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The Shapley value on convex geometries

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Abstract

A *game on a convex geometry* is a real-valued function defined on the family $\mathscr{L}$ of the closed sets of a closure operator which satisfies the finite Minkowski–Krein–Milman property. If $\mathscr{L}$ is the boolean algebra 2^N then we obtain an n-person cooperative game. Faigle and Kern investigated games where $\mathscr{L}$ is the distributive lattice of the order ideals of the poset of players. We obtain two classes of axioms that give rise to a unique Shapley value for games on convex geometries.

Keywords: Cooperative game; Convex geometry; Shapley value

1. Games on convex geometries

The goal of this paper is to develop a theoretical framework in which to analyze cooperative games in which only certain coalitions are allowed to form. We will axiomatize the structure of such allowable coalitions using the theory of *convex geometries*, a notion developed to combinatorially abstract geometric convexity. In this sense our model acts as a bridge between traditional cooperative game theory and spatial games, in which the (Euclidean) geometry controls the game.

There have been previous models developed to confront the problem of unallowable coalitions (see, for instance [7,8]). Most closely related to our work is the work of Faigle and Kern [4] on cooperative games under precedence constraints, that is, games on distributive lattices. Our model is a generalization of theirs and our analysis follows the arguments of Faigle and Kern quite closely.

The theory developed in this paper has already been applied in [1,2] to examine certain *voting games*. In the former, a new analysis of voting in a one-dimensional issue space is examined, as well as an alternative to the *Attitudinal-dependent index*

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of Shapley (see also [10,9]) for spatial games using the generalized Shapley value defined in Section 2. This leads to a reinterpretation of the model of voting on the Supreme Court discussed by Frank and Shapley [6]. In [2] an empirical approach to voting on the current Supreme Court is taken using the ideas of convex geometries and a generalization of the Banzhaf index. We think that these new approaches to old problems provide ample evidence that the theory developed in this paper is of more than aesthetic interest.

Example. A motivating example for our interest in convex geometries is the following spatial voting model. Let $N = \{1, 2, \ldots, n\}$ be voters and $\mathbb{E}^d$ be an d-dimensional Euclidean issue space. We will denote by x_i the ideal point of the voter i in this space. Let u: $\mathbb{E}^d \to \mathbb{E}$ be a convex function such that $u(x) = u(-x)$. If $y \in \mathbb{E}^d$ we will let $u_i(y) = u(y - x_i)$ be the utility of the outcome y to voter i. Assume that each voter i will vote in favor of an outcome y if $u_i(y) > \varepsilon_y$. Suppose that every voter $i \in A \subset N$ will vote in favor of y and that for some $j \notin A$ we have that x_j is in the convex hull of $\{x_i\colon i \in A\}$. It follows from the convexity of the function u that $u_j(y) > \varepsilon_y$ and hence j will vote in favor of y as well. Thus coalitions $A \subseteq N$ that form in this model have the property that x_j in the convex hull of $\{x_i{:}i \in A\}$ implies that $j \in A$. Since, we do not know very much about the function u or the threshold values ε_y, it would be reasonable to assume that all coalitions with this closure property might form. This collection of subsets is a convex geometry (see [3, Example I]).

Let N be a finite set of cardinality n and $\mathscr{L}$ be a collection of subsets of N with the properties

- $\emptyset \in \mathscr{L}$,
- if $A, B \in \mathscr{L}$, then $A \cap B \in \mathscr{L}$ (intersection-closed),
- if $A \in \mathscr{L}$, $A \neq N$, then there exists an $x \in N \setminus A$ such that $A \cup \{x\} \in \mathscr{L}$ (one-point extension).

Call $\mathscr{L}$ a *convex geometry* on N. Convex geometries are a combinatorial abstraction of convex sets. The reader should see [3] for a discussion of their properties and a myriad of examples. Alternatively, one can think of $\mathscr{L}$ as being a *closure operator*, i.e., for any subset $A \subseteq N$ define the *closure of* A, $\mathscr{L}(A)$ to be

$$\mathscr{L}(A) = \bigcap_{\{C \in \mathscr{L}\colon C \supseteq A\}} C.$$

It is easy to check that $\mathscr{L}$ is a closure operator on N, i.e., $\mathscr{L}$ is a function from 2^N, the set of all subsets of N, to itself satisfying

(C1) $A \subseteq \mathscr{L}(A)$,

(C2) $A \subseteq B$ implies that $\mathscr{L}(A) \subseteq \mathscr{L}(B)$,

(C3) $\mathscr{L}(\mathscr{L}(A)) = \mathscr{L}(A)$,

with the additional condition that $\mathscr{L}(\emptyset) = \emptyset$. The subsets in $\mathscr{L}$, or equivalently those subsets of N of the form $\mathscr{L}(A)$ for some $A \subseteq N$ are usually called *convex sets*.

If $A \in \mathscr{L}$ and $a \in A$, then call a an *extreme point* of A if $A \setminus a \in \mathscr{L}$. The set of extreme points of A will be denoted $ex(A)$.

Theorem 1. *A convex geometry $\mathscr{L}$ on a set N has the property that for every $A \in \mathscr{L}$, $A = \mathscr{L}(ex(A))$.*

Proof. See [3, Theorem 2.1]. $\square$

Example. A graph $G=(N,E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is a vertex whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G. A graph G is a *block graph* if every block is a complete subgraph of G. Clearly, if G is a disjoint union of trees, then G is a block graph. Jamison [3, Theorem 3.7] showed: *$G = (N,E)$ is a connected block graph if and only if the collection of subsets of N which induce connected subgraphs is a convex geometry.*

We can partially order the collection $\mathscr{L}$ by containment. The resulting poset is a *meet-distributive lattice* and, indeed, all meet-distributive lattices arise from convex geometries in this way [3, Theorem 4.1]. Given any pair of sets $S,T \in \mathscr{L}$ such that $S \subseteq T$, we define a *maximal chain between S and T* to be an ordered collection of subsets in $\mathscr{L}$

$$(S = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = T)$$

such that there does not exist a set $C \in \mathscr{L}$ such that

$$M_j \subsetneqq C \subsetneqq M_{j+1}$$

for any index $0 \leqslant j \leqslant k-1$. Denote by $c([S,T])$ the number of maximal chains from S to T, $c(T){:=}c([\emptyset,T])$, $T \neq \emptyset$, the number of maximal chains from $\emptyset$ to T and $c([T,T]) = 1$, $\forall T \in \mathscr{L}$. Then $c(N) = c(\mathscr{L})$ is the total number of maximal chains.

It follows from the axiom of one-point extension that in each maximal chain from S to T, $|M_j| = |S| + j$, for any $0 \leqslant j \leqslant k$. Thus, each maximal chain from S to T gives rise to a *compatible ordering* of the set $T \setminus S = \{a_1,\ldots,a_k\}$, where $M_j = S \cup \{a_1,\ldots,a_j\}$ for $1 \leqslant j \leqslant k$. We will denote the set of compatible orderings of $N = N \setminus \emptyset$ by $\mathscr{C}(\mathscr{L})$. If $C \in \mathscr{C}(\mathscr{L})$ with $C = (a_1,\ldots,a_n)$, then we will say that $i \leqslant j$ in C if $i = a_s$, $j = a_t$, and $s \leqslant t$.

A cooperative game is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The players are the elements of N and the coalitions are the elements $S \subseteq N$ of the boolean algebra 2^N.

Definition 1. A game on a convex geometry $\mathscr{L}$ is a function $v : \mathscr{L} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

We shall assume that the coalitions are the convex sets of $\mathscr{L}$, the players are the elements $i \in N$, and $\Gamma(\mathscr{L})$ is the vector space over $\mathbb{R}$ of all games on the convex geometry $\mathscr{L} \subseteq 2^N$.

Example. If $\mathscr{L}$ is a Boolean algebra of rank n then it is isomorphic to 2^N, where $N = \{1,2,\dots,n\}$ is the set of n atoms of $\mathscr{L}$. Thus, the game on $\mathscr{L}$ is an ordinary cooperative game.

Example. Let $(P, \leqslant)$ be a poset. For any $N \subseteq P$,

$$N \mapsto \bar{N} := \{y \in P\colon\ y \leqslant x \text{ for some } x \in N\},$$

defines a closure operator on P. Its closed sets are the *order ideals* (down sets) of P, and we denote this lattice $J(P)$. Since, the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a sublattice of 2^P. Then $J(P)$ is a distributive lattice and so, $J(P)$ is a convex geometry closed under set-union and $ex(S)$ is the set of all maximal points $Max(S)$ of the subposet $S \in J(P)$. When P is finite, there is a 1–1 correspondence between antichains of P and order ideals. Then the games $(\mathscr{C}, v)$ and $(\mathscr{A}, c)$, where $\mathscr{C}$ is the family of down sets of P [4] and $\mathscr{A}$ is the set of antichains of a *rooted tree* [5] are games on distributive lattices.

We consider the following $\{0,1\}$-value games on $\mathscr{L}$. For any $T \in \mathscr{L}$, the *upper game*, denoted $\zeta_T : \mathscr{L} \to \mathbb{R}$ is defined by $\zeta_T(S) := 1$, if $T \subseteq S$ and $\zeta_T(S) := 0$, otherwise.

Theorem 2. *Let $v : \mathscr{L} \to \mathbb{R}$ be a game on a convex geometry $\mathscr{L}$. Then there exists an unique set of coefficients $\{\Delta_v(T)\colon T \in \mathscr{L}\}$ such that*

$$v = \sum_{T \in \mathscr{L}} \Delta_v(T)\zeta_T.$$

Moreover,

$$\Delta_v(S) = \sum_{T \in [S \setminus ex(S),\, S]} (-1)^{|S|-|T|} v(T).$$

Proof. The family $\{\zeta_T\colon T \in \mathscr{L},\ T \neq \emptyset\}$ is a basis of the vector space $\Gamma(\mathscr{L})$. We have for all $S \in \mathscr{L}$,

$$v(S) = \sum_{T \in \mathscr{L}} \Delta_v(T)\zeta_T(S) = \sum_{\{T \in \mathscr{L}\colon T \subseteq S\}} \Delta_v(T).$$

The Möbius inversion formula [12, Chapter 3] for the lattice $\mathscr{L}$ implies

$$\Delta_v(S) = \sum_{\{T \in \mathscr{L}\colon\ T \subseteq S\}} \mu_{\mathscr{L}}(T,S) v(T) \quad \text{for all } S \in \mathscr{L},$$

where $\mu_{\mathscr{L}}$ is the Möbius function of the lattice. $\mathscr{L}$ is a convex geometry and its Möbius function satisfies [1, Theorem 4.3]

$$\mu_{\mathscr{L}}(T,S) = \begin{cases} (-1)^{|S|-|T|} & \text{if } S \setminus T \subseteq ex(S), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{T \in \mathscr{L}: T \subseteq S \text{ and } S \setminus T \subseteq ex(S)\} = [S \setminus ex(S), S]$, we obtain the required result. □

2. Axioms for the Shapley value

The classical characterization of the *Shapley value* is as the only value that satisfies the carrier, symmetry and additivity on the class of all superadditive games [11]. For the class of all games we add the linearity axiom. If (N, v) is a game then the Shapley value for the player $i \in N$ is

$$\Phi_i(N, v) = \sum_{\{S \in 2^N:\ i \in S\}} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \setminus i)],$$

where $n = |N|$ and $s = |S|$. We will follow the work of Faigle and Kern [4] to obtain an axiomatization of the Shapley value for games on convex geometries.

Given an element $i \in N$ and a compatible ordering C of $\mathscr{L}$, let

$$C(i){:=}\{j \in N:\ j \leqslant i \text{ in } C\}.$$

Let $S \in \mathscr{L}$ and $i \in S$. Following Faigle and Kern [4], we define the *hierarchical strength* $h_S(i)$ of i in S to be

$$h_S(i){:=}\frac{|\{C \in \mathscr{C}(\mathscr{L}){:}C(i) \cap S = S\}|}{|\mathscr{C}(\mathscr{L})|},$$

i.e., $h_S(i)$ is the average number of compatible orderings of $\mathscr{L}$ in which i is the last member of S in the ordering. Note that $h_S(i) \neq 0 \Leftrightarrow i \in ex(S)$.

Definition 2. A convex set $U \in \mathscr{L}$ is called a *carrier* for a game $v \in \Gamma(\mathscr{L})$ if $v(S) = v(S \cap U)$ for all $S \in \mathscr{L}$.

Let $\Phi{:}\Gamma(\mathscr{L}) \to \mathbb{R}^n : v \mapsto (\Phi_1(v), \ldots, \Phi_n(v))$, be a map satisfying the following axioms:

(A1) (*Linearity*). For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathscr{L})$ we have

$$\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(w).$$

(A2) (*Carrier*). If $U \in \mathscr{L}$ is a carrier of $v \in \Gamma(\mathscr{L})$ then

$$\sum_{i \in U} \Phi_i(v) = v(U).$$

(A3) (*Hierarchical strength*). For any $S \in \mathscr{L}$ and $i, j \in S$,

$$h_S(i)\Phi_j(\zeta_S) = h_S(j)\Phi_i(\zeta_S).$$

Proposition 1. *There is an unique function* $\Phi : \Gamma(\mathscr{L}) \to \mathbb{R}^n$ *that satisfies axioms* (A1)–(A3). *Moreover, for every* $i \in N$, *the Shapley value is*

$$\Phi_i(v) = \sum_{\{S \in \mathscr{L}:\ i \in ex(S)\}} h_S(i)\, \Delta_v(S).$$

Proof. From axiom (A1) of linearity and Theorem 2 it suffices to show that Φ exists and is unique for the upper games ζ_S, $S \in \mathscr{L}$. We have that S is a carrier for the game ζ_S hence axiom (A2) implies

$$\sum_{i \in S} \Phi_i(\zeta_S) = \zeta_S(S) = 1.$$

Moreover, it follows easily that $\Phi_j(\zeta_S) = 0$ if $j \notin S$. If we fix $i \in S$, then by axiom (A3) we have that

$$\Phi_j(\zeta_S) = \frac{h_S(j)}{h_S(i)} \Phi_i(\zeta_S)$$

and hence

$$1 = \sum_{i \in S} \Phi_i(\zeta_S) = \Phi_i(\zeta_S) + \sum_{j \in S \setminus i} \frac{h_S(j)}{h_S(i)} \Phi_i(\zeta_S) = \frac{\sum_{j \in S} h_S(j)}{h_S(i)} \Phi_i(\zeta_S).$$

Note that $\sum_{j \in S} h_S(j) = 1$ since in every compatible ordering $C \in \mathscr{C}(\mathscr{L})$ there will be a unique element $j \in S$ so that $C(j) \cap S = S$. Thus, the Shapley value of the games ζ_S is given by

$$\Phi_i(\zeta_S) = \begin{cases} h_S(i) & \text{if } i \in ex(S), \\ 0 & \text{otherwise.} \end{cases}$$

By linearity (A1) and Theorem 2 the Shapley value exists and is unique for every game $v \in \Gamma(\mathscr{L})$. $\square$

We can describe the Shapley value as the average of the marginal contributions of the player i in the set of all compatible orderings of $\mathscr{L}$.

Theorem 3. *For any game $v \in \Gamma(\mathscr{L})$ and any player $i \in N$ we have*

$$\Phi_i(v) = T_i(v) := \frac{1}{c(N)} \sum_{C \in \mathscr{C}(\mathscr{L})} [v(C(i)) - v(C(i) \setminus i)].$$

Proof. It is clear that the operator $T : \Gamma(\mathscr{L}) \to \mathbb{R}^n$ satisfies axiom (A1), so if we can show that T agrees with Φ on the upper games ζ_S then we will be done. Fix $S \in \mathscr{L}$ and $i \in N$. Given an ordering $C \in \mathscr{C}(\mathscr{L})$, we see that the term $\zeta_S(C(i)) - \zeta_S(C(i) \setminus i)$ will contribute the value 1 exactly when $S \subseteq C(i)$ and $i \in S$, and 0 otherwise. If $i \in S$ we have that

$$\zeta_S(C(i)) - \zeta_S(C(i) \setminus i) = 1 \Leftrightarrow C(i) \cap S = S.$$

Thus, we obtain

$$T_i(\zeta_S) = \frac{1}{c(N)} |\{C \in \mathscr{C}(\mathscr{L}) \colon C(i) \cap S = S\}| = h_S(i) = \Phi_i(\zeta_S),$$

for all $i \in S$. Otherwise, $T_i(\zeta_S) = \Phi_i(\zeta_S) = 0$. $\square$

We define the concept of *dummy player* [4].

Definition 3. The player $k \in N$ is a dummy in the game $v \in \Gamma(\mathscr{L})$ if, for every convex $S \in \mathscr{L}$ such that $k \notin S$ and $S \cup k \in \mathscr{L}$, we have $v(S \cup k) - v(S) = 0$.

Proposition 2. *Let $\mathscr{L}$ be a convex geometry and let $S \in \mathscr{L}$ be a convex set. If $i \notin ex(S)$ then i is a dummy player in the upper game ζ_S.*

Proof. If there exists $T \in \mathscr{L}$ such that $T \cup i \in \mathscr{L}$ and $\zeta_S(T \cup i) \neq \zeta_S(T)$ then $T = S \setminus i \in \mathscr{L}$. Thus we obtain that $i \in ex(S)$. $\square$

We now show that the Shapley value is the expected marginal contribution of a player i to the coalitions $S \in \mathscr{L}$ such that $S \setminus i \in \mathscr{L}$, when the players joining in a compatible ordering.

Theorem 4. *Let $E : \Gamma(\mathscr{L}) \to \mathbb{R}^n$ be the operator defined by*

$$E_i(v) := \sum_{\{S \in \mathscr{L}:\, i \in ex(S)\}} \frac{c(S \setminus i)c([S,N])}{c(N)} [v(S) - v(S \setminus i)], \quad i \in N.$$

Then the Shapley value satisfies $\Phi = E$.

Proof. The operator E is linear hence it suffices to show that Φ and E coincide on any upper game $\zeta_T, T \in \mathscr{L}$. Fix $T \in \mathscr{L}$ and $i \in N$. If $i \notin ex(T)$ then Proposition 2 implies that i is a dummy player in the game ζ_T. Then, for every $S \in \mathscr{L}$ such that $i \in ex(S)$ we have $\zeta_T(S) - \zeta_T(S \setminus i) = 0$ and $E_i(\zeta_T) = 0$. Now suppose that $i \in ex(T)$. Then $T \nsubseteq S \setminus i$ hence, $\zeta_T(S \setminus i) = 0$ and the marginal contributions of i satisfies

$$\zeta_T(S) - \zeta_T(S \setminus i) = \begin{cases} 1 & \text{if } S \in \mathscr{L} \text{ and } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we obtain

$$\begin{aligned} E_i(\zeta_T) &= \sum_{\{S \in \mathscr{L}:\, i \in ex(S),\ S \supseteq T\}} \frac{c(S \setminus i)c([S,N])}{c(N)} \\ &= \frac{1}{c(N)} \sum_{\{S \in \mathscr{L}:\, i \in ex(S),\ S \supseteq T\}} |\{C \in \mathscr{C}(\mathscr{L}) : C(i) = S\}| \\ &= \frac{1}{c(N)} |\{C \in \mathscr{C}(\mathscr{L}) \colon C(i) \cap T = T\}| \\ &= h_T(i). \qquad \square \end{aligned}$$

We can now give another axiomatization for the Shapley value of games on convex geometries. We consider the axioms:

(A1) (*Linearity*)

(A2a) (*Efficiency*). If N is the set of all players of $v \in \Gamma(\mathscr{L})$ then

$$\sum_{i \in N} \Phi_i(v) = v(N).$$

(A2b) (*Dummy*). If the player $k \in N$ is a dummy in $v \in \Gamma(\mathscr{L})$ then $\Phi_k(v) = 0$.
(A3) (*Hierarchical strength*)
It is easy to prove that these axioms also characterize the Shapley value.

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