Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 114–129
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Bounded Toeplitz products on Bergman spaces of the unit ball

Karel Stroethoff [a,*], Dechao Zheng [b,1]

[a] *Department of Mathematical Sciences, University of Montana, Missoula, MT 59812, USA*
[b] *Mathematics Department, Vanderbilt University, Nashville, TN 37240, USA*

Received 6 June 2005

Available online 17 February 2006

Submitted by J.H. Shapiro

Abstract

We consider the question for which square integrable analytic functions f and g on the unit ball the densely defined products $T_f T_{\bar{g}}$ are bounded on the weighted Bergman spaces. We prove results analogous to those we obtained in the setting of the unit disk and the polydisk.

Keywords: Toeplitz operator; Bergman spaces; Unit ball

1. Introduction

Throughout let n be a fixed integer $n \geqslant 2$. Denote the unit ball in $\mathbb{C}^n$ by $\mathbb{B}_n$, and let ν be Lebesgue volume measure on $\mathbb{B}_n$, normalized so that $\nu(\mathbb{B}_n) = 1$.

For $w \in \mathbb{B}_n$, let φ_w be the automorphism of $\mathbb{B}_n$ such that $\varphi_w(0) = w$ and $\varphi_w^{-1} = \varphi_w$. The mappings φ_w are described in [4, Section 2.2].

For $-1 < \alpha < \infty$, we denote by ν_α the measure on $\mathbb{B}_n$ defined by $d\nu_\alpha(z) = (1 - |z|^2)^\alpha \, d\nu(z)$. The weighted Bergman space $A^2_\alpha(\mathbb{B}_n)$ is the space of analytic functions h on $\mathbb{B}_n$ which are

* Corresponding author.
E-mail addresses: karel.stroethoff@umontana.edu (K. Stroethoff), zheng@math.vanderbilt.edu (D. Zheng).
[1] Partially supported by the National Science Foundation.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.01.009

square-integrable with respect to measure ν_α on $\mathbb{B}_n$. The reproducing kernel in $A^2_\alpha(\mathbb{B}_n)$ is given by

$$K_w^{(\alpha)}(z) = \frac{1}{(1 - \langle z, w\rangle)^{n+\alpha+1}},$$

for $z, w \in \mathbb{B}_n$. If $\langle \cdot,\cdot\rangle_\alpha$ denotes the inner product in $L^2(\mathbb{B}_n, d\nu_\alpha)$, then $\langle h, K_w^{(\alpha)}\rangle_\alpha = h(w)$, for every $h \in A^2_\alpha(\mathbb{B}_n)$ and $w \in \mathbb{B}_n$. In this paper we use $\|\cdot\|_\alpha$ to denote the norm in $L^2(\mathbb{B}_n, d\nu_\alpha)$. The orthogonal projection P_α of $L^2(\mathbb{B}_n, d\nu_\alpha)$ onto $A^2_\alpha(\mathbb{B}_n)$ is given by

$$(P_\alpha g)(w) = \langle g, K_w^{(\alpha)}\rangle_\alpha = \int_{\mathbb{B}_n} g(z) \frac{1}{(1 - \langle w, z\rangle)^{n+\alpha+1}}\, d\nu_\alpha(z),$$

for $g \in L^2(\mathbb{B}_n, d\nu_\alpha)$ and $w \in \mathbb{B}_n$. Given $f \in L^\infty(\mathbb{B}_n)$, the Toeplitz operator T_f is defined on $A^2_\alpha(\mathbb{B}_n)$ by $T_f h = P_\alpha(fh)$. We have

$$(T_f h)(w) = \int_{\mathbb{B}_n} \frac{f(z)h(z)}{(1 - \langle w, z\rangle)^{n+\alpha+1}}\, d\nu_\alpha(z),$$

for $h \in A^2_\alpha(\mathbb{B}_n)$ and $w \in \mathbb{B}_n$. Note that the above formula makes sense, and defines a function analytic on $\mathbb{B}_n$, also if $f \in L^2(\mathbb{B}_n, d\nu_\alpha)$. So, if $g \in A^2_\alpha(\mathbb{B}_n)$ we define $T_{\bar g}$ by the formula

$$(T_{\bar g} h)(w) = \int_{\mathbb{B}_n} \frac{\overline{g(z)}h(z)}{(1 - \langle w, z\rangle)^{n+\alpha+1}}\, d\nu_\alpha(z),$$

for $h \in A^2_\alpha(\mathbb{B}_n)$ and $w \in \mathbb{B}_n$. If also $f \in A^2_\alpha(\mathbb{B}_n)$, then $T_f T_{\bar g} h$ is the analytic function $f T_{\bar g} h$ for $h \in H^\infty(\mathbb{B}_n)$.

We will give a necessary condition for boundedness of the Toeplitz product $T_f T_{\bar g}$ in Section 3, and then show that this condition is very close to being sufficient in Section 4. Our conditions are formulated in terms of the (weighted) Berezin transform: for a function $u \in L^1(\mathbb{B}_n, d\nu_\alpha)$, the Berezin transform $B_\alpha[u]$ is the function on $\mathbb{B}_n$ defined by

$$B_\alpha[u](w) = \int_{\mathbb{B}_n} u(z) \frac{(1 - |w|^2)^{n+\alpha+1}}{|1 - \langle z, w\rangle|^{2n+2+2\alpha}}\, d\nu_\alpha(z), \quad w \in \mathbb{B}_n.$$

We prove that a necessary condition for boundedness of the Toeplitz product $T_f T_{\bar g}$ on $A^2_\alpha(\mathbb{B}_n)$ is that

$$\sup_{w \in \mathbb{B}_n} B_\alpha\big[|f|^2\big](w) B_\alpha\big[|g|^2\big](w) < \infty. \tag{1.1}$$

We also prove that a slightly stronger condition is sufficient: if for every $\varepsilon > 0$

$$\sup_{w \in \mathbb{B}_n} B_\alpha\big[|f|^{2+\varepsilon}\big](w) B_\alpha\big[|g|^{2+\varepsilon}\big](w) < \infty, \tag{1.2}$$

then the operator $T_f T_{\bar g}$ is bounded on $A^2_\alpha(\mathbb{B}_n)$. The study of this problem was initiated by Sarason [7] in the context of the Hardy space H^2 of the unit circle, after he had obtained examples of functions f and g in H^2 such that the product $T_f T_{\bar g}$ is bounded on H^2, while neither T_f nor T_g is bounded [5,6]. In the context of the Hardy space, the Poisson kernel plays the role of the Berezin transform. Treil showed that a condition analogous to (1.1) is necessary [7], while the second author proved that a condition analogous to (1.2) is sufficient [12].

The above results are analogous to [10], and generalize the results we obtained in [9,11]. However, the proofs require new tools to establish the necessary condition, as well as consideration of higher derivatives, and an inner product formula involving higher derivatives. Recently Park [3] has proved a necessary and close-to-sufficient condition for Toeplitz products on the Bergman space of the ball. In addition to consideration of higher-order derivatives, this required him to find a new way to rewrite the inner product formula. Park's method to obtain an inner product suitable to prove sufficiency does not work for weighted Bergman spaces on the ball. In this paper we will use a novel approach to obtain a suitable inner product formula. In extending the necessity result to the weighted setting on the ball, we make use of an identity that played a key role in the argument for weighted Bergman spaces on the disk [11] as well as an estimate inspired by recent results of Arazy and Engliš [1,2].

2. Preliminaries

In this section we give more preliminaries than already discussed above needed in the sequel. Using the reproducing property of $K_w^{(\alpha)}$, we have

$$\|K_w^{(\alpha)}\|_\alpha^2 = \langle K_w^{(\alpha)}, K_w^{(\alpha)}\rangle_\alpha = K_w^{(\alpha)}(w) = \frac{1}{(1-|w|^2)^{n+\alpha+1}},$$

thus the normalized reproducing kernel is given by

$$k_w^{(\alpha)}(z) = \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-\langle z, w\rangle)^{n+\alpha+1}}, \tag{2.1}$$

for $z, w \in \mathbb{B}_n$. For $w \in \mathbb{B}_n$ the function φ_w has real Jacobian equal to

$$\left|\varphi_w'(z)\right|^2 = \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2n+2}}.$$

Thus we have the change-of-variable formula

$$\int_{\mathbb{B}_n} h\big(\varphi_w(z)\big)\left|k_w^{(\alpha)}(z)\right|^2 d\nu_\alpha(z) = \int_{\mathbb{B}_n} h(u)\, d\nu_\alpha(u), \tag{2.2}$$

for every $h \in L^1(\mathbb{B}_n, d\nu_\alpha)$. It follows from (2.2) that the mapping $U_w^{(\alpha)} h = (h \circ \varphi_w) k_w^{(\alpha)}$ is an isometry on $A_\alpha^2(\mathbb{B}_n)$:

$$\|U_w^{(\alpha)} h\|_\alpha^2 = \int_{\mathbb{B}_n} \left|h\big(\varphi_w(z)\big)\right|^2 \left|k_w^{(\alpha)}(z)\right|^2 d\nu_\alpha(z) = \int_{\mathbb{B}_n} \left|h(u)\right|^2 d\nu_\alpha(u) = \|h\|_\alpha^2,$$

for all $h \in A_\alpha^2(\mathbb{B}_n)$. Using the identity

$$1 - \langle \varphi_w(z), w\rangle = \frac{1-|w|^2}{1-\langle z, w\rangle},$$

we have

$$k_w^{(\alpha)}\big(\varphi_w(z)\big) = \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-\langle \varphi_w(z), w\rangle)^{n+\alpha+1}} = \frac{(1-\langle z, w\rangle)^{n+\alpha+1}}{(1-|w|^2)^{(n+\alpha+1)/2}} = \frac{1}{k_w^{(\alpha)}(z)}.$$

Since $\varphi_w \circ \varphi_w = \mathrm{id}$, we see that

$$\big(U_w^{(\alpha)}\big(U_w^{(\alpha)} h\big)\big)(z) = \big(U_w^{(\alpha)} h\big)\big(\varphi_w(z)\big) k_w^{(\alpha)}(z) = h(z) k_w^{(\alpha)}\big(\varphi_w(z)\big) k_w^{(\alpha)}(z) = h(z),$$

for all $z \in \mathbb{B}_n$ and $h \in A^2_\alpha(\mathbb{B}_n)$. Thus $(U^{(\alpha)}_w)^{-1} = U^{(\alpha)}_w$, and hence $U^{(\alpha)}_w$ is unitary. Furthermore,

$$T_{f\circ\varphi_w} U^{(\alpha)}_w = U^{(\alpha)}_w T_f \tag{2.3}$$

holds for $f \in L^\infty(\mathbb{B}_n)$.

Proof. For $h \in H^\infty(\mathbb{B}_n)$ and $g \in A^2_\alpha(\mathbb{B}_n)$ we have

$$\begin{aligned}
\left\langle U^{(\alpha)}_w T_f h, U^{(\alpha)}_w g\right\rangle_\alpha &= \langle T_f h, g\rangle_\alpha = \langle fh, g\rangle_\alpha \\
&= \int_{\mathbb{B}_n} f(u)h(u)\overline{g(u)}\,d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} f\big(\varphi_w(z)\big)h\big(\varphi_w(z)\big)\overline{g\big(\varphi_w(z)\big)}\big|k^{(\alpha)}_w(z)\big|^2\,d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} f\big(\varphi_w(z)\big)h\big(\varphi_w(z)\big)k^{(\alpha)}_w(z)\overline{g\big(\varphi_w(z)\big)k^{(\alpha)}_w(z)}\,d\nu_\alpha(z) \\
&= \left\langle fU^{(\alpha)}_w h, U^{(\alpha)}_w g\right\rangle_\alpha = \left\langle T_{f\circ\varphi_w} U^{(\alpha)}_w h, U^{(\alpha)}_w g\right\rangle_\alpha,
\end{aligned}$$

establishing (2.3). □

3. Necessary condition for boundedness

In this section we prove the following necessary condition for boundedness of the Toeplitz product $T_f T_{\bar g}$ with f and g in $A^2_\alpha(\mathbb{B}_n)$.

Theorem 3.1. *Let $-1 < \alpha < \infty$, and let f and g be in $A^2_\alpha(\mathbb{B}_n)$. If $T_f T_{\bar g}$ is bounded on $A^2_\alpha(\mathbb{B}_n)$, then*

$$\sup_{w\in\mathbb{B}_n} B_\alpha\big[|f|^2\big](w) B_\alpha\big[|g|^2\big](w) < \infty.$$

Suppose f and g are in $A^2_\alpha(\mathbb{B}_n)$. Consider the operator $f \otimes g$ on $A^2_\alpha(\mathbb{B}_n)$ defined by

$$(f \otimes g)h = \langle h, g\rangle_\alpha f,$$

for $h \in A^2_\alpha(\mathbb{B}_n)$. It is easily proved that $f \otimes g$ is bounded on $A^2_\alpha(\mathbb{B}_n)$ with norm equal to $\|f \otimes g\| = \|f\|_\alpha \|g\|_\alpha$.

We will obtain an expression for the operator $f \otimes g$, where $f, g \in A^2_\alpha(\mathbb{B}_n)$, in terms of the Toeplitz product $T_f T_{\bar g}$, which we will be able to use to bound the norm of $f \otimes g$ by a constant multiple of the norm of $T_f T_{\bar g}$. To obtain a suitable operator identity, we will use the Berezin transform: writing $k^{(\alpha)}_w$ for the normalized reproducing kernels, we define the Berezin transform of a bounded linear operator S on $A^2_\alpha(\mathbb{B}_n)$ to be the function $B_\alpha[S]$ defined on $\mathbb{B}_n$ by

$$B_\alpha[S](w) = \left\langle Sk^{(\alpha)}_w, k^{(\alpha)}_w\right\rangle_\alpha,$$

for $w \in \mathbb{B}_n$. The boundedness of S implies that the function $B_\alpha[S]$ is bounded on $\mathbb{B}_n$. The Berezin transform is injective, for $B_\alpha[S](w) = 0$, for all $w \in \mathbb{B}_n$, implies that $S = 0$, the zero operator on

$A^2_\alpha(\mathbb{B}_n)$ (see [8] for a proof). We will also make use of the following continuity condition of the Berezin transform: if $S_N \to S$ in operator norm, then

$$B_\alpha[S](w) = \lim_{N\to\infty} B_\alpha[S_N](w), \tag{3.2}$$

for each $w \in \mathbb{B}_n$. The above statement is an immediate consequence of the following inequality:

$$\left|B_\alpha[S](w) - B_\alpha[S_N](w)\right| \leqslant \|S - S_N\|.$$

To prove a suitable operator identity, we need the following lemma, which is inspired by [1,2].

Lemma 3.3. *Let $-1 < \alpha < \infty$. If S is a bounded linear operator on $A^2_\alpha(\mathbb{B}_n)$, then*

$$\left\| \sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z^\gamma} S T_{\bar{z}^\gamma} \right\| \leqslant \|S\|,$$

for every positive integer m.

Proof. Given a positive integer m, let d_m denote the number of elements of the set $\{\gamma\colon |\gamma| = m\}$. Then $\{\gamma\colon |\gamma| = m\} = \{\gamma_1, \gamma_2, \dots, \gamma_{d_m}\}$. Observe that each γ_k is a multi-index, that is, $\gamma_k = (\gamma_{k,1}, \gamma_{k,2}, \dots, \gamma_{k,n}) \in \mathbb{N}^n$ with $|\gamma_k| = \gamma_{k,1} + \gamma_{k,2} + \cdots + \gamma_{k,n} = m$. For each k in $\{1, 2, \dots, d_m\}$ consider the monomials

$$\psi_k(z) = \sqrt{\frac{m!}{\gamma_k!}}\, z^{\gamma_k} = \sqrt{\frac{m!}{\gamma_k!}}\, z_1^{\gamma_{k,1}} z_2^{\gamma_{k,2}} \cdots z_n^{\gamma_{k,n}}.$$

Define the row block operator

$$\mathbf{M}_m = [T_{\psi_1}, T_{\psi_2}, \dots, T_{\psi_{d_m}}]\colon A^2_\alpha(B_n) \times A^2_\alpha(B_n) \times \cdots \times A^2_\alpha(B_n) \to A^2_\alpha(B_n)$$

by

$$\mathbf{M}_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} = \sum_{k=1}^{d_m} T_{\psi_k} f_k = P\left(\sum_{k=1}^{d_m} \psi_k f_k\right).$$

Then

$$\left\| \mathbf{M}_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} \right\|^2 \leqslant \left\| \sum_{k=1}^{d_m} \psi_k f_k \right\|^2 = \int_{B_n} \left| \sum_{k=1}^{d_m} \psi_k(z) f_k(z) \right|^2 d\nu_\alpha(z)$$

$$\leqslant \int_{B_n} \left(\sum_{k=1}^{d_m} \left|\psi_k(z)\right|^2 \right) \left(\sum_{k=1}^{d_m} \left|f_k(z)\right|^2 \right) d\nu_\alpha(z).$$

Using the multinomial formula,

$$\sum_{k=1}^{d_m} \left|\psi_k(z)\right|^2 = \sum_{k=1}^{d_m} \frac{m!}{\gamma_k!} \left|z^{\gamma_k}\right|^2 = \left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^m = |z|^{2m} \leqslant 1.$$

Thus

$$\left\| \mathbf{M}_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} \right\| \leqslant \left(\sum_{k=1}^{d_m} \int_{B_n} \left| f_k(z) \right|^2 d\nu_\alpha(z) \right)^{1/2} = \left\| (f_1, f_2, \dots, f_{d_m}) \right\|.$$

If S is a bounded linear operator on $A^2_\alpha(B_n)$, the operator

$$S_m : A^2_\alpha(B_n) \times A^2_\alpha(B_n) \times \cdots \times A^2_\alpha(B_n) \to A^2_\alpha(B_n) \times A^2_\alpha(B_n) \times \cdots \times A^2_\alpha(B_n)$$

defined by

$$S_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} = \begin{pmatrix} Sf_1 \\ Sf_2 \\ \vdots \\ Sf_{d_m} \end{pmatrix}$$

is bounded, in fact,

$$\|S_m\| \leqslant \|S\|.$$

It is easily seen that the adjoint of $\mathbf{M}_m$ is the column block operator

$$\mathbf{M}_m^* g = \begin{bmatrix} T_{\bar{\psi}_1} g \\ \vdots \\ T_{\bar{\psi}_{d_m}} g \end{bmatrix}.$$

For $g \in A^2_\alpha(B_n)$ we have

$$\mathbf{M}_m S_m \mathbf{M}_m^* g = \mathbf{M}_m S_m \begin{bmatrix} T_{\bar{\psi}_1} g \\ \vdots \\ T_{\bar{\psi}_{d_m}} g \end{bmatrix} = \mathbf{M}_m \begin{bmatrix} ST_{\bar{\psi}_1} g \\ ST_{\bar{\psi}_2} g \\ \vdots \\ ST_{\bar{\psi}_{d_m}} g \end{bmatrix} = \sum_{k=1}^{d_m} T_{\psi_k} (ST_{\bar{\psi}_k} g)$$
$$= \left(\sum_{k=1}^{d_m} T_{\psi_k} ST_{\bar{\psi}_k} \right) g,$$

so

$$\mathbf{M}_m S_m \mathbf{M}_m^* = \sum_{k=1}^{d_m} T_{\psi_k} ST_{\bar{\psi}_k},$$

that is

$$\sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z^\gamma} ST_{\bar{z}^\gamma} = \mathbf{M}_m S_m \mathbf{M}_m^*.$$

It follows that

$$\left\| \sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z^\gamma} ST_{\bar{z}^\gamma} \right\| = \left\| \mathbf{M}_m S_m \mathbf{M}_m^* \right\| \leqslant \|S_m\| \leqslant \|S\|,$$

as was to be shown. □

We will use the Berezin transform to derive an operator identity suitable for our purposes. It follows from (2.1) that

$$B_\alpha[S](w) = \left(1-|w|^2\right)^{n+\alpha+1}\left\langle SK_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha,$$

for $w \in \mathbb{B}_n$. It is easily seen that $T_{\bar g}K_w^{(\alpha)} = \overline{g(w)}K_w^{(\alpha)}$. Thus

$$\begin{aligned}\left\langle T_f T_{\bar g} K_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha &= \left\langle T_{\bar g} K_w^{(\alpha)}, T_{\bar f} K_w^{(\alpha)}\right\rangle_\alpha = \left\langle \overline{g(w)} K_w^{(\alpha)}, \overline{f(w)} K_w^{(\alpha)}\right\rangle_\alpha \\ &= f(w)\overline{g(w)}/\left(1-|w|^2\right)^{n+\alpha+1},\end{aligned}$$

and we see that

$$B_\alpha[T_f T_{\bar g}](w) = f(w)\overline{g(w)}.$$

We also have

$$\begin{aligned}B_\alpha[f\otimes g](w) &= \left(1-|w|^2\right)^{n+\alpha+1}\left\langle (f\otimes g)K_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha \\ &= \left(1-|w|^2\right)^{n+\alpha+1}\left\langle \left\langle K_w^{(\alpha)}, g\right\rangle f, K_w^{(\alpha)}\right\rangle_\alpha \\ &= \left(1-|w|^2\right)^{n+\alpha+1}\left\langle K_w^{(\alpha)}, g\right\rangle\left\langle f, K_w^{(\alpha)}\right\rangle_\alpha \\ &= \left(1-|w|^2\right)^{n+\alpha+1} f(w)\overline{g(w)}.\end{aligned}$$

We need the following lemma, which was proved in [11].

Lemma 3.4. *If* $0 < \beta < 1$ *and* k *is a positive integer, then*

$$(1-t)^{k-\beta} = \sum_{j=0}^{k-1}(-1)^j\frac{\Gamma(k+1-\beta)}{\Gamma(k+1-\beta-j)}\frac{t^j}{j!} + (-1)^k\frac{\Gamma(k+1-\beta)}{\Gamma(\beta)\Gamma(1-\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\beta)}{(n+k)!}t^{n+k},$$

for all $-1 < t < 1$.

Assuming α to be a non-integer, we apply the above lemma with $k = [\alpha]+n+2$ and $\beta = 1-\{\alpha\}$, where $\{\alpha\} = \alpha - [\alpha]$. Using that $\Gamma(\{\alpha\})\Gamma(1-\{\alpha\}) = \pi/\sin(\pi\{\alpha\})$, we get

$$\begin{aligned}(1-t)^{n+1+\alpha} = &\sum_{j=0}^{n+1+[\alpha]}(-1)^j\frac{\Gamma(n+2+\alpha)}{j!\Gamma(n+2+\alpha-j)}t^j \\ &+ (-1)^{n+[\alpha]}\frac{\Gamma(n+2+\alpha)\sin(\pi\{\alpha\})}{\pi}\sum_{j=0}^{\infty}\frac{\Gamma(j+1-\{\alpha\})}{(n+2+[\alpha]+j)!}t^{n+2+[\alpha]+j}.\end{aligned}$$

Note that the above formula is also valid in case α is an integer. Applying the above identity to $t = |w|^2 = w_1\bar w_1 + \cdots + w_n\bar w_n$, making use of the multinomial formula (see, for example, [13, Section 1.1]), we have

$$t^{n+2+[\alpha]+j} = \sum_{|\gamma|=n+2+[\alpha]+j}\frac{(n+2+[\alpha]+j)!}{\gamma!}w^\gamma\bar w^\gamma,$$

and we obtain

$$\begin{aligned}\left(1-|w|^2\right)^{n+1+\alpha} = &\sum_{j=0}^{n+1+[\alpha]}(-1)^j\sum_{|\gamma|=j}\frac{\Gamma(n+2+\alpha)}{\gamma!\Gamma(n+2+\alpha-j)}w^\gamma\bar{w}^\gamma\\ &+(-1)^{n+[\alpha]}\frac{\Gamma(n+2+\alpha)\sin(\pi\{\alpha\})}{\pi}\\ &\times\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}w^\gamma\bar{w}^\gamma.\end{aligned}$$

Combining the above formula with $B_\alpha[f\otimes g](w)=(1-|w|^2)^{n+1+\alpha}f(w)\overline{g(w)}$, we get

$$\begin{aligned}B_\alpha[f\otimes g](w) = &\sum_{j=0}^{n+1+[\alpha]}(-1)^j\sum_{|\gamma|=j}\frac{\Gamma(n+2+\alpha)}{\gamma!\,\Gamma(n+2+\alpha-j)}w^\gamma f(w)\overline{g(w)}\bar{w}^\gamma\\ &+(-1)^{n+[\alpha]}\frac{\Gamma(n+2+\alpha)\sin(\pi\{\alpha\})}{\pi}\\ &\times\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}w^\gamma f(w)\overline{g(w)}\overline{w}^\gamma.\end{aligned}\tag{3.5}$$

Thus the above identity and uniqueness of the Berezin transform imply the following operator identity:

$$\begin{aligned}f\otimes g = &\sum_{j=0}^{n+1+[\alpha]}(-1)^j\sum_{|\gamma|=j}\frac{\Gamma(n+2+\alpha)}{\gamma!\Gamma(n+2+\alpha-j)}T_{z^\gamma}T_fT_{\bar{g}}T_{\bar{z}^\gamma}\\ &+(-1)^{n+[\alpha]}\frac{\Gamma(n+2+\alpha)\sin(\pi\{\alpha\})}{\pi}\\ &\times\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}T_{z^\gamma}T_fT_{\bar{g}}T_{\bar{z}^\gamma}.\end{aligned}\tag{3.6}$$

Remark. That the operator on the right-hand side of (3.6) defines a bounded operator follows from Lemma 3.3, which implies

$$\sum_{j=0}^{\infty}\left\|\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}T_{z^\gamma}T_fT_{\bar{g}}T_{\bar{z}^\gamma}\right\|\leqslant\sum_{j=0}^{\infty}\frac{\Gamma(j+1-\{\alpha\})}{(n+2+[\alpha]+j)!}\|T_fT_{\bar{g}}\|.$$

Stirling's formula shows that the series

$$\sum_{j=0}^{\infty}\frac{\Gamma(j+1-\{\alpha\})}{(n+2+[\alpha]+j)!}$$

converges. Thus

$$\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}T_{z^\gamma}T_fT_{\bar{g}}T_{\bar{z}^\gamma}$$

is a bounded operator, which by (3.2) has Berezin transform equal to

$$\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\alpha]+j}\frac{\Gamma(j+1-\{\alpha\})}{\gamma!}B_\alpha[T_{z^\gamma}T_fT_{\bar{g}}T_{\bar{z}^\gamma}].$$

Thus the operator on the right-hand side of (3.6) has Berezin transform equal to the right-hand side of (3.5). By uniqueness of the Berezin transform the two operators must be equal, establishing operator identity (3.6).

We are now ready to prove the necessary condition of Theorem 3.1.

Proof Theorem 3.1. Suppose f and g are analytic on $A^2_\alpha(\mathbb{B}_n)$ such that the densely defined Toeplitz product $T_f T_{\bar{g}}$ is bounded on $A^2_\alpha(\mathbb{B}_n)$. Using identity (3.6) and the estimate in the above remark, we see that there exists a finite constant C_α such that

$$\|f \otimes g\| \leqslant C_\alpha \|T_f T_{\bar{g}}\|,$$

thus

$$\|f\|_2 \|g\|_2 \leqslant C_\alpha \|T_f T_{\bar{g}}\|. \tag{3.7}$$

It follows from (2.3), applied to f and $\bar{g}$, that

$$\begin{aligned} T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w} &= \big(T_{f\circ\varphi_w} U_w^{(\alpha)}\big) U_w^{(\alpha)} \big(T_{\bar{g}\circ\varphi_w} U_w^{(\alpha)}\big) U_w^{(\alpha)} \\ &= \big(U_w^{(\alpha)} T_f\big) U_w^{(\alpha)} \big(U_w^{(\alpha)} T_{\bar{g}}\big) U_w^{(\alpha)} = U_w^{(\alpha)} (T_f T_{\bar{g}}) U_w^{(\alpha)}, \end{aligned}$$

for all $w \in \mathbb{B}_n$. Inequality (3.7) applied to $f \circ \varphi_w$ and $g \circ \varphi_w$ gives

$$\|f \circ \varphi_w\|_2 \, \|g \circ \varphi_w\|_2 \leqslant C_\alpha \|T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w}\| = C_\alpha \|T_f T_{\bar{g}}\|,$$

hence

$$B_\alpha\big[|f|^2\big](w) B_\alpha\big[|g|^2\big](w) \leqslant C_\alpha^2 \|T_f T_{\bar{g}}\|^2,$$

for all $w \in \mathbb{B}_n$. So, for $f, g \in A^2_\alpha(\mathbb{B}_n)$, a necessary condition for the Toeplitz product $T_f T_{\bar{g}}$ to be bounded on $A^2_\alpha(\mathbb{B}_n)$ is

$$\sup_{w\in\mathbb{B}_n} B_\alpha\big[|f|^2\big](w) B_\alpha\big[|g|^2\big](w) < \infty. \tag{3.8}$$

This completes the proof of Theorem 3.1. □

4. Sufficient condition

In this section we will prove a condition slightly stronger than (3.8) in the following theorem.

Theorem 4.1. *Let $-1 < \alpha < \infty$, and let f and g be in $A^2_\alpha(\mathbb{B}_n)$. If for $\varepsilon > 0$,*

$$\sup_{w\in\mathbb{B}_n} B_\alpha\big[|f|^{2+\varepsilon}\big](w) B_\alpha\big[|g|^{2+\varepsilon}\big](w) < \infty, \tag{4.2}$$

then the operator $T_f T_{\bar{g}}$ is bounded on $A^2_\alpha(\mathbb{B}_n)$.

By Hölder's inequality,

$$\bigg(\int_{\mathbb{B}_n} |f|^2 \, d\nu_\alpha\bigg)^{1/2} \leqslant \bigg(\int_{\mathbb{B}_n} |f|^{2+\varepsilon} \, d\nu_\alpha\bigg)^{1/(2+\varepsilon)}.$$

Applying this to the function $f \circ \varphi_w$, making use of (2.2), it follows that

$$B_\alpha\big[|f|^2\big](w)^{1/2} \leqslant B_\alpha\big[|f|^{2+\varepsilon}\big](w)^{1/(2+\varepsilon)},$$

and thus

$$\big(B_\alpha\big[|f|^2\big](w) B_\alpha\big[|g|^2\big](w)\big)^{1/2} \leqslant \big(B_\alpha\big[|f|^{2+\varepsilon}\big](w) B_\alpha\big[|g|^{2+\varepsilon}\big](w)\big)^{1/(2+\varepsilon)}, \tag{4.3}$$

for all $w \in \mathbb{B}_n$, so condition (4.2) implies necessary condition (3.8).

In the proof that $T_f T_{\bar g}$ is bounded on $A^2_\alpha(\mathbb{B}_n)$ if condition (4.2) holds, we will need estimates on $T_{\bar f} h$ and its derivatives, as well as an alternative way to write the inner product formula in $A^2_\alpha(\mathbb{B}_n)$.

4.1. Two estimates

We will need the two estimates contained in the following lemmas.

Lemma 4.4. *Let $-1 < \alpha < \infty$. For $f \in L^2(\mathbb{B}_n, \nu_\alpha)$ and $h \in H^\infty(\mathbb{B}_n)$ we have*

$$\big|(T_{\bar f} h)(w)\big| \leqslant \frac{B_\alpha[|f|^2](w)^{1/2}}{(1-|w|^2)^{(n+\alpha+1)/2}} \|h\|_\alpha,$$

for all $w \in \mathbb{B}_n$.

Proof. By Cauchy–Schwarz's inequality,

$$\begin{aligned}
\big|(T_{\bar f} h)(w)\big|^2 &\leqslant \left(\int_{\mathbb{B}_n} \frac{|f(z)||h(z)|}{|1-\langle z, w\rangle|^{n+\alpha+1}}\, d\nu_\alpha(z)\right)^2 \\
&\leqslant \int_{\mathbb{B}_n} \frac{|f(z)|^2}{|1-\langle z, w\rangle|^{2n+2\alpha+2}}\, d\nu_\alpha(z) \int_{\mathbb{B}_n} \big|h(z)\big|^2\, d\nu_\alpha(z) \\
&= \frac{B_\alpha[|f|^2](w)}{(1-|w|^2)^{n+\alpha+1}} \|h\|_\alpha^2,
\end{aligned}$$

and the stated inequality follows. □

Lemma 4.5. *Let $-1 < \alpha < \infty$ and $\varepsilon > 0$. For $f \in L^2(\mathbb{B}_n, \nu_\alpha)$, $h \in H^\infty(\mathbb{B}_n)$, and multi-index γ with $|\gamma| = m \geqslant (n+\alpha+1)/2$ we have*

$$\big|\big(D^\gamma T_{\bar f} h\big)(w)\big| \leqslant C \frac{B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)}}{(1-|w|^2)^m} \left(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w, z\rangle|^{n+\alpha+1}}\, d\nu_\alpha(z)\right)^{1/\delta},$$

for all $w \in \mathbb{B}_n$, where $\delta = (2+\varepsilon)/(1+\varepsilon)$.

Proof. Let $\varepsilon > 0$. For $f \in A^2_\alpha(\mathbb{B}_n)$ and $h \in H^\infty(\mathbb{B}_n)$ we have

$$(T_{\bar f} h)(w) = \big\langle T_{\bar f} h, K_w^{(\alpha)}\big\rangle_\alpha = \int_{\mathbb{B}_n} \frac{\overline{f(z)} h(z)}{(1-\langle w, z\rangle)^{n+\alpha+1}}\, d\nu_\alpha(z), \quad w \in \mathbb{B}_n,$$

thus

$$\big(D^\gamma T_{\bar f} h\big)(w) = \frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}_n} \frac{\overline{z^\gamma f(z)}h(z)}{(1-\langle w,z\rangle)^{n+\alpha+m+1}}\, d\nu_\alpha(z),$$

for every multi-index γ with $|\gamma| = m$. Applying Hölder's inequality, we get

$$\begin{aligned}
&\big|\big(D^\gamma T_{\bar f} h\big)(w)\big| \\
&\quad\leqslant C\bigg(\int_{\mathbb{B}_n} \frac{|f(z)|^{2+\varepsilon}}{|1-\langle w,z\rangle|^{2n+2\alpha+2}}\, d\nu_\alpha(z)\bigg)^{1/(2+\varepsilon)} \\
&\qquad\times\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{(2m+(n+\alpha+m+1)\varepsilon)/(1+\varepsilon)}}\, d\nu_\alpha(z)\bigg)^{1/\delta} \\
&\quad= C\frac{B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)}}{(1-|w|^2)^{(n+\alpha+1)/(2+\varepsilon)}}\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{(2m+(n+\alpha+m+1)\varepsilon)/(1+\varepsilon)}}\, d\nu_\alpha(z)\bigg)^{1/\delta}.
\end{aligned}$$

Since $2m \geqslant n+\alpha+1$, we have

$$\begin{aligned}
\big|1-\langle w,z\rangle\big|^{(2m+(n+\alpha+m+1)\varepsilon)/(1+\varepsilon)} &\geqslant \big(1-|w|\big)^{(2m-n-\alpha-1+m\varepsilon)/(1+\varepsilon)}\big|1-\langle w,z\rangle\big|^{n+\alpha+1} \\
&\geqslant 2^{-\beta/(1+\varepsilon)}\big(1-|w|^2\big)^{\beta/(1+\varepsilon)}\big|1-\langle w,z\rangle\big|^{n+\alpha+1},
\end{aligned}$$

where $\beta = 2m-n-\alpha-1+m\varepsilon$, and thus

$$\begin{aligned}
&\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{(2m+(n+\alpha+m+1)\varepsilon)/(1+\varepsilon)}}\, d\nu_\alpha(z)\bigg)^{1/\delta} \\
&\quad\leqslant \frac{2^{\beta/(2+\varepsilon)}}{(1-|w|^2)^{\beta/(2+\varepsilon)}}\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{n+\alpha+1}}\, d\nu_\alpha(z)\bigg)^{1/\delta}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big|\big(D^\gamma T_{\bar f} h\big)(w)\big| \\
&\quad\leqslant C\frac{B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)}}{(1-|w|^2)^{(n+\alpha+1)/(2+\varepsilon)}}\,\frac{2^{\beta/(2+\varepsilon)}}{(1-|w|^2)^{\beta/(2+\varepsilon)}}\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{n+\alpha+1}}\, d\nu_\alpha(z)\bigg)^{1/\delta} \\
&\quad= C'\frac{B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)}}{(1-|w|^2)^{m}}\bigg(\int_{\mathbb{B}_n} \frac{|h(z)|^\delta}{|1-\langle w,z\rangle|^{n+\alpha+1}}\, d\nu_\alpha(z)\bigg)^{1/\delta}.
\end{aligned}$$

This proves the stated inequality. □

4.2. *Inner product formula in $A^2_\alpha(\mathbb{B}_n)$*

In this subsection we will establish a formula for the inner product in $A^2_\alpha(\mathbb{B}_n)$ needed to prove our sufficiency condition for boundedness of Toeplitz products. Let F and G be in $A^2_\alpha(\mathbb{B}_n)$. Then

$$\Delta\big[F(z)\overline{G(z)}\,\big] = 4\sum_{j=1}^n D_j F(z)\overline{D_j G(z)},$$

and, by induction,

$$\Delta^m\big[F(z)\overline{G(z)}\,\big] = 4^m \sum_{|\gamma|=m} D^\gamma F(z)\overline{D^\gamma G(z)}, \tag{4.6}$$

where the sum is over all multi-indices γ with $|\gamma| = m$. We have

$$\Delta\big[\big(1-|z|^2\big)^\beta\big] = 4\beta(\beta-1)\big(1-|z|^2\big)^{\beta-2} - 4\beta(\beta+n-1)\big(1-|z|^2\big)^{\beta-1}, \tag{4.7}$$

for all $\beta \geqslant 2$. To obtain an inner product formula suitable for the estimates we have established in the previous section, we will need the following lemma:

Lemma 4.8. *Let $-1 < \alpha < \infty$ and let m be a positive integer. There exist constants $a_1, a_2, \ldots, a_{2m-1}$ and $b_1, b_2, \ldots, b_m$ (depending on m, n and α) such that*

$$\Delta^m\left[\big(1-|z|^2\big)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j\big(1-|z|^2\big)^{\alpha+2m+j}\right]$$
$$= 4^m\frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+1)}\big(1-|z|^2\big)^\alpha - \sum_{k=1}^m b_k\big(1-|z|^2\big)^{\alpha+2m+k-1}. \tag{4.9}$$

Proof. Using (4.7) and induction it is easy to show that for every $-1 < \alpha < \infty$ and positive integer m, there exist scalars $\lambda_{\alpha,0}, \ldots, \lambda_{\alpha,m}$ (depending also on m and n) such that

$$\Delta^m\big[\big(1-|z|^2\big)^{\alpha+2m}\big] = \sum_{k=0}^m \lambda_{\alpha,k}\big(1-|z|^2\big)^{\alpha+k}, \tag{4.10}$$

where $\lambda_{\alpha,0} = 4^m\Gamma(\alpha+2m+1)/\Gamma(\alpha+1)$. Fix a positive integer m. For convenience of notation, define $\lambda_{\alpha,j} = 0$ if $j > m$ and α is arbitrary.

From (4.10) we subtract $\sum_{j=1}^{2m-1}\Delta^m[a_j(1-|z|^2)^{\alpha+2m+j}]$ for scalars $a_1, \ldots, a_{2m-1}$ that will be chosen such that all terms involving powers $(1-|z|^2)^{\alpha+k}$, with $1 \leqslant k \leqslant 2m-1$, drop out. To show that this can be done, we use (4.10) to note that

$$\begin{aligned}\sum_{j=1}^{2m-1}\Delta^m\big[a_j\big(1-|z|^2\big)^{\alpha+2m+j}\big] &= \sum_{j=1}^{2m-1}\sum_{k=0}^{m} a_j\lambda_{\alpha+j,k}\big(1-|z|^2\big)^{\alpha+k+j}\\ &= \sum_{j=1}^{2m-1}\sum_{\ell=j}^{j+m} a_j\lambda_{\alpha+j,\ell-j}\big(1-|z|^2\big)^{\alpha+\ell}\\ &= \sum_{j=1}^{2m-1}\sum_{\ell=j}^{3m-1} a_j\lambda_{\alpha+j,\ell-j}\big(1-|z|^2\big)^{\alpha+\ell},\end{aligned}$$

using that $\lambda_{\alpha+j,\ell-j} = 0$ if $\ell > j+m$. Interchanging the order of summation, we have

$$\sum_{j=1}^{2m-1}\Delta^m\big[a_j\big(1-|z|^2\big)^{\alpha+2m+j}\big]$$
$$= \sum_{j=1}^{2m-1}\sum_{k=j}^{3m-1} a_j\lambda_{\alpha+j,k-j}\big(1-|z|^2\big)^{\alpha+k}$$

$$= \sum_{k=1}^{2m-1} \sum_{j=1}^{k} a_j \lambda_{\alpha+j,k-j} \bigl(1-|z|^2\bigr)^{\alpha+k} + \sum_{k=2m}^{3m-1} \sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j,k-j} \bigl(1-|z|^2\bigr)^{\alpha+k}.$$

Hence

$$\Delta^m \Biggl[\bigl(1-|z|^2\bigr)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j \bigl(1-|z|^2\bigr)^{\alpha+2m+j} \Biggr]$$
$$= \lambda_{\alpha,0} \bigl(1-|z|^2\bigr)^{\alpha} + \sum_{k=1}^{2m-1} \Biggl(\lambda_{\alpha,k} - \sum_{j=1}^{k} a_j \lambda_{\alpha+j,k-j} \Biggr) \bigl(1-|z|^2\bigr)^{\alpha+k}$$
$$+ \sum_{k=2m}^{3m-1} \sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j,k-j} \bigl(1-|z|^2\bigr)^{\alpha+k}.$$

Choose $a_1, \ldots, a_{2m-1}$ such that

$$\lambda_{\alpha,k} = \sum_{j=1}^{k} a_j \lambda_{\alpha+j,k-j},$$

for $k = 1, \ldots, 2m-1$. This can be done since $(\lambda_{\alpha+j,i-1})_{i,j=1}^{2m-1}$ is a lower diagonal matrix with non-zero entries on its diagonal. Putting

$$b_k = -\sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j,2m+k-1-j},$$

for $k = 1, \ldots, m$, Eq. (4.9) follows. □

Let $-1 < \alpha < \infty$ and assume that $F, G \in A^2_\alpha(\mathbb{B}_n)$ are analytic on an open neighborhood of the closed unit ball $\bar{\mathbb{B}}_n$. Now, it is easily seen that

$$\Delta^k \bigl[\bigl(1-|z|^2\bigr)^{2\beta} \bigr] = \Delta^k \biggl[\frac{\partial}{\partial n} \bigl(1-|z|^2\bigr)^{2\beta} \biggr] = 0$$

on S, for $k < \beta$, where $\frac{\partial}{\partial n}$ denotes the normal derivative.

Repeatedly applying Green's formula and using Lemma 4.8, we get

$$\int_{\mathbb{B}_n} \Delta^m [F\bar{G}](z) \Biggl[\bigl(1-|z|^2\bigr)^{2m} - \sum_{j=1}^{2m-1} a_j \bigl(1-|z|^2\bigr)^{2m+j} \Biggr] d\nu_\alpha(z)$$
$$= \int_{\mathbb{B}_n} \Delta^m [F\bar{G}](z) \Biggl[\bigl(1-|z|^2\bigr)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j \bigl(1-|z|^2\bigr)^{\alpha+2m+j} \Biggr] d\nu(z)$$
$$= \int_{\mathbb{B}_n} F(z)\overline{G(z)} \Delta^m \Biggl[\bigl(1-|z|^2\bigr)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j \bigl(1-|z|^2\bigr)^{\alpha+2m+j} \Biggr] d\nu(z)$$
$$= \int_{\mathbb{B}_n} F(z)\overline{G(z)} \Biggl\{ 4^m \frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+1)} \bigl(1-|z|^2\bigr)^{\alpha} - \sum_{j=1}^{m} b_j \bigl(1-|z|^2\bigr)^{\alpha+2m+j-1} \Biggr\} d\nu(z)$$

$$= 4^m \frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+1)} \langle F, G\rangle_\alpha - \sum_{j=1}^{m} b_j \int_{\mathbb{B}_n} F(z)\overline{G(z)}\big(1-|z|^2\big)^{2m+j-1}\,d\nu_\alpha(z).$$

It follows that

$$\begin{aligned}\langle F, G\rangle_\alpha &\\ = \frac{\Gamma(\alpha+1)}{4^m\Gamma(\alpha+2m+1)} &\int_{\mathbb{B}_n} \Delta^m[F\bar{G}](z)\left[\big(1-|z|^2\big)^{2m} - \sum_{j=1}^{2m-1} a_j\big(1-|z|^2\big)^{2m+j}\right] d\nu_\alpha(z)\\ &+ \sum_{j=1}^{m} b'_j \int_{\mathbb{B}_n} F(z)\overline{G(z)}\big(1-|z|^2\big)^{2m+j-1}\,d\nu_\alpha(z).\end{aligned}$$

Combining the above formula with (4.6), we obtain the following formula for the inner product in $A^2_\alpha(\mathbb{B}_n)$:

$$\begin{aligned}\langle F, G\rangle_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)} &\sum_{|\gamma|=m} \int_{\mathbb{B}_n} D^\gamma F(z)\overline{D^\gamma G(z)}\big(1-|z|^2\big)^{2m}\,d\nu_\alpha(z)\\ &+ \sum_{j=1}^{2m-1} a'_j \sum_{|\gamma|=m} \int_{\mathbb{B}_n} D^\gamma F(z)\overline{D^\gamma G(z)}\big(1-|z|^2\big)^{2m+j}\,d\nu_\alpha(z)\\ &+ \sum_{j=1}^{m} b'_j \int_{\mathbb{B}_n} F(z)\overline{G(z)}\big(1-|z|^2\big)^{2m+j-1}\,d\nu_\alpha(z), \qquad (4.11)\end{aligned}$$

for any $m \in \mathbb{N}$. Given $F, G \in A^2_\alpha(\mathbb{B}_n)$, apply the above argument to the dilates F_r and G_r defined by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$, for $|z| < 1/r$. Using that $F_r \to F$ in $A^2_\alpha(\mathbb{B}_n)$ and $D^\gamma F_r \to D^\gamma F$ in $A^2_{\alpha+m}(\mathbb{B}_n)$, for every multi-index γ with $|\gamma| = m$, Eq. (4.11) for general $F, G \in A^2_\alpha(\mathbb{B}_n)$ is the limit of (4.11) for the dilates F_r and G_r as $r \to 1^-$ (by Theorems 2.16 and 2.17 in [13] the terms occurring in the inner product formula are all in $L^2(\mathbb{B}_n, d\mu_\alpha)$).

4.3. Sufficient condition for boundedness

We are now in a position to prove our sufficiency condition for boundedness of Toeplitz products.

Proof of Theorem 4.1. Assume that for $\varepsilon > 0$, M is a positive constant such that

$$B_\alpha\big[|f|^{2+\varepsilon}\big](w)B_\alpha\big[|g|^{2+\varepsilon}\big](w) \leqslant M^{2+\varepsilon},$$

for all $w \in \mathbb{B}_n$. By (4.3) we also have

$$B_\alpha\big[|f|^{2}\big](w)B_\alpha\big[|g|^{2}\big](w) \leqslant M^{2},$$

for all $w \in \mathbb{B}_n$. Let h and k be bounded analytic functions on $\mathbb{B}_n$. It follows from Lemma 4.4 that

$$\big|(T_{\bar{f}}h)(w)(T_{\bar{g}}k)(w)\big| \leqslant \frac{M}{(1-|w|^2)^{n+\alpha+1}}\|h\|_\alpha\|k\|_\alpha,$$

thus

$$\int_{\mathbb{B}_n} \left|(T_{\bar{f}}h)(z)(T_{\bar{g}}k)(z)\right|\left(1-|z|^2\right)^q \, d\nu_\alpha(z) \leqslant M\|h\|_\alpha \|k\|_\alpha,$$

for all $q \geqslant n+\alpha+1$. So if we choose a large m, such that $2m \geqslant n+\alpha+1$, then each of the terms

$$\int_{\mathbb{B}_n} \left|(T_{\bar{f}}h)(z)(T_{\bar{g}}k)(z)\right|\left(1-|z|^2\right)^{2m+j-1} \, d\nu_\alpha(z)$$

is bounded by $M\|h\|_\alpha \|k\|_\alpha$, for $j = 1, \ldots, m$.

Let Q_α be the integral operator on $L^2(\mathbb{B}_n, \nu_\alpha)$ defined by

$$Q_\alpha u(w) = \int_{\mathbb{B}_n} \frac{u(z)}{|1-\langle z, w\rangle|^{n+\alpha+1}} \, d\nu_\alpha(z), \quad w \in \mathbb{B}_n.$$

Using Lemma 4.5 for a multi-index γ with $|\gamma| = m \geqslant (n+\alpha+1)/2$, we have

$$\left|\left(D^\gamma T_{\bar{g}}k\right)(w)\overline{(D^\gamma T_{\bar{f}}h)(w)}\right| \leqslant C^2 \frac{M}{(1-|w|^2)^{2m}} \left(Q_\alpha |h|^\delta(w)\right)^{1/\delta} \left(Q_\alpha |k|^\delta(w)\right)^{1/\delta},$$

for all $w \in \mathbb{B}_n$, where $\delta = (2+\varepsilon)/(1+\varepsilon)$. Since $p = 2/\delta > 1$ and Q_α is L^p-bounded [13, Theorem 2.10] there exists a constant $N > 0$ such that

$$\int_{\mathbb{B}_n} \left(Q_\alpha |h|^\delta(w)\right)^{2/\delta} d\nu_\alpha(w) \leqslant N \int_{\mathbb{B}_n} \left(|h|^\delta(w)\right)^{2/\delta} d\nu_\alpha(w) = N\|h\|_\alpha^2,$$

and, likewise,

$$\int_{\mathbb{B}_n} \left(Q_\alpha |k|^\delta(w)\right)^{2/\delta} d\nu_\alpha(w) \leqslant N\|k\|_\alpha^2.$$

By the Cauchy–Schwarz inequality,

$$\int \left(Q_\alpha |h|^\delta(w)\right)^{1/\delta} \left(Q_\alpha |k|^\delta(w)\right)^{1/\delta} d\nu_\alpha(w) \leqslant N\|h\|_\alpha \|k\|_\alpha.$$

We conclude that

$$\left|\int_{\mathbb{B}_n} D^\gamma T_{\bar{g}}k(z)\overline{D^\gamma T_{\bar{f}}h(z)}\left(1-|z|^2\right)^{2m+j} d\nu_\alpha(z)\right| \leqslant MNC^2\|h\|_\alpha \|k\|_\alpha,$$

for $j = 0, 1, \ldots, 2m-1$. Using inner product formula (4.11) with $F = T_{\bar{g}}k$ and $G = T_{\bar{f}}h$ we conclude that there is a finite constant L such that

$$\left|\langle T_f T_{\bar{g}}k, h\rangle_\alpha\right| \leqslant L\|h\|_\alpha \|k\|_\alpha,$$

for all bounded analytic functions h and k on $\mathbb{B}_n$. Hence the operator $T_f T_{\bar{g}}$ is bounded on $A_\alpha^2(\mathbb{B}_n)$. This completes the proof of Theorem 4.1. □

4.4. Compact Toeplitz products

The following theorem states that the Toeplitz product $T_f T_{\bar{g}}$ is only compact in the trivial case that it is the zero operator.

Theorem 4.12. *Let $-1 < \alpha < \infty$, and f and g be in $A^2_\alpha(\mathbb{B}_n)$. Then $T_f T_{\bar{g}}$ is compact if and only if $f \equiv 0$ or $g \equiv 0$.*

Proof. If $T_f T_{\bar{g}}$ is compact on $A^2_\alpha(\mathbb{B}_n)$, then its Berezin transform vanishes near the unit sphere

$$B_\alpha[T_f T_{\bar{g}}](w) \to 0$$

as $|w| \to 1^-$. We have seen that $B_\alpha[T_f T_{\bar{g}}](w) = f(w)\overline{g(w)}$, so

$$\left|f(w)g(w)\right| = \left|B_\alpha[T_f T_{\bar{g}}](w)\right| \to 0$$

as $|w| \to 1^-$, and it follows from the Maximum Modulus Principle that $fg \equiv 0$.

Acknowledgment

We thank the referee for carefully reading the paper and providing corrections and suggestions for improvements.

References

[1] J. Arazy, M. Engliš, Analytic models for commuting operator tuples on bounded symmetric domains, Trans. Amer. Math. Soc. 355 (2) (2003) 837–864.
[2] M. Engliš, Some problems in operator theory on bounded symmetric domains, Acta Appl. Math. 81 (1–3) (2004) 51–71.
[3] J.-D. Park, Boundedness of Toeplitz products on the Bergman space of the unit ball in $\mathbb{C}^n$, preprint, 2004.
[4] W. Rudin, Function Theory in the Unit Ball in $\mathbb{C}^n$, Springer-Verlag, 1980.
[5] D. Sarason, Exposed points in H^1, I, in: Oper. Theory Adv. Appl., vol. 41, Birkhäuser-Verlag, Basel, 1989, pp. 485–496.
[6] D. Sarason, Exposed points in H^1, II, in: Oper. Theory Adv. Appl., vol. 48, Birkhäuser, Basel, 1990, pp. 333–347.
[7] D. Sarason, Products of Toeplitz operators, in: V.P. Havin, N.K. Nikolski (Eds.), Linear and Complex Analysis Problem Book 3, Part I, in: Lecture Notes in Math., vol. 1573, Springer-Verlag, Berlin, 1994, pp. 318–319.
[8] K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, in: J. Zemánek (Ed.), Linear Operators, in: Banach Center Publ., vol. 38, Polish Acad. Sci., Warsaw, 1997, pp. 361–380.
[9] K. Stroethoff, D. Zheng, Products of Hankel and Toeplitz operators on the Bergman space, J. Funct. Anal. 169 (1999) 289–313.
[10] K. Stroethoff, D. Zheng, Bounded Toeplitz products on the Bergman space of the polydisk, J. Math. Anal. Appl. 278 (2003) 125–135.
[11] K. Stroethoff, D. Zheng, Bounded Toeplitz products on Weighted Bergman spaces, preprint, http://www.math.umt.edu/stroethoff/BTPWBS.pdf, 2004.
[12] D. Zheng, The distribution function inequality and products of Toeplitz operators and Hankel operators, J. Funct. Anal. 138 (1996) 477–501.
[13] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Grad. Texts in Math., vol. 226, Springer-Verlag, New York, 2005.