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Inhomogeneous Diophantine approximation over the field of formal Laurent series ☆

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Abstract

De Mathan [B. de Mathan, Approximations diophantiennes dans un corps local, Bull. Soc. Math. France, Suppl. Mém. 21 (1970)] proved that Khintchine's theorem on homogeneous Diophantine approximation has an analogue in the field of formal Laurent series. Kristensen [S. Kristensen, On the well-approximable matrices over a field of formal series, Math. Proc. Cambridge Philos. Soc. 135 (2003) 255–268] extended this metric theorem to systems of linear forms and gave the exact Hausdorff dimension of the corresponding exceptional sets. In this paper, we study the inhomogeneous Diophantine approximation over a field of formal Laurent series, the analogue Khintchine's theorem and Jarnik–Besicovitch theorem are proved.

Keywords: Finite field; Inhomogeneous Diophantine approximation; Metric theory; Exceptional sets; Hausdorff dimension

1. Introduction and main results

The field of formal power series are analogues of the field of real numbers. Many problems in number theory which have been studied in the setting of the real numbers can be transposed to the case of the formal power series. Recently, the metric theory of Diophantine approximation attains considerable attentions, while most of them are concerned on homogeneous Diophantine approximation (see [1,2,7,10–14]).

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A well-known result (see [3–6]) in Diophantine approximation over the reals says that the Lebesgue measure of the pairs $(x,\alpha)\in\mathbb{R}^2$, such that $\|qx-\alpha\|<q^{-v}$ for infinitely many $q\in\mathbb{N}$, is full for $v\leqslant 1$ and null otherwise. Furthermore, the Hausdorff dimension of the set of these points is known to be $1+\frac{2}{v+1}$ for $v>1$. In this paper, we study the corresponding inhomogeneous Diophantine approximation over the field of formal Laurent series with coefficients in a given finite field.

Let $\mathbb{F}$ be a finite field with $k=p^l$ elements, where p is a prime and l is a natural number. Let X be an indeterminate and denote by $\mathbb{F}[X]$ and $\mathbb{F}(X)$ respectively the ring of polynomials with coefficients from $\mathbb{F}$ and the field of fractions over this ring. Define the field of formal Laurent series with coefficients from $\mathbb{F}$ to be

$$\mathbb{F}\big(\big(X^{-1}\big)\big)=\left\{\sum_{i=-n}^{\infty}a_{-i}X^{-i}\colon\ n\in\mathbb{Z},\ a_{-i}\in\mathbb{F},\ a_n\neq 0\right\}.$$

The norm of $x\in\mathbb{F}((X^{-1}))$ is defined by

$$|x|=\begin{cases}0, & \text{whenever } a_{-i}=0 \text{ for all } i\in\mathbb{Z},\\ k^n, & \text{whenever } a_{-n}\neq 0, \text{ and } a_{-i}=0 \text{ for all } i>n.\end{cases}$$

It is easy to verify that the above norm is non-Archimedean, i.e., for any $x,y\in\mathbb{F}((X^{-1}))$:

(a) $|x|\geqslant 0$, and $|x|=0$ if and only if $x=0$;
(b) $|xy|=|x||y|$;
(c) $|x+y|\leqslant\max\{|x|,|y|\}$.

Moreover, the equality in (c) holds whenever $|x|\neq|y|$. An important consequence of (c) is the “ball intersection” property: for any two balls C_1 and C_2 with the same radius, one has either $C_1\cap C_2=\phi$, or $C_1=C_2$.

The space $(\mathbb{F}((X^{-1})),d)$ is a complete metric space with the metric d induced by the norm $|\cdot|$. Here, comparing with the real case, $\mathbb{F}[X]$, $\mathbb{F}(X)$ and $\mathbb{F}((X^{-1}))$ play the roles of integers, rational numbers and real numbers, respectively.

Define

$$\lfloor x\rfloor=\left\lfloor\sum_{i=-n}^{\infty}a_{-i}X^{-i}\right\rfloor=\sum_{i=-n}^{0}a_{-i}X^{-i}\in\mathbb{F}[X],\qquad x\in\mathbb{F}\big(\big(X^{-1}\big)\big)$$

and call it the integer part of x. Put

$$I=\big\{x\in\mathbb{F}\big(\big(X^{-1}\big)\big)\colon\ [x]=0\big\}=\big\{x\in\mathbb{F}\big(\big(X^{-1}\big)\big)\colon\ |x|<1\big\}=B(0,1),$$

which plays the role of the unit interval $[0,1)$ in $\mathbb{R}$. The ideal I is compact because it can be identified with $\prod_{n=1}^{\infty}\mathbb{F}$. A natural measure on I is the unique normalized Haar measure on $\prod_{n=1}^{\infty}\mathbb{F}$, which we denote by μ. For any $b_{-1},b_{-2},\dots,b_{-m}\in\mathbb{F}$, we call the set

$$\begin{aligned}[b_{-1},b_{-2},\dots,b_{-m}]=\big\{&x=a_{-1}X^{-1}+a_{-2}X^{-2}+\cdots\in I\colon\\ &a_{-1}=b_{-1},a_{-2}=b_{-2},\dots,a_{-m}=b_{-m}\big\}\end{aligned}$$

a mth order cylinder. For the Haar measure μ, Sprindžuk (see [15]) gave an accurate characterization, where he showed that

$$\mu\big([b_{-1}, b_{-2}, \dots, b_{-m}]\big) = k^{-m}.$$

Definition 1.1. Fix $n \in \mathbb{N}$. For any $x = (x_1, x_2, \dots, x_n) \in (\mathbb{F}((X^{-1})))^n$, we define the height of x as

$$|x|_\infty = \max\big\{|x_j|:\ 1 \leqslant j \leqslant n\big\}.$$

It is easy to verify that (a) and (c) in the definition of non-Archimedean norm hold for $|\cdot|_\infty$.

Definition 1.2. For $x \in \mathbb{F}((X^{-1}))$, we define the distance from x to the polynomial lattice points as

$$\|x\| = \min\big\{|x - p|:\ p \in \mathbb{F}[X]\big\}.$$

In fact, we have that

$$\|x\| = \big|x - \lfloor x \rfloor\big|,$$

and $\|x\| = |x|$ for any $x \in I$; $\|x\| \leqslant |x|$ for any $x \in \mathbb{F}((X^{-1}))$.

For $q \in \mathbb{F}[X]$, let $\psi : \mathbb{F}[X] \to \mathbb{R}^+$ be a function satisfying $\psi(q) \leqslant \frac{1}{2}$ and

$$\psi(q_1) = \psi(q_2), \quad \text{if } |q_1| = |q_2|.$$

Then we can write $\psi(q) = \psi(|q|)$, i.e., ψ is radial. We consider the set of points $(x, \alpha) \in (\mathbb{F}((X^{-1})))^2$ for which the Diophantine inequality

$$\|qx - \alpha\| < \psi(q)$$

has for infinitely many solutions $q \in \mathbb{F}[X]$.

Remark 1.3. Since the norm $|\cdot|$ defined above takes value only on $\{k^r;\ r \in \mathbb{Z}\}$, then we can restrict $\psi : \mathbb{F}[X] \to \mathbb{R}^+$ to $\psi : \mathbb{F}[X] \to \{k^r;\ r \in \mathbb{Z}\}$. When ψ assumes on other values such as $\psi(q) = k^{-\lambda(q)}$, we replace the value of $\psi(q)$ by $k^{-\lfloor \lambda(q) \rfloor}$ and still denote it by $\psi(q)$ without causing any confusion.

Write $\Omega = I \times I$. Since $(x, \alpha) \to \|qx - \alpha\|$ is p-periodic for any $p \in \mathbb{F}[X]$, thus we only concentrate our attention on Ω instead of $\mathbb{F}((X^{-1}))^2$. Let

$$\Phi(\psi) = \big\{(x, \alpha) \in \Omega:\ \|qx - \alpha\| < \psi(q) \text{ for infinitely many } q \in \mathbb{F}[X]\big\}.$$

For the special case that $\psi(q) = |q|^{-v}$ with $v > 0$, we write $\Phi(\psi) = \Phi_v$.

Now we give our main results.

Theorem 1.4.

$$(\mu \times \mu)\big(\Phi(\psi)\big) = \begin{cases} 0, & \text{if } \sum_{q\in\mathbb{F}[X]} \psi(q) < \infty, \\ 1, & \text{if } \sum_{q\in\mathbb{F}[X]} \psi(q) = \infty. \end{cases}$$

Theorem 1.5. *For $v > 1$, $\dim_{\mathrm{H}} \Phi_v = 1 + \frac{2}{v+1}$.*

2. Metric theory

In this section, we will establish Theorem 1.4. First we fix some notation.

For $q \in \mathbb{F}[X]$ and $\alpha \in I$, denote

$$A_{q,\alpha} = \big\{x \in I\colon \ \|qx - \alpha\| < \psi(q)\big\}.$$

For $p, q \in \mathbb{F}[X], \alpha \in I$, let

$$A_{q,\alpha}(p) = \big\{x \in I\colon \ |qx - \alpha - p| < \psi(q)\big\}.$$

Clearly,

$$A_{q,\alpha} = \bigcup_{\deg(p)<\deg(q)} A_{q,\alpha}(p).$$

By the translate-invariant property of the Haar measure μ we have

$$\mu\big(A_{q,\alpha}(p)\big) = \mu\left(B\left(0, \frac{\psi(q)}{|q|}\right)\right) = \frac{\psi(q)}{|q|}.$$

Then it follows that

$$\mu(A_{q,\alpha}) = \sum_{\deg(p)<\deg(q)} \mu\big(A_{q,\alpha}(p)\big) = |q|\frac{\psi(q)}{|q|} = \psi(q). \tag{1}$$

Lemma 2.1. *Given $\alpha \in I$. If the sum*

$$\sum_{q\in\mathbb{F}[X]} \psi(q)$$

converges, then for μ a.e. $x \in I$, the inequality $\|qx - \alpha\| < \psi(q)$ holds for at most finitely many $q \in \mathbb{F}[X]$.

Proof. In the light of (1), this assertion is just a consequence of the Borel–Cantelli lemma [16]. $\square$

The following lemma is a standard inequality in probability theory (see [16]).

Lemma 2.2. *Assume $f(x,y):\Omega \to \mathbb{R}^+$ is a function such that $f(x,y) = f(|x|,|y|)$ and is square integrable on Ω. Denote*

$$M_1 = \iint_\Omega f(x,y)\,\mu(dx)\,\mu(dy), \qquad M_2 = \left(\iint_\Omega f(x,y)^2\,\mu(dx)\,\mu(dy)\right)^{1/2}.$$

If $M_1 \geqslant aM_2$, $0 < b \leqslant a$, and $A = \{(x,y) \in \Omega\colon\ f(x,y) \geqslant bM_2\}$, then $(\mu \times \mu)(A) \geqslant (b-a)^2$.

For $q \in \mathbb{F}[X]$, define $\delta_q(x): I \to \mathbb{R}^+$ as

$$\delta_q(x) = \begin{cases} 1, & \text{if } \|x\| < \psi(q), \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that $\delta_q(x)$ is defined in I with one variable, but we can look $\delta_q(qx-\alpha)$ as a composition of δ_q and f_q, where $f_q: I \times I \to \mathbb{R}^+$, $f_q(x,\alpha) = qx - \alpha$ for any $x, \alpha \in I$, when we encounter the function $\delta_q(qx-\alpha)$.

By the definition of $\delta_q(x)$, we have

Lemma 2.3. *The function $\delta_q(x)$ shares the following properties*:

(1) $\delta_q(-x) = \delta_q(x)$;
(2) *$\delta_q(x)$ is a periodic function, and any p in $\mathbb{F}[X]$ is a period of $\delta_q(x)$*;
(3) $\delta_q^2(x) = \delta_q(x)$.

Furthermore, we have

Lemma 2.4.

(1) $\int_I \delta_q(x)\,\mu(dx) = \psi(q)$;
(2) $\iint_\Omega \delta_q(qx-\alpha)\,\mu(dx)\,\mu(d\alpha) = \psi(q)$;
(3) $\iint_\Omega \delta_q(qx-\alpha)\delta_{q'}(q'x-\alpha)\,\mu(dx)\,\mu(d\alpha) = \begin{cases} \psi(q), & \textit{if } q = q', \\ \psi(q)\psi(q'), & \textit{if } q \neq q'. \end{cases}$

Proof. We show (3) only for the case when $q \neq q'$.

Let $r = q - q'$ and $\alpha'(\equiv \mathbb{T}\alpha) = \alpha - qx - [\alpha - qx]$. By Lemma 2.3 and the translate-invariant property of the Haar measure μ, we have

$$\begin{aligned} &\iint_\Omega \delta_q(qx-\alpha)\delta_{q'}(q'x-\alpha)\,\mu(dx)\,\mu(d\alpha) \\ &\quad = \int_I \left(\int_I \delta_q(-\alpha')\delta_{q'}\big(-(rx+\alpha')\mu \circ \mathbb{T}^{-1}(d\alpha')\big)\,\mu(dx)\right) \\ &\quad = \int_I \delta_q(\alpha')\left(\int_I \delta_{q'}(rx+\alpha')\,\mu(dx)\right)\mu(d\alpha') \\ &\quad = \psi(q)\psi(q'). \qquad \square \end{aligned}$$

Proof of Theorem 1.4. (I) *Convergence part*. Lemma 2.1 says that for fixed $\alpha \in I$,

$$\mu\{x \in I\colon \|qx - \alpha\| < \psi(q) \text{ for infinitely many } q \in \mathbb{F}[X]\} = 0.$$

Then the application of Fubini theorem yields that

$$(\mu \times \mu)\big(\Phi(\psi)\big) = 0.$$

(II) *Divergence part*. For any $N \in \mathbb{N}$, write

$$\Delta_N(x,\alpha) = \#\{q \in \mathbb{F}[X]\colon \|qx - \alpha\| < \psi(q),\ 0 < |q| \leqslant k^N\},$$

$$M_1(N) = \iint\limits_{\Omega} \Delta_N(x,\alpha)\,\mu(dx)\,\mu(d\alpha), \qquad M_2(N) = \Bigg(\iint\limits_{\Omega} \Delta_N^2(x,\alpha)\,\mu(dx)\,\mu(d\alpha)\Bigg)^{1/2}$$

and $\psi(N) = \sum_{|q| \leqslant k^N} \psi(q)$.

We claim that, for any $\epsilon > 0$,

$$\psi(N) = M_1(N) \geqslant (1-\epsilon)M_2(N) \tag{$*$}$$

holds for all N large enough.

Notice that

$$\Delta_N(x,\alpha) = \sum_{|q| \leqslant k^N} \delta_q(qx - \alpha).$$

Then by Lemma 2.4, we have

$$\begin{aligned}
M_1(N) &= \sum_{|q| \leqslant k^N} \iint\limits_{\Omega} \delta_q(qx-\alpha)\,\mu(dx)\,\mu(d\alpha) = \sum_{|q| \leqslant k^N} \psi(q) = \psi(N),\\
M_2(N)^2 &= \sum_{|q| \leqslant k^N,\, |q'| \leqslant k^N} \iint\limits_{\Omega} \delta_q(qx-\alpha)\delta_{q'}(q'x-\alpha)\,\mu(dx)\,\mu(d\alpha)\\
&= \sum_{|q| \leqslant k^N,\, |q'| \leqslant k^N,\, q \neq q'} \psi(q)\psi(q') + \sum_{|q| \leqslant k^N} \psi(q)\\
&\leqslant \psi(N)^2 + \psi(N).
\end{aligned}$$

Since $\lim_{N\to\infty} \frac{\psi(N)}{\psi(N)^2} = 0$, then for any $\epsilon > 0$, when N is large enough, we have

$$M_2(N)^2 \leqslant (1-\epsilon)^{-2}\psi(N)^2.$$

Therefore, $(*)$ holds.

Set

$$A_N(\epsilon) = \{(x,\alpha) \in \Omega\colon \Delta_N(x,\alpha) \geqslant \epsilon M_2(N)\}.$$

Then applying Lemma 2.2 to $f(x,\alpha) = \Delta_N(x,\alpha)$, $a = 1-\epsilon$, $b = \epsilon$, we have

$$(\mu \times \mu)\big(A_N(\epsilon)\big) \geqslant (1-2\epsilon)^2 \geqslant 1-4\epsilon. \tag{2}$$

Fix $\epsilon > 0$. Choose a subsequence N_k of $\mathbb{N}$ large enough such that (2) holds with respect to $A_{N_k}(\frac{\epsilon}{2^{k+2}})$, for all $k \geqslant 1$. Set

$$A(\epsilon) = \bigcap_{k \geqslant 1} A_{N_k}\left(\frac{\epsilon}{2^{k+2}}\right).$$

Then we have $\mu(A(\epsilon)) \geqslant 1-\epsilon$. It is evident that $A(\epsilon)$ is a subset of $\Psi(\psi)$. As a consequence, by the arbitrariness of ϵ, we have $\mu(\Psi(\psi)) = 1$. $\square$

As a special case of Theorem 1.4, we have

Corollary 2.5.

$$(\mu \times \mu)(\Phi_v) = \begin{cases} 0, & v > 1, \\ 1, & v \leqslant 1. \end{cases}$$

3. Hausdorff dimension of some exceptional sets

We now consider the Hausdorff dimensions of some exceptional sets. First we cite the definition of Hausdorff dimension in this setting. More detailed accounts can be found in [8,9].

Let E be a subset of Ω. For any $\delta > 0$, we call $\mathcal{C}_\delta$ with countable many balls $B_i = B(c_i, \rho_i)$ is a δ cover of E if $E \subset \bigcup_{B \in \mathcal{C}_\delta} B$ and $\rho_i < \delta$. Then for any $s \geqslant 0$, the s-dimensional Hausdorff dimension of E is defined as

$$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{C}_\delta} \left\{ \sum_{B(x_i,\rho_i) \in \mathcal{C}_\delta} \rho_i^s \right\}.$$

The Hausdorff dimension of E is defined as

$$\dim_{\mathrm{H}}(E) = \inf\big\{s \geqslant 0\colon \mathcal{H}^s(E) = 0\big\}.$$

Now we fix some notation before a lemma. For $q \in \mathbb{F}[X]$ and $\delta > 0$, consider the "resonant" sets

$$R_q = \big\{(x,\alpha) \in \Omega\colon qx - \alpha \in \mathbb{F}[X]\big\},$$

and their δ neighborhood

$$B_\delta(R_q) = \big\{(x,\alpha) \in \Omega\colon \|qx - \alpha\| < \delta\big\}.$$

We also define the set

$$B(R_q;\delta) = \big\{(x,\alpha) \in \Omega\colon \mathrm{dist}_\infty\big((x,\alpha), R_q\big) < \delta\big\},$$

where dist_∞ denotes the distance in the height-norm, i.e.,

$$\mathrm{dist}_\infty\big((x,\alpha), R_q\big) = \inf_{(x',\alpha')\in R_q} \max\big\{|x-x'|, |\alpha-\alpha'|\big\}.$$

Lemma 3.1.

(1) $R_q = \bigcup_{\deg(p)<\deg(q)}\{(x,\alpha)\in\Omega\colon\ qx-\alpha=p\}$.

(2) $B(R_q;\delta) = \bigcup_{(x_0,\alpha_0)\in R_q}\{(x,\alpha)\in\Omega\colon\ \max\{|x-x_0|, |\alpha-\alpha_0|\}<\delta\}$.

(3) $B_\delta(R_q) = B(R_q;\delta/|q|)$.

Proof. The assertion in (2) is the fact since R_q is compact. We prove (3) in two steps.

For any $(x,\alpha)\in B(R_q;\frac{\delta}{|q|})$, by assertion (2), there exists (x_0,α_0) such that

$$|x-x_0| < \frac{\delta}{|q|} \quad\text{and}\quad |\alpha-\alpha_0| < \frac{\delta}{|q|}.$$

As a consequence,

$$\|qx-\alpha\| = \big\|qx-\alpha-(qx_0-\alpha_0)\big\| \leqslant \max\big\{|q||x-x_0|, |\alpha-\alpha_0|\big\} < \delta.$$

This means that $(x,\alpha)\in B_\delta(R_q)$, which follows $B(R_q;\frac{\delta}{|q|})\subset B_\delta(R_q)$.

On the other hand, for any $(x,\alpha)\in B_\delta(R_q)$, let

$$\|qx-\alpha\| = k^{-t-1} < \delta,$$

with $t\in\mathbb{N}$. By Definition 1.2, there exists $p\in\mathbb{F}[X]$ such that

$$|qx-\alpha-p| = k^{-t-1} < \delta.$$

Thus α and $qx-p$ belong to the same tth order cylinder. Without loss of generality, we assume

$$\alpha\in[a_{-1},a_{-2},\dots,a_{-t}], \qquad qx-p\in[a_{-1},a_{-2},\dots,a_{-t}].$$

Hence x must lie in a well-determined $(t+\deg(q))$th order cylinder denoted by

$$[b_{-1},b_{-2},\dots,b_{-t-\deg(q)}].$$

Moreover,

$$q[b_{-1},b_{-2},\dots,b_{-t-\deg(q)}]-p = [a_{-1},a_{-2},\dots,a_{-t}].$$

Take $x_0\in[b_{-1},b_{-2},\dots,b_{-t-\deg(q)}]$, such that $qx_0-p=\alpha$, and take $\alpha_0=\alpha$. Clearly, $qx_0-\alpha_0\in\mathbb{F}[X]$, i.e., $(x_0,\alpha_0)\in R_q$. It is easy to see that

$$|x-x_0| < \frac{1}{|q|}k^{-t} < \frac{\delta}{|q|}, \qquad |\alpha-\alpha_0| = 0.$$

This means $(x,\alpha)\in B(R_q;\frac{\delta}{|q|})$, which gives $B_\delta(R_q)\subset B(R_q;\frac{\delta}{|q|})$. $\square$

Now, we begin with the upper bound estimation on the Hausdorff dimension of Φ_v.

Lemma 3.2. *Let* $v > 1$. *Then*

$$\dim_{\mathrm{H}} \Phi_v \leqslant 1 + \frac{2}{v+1}.$$

Proof. By definition

$$\Phi_v = \big\{(x,\alpha) \in \Omega\colon\ (x,\alpha) \in B_{|q|^{-v}}(R_q) \text{ for infinitely many } q \in \mathbb{F}[X]\big\}.$$

Furthermore, Φ_v can be expressed as a lim sup set

$$\Phi_v = \bigcap_{r \geqslant 1} \bigcup_{|q| \geqslant k^r} B_{|q|^{-v}}(R_q).$$

Clearly, $\bigcup_{|q| \geqslant k^r} B_{|q|^{-v}}(R_q)$ is a natural cover of Φ_v, denote this cover by $\mathcal{C}$, i.e., $\mathcal{C} = \{\bigcup_{q \in \mathbb{F}[X]} B_{|q|^{-v}}(R_q)\colon |q| \geqslant k^r\}$.

Fix $t \in \mathbb{N}$ and $t \geqslant r$. Fix a $q \in \mathbb{F}[X]$ of degree t, we will construct a natural covering of $B_{|q|^{-v}}(R_q)$.

It is evident that I can be expressed as the union of all $\lfloor tv \rfloor$th order cylinders, that is to say

$$I = \bigcup_{c_{-1}, c_{-2}, \ldots, c_{-\lfloor tv \rfloor}} [c_{-1}, c_{-2}, \ldots, c_{-\lfloor tv \rfloor}],$$

where $c_{-1}, c_{-2}, \ldots, c_{-\lfloor tv \rfloor} \in \mathbb{F}$. Then we refine $B_{|q|^{-v}}(R_q)$ to a sequence of "regular" sets:

$$\begin{aligned}
&B_{|q|^{-v}}(R_q) \\
&\quad= \bigcup_{\substack{c_{-1},\ldots,\\ c_{-\lfloor tv \rfloor}}} \big\{(x,\alpha) \in \Omega\colon\ \|qx - \alpha\| < k^{-\lfloor tv \rfloor},\ \alpha \in [c_{-1}, \ldots, c_{-\lfloor tv \rfloor}]\big\} \\
&\quad= \bigcup_{\substack{c_{-1},\ldots,\\ c_{-\lfloor tv \rfloor}}} \bigcup_{i=0}^{t-1} \bigcup_{|p| = k^i} \big\{(x,\alpha) \in \Omega\colon\ qx - p \in [c_{-1}, \ldots, c_{-\lfloor tv \rfloor}],\ |q| = k^t,\ \alpha \in [c_{-1}, \ldots, c_{-\lfloor tv \rfloor}]\big\} \\
&\quad\subset \bigcup_{\substack{c_{-1},\ldots,\\ c_{-\lfloor tv \rfloor - t}}} \bigcup_{i=0}^{t-1} \bigcup_{|p| = k^i} \big\{(x,\alpha) \in \Omega\colon\ x \in [b_{-1}, \ldots, b_{-\lfloor tv \rfloor - t}], \alpha \in [c_{-1}, \ldots, c_{-\lfloor tv \rfloor - t}]\big\}.
\end{aligned}$$

Thus

$$\bigcup_{t=r}^{\infty} \bigcup_{\deg(q)=t} \bigcup_{\substack{c_{-1},\ldots,\\ c_{-\lfloor tv \rfloor - t}}} \bigcup_{i=0}^{t-1} \bigcup_{|p| = k^i} \big\{(x,\alpha) \in \Omega\colon\ (x,\alpha) \in [b_{-1}, \ldots, b_{-\lfloor tv \rfloor - t}] \times [c_{-1}, \ldots, c_{-\lfloor tv \rfloor - t}]\big\}$$

forms a cover of Φ_v. Then by the definition of Hausdorff dime measure, we have

$$H^s(\Phi_v) \leqslant \liminf_{r\to\infty} \sum_{t=r}^{\infty} k^{-(\lfloor tv\rfloor+t)s} k^{\lfloor tv\rfloor+t} \sum_{i=0}^{t-1} (k-1)k^i (k-1)k^t$$
$$\leqslant \liminf_{r\to\infty} \sum_{t=r}^{\infty} k^{[(v+1)(1-s)+2]t} < \infty$$

for any $s > 1 + \frac{2}{v+1}$. Thus

$$\dim_{\mathrm{H}} \Phi_v \leqslant 1 + \frac{2}{v+1}. \qquad \square$$

Now we turn to the estimation of the lower bound of $\dim_{\mathrm{H}} \Phi_v$. In fact, we will give the lower bound of the Hausdorff dimension of a more general set $\Lambda(\psi)$.

$$\Lambda(\psi) = \big\{(x,\alpha) \in \Omega\colon \operatorname{dist}_\infty\big((x,\alpha), R_q\big) < \psi(q) \text{ for infinitely many } q \in \mathbb{F}[X]\big\},$$

where $\psi : \mathbb{F}[X] \to \{k^r\colon r \in \mathbb{Z}\}$ be a decreasing function such that $\psi(q) = \psi(|q|)$, $\psi(k^N) \leqslant k^{-\rho_N}$, and $\rho_N = \lfloor 2N - \log N\rfloor$.

For $N \in \mathbb{N}$, put

$$\Omega(N) = \bigcup_{|q|\leqslant k^N} B\big(R_q;\ k^{-\rho_N}\big).$$

Lemma 3.3. $\lim_{N\to\infty}(\mu\times\mu)(\Omega \setminus \Omega(N)) = 0$.

Proof. For $N \in \mathbb{N}$, set

$$\nu_N(x,\alpha) = \sum_{|q|\leqslant k^N} \chi_{B_{|q|k^{-\rho_N}}(R_q)}(x,\alpha).$$

It is natural that

$$\nu_N^{-1}(0) = \Omega \setminus \Omega_N = \Omega \setminus \bigcup_{|q|\leqslant k^N} B_{|q|k^{-\rho_N}}(R_q).$$

Denote by μ_N the mean of $\nu_N(x,\alpha)$. Then we have

$$\mu_N = \iint_{\Omega} \nu_N(x,\alpha)\,\mu(dx)\,\mu(d\alpha) = \sum_{|q|\leqslant k^N} (\mu\times\mu)\big(B_{|q|k^{-\rho_N}}(R_q)\big) = \sum_{|q|\leqslant k^N} |q|k^{-\rho_N}$$
$$= \sum_{1\leqslant r\leqslant N} \sum_{|q|=k^r} k^r k^{-\rho_N} = (k-1)\frac{k^2}{k^2-1}\frac{k^{2N}-1}{k^{\rho_N}} \to \infty.$$

Moreover, we can calculate the variance σ_N^2 of $\nu_N(x,\alpha)$. By Lemma 2.4, we get

$$\begin{aligned}
\sigma_N^2 &= \iint\limits_{\Omega} \nu_N(x,\alpha)^2\,\mu(dx)\,\mu(d\alpha) - \mu_N^2 \\
&= \sum_{|q|\leqslant k^N} \sum_{|q'|\leqslant k^N} \iint\limits_{\Omega} \chi_{B_{|q|k^{-\rho_N}}(R_q)}(x,\alpha)\chi_{B_{|q'|k^{-\rho_N}}(R_{q'})}(x,\alpha)\,\mu(dx)\,\mu(d\alpha) - \mu_N^2 \\
&= \sum_{|q|\leqslant k^N} |q|k^{-\rho_N} + \sum_{|q|,\,|q'|\leqslant k^N,\,q\neq q'} |q|k^{-\rho_N}|q'|k^{-\rho_N} - \mu_N^2 \\
&\leqslant \mu_N.
\end{aligned}$$

On the other hand, by the definition of $\nu_N^{-1}(0)$, we have

$$\sigma_N^2 \geqslant \iint\limits_{\nu_N^{-1}(0)} \bigl(\nu_N(x,\alpha)-\mu_N\bigr)^2\,\mu(dx)\,\mu(d\alpha) = \mu_N^2\cdot(\mu\times\mu)\bigl(\nu_N^{-1}(0)\bigr).$$

Therefore,

$$\lim_{N\to\infty}(\mu\times\mu)\bigl(\Omega\setminus\Omega(N)\bigr) = \lim_{N\to\infty}(\mu\times\mu)\bigl(\nu_N^{-1}(0)\bigr) \leqslant \lim_{N\to\infty}\frac{\sigma_N^2}{\mu_N^2} \leqslant \lim_{N\to\infty}\frac{1}{\mu_N} = 0. \qquad \square$$

For the above ρ_N and ψ, let

$$\gamma = \limsup_{N\to\infty}\frac{-\rho_N\log k}{\log\psi(k^N)}.$$

Then $0\leqslant\gamma\leqslant 1$.

Lemma 3.4. $\dim_{\mathrm{H}}\Lambda(\psi)\geqslant 1+\gamma$.

Proof. For $N\in\mathbb{N}$, let

$$\Gamma(\rho_N) = \left\{\left(\sum_{i=1}^{\rho_N} a_{-i}X^{-i}, \sum_{i=1}^{\rho_N} b_{-i}X^{-i}\right):\ a_{-i}, b_{-i}\in\mathbb{F},\ 1\leqslant i\leqslant\rho_N\right\}.$$

For any $x,y\in\Gamma(\rho_N)$ and $x\neq y$, it is noticed that $|x-y|_\infty\geqslant k^{-\rho_N}$. Then we can define a partition $\mathcal{P}(N)$ of Ω as

$$\mathcal{P}(N) = \bigl\{B\bigl(c,k^{-\rho_N}\bigr)\subset\Omega:\ c\in\Gamma(\rho_N)\bigr\}.$$

Now we define the family of bad balls and good balls in $\mathcal{P}(N)$, respectively.

$$\mathcal{B}(N) = \bigl\{B\in\mathcal{P}(N):\ B\cap\Omega(N)=\emptyset\bigr\}, \qquad \mathcal{G}(N) = \mathcal{P}(N)\setminus\mathcal{B}(N).$$

We see that $B\subset\Omega\setminus\Omega(N)$ for all $B\in\mathcal{B}(N)$. By Lemma 3.3, we have

$$1 = \lim_{N\to\infty}\mu\bigl(\mathcal{G}(N)\bigr) = \lim_{N\to\infty}\sharp\,\mathcal{G}(N)k^{-2\rho_N}.$$

Then for N large enough,

$$\frac{1}{2}k^{2\rho_N} \leqslant \sharp\mathcal{G}(N) \leqslant 2k^{2\rho_N}.$$

Fix a $|\cdot|_\infty$-ball $B \in \mathcal{G}(N)$, and let $q \in \mathbb{F}[X]$ with $|q| \leqslant k^N$ be the polynomial such that $B \cap B(R_q; k^{-\rho_N}) \neq \emptyset$. We construct a family $\mathcal{D}(B)$ of B as follows:

(1) choose points c_i in $B \cap B(R_q; k^{-\rho_N})$ such that for $i \neq j$,

$$|c_i - c_j|_\infty \geqslant \psi(k^N);$$

(2) take balls B_i with centers c_i and radii $\psi(k^N)$;
(3) remove all points belonging to R_q from the above balls. Denote

$$\mathcal{D}(B) = \{D\colon\ D \text{ is the “ball” obtained in the above fashion}\}.$$

Without confusion, any $D \in \mathcal{D}(B)$ is still called a “ball.” Clearly, the “balls” in $\mathcal{D}(B)$ are disjoint, and

$$(\mu \times \mu)(D) = (\mu \times \mu)(B_i) = \psi^2(k^N),$$

for $(\mu \times \mu)(R_q) = 0$. Furthermore, since ψ is decreasing, we have for each point $u \in D \in \mathcal{D}(B)$,

$$0 < \operatorname{dist}_\infty(u, R_q) < \psi(k^N) \leqslant \psi(q).$$

Since $B \cap B(R_q; k^{-\rho_N}) \neq \emptyset$ and $\psi(k^N) \leqslant k^{-\rho_N}$, then $\mathcal{D}(B)$ is nonempty.

Lemma 3.5 *(Estimate the number of “balls”).*

$$\sharp\mathcal{D}(B) = \frac{k^{-\rho_N}}{\psi(k^N)}.$$

Proof. It suffices to consider the number of pairs (x, α) in $B \cap R_q$ with the properties

$$\operatorname{dist}_\infty\big((x_1, \alpha_1), (x_2, \alpha_2)\big) \geqslant \psi(k^N).$$

Write $B = [a_{-1}, \ldots, a_{\rho_N}] \times [b_{-1}, \ldots, b_{-\rho_N}]$. Since the integer part of qx is wholly determined by the first $\deg(q)$ terms of the digits of x, thus, for some p_0,

$$\lfloor qx \rfloor = p_0 \tag{3}$$

for all (x, α) in B.

This can make us claim that, for any pair $(x_1, \alpha_1), (x_2, \alpha_2) \in B \cap R_q$

$$\operatorname{dist}_\infty\big((x_1, \alpha_1), (x_2, \alpha_2)\big) = |\alpha_1 - \alpha_2|.$$

Since $qx_i - \alpha_i \in F[X]$, $i = 1, 2$, and by (3), we know

$$x_i = \frac{\alpha_i + p_0}{q}. \tag{4}$$

Moreover, $x_i \in [a_{-1}, \ldots, a_{\rho_N}]$, $i = 0, 1, 2$, since it holds for $i = 0$. Equation (4) implies that for $(x, \alpha) \in B \cap R_q$, once α is given, x has been already determined.

As a consequence, we only need consider the number of $\alpha \in [b_{-1}, \ldots, b_{\rho_N}]$ such that $|\alpha_1 - \alpha_2| \geqslant \psi(k^N)$. It is quite straightforward that this number is just equal to $\frac{k^{-\rho_N}}{\psi(k^N)}$.

Thus

$$\sharp\,\mathcal{D}(B) = \frac{k^{-\rho_N}}{\psi(k^N)}. \qquad \square$$

For $N \in \mathbb{N}$, set

$$T_N = \bigcup_{B \in \mathcal{G}(N)} \bigcup_{D \in \mathcal{D}(B)} D, \qquad t_N = \sum_{B \in \mathcal{G}(N)} \sum_{D \in \mathcal{D}(B)} 1.$$

Then

$$\frac{1}{2}\frac{k^{\rho_N}}{\psi(k^N)} \leqslant t_N \leqslant 2\frac{k^{\rho_N}}{\psi(k^N)},$$

and

$$(\mu \times \mu)(T_N) = \sharp\,\mathcal{G}(N)\,\sharp\,\mathcal{D}(B)(\mu \times \mu)(D),$$

for any $D \in \mathcal{D}(B)$.

Lemma 3.6. *Let $Y \subset \Omega$ be a $|\cdot|_\infty$-ball. Then*

$$\frac{1}{4}(\mu \times \mu)(T_N)(\mu \times \mu)(Y) \leqslant (\mu \times \mu)(T_N \cap Y) \leqslant 4(\mu \times \mu)(T_N)(\mu \times \mu)(Y)$$

holds for N large enough.

Proof. For $Y \subset \Omega$, let $\mathcal{G}_Y(N) = \{B \in \mathcal{G}(N)\colon\ B \subset Y\}$ and $Y(N) = \bigcup_{B \in \mathcal{G}_Y(N)} B$. By Lemma 3.3, we have

$$\lim_{N\to\infty} (\mu \times \mu)\big(Y \setminus Y(N)\big) \leqslant \lim_{N\to\infty} (\mu \times \mu)\big(\Omega \setminus \Omega(N)\big) = 0.$$

Thus, for N large enough

$$\frac{1}{4}\,\sharp\,\mathcal{G}(N) \leqslant \frac{1}{2}k^{2\rho_N} \leqslant \frac{\sharp\,\mathcal{G}_Y(N)}{(\mu \times \mu)(Y)} \leqslant 2k^{2\rho_N} \leqslant 4\,\sharp\,\mathcal{G}(N).$$

By the definition of $\mathcal{G}_Y(N)$ and the construction of $\mathcal{D}(B)$, we have

$$\begin{aligned}(\mu\times\mu)(T_N\cap Y) &= (\mu\times\mu)\Big(\bigcup_{B\in\mathcal{G}(N)}\ \bigcup_{D\in\mathcal{D}(B)}(D\cap Y)\Big)\\ &= \sum_{B\in\mathcal{G}_Y(N)}\ \sum_{D\in\mathcal{D}(B)}(\mu\times\mu)(D)\\ &= \sharp\,\mathcal{G}_Y(N)\,\sharp\,\mathcal{D}(B)(\mu\times\mu)(D).\end{aligned}$$

Therefore Lemma 3.6 follows. □

Now we are ready to construct a Cantor set in $\Lambda(\psi)$.

First, recalling that $\gamma=\limsup_{N\to\infty}\frac{-\rho_N\log k}{\log\psi(k^N)}$, we choose an increasing sequence $\{M_r\colon r\in\mathbb{N}\}\subseteq\mathbb{N}$ such that

$$\gamma=\lim_{r\to\infty}\frac{-\rho_{M_r}\log k}{\log\psi(k^{M_r})}.$$

Choose $N_1\in\{M_r\colon r\in\mathbb{N}\}$ sufficiently large to guarantee that Lemma 3.6 holds for $Y=\Omega$. Let

$$C'(N_1)=\bigcup_{B\in\mathcal{G}(N_1)}\ \bigcup_{D\in\mathcal{D}(B)}D,\qquad c'(N_1)=\sum_{B\in\mathcal{G}(N_1)}\ \sum_{D\in\mathcal{D}(B)}1.$$

Then

$$\frac{1}{2}\frac{k^{\rho_{N_1}}}{\psi(k^{N_1})}\leqslant c'(N_1)\leqslant 2\frac{k^{\rho_{N_1}}}{\psi(k^{N_1})}.$$

As a consequence, we can choose $[\frac{1}{2}\frac{k^{\rho_{N_1}}}{\psi(k^{N_1})}]$ many "balls" D from $C'(N_1)$, and denote the union of these "balls" by $C(N_1)$. For convenience, let $c(N_1)=[\frac{1}{2}\frac{k^{\rho_{N_1}}}{\psi(k^{N_1})}]$. This completes the first level on the construction of the Cantor set. Clearly, $C(N_1)\subset C'(N_1)$.

For the second level, Ω is further sub-divided into balls with centers in $\Gamma(\rho_{N_2})$ and of radii $k^{-\rho_{N_2}}$. Choose $N_2\in\{M_r\colon r\in\mathbb{N}\}$, $N_2>N_1$ large enough such that Lemma 3.6 holds for all $Y=D\in C(N_1)$ and

$$\psi\big(k^{N_2}\big)\leqslant 2\bigg(\frac{\psi(k^{N_1})}{k^{-\rho_{N_1}}}\bigg)^2.$$

Then for such a D,

$$\frac{1}{2}\frac{(\mu\times\mu)(D)}{k^{-2\rho_{N_2}}}\leqslant\sharp\,\mathcal{G}_D(N_2)\leqslant 2\frac{(\mu\times\mu)(D)}{k^{-2\rho_{N_2}}}.$$

Let

$$C'(N_2)=\bigcup_{B\in\mathcal{G}_{C(N_1)}(N_2)}\ \bigcup_{D\in\mathcal{D}(B)}D,\qquad c'(N_2)=\sum_{B\in\mathcal{G}_{C(N_1)}(N_2)}\ \sum_{D\in\mathcal{D}(B)}1,$$

where $\mathcal{G}_{C(N_1)}(N_2) = \{B \in \mathcal{G}(N_2)\colon\ B \subset C(N_1)\}$. Then

$$\frac{1}{2}\frac{k^{\rho N_2}\psi^2(k^{N_1})}{\psi(k^{N_2})}c(N_1) \leqslant c'(N_2) \leqslant 2\frac{k^{\rho N_2}\psi^2(k^{N_1})}{\psi(k^{N_2})}c(N_1).$$

We choose $[\frac{1}{2}\frac{k^{\rho N_2}\psi^2(k^{N_1})}{\psi(k^{N_2})}c(N_1)]$ "balls" from $C'(N_2)$ and denote the union of these "balls" by $C(N_2)$. Clearly, $C(N_2) \subset C'(N_2)$. We also denote $c(N_2) = [\frac{1}{2}\frac{k^{\rho N_2}\psi^2(k^{N_1})}{\psi(k^{N_2})}c(N_1)]$.

Repeat this construction to obtain $N_r \in \{M_r\colon\ r \in \mathbb{N}\}$ large enough to ensure Lemma 3.6 holds for all $Y = D \in C(N_{r-1})$, and

$$\frac{1}{2}k^{2\rho N_r}\psi^2\left(k^{N_{r-1}}\right) \leqslant \sharp\,\mathcal{G}_D(N_r) \leqslant 2k^{2\rho N_r}\psi^2\left(k^{N_{r-1}}\right), \quad D \in C(N_{r-1})$$

and

$$\psi\left(k^{N_r}\right) \leqslant 2\prod_{i=1}^{r-1}\left(\frac{\psi(k^{N_i})}{k^{-\rho N_i}}\right)^r.$$

Set

$$C'(N_r) = \bigcup_{B\in\mathcal{G}_{C(N_{r-1})}(N_r)}\ \bigcup_{D\in\mathcal{D}(B)} D, \qquad c'(N_r) = \sum_{B\in\mathcal{G}_{C(N_{r-1})}(N_r)}\ \sum_{D\in\mathcal{D}(B)} 1,$$

where $\mathcal{G}_{C(N_{r-1})}(N_r) = \{B \in \mathcal{G}(N_r)\colon\ B \subset C(N_{r-1})\}$. Then

$$c'(N_r) = \sharp\,\mathcal{G}_{C(N_{r-1})}(N_r)\,\sharp\left\{D \in \mathcal{D}(B)\colon\ B \in \mathcal{G}_{C(N_{r-1})}(N_r)\right\}.$$

This gives

$$\frac{1}{2}\psi^2\left(k^{N_{r-1}}\right)k^{2\rho N_r}c(N_{r-1})\frac{k^{-\rho N_r}}{\psi(k^{N_r})} \leqslant c'(N_r) \leqslant 2\psi^2\left(k^{N_{r-1}}\right)k^{2\rho N_r}c(N_{r-1})\frac{k^{-\rho N_r}}{\psi(k^{N_r})}$$

and

$$\frac{1}{2^r}\prod_{i=1}^{r}k^{\rho N_i}\psi\left(k^{N_i}\right)\psi^{-2}\left(k^{N_r}\right) \leqslant c'(N_r) \leqslant 2^r\prod_{i=1}^{r}k^{\rho N_i}\psi\left(k^{N_i}\right)\psi^{-2}\left(k^{N_r}\right).$$

Following the above fashion, we choose $[\frac{1}{2^r}\prod_{i=1}^{r}k^{\rho N_i}\psi(k^{N_i})\psi^{-2}(k^{N_r})]$ "balls" D from $C'(N_r)$ and denote the union of these "balls" by $C(N_r)$, also we put

$$c(N_r) = \left[\frac{1}{2^r}\prod_{i=1}^{r}k^{\rho N_i}\psi\left(k^{N_i}\right)\psi^{-2}\left(k^{N_r}\right)\right].$$

Then we define

$$C_\infty = \bigcap_{r=1}^{\infty} C(N_r)$$

to be our desired Cantor set.

Given $u \in C_\infty$, for each $r \in \mathbb{N}$, there exists $q \in \mathbb{F}[X]$ with $|q| \leqslant k^{N_r}$ such that

$$0 < \mathrm{dist}_\infty(u, R_q) < \psi\big(k^{N_r}\big) \leqslant \psi(q),$$

since the ball $D \in \mathcal{D}(B)$, $D \cap R_q = \emptyset$ and $\psi(k^{N_r}) \to 0$ as $r \to \infty$, so, for each $u \in C_\infty$,

$$0 < \mathrm{dist}_\infty(u, R_q) < \psi\big(k^{N_r}\big) \leqslant \psi(q)$$

holds for infinitely many different $q \in \mathbb{F}[X]$. Hence

$$C_\infty \subset \Lambda(\psi).$$

We now give a mass distribution ν on C_∞. For any $D \in C(N_r)$, $r \geqslant 1$, we define

$$\nu(D) = \frac{1}{c(N_r)}.$$

Since

$$\nu(D) = \sum_{D' \subset C(N_{r+1}) \cap D} \nu(D'),$$

so ν is a mass distribution.

Notice that for any $D \in C(N_r), l(D) = \psi(k^{N_r})$, where $l(D)$ denotes the radius of D. Then

$$\begin{aligned}
\nu(D) = \frac{1}{c(N_r)} &\leqslant 2^{r-1} \prod_{i=1}^{r} k^{-\rho_{N_i}} \psi^{-1}\big(k^{N_i}\big)\psi^2\big(k^{N_r}\big) \\
&= 2^{r-1} k^{-\rho_{N_r}} \psi\big(k^{N_r}\big) \prod_{i=1}^{r-1} k^{-\rho_{N_i}} \psi^{-1}\big(k^{N_i}\big) \\
&\leqslant 2^{r-1} k^{-\rho_{N_r}} \psi\big(k^{N_r}\big) \psi^{-\frac{1}{r}}\big(k^{N_r}\big).
\end{aligned}$$

For any $\epsilon > 0$ and r large enough, we can get

$$\frac{\log \nu(D)}{\log l(D)} \geqslant 1 + \gamma - \epsilon,$$

where $\gamma = \lim_{r \to +\infty} \frac{-\rho_{N_r} \log k}{\log \psi(k^{N_r})}$. It follows that for each $D \in C(N_r)$, the measure ν satisfies

$$\nu(D) \leqslant \big(l(D)\big)^{1+\gamma}.$$

In order to obtain the Lemma 3.4, it suffices to show

Lemma 3.7. *For any small $|\cdot|_\infty$-ball C in Ω, $\nu(C) \leqslant (l(C))^{1+\gamma}$, where $l(C)$ is the radius of C.*

Proof.

$$\nu(C) = \nu(C_\infty \cap C) \leqslant \nu\big(C(N_r) \cap C\big).$$

Take r sufficiently large to make sure that

$$\psi\big(k^{N_r}\big) < l(C) \leqslant \psi\big(k^{N_{r-1}}\big).$$

We prove Lemma 3.7 in two steps.

(1) If $l(C) > k^{-\rho_{N_r}}$, then by Lemma 3.5 we have

$$\begin{aligned}
\nu(C) &\leqslant \nu\big(C(N_r) \cap C\big) \\
&\leqslant \frac{k^{-\rho_{N_r}}}{\psi(k^{N_r})}\left(\frac{l(C)}{k^{-\rho_{N_r}}}\right)^2 \nu(D) \\
&\leqslant \frac{1}{2}\frac{c(N_r)}{c(N_{r-1})}\left(\frac{k^{-\rho_{N_r}}}{\psi(k^{N_{r-1}})}\right)^2\left(\frac{l(C)}{k^{-\rho_{N_r}}}\right)^2 \nu(D) \\
&= \frac{1}{2}\frac{1}{c(N_{r-1})}\left(\frac{l(C)}{\psi(k^{N_{r-1}})}\right)^2 \quad \big(D \in C(N_r)\big) \\
&\leqslant \frac{1}{2} l^2(C)\psi^{-2}\big(k^{N_{r-1}}\big)\psi^{1+\gamma}\big(k^{N_{r-1}}\big) \quad \left(\nu(D) = \frac{1}{c(N_r)} \leqslant \psi^{1+\gamma}\big(k^{N_r}\big)\right) \\
&= \frac{1}{2} l^2(C)\psi^{\gamma-1}\big(k^{N_r}\big) \\
&\leqslant \frac{1}{2}\big(l(C)\big)^{1+\gamma}.
\end{aligned}$$

(2) If $l(C) \leqslant k^{-\rho_{N_r}}$, then we have

$$\nu(C) \leqslant \nu\big(C(N_r) \cap C\big) \leqslant \frac{l(C)}{\psi(k^{N_r})}\nu(D) \leqslant l(C)\psi^{\gamma}\big(k^{N_r}\big) < \big(l(C)\big)^{1+\gamma}. \qquad \square$$

Now Lemma 3.4 follows from the preceding lemma by the mass distribution principle [8]. For $\rho_N = \lfloor 2N - \log N \rfloor$, $\psi(q) = |q|^{-(v+1)}$, we have $\gamma = \frac{2}{v+1}$ and $\Lambda(\psi) = \Phi_v$. Therefore

$$\dim_{\mathrm{H}} \Phi_v \geqslant 1 + \gamma.$$

Combining Lemmas 3.2 and 3.4, we get Theorem 1.5.

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