Journal of Pure and Applied Algebra 45 (1987) 99–100
North-Holland


# ADDENDUM TO "EXACT EMBEDDING FUNCTORS BETWEEN CATEGORIES OF MODULES"

George HUTCHINSON
*Division of Computer Research and Technology, National Institutes of Health, Bethesda, MD 20892, U.S.A.*

Communicated by P.J. Freyd
Received 9 August 1986

In [2], conditions equivalent to the existence of exact embedding functors $R$-**Mod** $\to$ $S$-**Mod** were studied. Another special result of this type is worth observing.

**Theorem.** *Let $R$ and $S$ be nontrivial rings with 1, with $R$ left artinian. Then the following conditions are equivalent:*

(1) *There exists an exact embedding functor $R$-**Mod** $\to$ $S$-**Mod**.*

(2) *There exists a bimodule ${}_SA_R$ such that $A_R$ is faithfully flat.*

(3) *There exists a bimodule ${}_SB_R$ with $B_R$ a (nontrivial) free $R$-module.*

(4) *There exists a ring homomorphism $S \to \mathrm{CFM}_\beta(R)$ preserving 1, where $\mathrm{CFM}_\beta(R)$ is the ring of (possibly infinite) $\beta \times \beta$ column-finite matrices on $R$, $\beta$ a nonzero cardinal number.*

**Proof.** For $J$ the Jacobson radical of $R$, $R$ left artinian implies $R$ is left noetherian, $J$ is nilpotent and $R/J$ is semisimple by Hopkin's theorem [1, 15.20, p. 172]. In particular, conditions (1) and (2) are equivalent by [2, Theorem 2, p. 110]. Furthermore, conditions (3) and (4) are equivalent, since each is equivalent to the existence of a ring homomorphism preserving 1 from $S$ into the ring of $R$-linear endomorphisms of $B_R$ (see [1, 4.10, p. 59] and [1, Exercises 11-12, p. 113]). Clearly (3) $\Rightarrow$ (2), and so we assume (2) and prove (3).

Since $J$ is nilpotent, hence right $T$-nilpotent, and $R/J$ is semisimple, $R$ is right perfect and every flat right $R$-module is projective by Bass' theorem (use the left-right dual result [1, 28.4*, p. 315]). In particular, $A_R$ is projective. But $R$ is semiperfect [1, 27.6*, p. 304], and so has a basic set of primitive idempotents $e_1, e_2, \dots, e_n$, $n \geq 1$ [1, 27.10, p. 306] such that

$$A_R \approx (e_1R)^{(\alpha_1)} \oplus (e_2R)^{(\alpha_2)} \oplus \cdots \oplus (e_nR)^{(\alpha_n)},$$

and

$$R_R \approx (e_1R)^{(\beta_1)} \oplus (e_2R)^{(\beta_2)} \oplus \cdots \oplus (e_nR)^{(\beta_n)},$$

for unique cardinals $\alpha_i$ and $\beta_i$, $i=1,2,\ldots,n$, by [1, 27.11*, p. 306].

Suppose some $\alpha_k=0$. Now, $T_k=Re_k/Je_k\neq 0$ by [1, 27.10, p. 306], and $A_R\otimes_R T_k$ is isomorphic to a direct sum of terms $e_iR\otimes_R T_k$ for $i\neq k$ by [1, 19.9, p. 223]. Also, $e_iRe_k\leq Je_k$ by the standard argument. (If $e_ise_k$ in $e_iRe_k$ is not in $Je_k$, then $\sigma(re_i+Je_i)=re_ise_k+Je_k$ defines a nonzero $R$-linear map $\sigma$ from $Re_i/Je_i$ into $Re_k/Je_k$. Now, $\sigma$ is an isomorphism because its domain and codomain are simple [1, 27.10, p. 306], so $Re_i\approx Re_k$ by [1, 17.18, p. 200] with $i\neq k$, contradicting the irredundancy of $Re_1,Re_2,\ldots,Re_n$.) Then, elements

$$e_iv\otimes(we_k+Je_k)=e_i\otimes(e_ivwe_k+Je_k)=e_i\otimes 0=0$$

generate $e_iR\otimes_R T_k$. So, ${}_SA\otimes_R T_k=0$, contradicting $A_R$ faithfully flat.

Therefore, all $\alpha_k>0$, and similarly all $\beta_k>0$. Then $\alpha=\alpha_k\alpha=\beta_k\alpha$, $k\leq n$, for a sufficiently large infinite cardinal $\alpha$, and (3) follows because ${}_SA_R^{(\alpha)}$ is a bimodule with $A_R^{(\alpha)}\approx R_R^{(\alpha)}$. $\square$

## References

[1] F. Anderson and K. Fuller, Rings and Categories of Modules (Springer, Berlin, 1974).

[2] G. Hutchinson, Exact embedding functors between categories of modules, J. Pure Appl. Algebra 25 (1982) 107–111.