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Isoperimetric functions of finitely generated nilpotent groups

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Abstract

We show that the isoperimetric function of a finitely generated nilpotent group of class c is bounded above by a polynomial of degree $2c$. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Isoperimetric functions

The isoperimetric function of a finitely presented group G limits the number of defining relators needed to show that a word represents the identity in G. Hence the isoperimetric function is a measure for the complexity of the word problem. Suppose $G=F/R$ where F is a free group freely generated by the finite set $\mathscr{F}$, and R is the normal closure of a finite set of relators $\mathscr{R}\subset F$. Thus $P=\langle\mathscr{F}\mid\mathscr{R}\rangle$ is a finite presentation of G. For short we identify words $w\in F$ with their residue classes $wR\in G$. A word w is equal to 1 in G if and only if w is freely equal to a word of the form

$$\prod_{i=1}^{m} u_i^{-1}r_i^{\varepsilon_i}u_i \quad \text{with } u_i\in F,\ r_i\in\mathscr{R} \text{ and } \varepsilon_i=\pm 1.$$

Let $\Delta_P : R\to\mathbb{N}$ be the so-called *area function* defined by

$$\Delta_P(w)=\min\left\{m\in\mathbb{N}\mid w=\prod_{i=1}^{m}u_i^{-1}r_i^{\varepsilon_i}u_i \text{ for } u_i\in F,\ r_i\in\mathscr{R},\varepsilon_i=\pm 1\right\}$$

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for $w \in R$. We denote by $|w|$ the length of the word w. Associated with Δ_P is the *isoperimetric function* Φ_P *of the finite presentation* P defined by

$$\Phi_P(n) = \max\{\Delta_P(w) \mid w \in R \text{ and } |w| \leq n\}.$$

A partial ordering $\preceq$ on functions on the natural numbers is used to compare isoperimetric functions. For $f, g : \mathbb{N} \to \mathbb{N}$ let $f \preceq g$ if and only if there exists a constant K such that $f(n) \leq Kg(Kn) + Kn$ for all $n \in \mathbb{N}$. Hence, we get an equivalence relation $\cong$ where $f \cong g$ if and only if $f \preceq g$ and $g \preceq f$.

If P and Q are different finite presentations of the same group then $\Phi_P \cong \Phi_Q$, cf. [2]. Any $\mathbb{N} \to \mathbb{N}$ function equivalent to Φ_P is called an *isoperimetric function of* G, denoted by Φ_G.

For any natural number k there exists a finitely presented group whose isoperimetric function is equivalent to n^k [3, 5]. There also exist finitely presented groups whose isoperimetric function is equivalent to n^r, where r is a fraction [4]. In fact, such groups exist for all rationals $r \geq 3$ [6]. A finitely presented group G is said to satisfy a linear, quadratic or exponential isoperimetric inequality if $\Phi_G \preceq n$, n^2 or 2^n, respectively. Automatic groups satisfy a quadratic and asynchronously automatic an exponential isoperimetric inequality [7]. Polycyclic groups satisfy an exponential isoperimetric inequality [12].

An isoperimetric function Φ_G is called superadditive if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $f \cong \Phi_G$ and $f(n) + f(m) \leq f(n+m)$. Non-trivial free products of finitely presented groups have a superadditive isoperimetric function [16]. Sapir conjectures that all finitely presented groups have a superadditive isoperimetric function.

1.2. Nilpotent groups

Let G be a finitely presented nilpotent group. In [12] it is proved that Φ_G is bounded above by a polynomial of degree 2^h, where h is the Hirsch number. In [8, 9] the bound on the degree was improved to $2 \cdot 3^c$, where c is the nilpotency class of G. Our main objective is to improve the bound on the degree to $2c$. It is not known if Φ_G is always equivalent to a polynomial. Likewise it is not known if Φ_G is superadditive in general. However, Gersten conjectures that $\Phi_G \preceq n^{c+1}$ for finitely generated nilpotent groups and Gromov asserts in [15, $5.A_5'$], without proof, that Gersten's conjecture holds.

Let G be a finitely generated free nilpotent group. If G is of class 2 and rank 2, i.e. the three-dimensional Heisenberg group, then $\Phi_G \cong n^3$ [13, 7]. In [3, 14] it is shown that $n^{c+1} \preceq \Phi_G$, where c is the nilpotency class of G. Pittet shows in [17], based on [15, $5.A_2'$], that $\Phi_G \preceq n^{c+1}$. Hence we have $\Phi_G \cong n^{c+1}$ for finitely generated free nilpotent groups. Heisenberg groups of dimension five or higher satisfy a quadratic isoperimetric inequality [1]. This is in contrast to the cubic isoperimetric function in the three-dimensional case mentioned above.

1.3. Rewriting process

Let G be a finitely presented group, H a finitely presented subgroup of G and w a word of length n equal to 1 in G. Suppose that we already know Φ_H or an upper bound thereof. To compute an upper bound for Φ_G we use the following approach. We rewrite w to a word $\rho(w)$ in the generators of H. We then compute an upper bound $\Phi_\rho(n)$ for the number of relators needed to rewrite w to $\rho(w)$ and an upper bound $\delta_\rho(n)$ for the length of $\rho(w)$. Since $\rho(w) =_G w$ the word $\rho(w)$ is equal to 1 in H as well. Thus the area of $\rho(w)$ is bounded above by $\Phi_H(\delta_\rho(n))$. Therefore, the area of w is bounded above by $\Phi_H(\delta_\rho(n))$ plus the number of relators needed to rewrite w to $\rho(w)$. Hence $\Phi_H(\delta_\rho(n)) + \Phi_\rho(n)$ is an upper bound for the isoperimetric function of G.

More precisely, let $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ be a finite presentation of the group G, F the free group freely generated by $\mathscr{F}$ and H a finitely generated subgroup of G. We may assume, without loss of generality, that H is generated by a subset $\mathscr{E} \subseteq \mathscr{F}$. Let E be the subgroup of F generated by $\mathscr{E}$. A *rewriting process ρ from G to H relative to P, $\mathscr{E}$* is a partial map $F \xrightarrow{\rho} E$ defined on all words $w \in H$ such that $\rho(w) =_G w$ and $\rho(1) = 1$. In general, ρ is not a homomorphism. Define $\delta_\rho(n)$ by the maximal length of $\rho(w)$ for all $w \in H$ with $|w| \le n$. We call δ_ρ the *distortion of the rewriting process ρ*. In analogy to Φ_P let $\Phi_\rho(n) = \max\{\Delta_P(w^{-1}\rho(w)) \mid w \in H \text{ and } |w| \le n\}$. We call Φ_ρ the *isoperimetric function of the rewriting process ρ*.

If a rewriting process ρ minimises the word length, i.e. $|\rho(w)| = \min\{|v| \text{ for } v \in E \text{ and } v =_G w\}$ for all $w \in H$, then δ_ρ is called the *distortion of H in G*. Analogously, if ρ minimises the area, i.e. $\Delta_P(w^{-1}\rho(w)) = \min\{\Delta_P(w^{-1}v) \text{ for } v \in E \text{ and } v =_G w\}$ for all $w \in H$, then Φ_ρ is called the *generalised isoperimetric function of H in G*, cf. [10].

1.4. Main result

Let G be a finitely presented nilpotent group and H a subgroup of G. The ith *term of the lower central series of a group G* is denoted by $\gamma_i G$. In Sections 2 and 3 we construct a rewriting process ρ from G to $H\gamma_{i+1}G$ relative to a particular finite presentation of G and establish upper bounds on Φ_ρ and δ_ρ. In Section 4 we prove our main result.

Theorem 2. *Let G be a finitely presented nilpotent group of class c and H a subgroup of G. There exists a rewriting process ρ from G to H, relative to some finite presentation of G and some finite set of generators of H, such that*

$$\delta_\rho(n) \preceq n^c \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2c}.$$

By Theorem 2 the distortion and the generalised isoperimetric function of a subgroup of a finitely generated nilpotent group of class c is bounded above by a polynomial of degree c and $2c$, respectively. Hence we have:

Theorem 3. *Let G be a finitely presented nilpotent group of class c. Then*

$$\Phi_G(n) \preceq n^{2c}.$$

In a subsequent paper the author will use rewriting processes to compute isoperimetric functions for amalgamated products of nilpotent groups.

2. Collection to the left

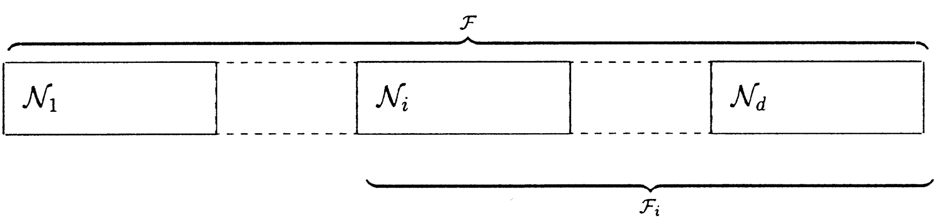
For convenience we introduce the following convention: For a finite presentation $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ we denote by F the free group freely generated by $\mathscr{F}$ and by R the normal closure of $\mathscr{R}$ in F. Analogously, if $\mathscr{E}$ is a subset of $\mathscr{F}$ we denote by E the subgroup of F generated by $\mathscr{E}$. If $\mathscr{U}$ is a set of words we denote by $\mathscr{U}^{\pm 1}$ the set $\{u, u^{-1} \mid u \in \mathscr{U}\}$. For a word $w \in F$ we denote the number of letters in $\mathscr{E}$ by $|w|_{\mathscr{E}}$ and call it the *relative length of w with respect to $\mathscr{E}$*. For words $v, w \in F$ we denote by $[v, w]$ the *commutator* $v^{-1}w^{-1}vw$. Let $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ be a finite presentation for a group G and w, v words in the generators $\mathscr{F}$. By $w = v$ we denote equality in the word-monoid generated by $\mathscr{F}$, by $w =_F v$ equality in the free group F and by $w =_G v$ equality in G.

Let $\mathscr{E} = \{e_1, \ldots, e_k\}$ be a subset of $\mathscr{F}$ and order the generators in $\mathscr{E}$ by their subscripts, i.e. $e_i \leq e_j$ if and only if $i \leq j$. A word $w \in F$ is *collected to the left with respect to $\mathscr{E}$* if and only if $w = e_1^{q_1} e_2^{q_2} \cdots e_k^{q_k} v$ where v is a word in the generators $\mathscr{F} \backslash \mathscr{E}$ and $q_i \in \mathbb{Z}$ for $1 \leq i \leq k$.

Let G be a finitely presented nilpotent group and let

$$G = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_d \supseteq N_{d+1} = \{1\}$$

be a central series of G such that $[N_i, N_j] \subseteq N_{i+j}$ for all i and j, e.g. the lower central series of G. We may assume, without loss of generality, that G has a finite presentation $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ of the following form (see the figure below): Let $\mathscr{F}$ be the disjoint union of $\mathscr{N}_i$ for $i = 1, \ldots, d$ such that $\mathscr{N}_i$ generates N_i and let $\mathscr{F}_i = \bigcup_{j=i}^{d} \mathscr{N}_j$.



Given a word $w \in F_i$ we construct in Lemma 1 a word $\eta(w) \in F_i$ such that $\eta(w) =_G w$ and $\eta(w)$ is collected to the left with respect to $\mathscr{N}_i$. To construct $\eta(w)$ we repeatedly move the smallest, leftmost generator $e \in \mathscr{N}_i$ in w to the left by inserting commutators of the form $[f, e]$ with $f \in \mathscr{N}_j$ for some j. Thus $[f, e] \in N_{i+j}$. Since N_{i+j} is generated by $\mathscr{N}_{i+j}$ we write $[f, e]$ as a word in the generators $\mathscr{N}_{i+j}$. Hence $|\eta(w)|_{\mathscr{N}_i} \leq |w|_{\mathscr{N}_i}$. For $|\eta(w)|_{\mathscr{N}_j}$ with $j > i$ and $\Delta_P(w^{-1}\eta(w))$ we establish upper bounds in terms of $|w|_{\mathscr{N}_k}$ for

$k=i,\dots,d$. It will be crucial for the following Section 3 to express these upper bounds in terms of $|w|_{\mathcal{N}_k}$ and not in terms of the full word length $|w|$.

Lemma 1. *There exists a map $\eta : F_i \to F_i$, $w \mapsto \eta(w)$ such that $\eta(w) =_G w$ and $\eta(w)$ is collected to the left with respect to $\mathcal{N}_i$. There exist positive integers A and D such that for $j=i,\dots,d$*

$$|\eta(w)|_{\mathcal{N}_j} \le \sum_{k=0}^{\lfloor (j-1)/i \rfloor} D^k\, n_i^k\, n_{j-ik}, \tag{1}$$

$$\Delta_P(w^{-1}\eta(w)) \le A \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} D^k\, n_i^{k+1}\, n_{j-ik}, \tag{2}$$

where $n_k = |w|_{\mathcal{N}_k}$ for $k=1,\dots,d$ and $n_k=0$ otherwise.

Proof. Let $w \in F_i$ and $n_j = |w|_{\mathcal{N}_j}$. We define $\eta(w)$ by induction on n_i.

For $n_i = 0$ the word w contains no letter in $\mathcal{N}_i$. Thus, w is already collected to the left with respect to $\mathcal{N}_i$. We define $\eta(w)$ by w.

Suppose $n_i > 0$ and we have defined η for all words with less than n_i letters in $\mathcal{N}_i$. Let $\mathcal{N}_i = \{e_1, e_2, \dots\}$. We may assume, without loss of generality, that $w = f_1 \cdots f_r e_1 f_{r+1} \cdots f_s$ with $f_l \in \mathcal{F}_i^{\pm 1}$ for $l = 1,\dots,s$ such that e_1 is the leftmost generator in $\mathcal{N}_i^{\pm 1}$. For $f \in \mathcal{N}_j^{\pm 1}$, $e \in \mathcal{N}_i^{\pm 1}$ there exists a word $u_{f,e} \in N_{i+j}$ such that $u_{f,e} =_G [f,e]$. With

$$\tilde{w} = f_1 u_{f_1,e_1} \cdots f_r u_{f_r,e_1} f_{r+1} f_{r+2} \cdots f_s \tag{3}$$

we get $w =_G e_1 \tilde{w}$. By $|\tilde{w}|_{\mathcal{N}_i} = n_i - 1$ and the induction hypothesis $e_1 \eta(\tilde{w})$ is collected to the left with respect to $\mathcal{N}_i$. We define $\eta(w)$ by $e_1 \eta(\tilde{w})$.

Let $D = \max\{|u_{f,e}|$ for $f \in \mathcal{F}_i^{\pm 1}$ and $e \in \mathcal{N}_i^{\pm 1}\}$. We prove inequality (1) by induction on n_i.

For $n_i = 0$ we have $\eta(w) = w$. Hence inequality (1) holds.

Suppose $n_i > 0$ and (1) holds for all words in F_i with less than n_i letters in $\mathcal{N}_i$. Suppose $|w|_{\mathcal{N}_i} = n_i$. Since $u_{f,e_1} \in N_j$ for $f \in \mathcal{N}_{j-i}$ we have $|\tilde{w}|_{\mathcal{N}_j} \le n_j + D n_{j-i}$ by (3). By $|\tilde{w}|_{\mathcal{N}_i} = n_i - 1$ and the induction hypothesis we get for $j > i$

$$\begin{aligned}|\eta(\tilde{w})|_{\mathcal{N}_j} &\le \sum_{k=0}^{\lfloor (j-1)/i \rfloor} D^k\,(n_i-1)^k\,(n_{j-ik} + D n_{j-ik-i}) \\ &= n_j + \sum_{k=1}^{\lfloor (j-1)/i \rfloor} D^k\,(n_i-1)^k\, n_{j-ik} + \sum_{k=0}^{\lfloor (j-1)/i \rfloor} D^{k+1}\,(n_i-1)^k\, n_{j-i(k+1)}.\end{aligned}$$

Since $n_{j-i(k+1)} = 0$ for $k = \lfloor (j-1)/i \rfloor$ we have

$$|\eta(\tilde{w})|_{\mathcal{N}_j} \le n_j + \sum_{k=1}^{\lfloor (j-1)/i \rfloor} D^k\,(n_i-1)^{k-1}\, n_i\, n_{j-ik} \le \sum_{k=0}^{\lfloor (j-1)/i \rfloor} D^k n_i^k\, n_{j-ik}.$$

By $|\eta(w)|_{\mathcal{N}_i} \le n_i$ and $|\eta(w)|_{\mathcal{N}_j} = |\eta(\tilde{w})|_{\mathcal{N}_j}$ for $j > i$ inequality (1) follows.

Let $A = \max\{\Delta_P([f,e]^{-1}u_{f,e})$ for $f \in \mathscr{F}_i^{\pm 1}$ and $e \in \mathscr{N}_i^{\pm 1}\}$. We prove inequality (2) by induction on n_i.

For $n_i = 0$ we have $\eta(w) = w$. Hence inequality (2) holds.

Suppose $n_i > 0$ and (2) holds for all words in F_i with less than n_i letters in $\mathscr{N}_i$. Suppose $|w|_{\mathscr{N}_i} = n_i$. By (3) we have $\eta(w) = e_1\eta(\tilde{w})$ and $|\tilde{w}|_{\mathscr{N}_j} \le n_j + Dn_{j-i}$. Therefore, we get by the induction hypothesis

$$\begin{aligned}\Delta_P(w^{-1}\eta(w)) &\le \Delta_P(w^{-1}e_1\tilde{w}) + \Delta_P(\tilde{w}^{-1}\eta(\tilde{w}))\\ &\le \sum_{l=1}^{r} \Delta_P([f_l,e_1]^{-1}u_{f_l,e_1})\\ &\quad + A\sum_{j=i}^{d}\sum_{k=0}^{\lfloor (j-1)/i\rfloor} D^k\,(n_i-1)^{k+1}\,(n_{j-ik}+Dn_{j-ik-i}).\end{aligned}$$

Since $\sum_{l=1}^{r} \Delta_P([f_l,e_1]^{-1}u_{f_l,e_1}) \le A\cdot r \le A\sum_{j=i}^{d} n_j$ we get

$$\begin{aligned}\Delta_P(w^{-1}\eta(w)) \le A\sum_{j=i}^{d}\Bigg(& n_j + (n_i-1)n_j + \sum_{k=1}^{\lfloor (j-1)/i\rfloor} D^k\,(n_i-1)^{k+1}\,n_{j-ik} +\\ & \sum_{k=0}^{\lfloor (j-1)/i\rfloor} D^{k+1}\,(n_i-1)^{k+1}\,n_{j-i(k+1)}\Bigg).\end{aligned}$$

By $n_{j-i(k+1)} = 0$ for $k = \lfloor (j-1)/i\rfloor$ we have

$$\begin{aligned}\Delta_P(w^{-1}\eta(w)) &\le A\sum_{j=i}^{d}\left(n_i\,n_j + \sum_{k=1}^{\lfloor (j-1)/i\rfloor} D^k\,(n_i-1)^k n_i\,n_{j-ik}\right)\\ &\le A\sum_{j=i}^{d}\sum_{k=0}^{\lfloor (j-1)/i\rfloor} D^k\,n_i^{k+1}\,n_{j-ik}.\end{aligned}$$

Thus inequality (2) holds. □

3. Rewriting along the lower central series

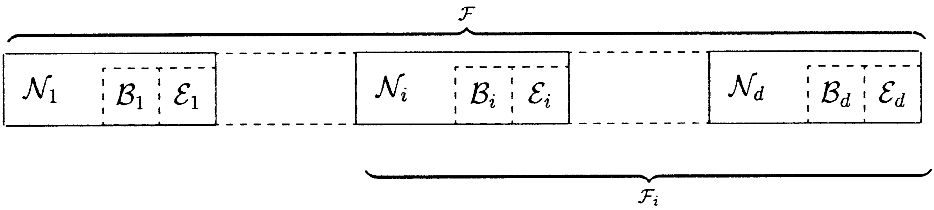
Let G be a finitely presented nilpotent group, H a subgroup of G and let

$$G = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_d \supseteq N_{d+1} = \{1\}$$

be a central series of G such that $[N_i,N_j] \subseteq N_{i+j}$ for all i and j, e.g. the lower central series. Using a particular finite presentation of G, we construct in Lemma 2 for each $i = 1,\ldots,d$ a rewriting process σ_i from N_i to $(H\cap N_i)N_{i+1}$. Using σ_i we then construct in Proposition 1 for each i a rewriting process ρ_i from G to HN_{i+1}. By setting $i = d$ we get in the following section our main result, i.e. a rewriting process from G to H.

Recall that any finitely generated abelian group A can be represented as a direct product $A_1 \times \cdots \times A_k$ of cyclic groups A_i. The set $\{a_1,\dots,a_k\}$ of generators a_i of A_i is called *a basis of A*. We note that given a basis $\{a_1,\dots,a_k\}$ of A then $a_1^{j_1}\cdots a_k^{j_k} =_A 1$ implies $a_1^{j_1} =_A \cdots =_A a_k^{j_k} =_A 1$, i.e. if a_i is of infinite order then $j_i = 0$ and if a_i is of finite order then the order of a_i is a divisor of j_i, cf. [11].

We may assume, without loss of generality, that G has a finite presentation $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ of the following form (see the figure below): $\mathscr{F}$ is the disjoint union of $\mathscr{N}_i$ for $1 \le i \le d$ such that for each i



1. $\mathscr{N}_i$ generates N_i,
2. $P_i = \langle \mathscr{F}_i \mid \mathscr{R}_i \rangle$ with $\mathscr{F}_i = \bigcup_{j=i}^{d} \mathscr{N}_j$ and $\mathscr{R}_i \subseteq \mathscr{R}$ is a finite presentation of N_i,
3. $\mathscr{B}_i \subseteq \mathscr{N}_i$ such that $\mathscr{B}_i$ is a basis of the abelian factor group N_i/N_{i+1} with respect to its presentation $\langle \mathscr{F}_i \mid \mathscr{R}_i \cup \mathscr{F}_{i+1} \rangle$,
4. $\mathscr{E}_i \subseteq \mathscr{N}_i$ such that $\mathscr{E}_i$ generates $(H \cap N_i)N_{i+1}/N_{i+1}$ with respect to the presentation $\langle \mathscr{F}_i \mid \mathscr{R}_i \cup \mathscr{F}_{i+1} \rangle$ for N_i/N_{i+1},
5. $\mathscr{N}_i \backslash (\mathscr{B}_i \cup \mathscr{E}_i)$ generates N_{i+1}.

We illustrate the construction of the rewriting process σ_i from N_i to $(H \cap N_i)N_{i+1}$ for the case $i = 1$: Let $w \in F$ represent an element in HN_2. Since $\mathscr{N}_1 \backslash (\mathscr{B}_1 \cup \mathscr{E}_1)$ generates N_2 we substitute the letters of w in $\mathscr{N}_1 \backslash (\mathscr{B}_1 \cup \mathscr{E}_1)$ by suitable words in N_2. Thus we may assume, without loss of generality, that w is a word in the generators $\mathscr{B}_1 \cup \mathscr{E}_1 \cup \mathscr{F}_2$. Using Lemma 1 we collect w to the left with respect to $\mathscr{B}_1 \cup \mathscr{E}_1$. Thereby we get a word of the form uv with $uv =_G w$, u a word in the generators $\mathscr{B}_1 \cup \mathscr{E}_1$ and v a word in F_2. Since $w \in HN_2$ and $v \in N_2$ we have $u \in HN_2$. Because $\mathscr{E}_1$ generates HN_2/N_2 there exists a word $h \in E_1$ such that h is equal to u in HN_2/N_2. By $h^{-1}u \in N_2$ we then construct a word $\tilde{v} \in F_2$ such that $\tilde{v} =_{N_2} h^{-1}u$. Thus, we get

$$w =_G uv =_G hh^{-1}uv =_G h\tilde{v}v$$

and define $\sigma_1(w)$ by $h\tilde{v}v$.

Lemma 2. *There exists for each $i = 1,\dots,d$ a rewriting process σ_i from N_i to $(H \cap N_i)N_{i+1}$ relative to P_i, $(\mathscr{E}_i \cup \mathscr{F}_{i+1})$ and positive integers L_i such that for $j = i,\dots,d$*

$$|\sigma_i(w)|_{\mathscr{N}_j} \le L_i \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^k m_{j-ik},$$

$$\Delta_P(w^{-1}\sigma_i(w)) \le L_i \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik}$$

where $w \in F_i$ represents an element in $(H \cap N_i)N_{i+1}$, $m_j = \sum_{k=i}^{j} |w|_{\mathcal{N}_k}$ for $j = i, \ldots, d$ and $m_j = 0$ otherwise. Moreover, $\sigma_i(w)$ is of the form hv with $h \in E_i$ and $v \in F_{i+1}$.

Proof. Let $w \in F_i$ represent an element in $(H \cap N_i)N_{i+1}$ and $m_j = \sum_{k=i}^{j} |w|_{\mathcal{N}_k}$. We substitute in w each generator in $\mathcal{N}_i \backslash (\mathcal{B}_i \cup \mathcal{E}_i)$ by a suitable word in N_{i+1}. Thus, we get a word $\tilde{w}$ in the generators $\mathcal{B}_i \cup \mathcal{E}_i \cup \mathcal{F}_{i+1}$ with $\tilde{w} =_G w$. Since $m_{i+1} = |w|_{\mathcal{N}_i} + |w|_{\mathcal{N}_{i+1}}$ there exists a positive integer K_1 such that

$$|\tilde{w}|_{\mathcal{N}_j} \le K_1 m_j \quad \text{and} \quad \Delta_P(w^{-1}\tilde{w}) \le K_1 m_i. \tag{4}$$

We note that K_1 as well as the constants $K_2, \ldots, K_6$, which we introduce below, depend on i but not on w. By Lemma 1 and (4) there exists a word u_1 in the generators $\mathcal{B}_i \cup \mathcal{E}_i$, a word $v_1 \in F_{i+1}$ and a positive integer K_2 such that $u_1 v_1 =_G \tilde{w} =_G w$ and

$$|u_1 v_1|_{\mathcal{N}_j} \le K_2 \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^k m_{j-ik}, \tag{5}$$

$$\Delta_P(\tilde{w}^{-1} u_1 v_1) \le K_2 \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik}. \tag{6}$$

Because N_i/N_{i+1} is a finitely generated abelian group, its subgroup $(H \cap N_i)N_{i+1}/N_{i+1}$ is linearly distorted. Hence there exists a word $h \in E_i$ and a positive integer K_3 such that

$$h =_{N_i/N_{i+1}} u_1 \quad \text{and} \quad |h| \le K_3 m_i \tag{7}$$

by $u_1 v_1 \in H \cap N_i$, $v_1 \in N_{i+1}$ and $|u_1| \le K_2 m_i$. In $h^{-1}u_1$ we substitute each generator in $\mathcal{E}_i$ by a corresponding word in the generators $\mathcal{B}_i \cup \mathcal{N}_{i+1}$. By (5)–(7) we get therefore a word u_2 in the generators $\mathcal{B}_i \cup \mathcal{N}_{i+1}$ such that

$$u_2 =_G h^{-1} u_1 \in N_{i+1}, \qquad |u_2| \le K_4 m_i \quad \text{and} \quad \Delta_P(u_1^{-1} h u_2) \le K_4 m_i \tag{8}$$

for a suitable positive integer K_4. By Lemma 1 there exist words $u_3 \in B_i$, $v_3 \in F_{i+1}$ such that $u_3 v_3 =_G u_2$, u_3 is collected to the left with respect to $\mathcal{B}_i$ and

$$|u_3 v_3|_{\mathcal{N}_j} \le K_5 \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^k m_{j-ik}, \tag{9}$$

$$\Delta_P(u_2^{-1} u_3 v_3) \le K_5 \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik} \tag{10}$$

for a suitable positive integer K_5.

We now have

$$w =_G h h^{-1} u_1 v_1 =_G h u_2 v_1 =_G h u_3 v_3 v_1$$

with $h \in E_i$, $u_3 \in B_i$ collected to the left with respect to $\mathscr{B}_i$ and $v_3 v_1 \in F_{i+1}$. Note that $u_3 \in N_{i+1}$ since

$$u_3 =_{N_i/N_{i+1}} u_2 =_{N_i/N_{i+1}} 1.$$

Hence the remaining task is to rewrite u_3 to a word in F_{i+1}. Let $\mathscr{B}_i = \{b_1, \ldots, b_r\}$. Since u_3 is collected to the left, u_3 is of the form $b_1^{m_1} \cdots b_r^{m_r}$. Because $\mathscr{B}_i$ is a basis for N_i/N_{i+1} and $u_3 =_{N_i/N_{i+1}} 1$ we have $b_l^{m_l} =_{N_i/N_{i+1}} 1$ for $l = 1, \ldots, r$. Thus m_l is either equal to 0, if b_l is of infinite order in N_i/N_{i+1}, or a multiple of the order of b_l if b_l is of finite order. Hence there exists a word $v_4 \in N_{i+1}$ and by inequalities (9) and (10) a positive integer K_6 such that

$$u_3 =_G v_4, \qquad |v_4| \le K_6 m_i \quad \text{and} \quad \Delta_P(u_3^{-1} v_4) \le K_6 m_i \tag{11}$$

Thus, we get $w =_G h u_3 v_3 v_1 =_G h v_4 v_3 v_1$ with $h \in E_i$ and $v_4 v_3 v_1 \in F_{i+1}$. We define $\sigma_i(w)$ by $h v_4 v_3 v_1$.

Since $K_1, \ldots, K_6$ depend on i but not on w, $L_i = K_1 + \cdots + K_6$ also depends on i but not on w. We get by (5), (7), (9) and (11)

$$|\sigma_i(w)|_{\mathscr{N}_j} \le |h|_{\mathscr{N}_j} + |v_4|_{\mathscr{N}_j} + |v_3|_{\mathscr{N}_j} + |v_1|_{\mathscr{N}_j} \le L_i \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^k m_{j-ik}.$$

By (4), (6) and (8) we get

$$\begin{aligned}\Delta_P(w^{-1}\sigma_i(w)) &\le \Delta_P(w^{-1}\tilde{w}) + \Delta_P(\tilde{w}^{-1}u_1 v_1) + \Delta_P(v_1^{-1}u_1^{-1}h v_4 v_3 v_1)\\ &\le K_1 m_i + K_2 \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik}\\ &\quad + \Delta_P(u_1^{-1} h u_2) + \Delta_P(u_2^{-1} v_4 v_3)\\ &\le (K_1 + K_2 + K_4) \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik}\\ &\quad + \Delta_P(u_2^{-1} u_3 v_3) + \Delta_P(v_3^{-1} u_3^{-1} v_4 v_3).\end{aligned}$$

Inequalities (10) and (11) now yield

$$\Delta_P(w^{-1}\sigma_i(w)) \le L_i \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} m_i^{k+1} m_{j-ik}. \qquad \square$$

In Proposition 1 we construct a rewriting process from G to HN_{i+1} using Lemma 2 and induction on i.

Let m_j for $j = 1, \ldots, d$ be non-negative integers. By

$$\sum_{\sum_{r=1}^{d} r p_r \le j} m_1^{p_1} \cdots m_d^{p_d}$$

we denote the finite sum of $m_1^{p_1}\cdots m_d^{p_d}$ over all d-tuples $(p_1,\ldots,p_d)$ of non-negative integers p_r such that $\sum_{r=1}^d rp_r \le j$.

Proposition 1. *There exists a rewriting process* ρ_i *for* $i=1,\ldots,d$ *from* G *to* HN_{i+1} *relative to* P, $(\bigcup_{j=1}^i \mathscr{E}_j \cup \mathscr{F}_{i+1})$ *and a positive integer* K_i *such that for* $j=1,\ldots,d$

$$|\rho_i(w)|_{\mathscr{N}_j} \le K_i \sum_{\sum_{r=1}^d rp_r \le j} m_1^{p_1}\cdots m_d^{p_d} \tag{12}$$

and

$$\varDelta_P(w^{-1}\rho_i(w)) \le K_i \sum_{\sum_{r=1}^d rp_r \le d+i} m_1^{p_1}\cdots m_d^{p_d} \tag{13}$$

with $w\in HN_{i+1}$, $m_j=\sum_{k=1}^j |w|_{\mathscr{N}_k}$ *and* $\mathscr{F}_{d+1}=\emptyset$. *Moreover*, $\rho_i(w)=hv$ *with* h *a word in the generators* $\bigcup_{j=1}^i \mathscr{E}_j$ *and* $v\in F_{i+1}$.

Proof. We proceed by induction on i. The case $i=1$ is implied by Lemma 2.

Suppose $i>1$ and there exists a rewriting process ρ_{i-1} from G to HN_i relative to P, $\bigcup_{j=1}^{i-1}\mathscr{E}_j\cup\mathscr{F}_i$ such that (12) and (13) hold. Let $w\in F$ represent an element in HN_{i+1}, $m_j=\sum_{k=1}^j |w|_{\mathscr{N}_k}$, $\rho_{i-1}(w)=hv$ where h is a word in the generators $\bigcup_{j=1}^{i-1}\mathscr{E}_j$ and $v\in F_i$. Hence $v\in HN_{i+1}\cap N_i$. By Lemma 2 there exists a rewriting process σ_i from N_i to $(N_i\cap H)N_{i+1}$ relative to P_i, $(\mathscr{E}_i\cup\mathscr{F}_{i+1})$ and a positive integer L_i such that $\sigma_i(v)=_G v$,

$$|\sigma_i(v)|_{\mathscr{N}_j} \le L_i \sum_{k=0}^{\lfloor (j-1)/i\rfloor} \tilde{m}_i^k \tilde{m}_{j-ik}, \tag{14}$$

$$\varDelta_P(v^{-1}\sigma_i(v)) \le L_i \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i\rfloor} \tilde{m}_i^{k+1}\tilde{m}_{j-ik} \tag{15}$$

with $\tilde{m}_j=\sum_{k=i}^j |v|_{\mathscr{N}_k}$ for $j=i,\ldots,d$ and $\tilde{m}_j=0$ otherwise. We define $\rho_i(w)$ by $h\sigma_i(v)$, since $\sigma_i(v)$ is of the form $\tilde{h}\tilde{v}$ with $\tilde{h}\in E_i$ and $\tilde{v}\in F_{i+1}$.

By the induction hypothesis we have

$$\tilde{m}_j \le \sum_{k=1}^{j} |\rho_{i-1}(w)|_{\mathscr{N}_k} \le jK_{i-1} \sum_{\sum_{r=1}^d rp_r \le j} m_1^{p_1}\cdots m_d^{p_d}. \tag{16}$$

Substituting $\tilde{m}$ in (14) by (16) yields

$$\begin{aligned}|\rho_i(w)|_{\mathscr{N}_j} &\le |h|_{\mathscr{N}_j} + |\sigma_i(v)|_{\mathscr{N}_j}\\ &\le |h|_{\mathscr{N}_j} + L_i \sum_{k=0}^{\lfloor (j-1)/i\rfloor}\left(\left(iK_{i-1}\sum_{\sum_{r=1}^d rp_r\le i} m_1^{p_1}\cdots m_d^{p_d}\right)^k\right.\\ &\qquad\left.\cdot (j-ik)K_{i-1}\left(\sum_{\sum_{s=1}^j sq_s \le j-ik} m_1^{q_1}\cdots m_d^{q_d}\right)\right).\end{aligned}$$

Thus, we get for suitable positive integers D_1 and D_2, which depend on i but not on w,

$$
\begin{aligned}
&|\rho_i(w)|_{\mathcal{N}_j}\\
&\le |h|_{\mathcal{N}_j} + D_1 \sum_{k=0}^{\lfloor (j-1)/i \rfloor} \left(\sum_{\sum_{r=1}^{d} r p_r \le ik} m_1^{p_1} \cdots m_d^{p_d} \right) \left(\sum_{\sum_{s=1}^{j} s q_s \le j-ik} m_1^{q_1} \cdots m_d^{q_d} \right)\\
&\le |h|_{\mathcal{N}_j} + D_2 \sum_{k=0}^{\lfloor (j-1)/i \rfloor} \sum_{\sum_{r=1}^{j} r p_r \le j} m_1^{p_1} \cdots m_d^{p_d}\\
&\le (K_{i-1} + D_2 j) \sum_{\sum_{r=1}^{j} r p_r \le j} m_1^{p_1} \cdots m_d^{p_d}.
\end{aligned}
$$

With $K_i = K_{i-1} + D_2 j$ inequality (12) holds.

Substituting $\tilde{m}$ in (15) by (16) yields

$$
\begin{aligned}
\Delta_P(w^{-1}\rho_i(w)) &\le \Delta_P(w^{-1}hv) + \Delta_P(v^{-1}\sigma_i(v))\\
&\le \Delta_P(w^{-1}hv)\\
&\quad + L_i \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} \left(\left(iK_{i-1} \sum_{\sum_{r=1}^{d} r p_r \le i} m_1^{p_1} \cdots m_d^{p_d} \right)^{k+1} \right.\\
&\quad \left. \cdot \left((j-ik) K_{i-1} \sum_{\sum_{s=1}^{d} s q_s \le j-ik} m_1^{q_1} \cdots m_d^{q_d} \right) \right).
\end{aligned}
$$

Thus, we get for suitable positive integers A_1 and A_2, which depend on i but not on w,

$$
\begin{aligned}
&\Delta_P(w^{-1}\rho_i(w))\\
&\le \Delta_P(w^{-1}hv)\\
&\quad + A_1 \sum_{j=i}^{d} \sum_{k=0}^{\lfloor (j-1)/i \rfloor} \left(\left(\sum_{\sum_{r=1}^{d} r p_r \le i(k+1)} m_1^{p_1} \cdots m_d^{p_d} \right) \left(\sum_{\sum_{s=1}^{d} s q_s \le j-ik} m_1^{q_1} \cdots m_d^{q_d} \right) \right)\\
&\le \Delta_P(w^{-1}hv) + A_2 \sum_{j=i}^{d} j \sum_{\sum_{r=1}^{d} r p_r \le j+i} m_1^{p_1} \cdots m_d^{p_d}.
\end{aligned}
$$

Together with the induction hypothesis we get

$$
\Delta_P(w^{-1}\rho_i(w)) \le (K_{i-1} + A_2 d^2) \sum_{\sum_{r=1}^{d} r p_r \le d+i} m_1^{p_1} \cdots m_d^{p_d}.
$$

We may assume, without loss of generality, that $K_{i-1} + A_2 d^2 \le K_i$. Thus inequality (13) holds. □

Theorem 1. *Let G be a finitely presented nilpotent group, H a subgroup of G and let*

$$G = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_d \supseteq N_{d+1} = \{1\}$$

be a central series of G such that $[N_r, N_s] \subseteq N_{r+s}$ for all positive integers r and s.
- *There exists a finite presentation $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ for G such that $\mathscr{F}$ is the disjoint union of $\mathscr{N}_j$ for $j = 1, \ldots, d$ and $\mathscr{N}_j$ generates N_j. Each $\mathscr{N}_j$ contains a subset $\mathscr{E}_j$ which generates $H \cap N_j$.*
- *Let $\mathscr{E} = \bigcup_{j=1}^{d} \mathscr{E}_j$. Thus $\mathscr{E}$ generates H. There exists a rewriting process ρ from G to H relative to P, $\mathscr{E}$ and a positive integer K such that for $j = 1, \ldots, d$*

$$|\rho(w)|_{\mathscr{N}_j} \leq K \sum_{\sum_{r=1}^{d} r p_r \leq j} n_1^{p_1} \cdots n_d^{p_d} \tag{17}$$

and

$$\Delta_P(w^{-1}\rho(w)) \leq K \sum_{\sum_{r=1}^{d} r p_r \leq 2d} n_1^{p_1} \cdots n_d^{p_d} \tag{18}$$

with $w \in H$ and $n_j = \sum_{k=1}^{j} |w|_{\mathscr{N}_k}$.

Proof. By Proposition 1 with $i = d$. $\square$

4. Main result

Theorem 2. *Let G be a finitely presented nilpotent group of class c and H a subgroup of G. There exists a rewriting process ρ from G to H, relative to some finite presentation of G and finite set of generators of H, such that*

$$\delta_\rho(n) \preceq n^c \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2c}.$$

Proof. Let $N_i = \gamma_i G$ for $i \geq 1$. Thus $N_{c+1} = \{1\}$ and $[N_i, N_j] \subseteq N_{i+j}$ for all i and j. Let $P = \langle \mathscr{F} \mid \mathscr{R} \rangle$ be a finite presentation of G of the form given in Section 3. Let $w \in H$ and $n = |w|$. We may assume, without loss of generality, that w is a word in the generators $\mathscr{N}_1$. By Theorem 1 there exists a rewriting process ρ from G to H relative to P, E and a positive integer K such that

$$|\rho(w)|_{\mathscr{N}_j} \leq K \sum_{\sum_{r=1}^{c} r p_r \leq j} m_1^{p_1} \cdots m_c^{p_c}$$

and

$$\Delta_P(w^{-1}\rho_i(w)) \leq K \sum_{\sum_{r=1}^{c} r p_r \leq c+i} m_1^{p_1} \cdots m_c^{p_c}$$

with $m_j = \sum_{k=1}^{j} |w|_{\mathscr{N}_k}$. Since $w \in N_1$ we have $|w|_{\mathscr{N}_1} = n$ and $|w|_{\mathscr{N}_k} = 0$ for $k > 1$. Hence,

$$|\rho(w)|_{\mathscr{N}_j} \leq K \sum_{\sum_{r=1}^{c} r p_r \leq j} n^{p_1} \cdots n^{p_c} \leq L n^j$$

and

$$\Delta_P(w^{-1}\rho(w)) \le K \sum_{\sum_{r=1}^{c} rp_r \le 2c} n^{p_1} \cdots n^{p_c} \le Ln^{2c}$$

for a suitable positive integer L. By $j \le c$ we get

$$\delta_\rho(n) \preceq n^c \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2c}. \qquad \square$$

Theorem 3. *Let G be a finitely presented nilpotent group of class c. Then*

$$\Phi_G(n) \preceq n^{2c}.$$

Proof. By Theorem 2 there exists a rewriting process ρ from G to $H = \{1\}$ such that $\Phi_\rho(n) \preceq n^{2c}$. Since $H = \{1\}$ we get $\Phi_G(n) \preceq n^{2c}$. $\square$

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