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On vertex-degree restricted subgraphs in polyhedral graphs

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Abstract

First a brief survey of known facts is given. Main result of this paper: every polyhedral (i.e. 3-connected planar) graph G with minimum degree at least 4 and order at least k ($k \geqslant 4$) contains a connected subgraph on k vertices having degrees (in G) at most $4k-1$, the bound $4k-1$ being best possible.

Keywords: 3-connected planar graph; Light graph; Subgraph with restricted degrees; Path

1. Introduction

Throughout this paper we consider planar graphs without loops or multiple edges. For a plane graph G let $V(G), E(G)$ and $F(G)$ denote the *vertex set*, *edge set* and *face set*, respectively, and let $v(G) := |V(G)|$, $e(G) := |E(G)|$ and $f(G) := |F(G)|$. For a face $\alpha \in F(G)$ let $V(\alpha)$ denote the set of vertices of G that are incident upon α. The *degree* of a face $\alpha \in F(G)$ is the number of edges incident upon α where each bridge is counted twice. For a graph G let $\delta(G)$ and $\Delta(G)$ denote the *minimum* and *maximum* (*vertex*) *degree* of G, respectively. Vertices and faces of degree i are called *i-vertices* and *i-faces*, respectively. For $i \geqslant 0$ let $v_i(G)$ denote the number of i-vertices of G and similarly for $i \geqslant 3$ let $f_i(G)$ denote the number of i-faces of G. A path (a cycle) on k vertices is called a *k-path* (a *k-cycle*) and is denoted by $P_k = [x_1, \ldots, x_k]$ ($C_k = [y_1, \ldots, y_k]$). For graphs G and H, $G \cong H$ means that G and H are isomorphic. If H is a (not necessarily induced) subgraph of G and $H \ncong G$ then H is called a *proper subgraph* of G (and G a *proper supergraph* of H).

It is well known that every planar graph contains a vertex of degree at most 5. Kotzig [15,16] proved that every 3-connected planar (i.e. polyhedral) graph G contains

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an edge with degree sum of its endvertices at most 13 in general and at most 11 if $\delta(G)\geqslant 4$; moreover these bounds are best possible. These results have been further extended in various directions, see, e.g., [6,9,19].

In generalizing Kotzig's theorem, there are, among others, the following two possibilities.

1.1. Fixed subgraphs

Let $\mathscr{G}$ be a family of polyhedral graphs and let H be a connected planar graph that is isomorphic to a proper subgraph of at least one member of $\mathscr{G}$. Let $\varphi(\mathscr{G},H)$ be the smallest integer such that every $G\in\mathscr{G}$ that is a proper supergraph of H contains a subgraph K isomorphic to H with

$$\deg_G(x)\leqslant\varphi(\mathscr{G},H) \quad \text{for every vertex } x\in V(K).$$

If such a $\varphi(\mathscr{G},H)$ exists we call the graph H *light in the family* $\mathscr{G}$, otherwise we say that H is *not light in* $\mathscr{G}$ and write $\varphi(\mathscr{G},H)=\infty$.

Let $\mathscr{G}(c,\delta)$ be the family of all c-connected planar graphs with minimum degree at least δ and set $\varphi(c,\delta;H)=\varphi(\mathscr{G}(c,\delta),H)$.

The following is a brief survey of known results; Theorems 6 and 7(i)–(iii) belong to this survey too.

Theorem 1 (Fabrici and Jendrol' [4]). (i) *H is light in $\mathscr{G}(3,3)$ if and only if $H\cong P_k$, $k\geqslant 1$.*

(ii) $\varphi(3,3;P_k)=5k$, $k\geqslant 1$.

Theorem 2 (Fabrici et al. [3]). (i) *H is light in $\mathscr{G}(3,4)$ if and only if $H\cong P_k$, $k\geqslant 1$.*

(ii) $\varphi(3,4;P_k)=5k-7$, $k\geqslant 8$.

Theorem 3. (i) ([4]) *P_k ($k\geqslant 1$) is light in $\mathscr{G}(3,5)$.*

(ii) ([17]) *C_3 is light in $\mathscr{G}(3,5)$.*

(iii) ([11]) *$K_{1,3}$ and $K_{1,4}$ are light in $\mathscr{G}(3,5)$.*

(iv) ([12]) *C_k ($k\geqslant 11$) is not light in $\mathscr{G}(3,5)$.*

(v) ([12]) *If $\Delta(H)\geqslant 6$ or H contains a block on at least 11 vertices then H is not light in $\mathscr{G}(3,5)$.*

(vi) ([14]) $5k-235\leqslant\varphi(3,5;P_k)\leqslant 5k-7$, $k\geqslant 68$.

(vii) ([1]) $\varphi(3,5;C_3)=7$.

(viii) ([11]) $\varphi(3,5;K_{1,3})=7$, $\varphi(3,5;K_{1,4})=10$.

Theorem 4. (i) ([18]) *H is light in $\mathscr{G}(4,4)$ if and only if $H\cong P_k$, $k\geqslant 1$.*

(ii) ([8]) $2k+2\leqslant\varphi(4,4;P_k)\leqslant 2k+3$, $k\geqslant 1$.

Theorem 5. (i) ([4]) *P_k ($k\geqslant 1$) is light in $\mathscr{G}(4,5)$ and $\mathscr{G}(5,5)$.*

(ii) ([17]) *C_3 is light in $\mathscr{G}(4,5)$ and $\mathscr{G}(5,5)$.*

(iii) ([11]) *$K_{1,3}$ and $K_{1,4}$ are light in $\mathscr{G}(4,5)$ and $\mathscr{G}(5,5)$.*

(iv) ([8]) $k+1 \leqslant \varphi(4,5;P_k) \leqslant k+4,\ k \geqslant 1$.
(v) ([8]) $\lfloor (2k+8)/3 \rfloor \leqslant \varphi(5,5;P_k) \leqslant k+4,\ k \geqslant 1$.

For similar results about planar graphs with restricted minimum face size see, e.g., [7] and [13].

1.2. Connected subgraphs of order k

Let $k \geqslant 1$ be an integer and let $\mathscr{G}$ be a family of polyhedral graphs with at least one member of order at least k. Let $\tau(\mathscr{G},k)$ be the smallest integer such that every graph $G \in \mathscr{G}$ of order at least k contains a connected subgraph H of order k with

$$\deg_G(x) \leqslant \tau(\mathscr{G},k) \quad \text{for every vertex } x \in V(H).$$

Set $\tau(c,\delta;k) = \tau(\mathscr{G}(c,\delta),k)$. The following is known:

Theorem 6 (Fabrici and Jendrol' [5]). (i) $\tau(3,3;1)=5$,
(ii) $\tau(3,3;2)=10$,
(iii) $\tau(3,3;k)=4k+3,\ k \geqslant 3$.

Instead of asking for bounds on the degrees of all vertices (i.e. on the maximum degree), one may ask for similar bounds on the degree sum of a fixed subgraph H or a connected subgraph of order k. Several cases of such problems have been solved in [2,4,11,18].

2. Results

The first three values of τ for the family $\mathscr{G}(3,4)$ are already known, for $k=1$ and $k=2$ these are easy consequences of Euler's formula and Kotzig's result, respectively, and the third value was found by Jendrol' [10]. In (iv) the remaining values are determined.

Theorem 7. (i) $\tau(3,4;1)=5$,
(ii) $\tau(3,4;2)=7$,
(iii) $\tau(3,4;3)=9$,
(iv) $\tau(3,4;k)=4k-1,\ k \geqslant 4$.

Theorem 3(v) is improved by the next result.

Theorem 8. *If H is a connected planar graph with $\Delta(H) \geqslant 5$ then it is not light in* $\mathscr{G}(3,5)$.

From Theorem 5(iii) it follows that the lower bound 5 in the above inequality is best possible.

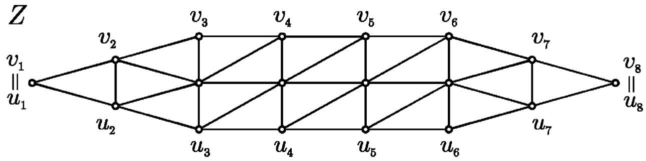

Fig. 1.

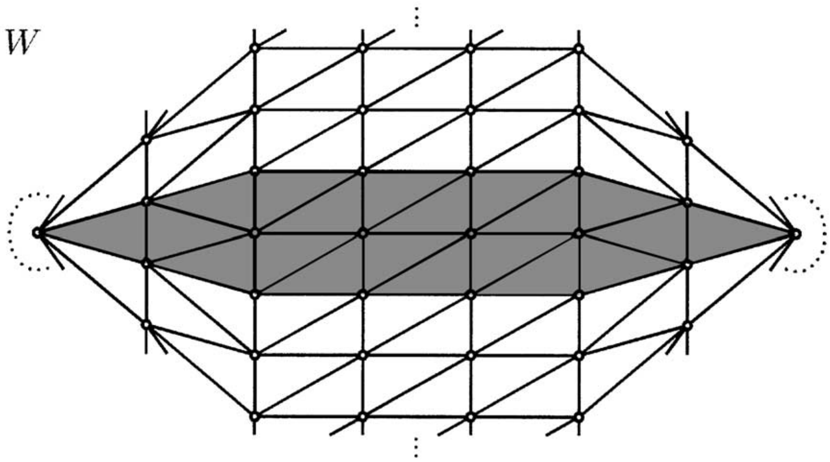

Fig. 2.

3. Proofs

Proof of Theorem 7(iv). I. To prove $\tau(3,4;k)\geqslant 4k-1$ it is enough to exhibit a 3-connected plane graph $\tilde{W}$ of order at least k with $\delta(\tilde{W})\geqslant 4$ in which any connected subgraph of order k contains a vertex of degree at least $4k-1$.

The construction starts with the graph Z (Fig. 1). Let $P:=[u_1,\ldots,u_8]$ and $Q:=[v_1,\ldots,v_8]$ be paths of Z as drawn in Fig. 1. Let W (Fig. 2) be the graph obtained from $2k$ copies $(Z_1,\ldots,Z_{2k})$ of Z by identifying the path Q of Z_i $(i=1,\ldots,2k)$ with P of Z_{i+1} (subscript addition is taken modulo $2k$). W has only vertices of degree 5 or 6, except for two vertices of degree $2k$.

Now—using the configurations R_k and S_k (Figs. 3(a), (b) and, for small k, Figs. 3(c)–(e)), both containing $k-1$ (black) vertices—we replace each 3-face of W with one of these configurations as shown for one copy of Z in Fig. 4 (for $k\geqslant 6$) to obtain the graph $\tilde{W}$. The numbers in Fig. 4 determine the configurations that replace (i.e. are inserted into) the 3-faces of W and their orientations. The numbers in each triangle denote the numbers of edges of R_k (or S_k) incident at the vertices of the triangle. There is an analogous construction when $k=4$ or 5, involving Figs. 3(c), (d).

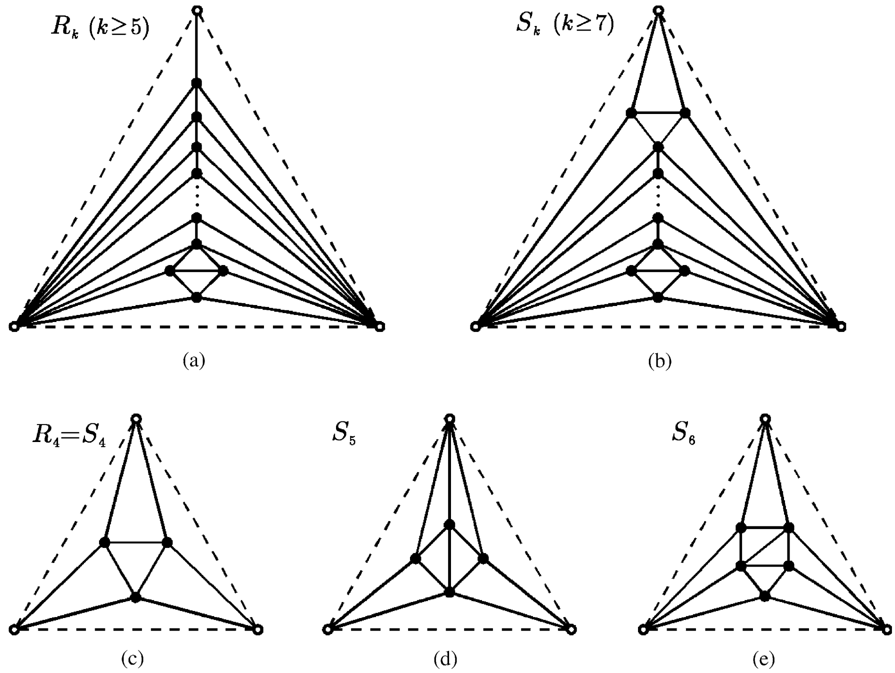

Fig. 3.

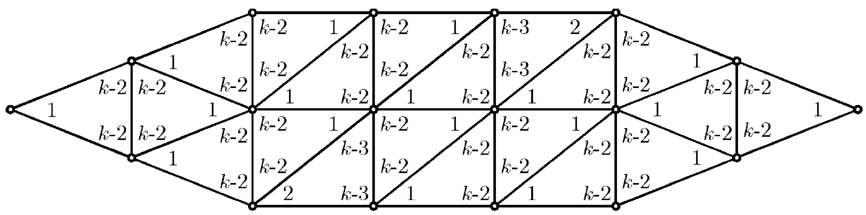

Fig. 4.

Every connected subgraph H of $\tilde{W}$ of order k contains a (white) vertex of W and it is easy to check that all vertices of W have degree (in $\tilde{W}$) at least $4k-1$.

II. It remains to show that $\tau(3,4;k) \leqslant 4k-1$ for $k \geqslant 4$. Suppose there is a $k \geqslant 4$ such that $\tau(3,4;k) \geqslant 4k$. Let G be a counterexample with n vertices and a maximum number of edges, say m. A vertex $x \in V(G)$ is said to be a *major vertex* or a *minor vertex* if $\deg_G(x) \geqslant 4k$ or $\deg_G(x) < 4k$, respectively.

Property 1. *G is a triangulation.*

Proof. Assume that G contains an r-face α, $r \geqslant 4$. If α is incident upon a major vertex x we add a diagonal xy into α, where y is a vertex incident upon α but not adjacent

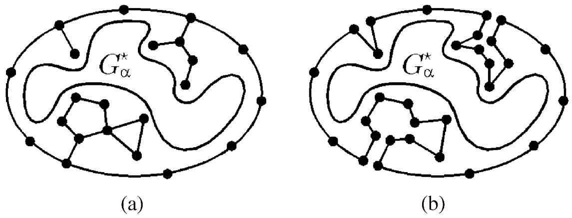

Fig. 5.

to x. Since the insertion of the diagonal xy cannot create a minor component (i.e. a component of the graph induced by the set of minor vertices) of order at least k, we get a counterexample with $m+1$ edges, a contradiction. If α is incident only upon minor vertices, all these belong to the same minor component and we can again add a diagonal into α and obtain a counterexample with $m+1$ edges. This is a contradiction. □

Let $M = M(G)$ be the plane graph induced by the set of major vertices of G and let $M_1,\dots,M_\omega$ $(\omega \geqslant 1)$ be the components of M.

Property 2 (almost obvious). *There is some component of M, say M_1, such that all other components of M are subgraphs of the same component of $G - V(M_1)$.*

Let $G^\star$ be the component of $G - (V(M_2)\cup\cdots\cup V(M_\omega))$ that contains M_1 as a subgraph (where, possibly, $G^\star \notin \mathcal{G}(3,4)$).

Property 3. *For each vertex x of M_1 we have $\deg_{G^\star}(x) = \deg_G(x)$.*

For $\alpha \in F(M_1)$ let $G^\star_\alpha$ be the (minor) subgraph induced by all those vertices of $V(G^\star)\backslash V(M_1)$ that lie in the interior of α; clearly, G being a triangulation, $G^\star_\alpha$ is connected. Let T_α be the subgraph of $G^\star$ induced by $V(G^\star_\alpha)\cup V(\alpha)$.

Property 4. *For each face α of M_1 we have $v(G^\star_\alpha) \leqslant k-1$.*

For $\alpha \in F(M_1)$, $x \in V(M_1)$ and $x \in V(\alpha)$, we define

$$m(x,\alpha) = |\{u \in V(G^\star_\alpha):\ ux \in E(G^\star)\}|;$$

$$m(\alpha) = \sum_{x\in V(\alpha)} m(x,\alpha).$$

Lemma 1. *For any $\alpha \in F(M_1)$ we have $m(\alpha) \leqslant 2k-4+\deg_{M_1}(\alpha)$.*

Proof. We can assume that the boundary of α is a cycle. Otherwise (Fig. 5(a)) we can cut every bridge and every articulation point of α (only for the purposes of this proof), as shown in Fig. 5(b). The graph T_α has $v(T_\alpha) = v(G^\star_\alpha) + \deg_{M_1}(\alpha)$ vertices and

$e(T_\alpha) \leqslant 3v(T_\alpha) - 6 - (\deg_{M_1}(\alpha) - 3) = 3v(G_\alpha^\star) - 3 + 2\deg_{M_1}(\alpha)$ edges. Hence, $m(\alpha) = e(T_\alpha) - \deg_{M_1}(\alpha) - e(G_\alpha^\star) \leqslant 3v(G_\alpha^\star) - 3 + 2\deg_{M_1}(\alpha) - \deg_{M_1}(\alpha) - (v(G_\alpha^\star) - 1) = 2v(G_\alpha^\star) - 2 + \deg_{M_1}(\alpha) \leqslant 2k - 4 + \deg_{M_1}(\alpha)$. □

Lemma 2 (Fabrici et al. [3]). *For any triangle* $\alpha = x_1x_2x_3 \in F(M_1)$, *such that* T_α *is a triangulation, we have* $m(x_i, \alpha) \leqslant k - 2$, *for all* $i \in \{1, 2, 3\}$.

We omit the (easy but somewhat lengthy) proof of this fact. The main idea is this: if there were a vertex $x_i \in V(\alpha)$ adjacent to all vertices of $G_\alpha^\star$, then there would be a vertex $y \in G_\alpha^\star$ with $\deg_{G_\alpha^\star}(y) = \deg_G(y) \leqslant 3$, contradicting $\delta(G) \geqslant 4$.

Lemma 3. *For any triangle* $\alpha = x_1x_2x_3 \in F(M_1)$ *we have* $m(\alpha) \leqslant 2k - 3$, *for* $k \geqslant 5$.

Proof. If $v(G_\alpha^\star) \leqslant 3$ then $m(\alpha) \leqslant 6 < 2k - 3$. So suppose $v(G_\alpha^\star) \geqslant 4$. We consider the following two cases.

Case 1: If T_α is not a triangulation, then $e(G_\alpha^\star) \geqslant v(G_\alpha^\star)$ (because the graph $G_\alpha^\star$ is connected and contains a cycle). It has $v(T_\alpha) = v(G_\alpha^\star) + 3$ vertices and $e(T_\alpha) \leqslant 3v(T_\alpha) - 7 = 3v(G_\alpha^\star) + 2$ edges, which yields $m(\alpha) = e(T_\alpha) - 3 - e(G_\alpha^\star) \leqslant 3v(G_\alpha^\star) + 2 - 3 - v(G_\alpha^\star) = 2v(G_\alpha^\star) - 1 \leqslant 2k - 3$ (by Property 4).

Case 2: If T_α is a triangulation, each 1-vertex u of $G_\alpha^\star$ (i.e. $\deg_{G_\alpha^\star}(u) = 1$) is adjacent to all three vertices of α (because of $\delta(G) \geqslant 4$). Hence, there is at most one 1-vertex in $G_\alpha^\star$.

2(a). Is there some 1-vertex in $G_\alpha^\star$, then (without loss of generality) $m(x_1, \alpha) = 1$ and by Lemma 2 we have $m(x_2, \alpha), m(x_3, \alpha) \leqslant k - 2$ which implies $m(\alpha) \leqslant 2k - 3$.

2(b). Is there no 1-vertex in $G_\alpha^\star$, then $e(G_\alpha^\star) \geqslant v(G_\alpha^\star) + 1$ (because of Property 1). The graph T_α has now $v(T_\alpha) = v(G_\alpha^\star) + 3$ vertices and $e(T_\alpha) = 3v(T_\alpha) - 6 = 3v(G_\alpha^\star) + 3$ edges, which yields $m(\alpha) = e(T_\alpha) - 3 - e(G_\alpha^\star) \leqslant 3v(G_\alpha^\star) + 3 - 3 - (v(G_\alpha^\star) + 1) = 2v(G_\alpha^\star) - 1 \leqslant 2k - 3$ (by Property 4). □

Clearly

$$\sum_{x \in V(M_1)} \deg_{M_1}(x) = 2e(M_1) = \sum_{\alpha \in F(M_1)} \deg_{M_1}(\alpha) = \sum_{i \geqslant 3} i f_i(M_1). \tag{1}$$

Euler's polyhedral formula provides

$$e(M_1) \leqslant 3v(M_1) - 6, \quad f(M_1) \leqslant 2v(M_1) - 4. \tag{2}$$

First consider the case $k \geqslant 5$. Using (1), (2) and Lemmas 1 and 3 we have

$$\sum_{x \in V(M_1)} \deg_G(x)$$

$$= \sum_{x \in V(M_1)} \deg_{G^\star}(x) = \sum_{x \in V(M_1)} \deg_{M_1}(x) + \sum_{\alpha \in F(M_1)} m(\alpha)$$

$$\leqslant 2e(M_1) + \sum_{\substack{\alpha\in F(M_1)\\ \deg_{M_1}(\alpha)\geqslant 4}} (2k-4+\deg_{M_1}(\alpha)) + \sum_{\substack{\alpha\in F(M_1)\\ \deg_{M_1}(\alpha)=3}} (2k-3)$$

$$= 2e(M_1) + (2k-4)\sum_{i\geqslant 3} f_i(M_1) + \sum_{i\geqslant 3} i f_i(M_1) - 2f_3(M_1)$$

$$= 4(3e(M_1) - 2e(M_1)) + (2k-4)f(M_1) - 2f_3(M_1)$$

$$= 4(3v(M_1) + 3f(M_1) - 6 - \sum_{i\geqslant 3} i f_i(M_1)) + (2k-4)f(M_1) - 2f_3(M_1)$$

$$= 12v(M_1) - 24 - 4\sum_{i\geqslant 3}(i-3)f_i(M_1) + (2k-4)f(M_1) - 2f_3(M_1)$$

$$\leqslant 12v(M_1) - 24 - 2\sum_{i\geqslant 4} f_i(M_1) + (2k-4)f(M_1) - 2f_3(M_1)$$

$$= 12v(M_1) - 24 + (2k-6)f(M_1)$$

$$\leqslant 12v(M_1) - 24 + (2k-6)(2v(M_1)-4)$$

$$= 4kv(M_1) - 8k.$$

This implies that there is a vertex $\tilde{x}\in V(M_1)$ such that $\deg_G(\tilde{x})\leqslant(4kv(M_1)-8k)/v(M_1)=4k-(8k)/v(M_1)<4k$, which is a contradiction because $\tilde{x}$ is a major vertex.

Finally consider the case $k=4$. For each triangle $\alpha=x_1x_2x_3\in F(M_1)$ Lemma 2 implies $m(x_i,\alpha)\leqslant 2$. For any r-face $\alpha\in F(M_1)$, $r\geqslant 4$, there are at most 2 vertices of α with $m(x,\alpha)=3$ (otherwise there is a $K_{3,3}\subseteq G^{\star}$, a contradiction to planarity) and for each other vertex y of α we have $m(y,\alpha)\leqslant 2$. Each major vertex x of G has $\deg_G(x)\geqslant 4k=16$. That means, there is no 3-vertex in M_1, for each 4-vertex x of M_1 and each of the four faces incident upon x we must have $m(x,\alpha)=3$ and, eventually, each 5-vertex x of M_1 must be incident upon some face $\alpha\in F(M_1)$, such that $m(x,\alpha)=3$. Let A denote the number of pairs (x,α) such that $x\in V(\alpha)$ and $m(x,\alpha)=3$. By the preceding arguments

$$2\sum_{i\geqslant 4} f_i(M_1)\geqslant A\geqslant 4v_4(M_1)+v_5(M_1). \tag{3}$$

A well-known formula for connected planar graphs says that

$$\sum_{i\geqslant 3}(6-i)v_i + 2\sum_{i\geqslant 3}(3-i)f_i = 12 \tag{4}$$

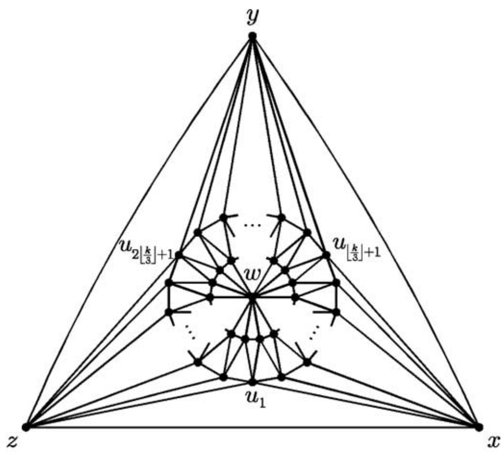

Fig. 6.

which, in connection with $v_3(M_1)=0$, implies

$$
\begin{aligned}
4v_4(M_1)+v_5(M_1) &\geqslant 2v_4(M_1)+v_5(M_1)\\
&= 12+\sum_{i\geqslant 6}(i-6)v_i(M_1)+2\sum_{i\geqslant 3}(i-3)f_i(M_1)\\
&> 2\sum_{i\geqslant 4} f_i(M_1)
\end{aligned}
$$

contradicting inequality (3). □

Proof of Theorem 8. For each connected planar graph H with $\Delta(H)\geqslant 5$ and for each integer $k\geqslant 6$ we shall find a graph $G\in\mathcal{G}(3,5)$, a proper supergraph of H, such that each subgraph of G isomorphic to H contains a vertex x with $\deg_G(x)\geqslant k$.

The construction starts with any triangulation T_H of the graph H. Into each 3-face $\alpha=xyz\in F(T_H)$ we insert two vertex-disjoint cycles $C_k=[u_1,\ldots,u_k]$ and $\tilde{C}_k=[\tilde{u}_1,\ldots,\tilde{u}_k]$, a new vertex w and edges $u_i\tilde{u}_i, u_i\tilde{u}_{i+1}, w\tilde{u}_i$ $(i=1,\ldots,k$, subscript addition is taken modulo $k)$ and $xu_1,\ldots,xu_{\lfloor k/3\rfloor+1}, yu_{\lfloor k/3\rfloor+1},\ldots,yu_{2\lfloor k/3\rfloor+1}, zu_{2\lfloor k/3\rfloor+1},\ldots,zu_k,zu_1$. The resulting graph is denoted by G (Fig. 6).

It is easy to see that for the graph N induced by the set of all vertices of G whose degree does not exceed $k-1$ we have $\Delta(N)\leqslant 4$, which means that any subgraph of G isomorphic to H contains a vertex of $G-N$. □

References

[1] O.V. Borodin, Solution of problems of Kotzig and Grünbaum concerning the isolation of cycles in planar graphs, Math. Notes 46 (1989) 835–837 (English translation), Mat. Zametki 46 (1989) 9–12 (in Russian).

[2] H. Enomoto, K. Ota, Connected subgraphs with small degree sums in 3-connected planar graphs, J. Graph Theory 30 (1999) 191–203.
[3] I. Fabrici, E. Hexel, S. Jendrol', H. Walther, On vertex-degree restricted paths in polyhedral graphs, Discrete Math. 212 (2000) 61–73.
[4] I. Fabrici, S. Jendrol', Subgraphs with restricted degrees of their vertices in planar 3-connected graphs, Graphs Combin. 13 (1997) 245–250.
[5] I. Fabrici, S. Jendrol', Subgraphs with restricted degrees of their vertices in planar graphs, Discrete Math. 191 (1998) 83–90.
[6] B. Grünbaum, G.C. Shephard, Analogues for tiling of Kotzig's theorem on minimal weights of edges, Ann. Discrete Math. 12 (1982) 129–140.
[7] J. Harant, S. Jendrol', M. Tkáč, On 3-connected plane graphs without triangular faces, J. Combin. Theory Ser. B 77 (1999) 150–161.
[8] E. Hexel, H. Walther, On vertex-degree restricted paths in 4-connected planar graphs, Tatra Mountains 18 (1999) 1–13.
[9] J. Ivančo, The weight of a graph, Ann. Discrete Math. 51 (1992) 113–116.
[10] S. Jendrol', Paths with restricted degrees of their vertices in planar graphs, Czechoslovak Math. J. 49 (1999) 481–490.
[11] S. Jendrol', T. Madaras, On light subgraphs in plane graphs with minimum degree five, Discuss. Math. Graph Theory 16 (1996) 207–217.
[12] S. Jendrol', T. Madaras, R. Sotak, Zs. Tuza, On light cycles in planar triangulations, Discrete Math. 197/198 (1999) 453–467.
[13] S. Jendrol', P.J. Owens, On light graphs in 3-connected plane graphs without triangular or quadrangular faces, preprint 1998.
[14] S. Jendrol', H.-J. Voss, Light paths in large polyhedral maps with prescribed minimum degree, preprint 1999.
[15] A. Kotzig, Contribution to the theory of Eulerian polyhedra, Mat.-Fyz. Čas. SAV (Math. Slovaca) 5 (1955) 111–113 (Slovak).
[16] A. Kotzig, On the theory of Euler's polyhedra, Mat.-Fyz. Čas. SAV (Math. Slovaca) 13 (1963) 20–31 (in Russian).
[17] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J. Math. Pures Appl. 19 (1940) 27–43.
[18] B. Mohar, Light paths in 4-connected graphs in the plane and other surfaces, J. Graph Theory 34 (2000) 170–179.
[19] J. Zaks, Extending Kotzig's theorem, Israel J. Math. 45 (1983) 281–296.