\) there is a closed set containing this point. By definition \(y\) is in the homological support of some object \(\mathcal{F} \in \mathcal{I}\). Hence \(y \in \text{supph}(ff(\mathcal{F}))\) which is a closed subset.

It is easy to check that \(\eta \circ \theta(Y) \subset Y\) and \(\mathcal{I} \subset \theta \circ \eta(\mathcal{I})\). To prove that \(Y \subset \eta \circ \theta(Y)\) it is enough to show that for any closed subset \(Z\) there exists an object with support \(Z\). But there exists an \(O_{X\text{id}}\) perfect sheaf with support \(Z\) and hence via the natural map \(\mathcal{O}_X \to O_{X\text{id}}\) we get a perfect sheaf with support \(Z\).

Finally to prove that \(\theta \circ \eta(\mathcal{I}) \subset \mathcal{I}\) it is enough to prove that for any \(\mathcal{F} \in \theta \circ \eta(\mathcal{I})\) the object \(\mathcal{F} \in \mathcal{I}\). Now following the proof of Theorem 3.15 of Thomason [26] we reduce to proving that if \(\text{supph}(\mathcal{F}^i) \subset \text{supph}(\mathcal{G}^i)\) for some object \(\mathcal{G}^i \in \mathcal{I}\), then \(\mathcal{F} \in \mathcal{I}\). By 4.10(1) we have that \(\text{supph}(\iota^*\mathcal{F}) \subset \text{supph}(\iota^*\mathcal{G})\). Now by Thomason [26] \(\iota^*\mathcal{F} \in (\iota^*\mathcal{G})\). Therefore by 4.10(2) \(\mathcal{I}_{\text{id}}\iota^*\mathcal{F} \in (\mathcal{G})\). Again using 4.10(1), \(\mathcal{F} \otimes_{\mathcal{O}_X} O_{X\text{id}} \in (\mathcal{G}) \subset \mathcal{I}, O_{X\text{id}}\) does not belong to any prime ideal since \(\mathcal{I}_{\text{id}}\) is dominant. Thus, using the fact that \(I\) is intersection of all primes containing it, \(\mathcal{F} \in \mathcal{I}\).

With this result it follows that \((X, \text{supph})\) is a classifying support data on the tensor triangulated category \(\mathbf{D}^{\text{per}}(X)\) for a quasi-compact and quasi-separated split superscheme \(X\), see [8, Definition 6.9] (and also Balmer [3, Definition 5.1] for the simpler noetherian case) for definitions. The following corollary is a restatement of the first part of Theorem 8.5 of [8].

**Corollary 4.17.** The canonical map \(f : X \to \text{Spc}(\mathbf{D}^{\text{per}}(X))\) given by \(x \mapsto \{\mathcal{F} : \mathcal{F} \in \mathbf{D}^{\text{per}}(X) \mid x \notin \text{supph}(\mathcal{F})\}\) is a homeomorphism.

**Remark 4.18.** One can use the classification of thick tensor ideals of the category of perfect complexes over quasi-compact and quasi-separated schemes to give a classification of strictly full tensor ideals, in the same way as in [26, Theorem 4.1].

## 5. Localization theorem and spectrum for a split superscheme

We shall prove a localization theorem (similar to that proved by Thomason) for split superschemes by using the generalisation of Thomason’s result proved by Neeman [21]. First we recall some notation. Given a closed subset \(Z\) of \(X\) we can define the full triangulated subcategory \(\mathbf{D}_{\text{qc}}(X) \subset \mathbf{D}_{\text{qc}}(Z)\) consisting of all objects with homological support contained in \(Z\). Suppose \(U\) is the open complement of closed subset \(Z\). There is a canonical restriction functor \(j^* : \mathbf{D}_{\text{qc}}(X) \to \mathbf{D}_{\text{qc}}(U)\) and clearly it will be the trivial functor on the thick subcategory \(\mathbf{D}_{\text{qc},Z}(X)\).

We have the following result whose proof is similar to the case of schemes.

**Proposition 5.1.** The canonical functor induced from the functor \(j^*\), which by abuse of notation we call \(j^* : \mathbf{D}_{\text{qc}}(X)/\mathbf{D}_{\text{qc},Z}(X) \to \mathbf{D}_{\text{qc}}(U)\) is an equivalence.

**Proof.** Using K-injective resolution we can derive \(j_*\) to unbounded derived category and we can prove, in a way similar to the scheme case, that it gives the inverse to the functor \(j^*\). □

**Definition 5.2.** Suppose \(\mathcal{T}\) is a triangulated category which is closed under formation of arbitrary small coproducts. \(\mathcal{T}\) is said to be **compactly generated** if there exists a set \(\mathcal{T}\) of compact
objects (Definition 4.12) such that $\mathcal{T}$ is a smallest triangulated subcategory containing $T$ which is closed under coproducts and distinguished triangles. Equivalently, $\mathcal{T}$ is compactly generated iff $T_\perp := \{ x \in T \mid \text{Hom}_\mathcal{T}(t,x) = 0 \text{ for all } t \in T \} = 0$. The set of compact objects $T$ is called generating set if further $T$ is closed under suspension or translation.

An example of such triangulated category can be given using derived category of left $R$-modules and category of quasi-coherent sheaves over superschemes. A result [23, Remark 1.2.2] of Neeman says that distinguished triangles are preserved under coproducts i.e. in a cocomplete triangulated category coproduct of distinguished triangle is distinguished.

Now we shall recall Theorem 2.1 of Neeman [22] which is proved in great generality and is a slight strengthening of Theorem 2.1 of Neeman [21].

**Theorem 5.3.** (See Neeman [21,22].) Let $S$ be a compactly generated triangulated category. Let $R$ be a set of compact objects of $S$ closed under suspension. Let $\mathcal{R}$ be the smallest full subcategory of $S$ containing $R$ and closed with respect to coproducts and triangles. Let $\mathcal{T}$ be the Verdier quotient $S/\mathcal{R}$. Then we know:

1. The category $\mathcal{R}$ is compactly generated, with $R$ as a generating set.
2. If $R$ happens to be a generating set for all of $S$, then $\mathcal{R} = S$.
3. If $R \subset \mathcal{R}$ is closed under the formation of triangles and direct summands, then it is all of $\mathcal{R}^C$. In any case $\mathcal{R}^C = \mathcal{R} \cap S^C$.
4. The induced functor $F : S^C/\mathcal{R}^C \to \mathcal{T}^C$ is fully faithful and every object of $\mathcal{T}^C$ is isomorphic to direct summand of image of the functor $F$. In particular, if $\mathcal{T}^C$ is an idempotent complete then we get an equivalence from idempotent completion $\tilde{S}^C/\mathcal{R}^C$ to the triangulated category $\mathcal{T}^C$.

In our particular situation we take $S := D^{qc}(X), \mathcal{R} := D^{qc,Z}(X)$ and as we proved above in 5.1 the quotient will be $\mathcal{T} := D^{qc}(U)$. We shall now prove the following result which will provide all hypothesis required for the application of Neeman’s theorem.

**Proposition 5.4.** The following statements are true for any split superscheme $(X, \mathcal{O}_X)$:

1. The triangulated category $D^{qc}(X)$ is closed under the formation of arbitrary small coproducts.
2. The triangulated category $D^{qc}(X)$ is a compactly generated category.
3. $D^{qc,Z}(X)^C \cong D^{ref}_Z(X)$ for any closed subset $Z$ of $X$.

**Proof.** Proof of 1. This is similar to the scheme case, as in Example 1.3 of Neeman [22].

Proof of 2. Suppose $T \subset D^{qc}(X)$ denotes the set of objects obtained by taking the image of all perfect complexes of $\mathcal{O}_{X_{rd}}$ under the functors $i_{rd}$ and $\Pi$ applied in that order. Let $\mathcal{F} \in D^{qc}(X)$. Since every unbounded complex of $\mathcal{O}_X$-modules over a superscheme $X$ has $K$-flat resolution, we can assume that $\mathcal{F}$ is a $K$-flat. Now using the tower in the proof of Proposition 4.10 of structure sheaf $\mathcal{O}_X$ we have the following tower for $\mathcal{F} \in D^{qc}(X)$:

$$
\begin{array}{cccc}
\mathcal{F} & \to & \mathcal{G}^1 & \cdots & \mathcal{G}_{n-1} & \to & \mathcal{G}_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_1 & \cdots & \mathcal{F}_{n-1} & \to & \mathcal{F}_n \\
\end{array}
$$

The base of above tower, $\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \Pi^i \Lambda^i(V) \in \text{Im}(i_{rd})$, is generated by objects of the set $T$. Hence every object $\mathcal{F} \in D^{qc}(X)$ is generated by the set $T$. It is now enough to prove that all objects of the set $T$ are compact in $D^{qc}(X)$. Since $\Pi$ commutes with coproducts it is enough to prove compactness of
the image of the functor $i_{\text{rd}}$ restricted to compact objects. Let $S$ be image of an $\mathcal{O}_{X_{\text{rd}}}$ perfect complex. We want to prove that $\text{Hom}(S, \_)$ commutes with small coproducts, that is,

$$\text{Hom}\left(S, \bigoplus_{\alpha \in A} \mathcal{F}_{\alpha}\right) \simeq \bigoplus_{\alpha \in A} \text{Hom}(S, \mathcal{F}_{\alpha}).$$

Considering above tower for each $\mathcal{F}_{\alpha}$ we get coproduct of tower as above. Using remark that small coproducts preserve distinguished triangles, [23, Remark 1.2.2], we get tower of distinguished triangles for $\bigoplus_{\alpha \in A} \mathcal{F}_{\alpha}$. If we denote by $\mathcal{F}_{\alpha,i}$ the lower terms of the corresponding towers then we have the following isomorphism using functor $i_{\text{rd}}$

$$\text{Hom}\left(S, \bigoplus_{\alpha \in A} \mathcal{F}_{\alpha,i}\right) \simeq \bigoplus_{\alpha \in A} \text{Hom}(S, \mathcal{F}_{\alpha,i}).$$

Using dévissage the proof follows from long exact sequence associated to $\text{Hom}(S, \_)$ and five lemma.

**Proof of 3.** It is enough to prove that all perfect complexes are compact objects. Indeed, the full subcategory of perfect complexes is closed under triangles and direct summands as in the case of schemes. Hence by taking $R$ to be all perfect complexes the above result of Neeman proves that all compact objects are perfect complexes.

Now to prove that every perfect complex is a compact object, we use the facts listed after Definition 4.12. First, we observe that the reduction principles 4 and 3 imply that it is enough to prove the statement in the affine case. But over affine $X$, there is an equivalence between $\mathcal{D}(\text{qc}/X)$ and $\mathcal{D}_{\text{qc}}(X)$ (see 2). In this case note that,

$$(H^0(\mathcal{R}\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})))^0 = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Here $\mathcal{R}\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})$ is the (internal) homomorphism between $\mathcal{F}$ and $\mathcal{G}$. Now the rest of the proof is similar to the proof given in Example 1.13 of Neeman [22].

Using the above result it is easy to deduce the following corollary:

**Corollary 5.5.** Given a split superscheme $(X, \mathcal{O}_X)$ we have an equivalence of tensor triangulated categories, $\tilde{j}^* : \mathcal{D}_{\text{per}}(X)/\mathcal{D}_{\text{per}}^Z(X) \xrightarrow{\sim} \mathcal{D}_{\text{per}}(U)$.

**Proof.** In the set up of Theorem 5.3, suppose $\mathcal{R} = \mathcal{D}_{\text{qc},Z}(X)$, $\mathcal{S} = \mathcal{D}_{\text{qc}}(X)$ and $T = \mathcal{D}_{\text{qc}}(U)$. Then Propositions 5.1 and 5.4 imply that the conditions for Theorem 5.3 are satisfied. Therefore, by 5.3(4), $\tilde{j}^*$ is an equivalence. This proves the result as $\tilde{j}^*$ is a tensor functor.

As in Balmer [3] we shall use this localization result to give a relation between structure sheaves. Balmer [3] has defined structure sheaf of $\text{Spc}(\mathcal{K})$ for any tensor triangulated category $\mathcal{K}$ as a sheaf associated to the presheaf given by $U \mapsto \text{End}_{\mathcal{K}/\mathcal{K}_Z}(1_U)$ where $U$ is an open set and $1_U \in (\mathcal{K}/\mathcal{K}_Z)$ is the image of tensor unit $1 \in \mathcal{K}$. Define $\text{Spec}(\mathcal{D}_{\text{per}}(X)) := (\text{Spc}(\mathcal{D}_{\text{per}}(X)), \mathcal{O}_{\mathcal{D}_{\text{per}}(X)})$ the locally ringed space associated to the tensor triangulated category $\mathcal{D}_{\text{per}}(X)$. Now the homeomorphism $f$ defined in 4.17 above for a split superscheme gives a map of locally ringed spaces, $f : (X \simeq X^0, \mathcal{O}_{X^0}) \rightarrow \text{Spec}(\mathcal{D}_{\text{per}}(X))$. Here the map of structure sheaves comes from the identification given in Corollary 5.5. We have the following result similar to Theorem 6.3 of Balmer [3].

**Theorem 5.6.** Suppose $X$ is a quasi-compact and quasi-separated split superscheme. The map $f : X^0 \simeq \text{Spec}(\mathcal{D}_{\text{per}}(X))$ defined above is an isomorphism of locally ringed spaces.
Proof. Using the homeomorphism $f$ it is enough to prove isomorphism of structure sheaves. Hence we can assume that the superscheme is affine. Now using Remark 8.2 of Balmer [2] and localization Theorem 5.5 we can prove that the induced map of sheaves is an isomorphism. □

Acknowledgments

We would like to take this opportunity to thank Prof. Kapil Paranjape and Prof. V. Srinivas for their encouragement and insightful comments.

References