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Prehomogeneous Vector Spaces and Ergodic Theory III

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INTRODUCTION

This is part three in this series of papers. Throughout this paper, k is a

Definition (0.1). Let G be a connected reductive group, V a representation of G, and χ a non-trivial character of G, all defined over k. Then (G, V, χ) is called a prehomogeneous vector space if it satisfies the following properties.

(1) There exists a Zariski open orbit.

(2) There exists a non-zero polynomial $\Delta(x) \in k[V]$ such that $\Delta(gx) = \chi(g)\, \Delta(x)$.

Such $\Delta(x)$ is called a relative invariant polynomial. We define $V^{ss} = \{x \in V \mid \Delta(x) \neq 0\}$ and call it the set of semi-stable points. If (G, V, χ) is an irreducible representation, the choice of χ is essentially unique and we may write (G, V) as well. The theory of prehomogeneous vector spaces was initiated by Sato–Shintani [21] and Shintani [24]. If (G, V) is irreducible, the classification is known (see [20]).

In parts one and two [27, 25], we considered cases (2), (5), (6), (7) in the Sato–Kimura classification [20]. In this part, we consider the prehomogeneous vector space $G = \mathrm{GL}(5) \times \mathrm{GL}(3)$, $V = \wedge^2 k^5 \otimes k^3$. Except for the case (29) in [20] (for which we haven't carried out our program), the remaining applicable cases are either very easy or closely related to the spin or half spin representations of the spin groups. In this sense, the present case is rather an isolated case and that's why we consider this single case separately here. However, this case is quite interesting, because it

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produces a family of (irrational) cubic forms in five variables whose values at integer points are dense in $\mathbb{R}$. This may be the family of cubic forms in the lowest number of variables we can achieve by the theory of prehomogeneous vector spaces. We consider most of the remaining irreducible split cases in part four.

Oppenheim conjectured in [13] that if $Q(x)$ is a real non-degenerate indefinite quadratic form in $n \geqslant 5$ variables such that the ratio of at least one pair of coefficients is irrational, for any $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that $0 < |Q(x)| < \varepsilon$. Due to the result of Lewis [9], this is equivalent to saying the set $\{Q(x) \mid x \in \mathbb{Z}^n\}$ is dense in $\mathbb{R}$. There were many partial results including the one by Davenport with the collaboration with others [4, 5, 6, 2, 19] for $n \geqslant 21$. It was proved in the final form by Margulis (see [11]) for $n \geqslant 3$ using ergodic theory.

We posed the question of generalizing the Oppenheim conjecture from the viewpoint of prehomogeneous vector spaces in [27]. For more detailed comments, the reader should see the introduction of [27]. Here, we briefly state what we are going to prove.

Let $H_1 = \mathrm{SL}(5)$, $H_2 = \mathrm{SL}(3)$, $H = H_1 \times H_2$. It is known that (G, V) is a prehomogeneous vector space (see [24, 29, 27] for the definition of prehomogeneous vector spaces). A non-constant polynomial $\Delta(x)$ on V is called a relative invariant polynomial if there exists a character χ such that $\Delta(gx) = \chi(g)\,\Delta(x)$. Such $\Delta(x)$ exists for our case and is essentially unique. So we define $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$. For $x \in V^{\mathrm{ss}}_{\mathbb{R}}$, let $H^{\circ}_{x\mathbb{R}+}$ be the identity component in classical topology of the stabilizer $H_{x\mathbb{R}}$. We will prove that if $x \in V^{\mathrm{ss}}_{\mathbb{R}}$ is "sufficiently irrational" (see Theorem (6.2) for the precise definition), $H^{\circ}_{x\mathbb{R}+} H_{\mathbb{Z}}$ is dense in $H_{\mathbb{R}}$.

What Margulis did was to prove the above statement for the case $H = \mathrm{SL}(3)$, $V = \mathrm{Sym}^2(\mathbb{R}^3)^*$. Our method is based on the following theorem due to Ratner.

THEOREM (0.2) (Ratner). *Let G be a connected Lie group and U a connected subgroup of G generated by unipotent elements of G. Then given any lattice $\Gamma \subset G$ and $x \in G/\Gamma$, there exists a connected closed subgroup $U \subset F \subset G$ such that $\overline{Ux\Gamma} = Fx\Gamma$. Moreover, $F/F \cap x\Gamma x^{-1}$ has a finite invariant measure.*

Note that in the above theorem, the definition of a lattice contains the condition that G/Γ has a finite volume. The first statement was called Raghunathan's topological conjecture, and the second statement was proved by Ratner in conjunction with Raghunathan's topological conjecture. Raghunathan's topological conjecture was published by Dani [3] for one dimensional unipotent groups and was generalized to groups generated by unipotent elements by Margulis [10]. The proof for the general case was given by Ratner in a series of papers [14–17]. For these, there is an excellent survey article by Ratner [18].

Note that in the above theorem, if G is an algebraic group over $\mathbb{Q}$ and Γ is an arithmetic lattice, the group F becomes an algebraic group defined over $\mathbb{Q}$. For this the reader should see Proposition (3.2) [22, pp. 321–322]. It is also proved in Proposition (3.2) [22, pp. 321–322] that the radical of F is a unipotent subgroup. In [22], only one lattice is considered, but one can deduce the above statement for any lattice commensurable with the lattice in [22] by a simple argument using Ratner's theorem.

We describe an application of the density of $H^{\circ}_{x\mathbb{R}+} H_{\mathbb{Z}}$ in $H_{\mathbb{R}}$. For any non-zero point x in a vector space, we denote the point in the corresponding projective space determined by x by $[x]$. Let V_1 be a five dimensional vector space defined over $\mathbb{Q}$. We fix a rational basis $\{m_0, ..., m_4\}$ for V_1.

Let

$$Q(a) = a_0 a_4 - \tfrac{1}{4} a_1 a_3 + \tfrac{1}{12} a_2^2,$$

$$F(a) = 72 a_0 a_2 a_4 + 9 a_1 a_2 a_3 - 2 a_2^3 - 27 a_0 a_3^2 - 27 a_1^2 a_4$$

for $a = \sum_{i=0}^{4} a_i m_i$. If we identify V_1 with the space of binary quartic forms by $a \to a_0 v_1^4 + \cdots + a_4 v_2^4$ (v_1, v_2 are variables), Q, F correspond to quadratic and cubic $\mathrm{SL}(2)$-invariant polynomials.

If $g \in \mathrm{GL}(V_1)_{\mathbb{R}} \cong \mathrm{GL}(5)_{\mathbb{R}}$, it naturally acts on $\mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{R}}$ and $\mathbb{P}(\mathrm{Sym}^3 V_1^*)_{\mathbb{R}}$. Note that $(gQ)(a) = Q(g^{-1}a)$, $(gF)(a) = F(g^{-1}a)$. Then the following theorem follows from the consideration of $\overline{H^{\circ}_{x\mathbb{R}+} H_{\mathbb{Z}}}$.

THEOREM (0.3). *Suppose $g[Q] \notin \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$. Then the set of values of the cubic polynomial $F(g^{-1}a)$ at primitive integer points in $\mathbb{Z}^5$ is dense in $\mathbb{R}$.*

In Section 1, we consider various identifications concerning tensor products of vector spaces. If $x \in V^{ss}_{\mathbb{R}}$, by Ratner's theorem (Theorem (0.2)), there exists a closed connected subgroup $H^{\circ}_{x\mathbb{R}+} \subset F \subset H_{\mathbb{R}}$ such that $\overline{H^{\circ}_{x\mathbb{R}+} H_{\mathbb{Z}}} = F H_{\mathbb{Z}}$. In Section 2, we construct equivariant maps from V^{ss} to various H-varieties.

We can summarize how we construct equivariant maps in Section 2 in the following manner.

$$\begin{tikzcd}
V^{ss} \arrow[r] & V_1^* \otimes \mathrm{Sym}^2 V_2 \arrow[r] \arrow[d] & \bigwedge_5 V_1^* \otimes {}^5\, \mathrm{Sym}^2 V_2 \cong \mathrm{Sym}^2 V_2^* \\
 & V_1^* \otimes \mathrm{Hom}(V_2, V_2) \arrow[r] \arrow[d] & \mathrm{Sym}^2 V_1^* \\
 & \mathrm{Sym}^3 V_1^* &
\end{tikzcd}$$

The map $V^{ss} \to V_1^* \otimes \mathrm{Sym}^2 V_2$ is similar to the one in [26], and $V_1^* \otimes \mathrm{Sym}^2 V_2 \to \mathrm{Sym}^2 V_2^*$ is simply the Castling transform in [20]. It turns out that if $x \in V^{ss}$, the corresponding quadratic form in three variables is non-degenerate. Using this quadratic form, we can identify V_2 with its dual, and hence getting a map $V_1^* \otimes \mathrm{Sym}^2 V_2 \to V_1^* \otimes \mathrm{Hom}(V_2, V_2)$. Regarding an element of $V_1^* \otimes \mathrm{Hom}(V_2, V_2)$ as a 3×3 matrix $M(v)$ with entries in the space of linear forms in five variables $v = (v_1, ..., v_5)$, we can consider $\mathrm{tr}(M(v)^2)$, $\det M(v)$, which are a quadratic form and a cubic form in five variables respectively. This rather indirect way is how cubic forms in five variables arise from points in V^{ss}.

In Section 3, we prove that these equivariant maps are well defined and are non-trivial. These equivariant maps correspond to families of such F's with the property that it X_F is the corresponding H-variety, F has a unique fixed point in X_F. Part of our consideration resembles the argument in [23]. In Section 4, we describe the orbit space to determine when $H^\circ_{x\mathbb{R}+}$ is generated by unipotent elements. In Section 5, we classify all F's as above. In Section 6, we prove Theorem (0.3).

1. PRELIMINARIES

We are going to do a lot of computations in Section 3 regarding symmetric tensor products of vector spaces. We fix various normalizations for that purpose in this section.

Let W be a vector space over k with a basis $\{e_1, ..., e_n\}$. Let W^* be the dual space with the dual basis $\{f_1, ..., f_n\}$. For $a_1, ..., a_d \in W$, we define

$$[a_1, ..., a_d]_d = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(d)},$$

where $\mathfrak{S}_d$ is the group of permutations of $\{1, ..., d\}$. We identify $\mathrm{Sym}^d W$ with the subspace of $W^{\otimes d}$ spanned by elements of the form $[a_1, ..., a_d]_d$. Similarly, we identify $\mathrm{Sym}^d W^*$ with a subspace of $(W^*)^{\otimes d}$. For $a_1, ..., a_{d_1}, a_{d_1+1}, ..., a_{d_1+d_2} \in W$, we define

$$[a_1, ..., a_{d_1}]_{d_1} [a_{d_1+1}, ..., a_{d_1+d_2}]_{d_2} = [a_1, ..., a_{d_1+d_2}]_{d_1+d_2}.$$

By this product, $\oplus \mathrm{Sym}^* W$ becomes an associative algebra. Since $[a_1, ..., a_d]_d = a_1 \cdots a_d$, we use this usual notation of product from now on.

Since $(W^*)^{\otimes d}$ and $W^{\otimes d}$ are dual spaces of each other, there is a natural pairing between $\mathrm{Sym}^d W$ and $\mathrm{Sym}^d W^*$. If $a = a_1 \cdots a_d \in \mathrm{Sym}^d W$, $b = b_1 \cdots b_d \in \mathrm{Sym}^d W^*$, we normalize this pairing by

$$(a, b)_d = (b, a)_d = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} b_1(a_{\sigma(1)}) \cdots b_d(a_{\sigma(d)}).$$

Then if $i_1 + \cdots + i_n = d$,

$$(e_1^{i_1} \cdots e_n^{i_n}, f_1^{i_1} \cdots f_n^{i_n})_d = \frac{i_1! \cdots i_n!}{d!}.$$

Therefore, $\mathrm{Sym}^d W^*$ can be identified with the dual space of $\mathrm{Sym}^d W$.

The map

$$W \ni a \to i_d(a) = a \otimes a \otimes \cdots \otimes a \in \mathrm{Sym}^d W$$

is a polynomial map. So if $f \in \mathrm{Sym}^d W^*$, $f(a) = f(i_d(a))$ is a polynomial map from W to k and is homogeneous of degree d. We can identify $\mathrm{Sym}^d W^*$ with the space of degree k forms on W by this correspondence. If $a = \sum_{i=1}^n a_i e_i$,

$$i_d(a) = \sum \frac{d!}{i_1! \cdots i_n!} a_1^{i_1} \cdots a_n^{i_n} e_1^{i_1} \cdots e_n^{i_n},$$

where the sum is over all $(i_1, ..., i_n)$ such that $i_1 + \cdots + i_n = d$. So if $f = f_1^{i_1} \cdots f_n^{i_n}$, $f(a) = a_1^{i_1} \cdots a_n^{i_n}$. Therefore, f corresponds to the monomial $a_1^{i_1} \cdots a_n^{i_n}$.

If $G \subset \mathrm{GL}(W)$ is a subgroup, G acts on W^* by $(gf)(v) = f(g^{-1}v)$ for $g \in G$, $f \in W^*$. Whenever we consider the contragredient representation, we consider this action.

2. DEFINITIONS OF EQUIVARIANT MAPS

Let G, V, H be as in the introduction. We construct H-equivariant maps from V or V^{ss} to various H-varieties in this section.

Let $W = k^2$ be the space of two dimensional column vectors. Let $\{e_1, e_2\}$ be the standard basis of W. Consider the usual action of $\mathrm{GL}(2)$ on W. This induces an action of $\mathrm{GL}(2)$ on $\mathrm{Sym}^d W$ and $\mathrm{Sym}^d W^*$ for any d. We define new actions of $\mathrm{GL}(2)$ on $\mathrm{Sym}^2 W, \mathrm{Sym}^4 W$ by $g \cdot x = (\det g)^{-1} gx$, $(\det g)^{-2} gx$ where $g \cdot x$ is the new action and gx is the usual action. Note that scalar matrices act trivially and therefore, this defines an action of $\mathrm{PGL}(2)$ on $\mathrm{Sym}^2 W, \mathrm{Sym}^4 W$.

Let $V_1 = \mathrm{Sym}^4 W$, $V_2 = \mathrm{Sym}^2 W$, $V = \bigwedge^2 V_1 \otimes V_2$ and $G_1 = \mathrm{GL}(V_1) \cong \mathrm{GL}(5)$, $G_2 = \mathrm{GL}(V_2) \cong \mathrm{GL}(3)$, $G = G_1 \times G_2$. Then G acts on V in the usual manner and the above action of PGL(2) defines a homomorphism $\mathrm{PGL}(2) \to G$. By Schur's lemma, this is an imbedding. In fact, $\mathrm{Ker}(\mathrm{PGL}(2) \to G_1) = \mathrm{Ker}(\mathrm{PGL}(2) \to G_2) = \{1\}$. So we regard PGL(2) as a subgroup of G. Let $\tilde{T} = \mathrm{Ker}(G \to \mathrm{GL}(V))$. By Schur's lemma again,

$$\tilde{T} = \{(tI_5, t^{-2}I_3) \mid t \in \mathrm{GL}(1)\} \cong \mathrm{GL}(1).$$

If $(tI_5, t^{-2}I_3) \in \mathrm{PGL}(2)$, it acts trivially on V_1, V_2. So $t = 1$. Therefore, $\mathrm{PGL}(2) \cap \tilde{T} = \{1\}$.

Let $l_0 = e_1^2, l_1 = e_1 e_2$, $l_2 = e_2^2$ and $m_0 = e_1^4, ..., m_4 = e_2^4$. Then $\{l_0, l_1, l_2\}$, $\{m_0, ..., m_4\}$ are bases of V_2, V_1 respectively.

We define a linear map $\phi_1 \colon V \to \bigwedge^4 V_1 \otimes \mathrm{Sym}^2 V_2 \cong V_1^* \otimes \mathrm{Sym}^2 V_2$ by

$$V \ni \sum_{i=1}^{N} p_i \otimes q_i \to \tfrac{1}{2} \sum_{i,\, j=1}^{N} p_i \wedge p_j \otimes q_i q_j \tag{2.1}$$

for $p_1, ..., p_N \in \bigwedge^2 V_1, q_1, ..., q_N \in V_2$. Regarding $\mathrm{Sym}^2 V_2$ as a subspace of $V_2 \otimes V_2$, we denote the element of $V_1^* \otimes V_2 \otimes V_2$ which corresponds to $\phi_1(x)$ by $\bar{\phi}_1(x)$. Regarding V_1^* as the contragredient representation of V_1, $V_1^* \otimes \mathrm{Sym}^2 V_1$ is a representation of G.

The following lemma can be proved as in [20, p. 80], and the proof is left to the reader.

LEMMA (2.2). *For* $g = (g_1, g_2) \in G$, $\phi_1(gx) = \det g_1 g\phi_1(x)$, $\bar{\phi}_1(gx) = \det g_1 g\bar{\phi}_1(x)$.

If $x = \sum_{0 \leqslant i < j \leqslant 4} m_i \wedge m_j \otimes x_{ij}$ *with* $x_{ij} \in V_2$,

$$\phi_1(x) = \tfrac{1}{2} \sum_{0 \leqslant i < j \leqslant 4} \; \sum_{0 \leqslant k < l \leqslant 4} m_i \wedge m_j \wedge m_k \wedge m_l \otimes x_{ij} x_{kl}.$$

Let $m_0^*, ..., m_4^* \in \bigwedge^4 V_1$ be elements such that $m_i \wedge m_j^* = \delta_{ij} m_0 \wedge \cdots \wedge m_4$ (δ_{ij} is Kronecker's delta). Explicitly,

$$m_0 = m_1 \wedge m_2 \wedge m_3 \wedge m_4, \qquad m_1 = -m_0 \wedge m_2 \wedge m_3 \wedge m_4, \text{etc.}$$

We identify $\bigwedge^4 V_1$ with the dual space of V_1 by the pairing

$$V_1 \times \bigwedge^4 V_1 \ni (a, b) \to a \wedge b$$

and choosing $m_0 \wedge \cdots \wedge m_4$ as the basis element of $\bigwedge^5 V_1$. Then $\{m_0^*, ..., m_4^*\}$ can be regarded as the dual basis of $\{m_0, ..., m_4\}$.

Let

$$\begin{aligned} \mathrm{Pfaff}_0(x) &= x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, \\ \mathrm{Pfaff}_1(x) &= -(x_{02}x_{34} - x_{03}x_{24} + x_{04}x_{23}), \\ \mathrm{Pfaff}_2(x) &= x_{01}x_{34} - x_{03}x_{14} + x_{04}x_{13}, \\ \mathrm{Pfaff}_3(x) &= -(x_{01}x_{24} - x_{02}x_{14} + x_{04}x_{12}) \\ \mathrm{Pfaff}_4(x) &= x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}. \end{aligned} \tag{2.3}$$

Then

$$\phi_1(x) = \sum_{i=0}^{4} m_i^* \otimes \mathrm{Pfaff}_i(x). \tag{2.4}$$

The quadratic polynomials $\mathrm{Pfaff}_0(x), \ldots, \mathrm{Pfaff}_4(x)$ are the Pfaffians of 4×4 main minors of x if we regard x as an alternating 5×5 matrix with entries in V_1. This idea was used in [26] for the case $G = \mathrm{GL}(5) \times \mathrm{GL}(4)$, $V = \bigwedge^2 k^5 \otimes k^4$ to parametrize quintic extensions of a given ground field.

Next we consider a linear map $\phi_2 \colon V_1^* \otimes \mathrm{Sym}^2 V_2 \to \bigwedge^5 \mathrm{Sym}^2 V_2 \cong \mathrm{Sym}^2 V_2^*$ *defined by*

$$V_1^* \otimes \mathrm{Sym}^2 V_2 \ni \sum_{i=0}^{4} m_i^* \otimes p_i \to p_0 \wedge \cdots \wedge p_4 \in \bigwedge^5 \mathrm{Sym}^2 V_2 \cong \mathrm{Sym}^2 V_2^*. \tag{2.5}$$

Note that ϕ_2 is the Castling transform discussed in [20].

DEFINITION (2.6). $\Phi_1 = (1/3^4)\, \phi_2 \circ \phi_1$.

Φ_1 is a map from V to $\mathrm{Sym}^2 V_2^*$. The following lemma can also be proved as in [20, p. 80], and the proof is left to the reader.

LEMMA (2.7). $\Phi_1(gx) = (\det g_1)^4 (\det g_2)\, g_2 \Phi_1(x)$.

This map Φ_1 was also considered in [12] (using ϕ_1 also) for a different purpose. We will show in Section 3 that the discriminant of $\Phi_1(x)$ is not identically zero and V^{ss} consists of x's such that $\Phi_1(x)$ is non-degenerate.

For $x \in V$, let $\bar{\Phi}_1(x)(\alpha, \beta)$ be the symmetric bilinear form on V_2 associated with $\Phi_1(x)$. In other words,

$$\bar{\Phi}_1(x)(\alpha, \beta) = \tfrac{1}{2}(\Phi_1(x)(\alpha + \beta) - \Phi_1(x)(\alpha) - \Phi_1(x)(\beta))$$

for $\alpha, \beta \in V_2$.

We define a linear map $j_x \colon V_1^* \otimes V_2 \otimes V_2 \to \mathrm{Hom}(V_2, V_1^* \otimes V_2)$ by

$$j_x(a \otimes \alpha \otimes \beta)(\gamma) = \Phi_1(x)(\alpha, \gamma)\, a \otimes \beta \tag{2.8}$$

for $a \in V_1^*$, $\alpha, \beta, \gamma \in V_2$. If $f \in \mathrm{Hom}(V_2, V_1^* \otimes V_2)$, we define $gf \in \mathrm{Hom}(V_2, V_1^* \otimes V_2)$ by

$$(gf)(\alpha) = gf(g_2^{-1}\alpha)$$

where we are considering the action of g on the element $f(g_2^{-1}\alpha)$.

LEMMA (2.9). *$j_{gx}(g(a \otimes \alpha \otimes \beta)) = (\det g_1)^4 (\det g_2)\, g j_x(a \otimes \alpha \otimes \beta)$ for all $a \in V_1^*$, $\alpha, \beta \in V_2$.*

Proof. Let $\gamma \in V_2$. Then by Lemma (2.7),

$$\begin{aligned} j_{gx}(g(a \otimes \alpha \otimes \beta))(\gamma) &= \bar{\Phi}_1(gx)(g_2\alpha, \gamma)\, g_1 a \otimes g_2 \beta \\ &= (\det g_1)^4 (\det g_2)(g_2 \bar{\Phi}_1(x))(g_2\alpha, \gamma)\, g_1 a \otimes g_2\beta \\ &= (\det g_1)^4 (\det g_2)\, \bar{\Phi}_1(x)(\alpha, g_2^{-1}\gamma)\, g_1 a \otimes g_2 \beta \\ &= (\det g_1)^4 (\det g_2)\, g(\bar{\Phi}_1(x)(\alpha, g_2^{-1}\gamma)\, a \otimes \beta) \\ &= (\det g_1)^4 (\det g_2)\, g(j_x(a \otimes \alpha \otimes \beta)(g_2^{-1}\gamma)) \\ &= (\det g_1)^4 (\det g_2)(g(j_x(a \otimes \alpha \otimes \beta)))(\gamma). \end{aligned}$$

This proves the lemma. ∎

DEFINITION (2.10). $\phi_3(x) = j_x(\bar{\phi}_1(x))$.

Apparently, ϕ_3 is a map from V to $\mathrm{Hom}(V_2, V_1^* \otimes V_2)$.

LEMMA (2.11). *$\phi_3(gx) = (\det g_1)^5 (\det g_2)\, g\phi_3(x)$.*

Proof.

$$\begin{aligned} \phi_3(gx) &= j_{gx}(\bar{\phi}_1(gx)) \\ &= \det g_1 j_{gx}(g\bar{\phi}_1(x)) \\ &= (\det g_1)^5 (\det g_2)\, g j_x(\bar{\phi}_1(x)) \\ &= (\det g_1)^5 (\det g_2)\, g\phi_3(x). \end{aligned}$$ ∎

By the basis $\{l_0, l_1, l_2\}$ for V_2, we can regard $\phi_3(x)$ as a 3×3 matrix with entries in V_1^*. Then the action of $g = (g_1, g_2) \in \mathrm{GL}(V_1) \times \mathrm{GL}(3)$ is obtained by considering $g_2 \phi_3(x)\, g_2^{-1}$ and then applying g_1 entry-wise. Therefore,

$$\Phi_2(x) = \mathrm{tr}(\phi_3(x)^2) \in \mathrm{Sym}^2 V_1^*, \quad F_x = \det \phi_3(x) \in \mathrm{Sym}^3 V_1^* \qquad (2.12)$$

define maps $x \to \Phi_2(x)$, F_x from V to $\mathrm{Sym}^2 V_1^*$, $\mathrm{Sym}^3 V_1^*$.

The following lemma is an easy corollary of Lemma (2.11).

LEMMA (2.13). (1) $\Phi_2(gx) = (\det g_1)^{10} (\det g_2)^2 \, g_1 \Phi_2(x)$.

(2) $F_{gx}(a) = (\det g_1)^{15} (\det g_2)^3 \, F_x(g_1^{-1} a)$ *for all* $a \in V_1$.

For later purposes, we describe how to compute $\Phi_2(x), F_x$. We have already described how to compute $\phi_1(x), \Phi_1(x)$ in (2.4), (2.5). Let $\{p_0, p_1, p_2\}$ be the dual basis of $\{l_0, l_1, l_2\}$. Suppose

$$\bar{\phi}_1(x) = \sum_{i,\, j=0}^{2} a_{ij} \otimes l_i \otimes l_j, \; \overline{\Phi_1}(x) = \sum_{t,\, s=0}^{2} b_{ts} p_t \otimes p_s$$

with $a_{ij} \in V_1^*$, $b_{ts} \in k$ for all i, j, t, s and $a_{ij} = a_{ji}$, $b_{ts} = b_{st}$. We denote the matrices (a_{ij}), (b_{ts}) also by $\bar{\phi}_1(x)$, $\bar{\Phi}_1(x)$. Let $\gamma = \sum_{s=0}^{2} \gamma_s l_s$. Then

$$\begin{aligned}
\phi_3(x)(\gamma) &= j_x(\bar{\phi}_1(x))(\gamma) \\
&= \sum_{i,\, j,\, s=0}^{2} j_x(a_{ij} \otimes l_i \otimes l_j)(\gamma_s l_s) \\
&= \sum_{i,\, j,\, s=0}^{2} \bar{\Phi}_1(x)(l_i \otimes l_s) \, a_{ij} \gamma_s \otimes l_j \\
&= \sum_{i,\, j,\, s=0}^{2} a_{ij} b_{is} \gamma_s \otimes l_j \\
&= \sum_{i,\, j,\, s=0}^{2} a_{ji} b_{is} \gamma_s \otimes l_j.
\end{aligned}$$

Note that $\sum_{i=0}^{2} a_{ji} b_{is}$ is the (j, s)-entry of the matrix product $\bar{\phi}_1(x)$ $\bar{\Phi}_1(x)$. So if we regard $\phi_3(x)$ as a 3×3 matrix with entries in V_1^*, we get the following relation

$$\phi_3(x) = \bar{\phi}_1(x) \, \bar{\Phi}_1(x). \tag{2.14}$$

3. EQUIVARIANT MAPS AT W

In this section, we prove that the equivariant maps we constructed in Section 2 are well defined and are non-trivial by evaluating them at a point w, which we will define in (3.10).

Let $\{l_0, l_1, l_2\}$, $\{m_0, ..., m_4\}$ be the bases of $\mathrm{Sym}^2 W$, $\mathrm{Sym}^4 W$ we defined in Section 2. Let $\{p_0, p_1, p_2\}$ be the dual basis of $\{l_0, l_1, l_2\}$. Let $\mathfrak{h}$ be the Lie algebra of PGL(2). Then $\mathfrak{h}$ is the Lie algebra of SL(2) also. We consider V as a representation of $\mathfrak{h}$ also. Let Λ be the fundamental dominant weight of $\mathfrak{h}$. We denote the irreducible representation of $\mathfrak{h}$ with highest weight $d\Lambda$ also by $d\Lambda$. Then by considering weights, $\wedge^2 V_1 \cong 6\Lambda \oplus 2\Lambda$ and

$$6\Lambda \otimes 2\Lambda \cong 8\Lambda \oplus 6\Lambda \oplus 4\Lambda, \qquad 2\Lambda \otimes 2\Lambda \cong 4\Lambda \oplus 2\Lambda \oplus k, \tag{3.1}$$

where k is the trivial representation.

Therefore, V contains the trivial representation precisely once.

Note that $V_2^* \cong \mathfrak{h}$ as an $\mathfrak{h}$-module. We identify $\mathrm{Sym}^4 W^*$ as the space of homogeneous polynomials of degree four in two variables $v = {}^t(v_1 \quad v_2)$ (v corresponds to $v_1 e_1 + v_2 e_2$). We identify $a = a(v) = a_0 v_1^4 + \cdots + a_4 v_2^4$ with $(a_0, ..., a_4)$.

Let

$$H_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

An easy way to compute Lie algebra actions is to consider values in the ring of dual numbers $k[\varepsilon]/(\varepsilon^2)$.

Let

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_0' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \qquad A_1' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{3.3}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the actions of $\mathfrak{h}$ on V_1, V_2 are easy to describe and with respect to the bases $\{e_0, ..., e_4\}$, $\{l_0, l_1, l_2\}$, H_0, H_1, H_2 map to

$$(A_0, A_0'), (A_1, A_1'), -(A_2, A_2')$$

respectively.

Since the action of $g \in \mathrm{PGL}(2)$ on $a(v) \in \mathrm{Sym}^4 W^*$ is $a(g^{-1}v)$, the action of $H \in \mathfrak{h}$ on $a = a(v)$ is given by

$$a((1 - \varepsilon H)\, v) = a(v) + \varepsilon (Ha)(v).$$

Then easy computations show that

$$\begin{aligned} H_0 a &= (0, -4a_0, -3a_1, -2a_2, -a_3), \\ H_1 a &= (-4a_0, -2a_1, 0, 2a_3, 4a_4), \\ H_2 a &= (a_1, 2a_2, 3a_3, 4a_4, 0). \end{aligned} \tag{3.4}$$

For $a = (a_0, ..., a_4)$, $b = (b_0, ..., b_4)$, we define

$$Q(a, b) = a_0 b_4 - \tfrac{1}{4} a_1 b_3 + \tfrac{1}{6} a_2 b_2 - \tfrac{1}{4} a_3 b_1 + a_4 b_0. \tag{3.5}$$

Then Q is a non-degenerate symmetric bilinear form, invariant under the action of $\mathrm{PGL}(2)$. This implies $Q(Ha, b)$ is an alternating form for any $H \in \mathfrak{h}$. We regard this alternating form as an element of $\bigwedge^2 (\mathrm{Sym}^4 W^*)^* \cong \bigwedge^2 \mathrm{Sym}^4 W$.

Lemma (3.6). *The map $H \to f_H(a, b) = Q(Ha, b)$ is an $\mathfrak{h}$-homomorphism from $\mathfrak{h}$ to $\bigwedge^2 \mathrm{Sym}^4 W$.*

Proof. Note that the action of $H \in \mathfrak{h}$ on an element $f(a, b)$ in $\bigwedge^2 (\mathrm{Sym}^4 W^*)^*$ is given by $(Hf)(a, b) = -f(Ha, b) - f(a, Hb)$. So if $H, H' \in \mathfrak{h}$,

$$\begin{aligned} (H' f_H)(a, b) &= -f_H(H'a, b) - f_H(a, H'b) \\ &= -Q(HH'a, b) - Q(Ha, H'b) \\ &= -Q(HH'a, b) + Q(H'Ha, b) \\ &= Q([H', H]\, a, b) \\ &= f_{[H', H]}(a, b). \quad \blacksquare \end{aligned}$$

This defines an $\mathfrak{h}$-homomorphism $\mathfrak{h} \to \bigwedge^2 V_1$. Regarding this homomorphism as an element of $\bigwedge^2 V_1 \otimes \mathfrak{h}^* \cong \bigwedge^2 V_1 \otimes V_2 = V$, we get a fixed point of V under the action of $\mathrm{PGL}(2)$. We compute this element explicitly.

Note that the linear map defined by

$$H_0 \to \tfrac{1}{2} v_2^2, \qquad H_1 \to v_1 v_2, \qquad H_2 \to \tfrac{1}{2} v_1^2 \tag{3.7}$$

is an $\mathfrak{h}$-homomorphism.

By (3.4),

$$\begin{aligned} Q(H_0 a, b) &= a_0 b_3 - \tfrac{1}{2} a_1 b_2 + \tfrac{1}{2} a_2 b_1 - a_3 b_0, \\ Q(H_1 a, b) &= -4 a_0 b_4 + \tfrac{1}{2} a_1 b_3 - \tfrac{1}{2} a_3 b_1 + 4 a_4 b_0, \\ Q(H_2 a, b) &= a_1 b_4 - \tfrac{1}{2} a_2 b_3 + \tfrac{1}{2} a_3 b_2 - a_4 b_1. \end{aligned} \tag{3.8}$$

Note that $\{e_1^2, 2e_1e_2, e_2^2\}$ is the dual basis of $\{v_1^2, v_1 v_2, v_2^2\}$, and $(m_0, v_1^4)_4 = 1$, $(m_1, v_1^3 v_2)_4 = \frac{1}{4}$, etc. We identify $\wedge^2 V_1$ with the space of alternating bilinear forms on V_1^* by assuming

$$m \wedge m'(a, b) = (m, a)_1 \, (m', b)_1 - (m, b)_1 \, (m', a)_1$$

for $m, m' \in V_1$, $a, b \in V_1^*$. So by the corresponding $H \to f_H$,

$$\begin{aligned} H_0 &\to 4m_0 \wedge m_3 - 12 m_1 \wedge m_2, \\ H_1 &\to -4m_0 \wedge m_4 + 8m_1 \wedge m_3, \\ H_2 &\to 4m_1 \wedge m_4 - 12 m_2 \wedge m_3. \end{aligned} \tag{3.9}$$

Since $\{H_0, H_1, H_2\}$ corresponds to $\{\frac{1}{2}v_2^2, v_1 v_2, \frac{1}{2}v_1^2\}$ and $\{2l_2, 2l_1, 2l_0\}$ is its dual basis, this correspondence can be regarded as the element $8w$ where

$$\begin{aligned} w = {} & (m_0 \wedge m_3 - 3m_1 \wedge m_2) \otimes l_2 + (-m_0 \wedge m_4 + 2m_1 \wedge m_3) \otimes l_1 \\ & + (m_1 \wedge m_4 - 3m_2 \wedge m_3) \otimes l_0. \end{aligned} \tag{3.10}$$

These considerations show the following proposition.

PROPOSITION (3.11). *The element $w \in V$ is fixed by* PGL(2).

In [20, p. 95], instead of w, the element

$$w' = (m_0 \wedge m_1 + m_2 \wedge m_3, m_1 \wedge m_2 + m_3 \wedge m_4, m_0 \wedge m_2 + m_1 \wedge m_4)$$

was considered (we shifted the indices in [20] because we are using indices $0, ..., 4$). However, by replacing $m_0, ..., m_4$ in w by m_4, m_2, m_0, m_1, m_3 respectively, and multiplying scalars to basis elements of $\bigwedge^2 V_1$, we get the above element w'. Therefore, we are considering essentially the same element as in [20].

In [20, p. 96], the Lie algebra of $G_{w'}^\circ / \tilde{T}$ ($G_{w'}^\circ$ is the identity component of the stabilizer) is computed and is isomorphic to the Lie algebra of PGL(2). Therefore, this is the case for w also. Since we are assuming $\mathrm{ch}\, k = 0$, this implies $G_w^\circ = \mathrm{PGL}(2) \times \tilde{T}$ if k is algebraically closed (see [7]). For arbitrary k, we still have the inclusion $\mathrm{PGL}(2) \times \tilde{T} \subset G_w^\circ$. Since this is an isomorphism over $\bar{k}$, we get the following proposition

PROPOSITION (3.12). $G_w^\circ = \mathrm{PGL}(2) \times \tilde{T}$.

By the basis $\{e_0, ..., e_4\}$, we regard V as the space of 5×5 alternating matrices with entries in V_2. Then

$$w = \begin{pmatrix} 0 & 0 & 0 & l_2 & -l_1 \\ 0 & 0 & -3l_2 & 2l_1 & l_0 \\ 0 & 3l_2 & 0 & -3l_0 & 0 \\ -l_2 & -2l_1 & 3l_0 & 0 & 0 \\ l_1 & -l_0 & 0 & 0 & 0 \end{pmatrix}.$$

By the definition (2.3),

$$\begin{aligned} &\mathrm{Pfaff}_0(w) = -3l_0^2, \qquad \mathrm{Pfaff}_1(w) = -3l_0 l_1, \\ &\mathrm{Pfaff}_2(w) = -2l_1^2 - l_0 l_2, \\ &\mathrm{Pfaff}_3(w) = -3l_1 l_2, \qquad \mathrm{Pfaff}_4(w) = -3l_2^2. \end{aligned} \tag{3.13}$$

Note that we are regarding them as elements of $\mathrm{Sym}^2 V_2$ and not $\mathrm{Sym}^4 W$.

By the basis $\{l_0, l_1, l_2\}$, we regard $\bar{\phi}_1(w)$ as a 3×3 matrix with entries in V_1^* as in Section 2. Then

$$\bar{\phi}_1(w) = -\begin{pmatrix} 3m_0^* & \frac{3}{2}m_1^* & \frac{1}{2}m_2^* \\ \frac{3}{2}m_1^* & 2m_2^* & \frac{3}{2}m_3^* \\ \frac{1}{2}m_2^* & \frac{3}{2}m_3^* & 3m_4^* \end{pmatrix}. \tag{3.14}$$

Let

$$n_0 = l_0^2, \quad n_1 = l_1^2, \quad n_2 = l_2^2, \quad n_3 = l_0 l_1, \quad n_4 = l_1 l_2, \quad n_5 = l_0 l_2 \in \mathrm{Sym}^2 V_2. \tag{3.15}$$

Then $\{n_0, ..., n_5\}$ is a basis of $\mathrm{Sym}^2 V_2$. Let $n_0^*, ..., n_5^*$ be elements of $\bigwedge^5 \mathrm{Sym}^2 V_2 \cong \mathrm{Sym}^2 V_2^*$ such that $n_i \wedge n_j^* = \delta_{ij} n_0 \wedge \cdots \wedge n_5$. Then

$$\begin{aligned} &\mathrm{Pfaff}_0(w) \wedge \mathrm{Pfaff}_1(w) \wedge \mathrm{Pfaff}_2(w) \wedge \mathrm{Pfaff}_3(w) \wedge \mathrm{Pfaff}_4(w) \\ &\quad = (-3n_0) \wedge (-3n_3) \wedge (-2n_1 - n_5) \wedge (-3n_4) \wedge (-3n_2) \\ &\quad = -3^4 n_0 \wedge n_3 \wedge (2n_1 + n_5) \wedge n_4 \wedge n_2 \\ &\quad = -3^4(2n_0 \wedge n_3 \wedge n_1 \wedge n_4 \wedge n_2 + n_0 \wedge n_3 \wedge n_5 \wedge n_4 \wedge n_2) \\ &\quad = 3^4(n_1^* - 2n_5^*). \end{aligned}$$

Therefore,

$$\Phi_1(w) = n_1^* - 2n_5^*. \tag{3.15}$$

We identify $\mathrm{Sym}^2 V_2^*$ with the dual space of $\mathrm{Sym}^2 V_2$. Then with respect to the basis $\{p_0, p_1, p_2\}$, n_1^*, n_5^* correspond to $p_1^2, 2p_0p_2$. Therefore, $\Phi_1(w) = p_1^2 - 4p_0p_2$. We regard $\bar{\Phi}_1(w)$ as a 3×3 matrix as in Section 2. Then

$$\bar{\Phi}_1(w) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Therefore,

$$\phi_3(w) = \bar{\phi}_1(w)\,\bar{\Phi}_1(w) = \begin{pmatrix} m_2^* & -\frac{3}{2}m_1^* & 6m_0^* \\ 3m_3^* & -2m_2^* & 3m_1^* \\ 6m_4^* & -\frac{3}{2}m_3^* & m_2^* \end{pmatrix}. \tag{3.17}$$

So, we get the following proposition easily.

Proposition (3.18). *Let* $a = a_0m_0 + \cdots + a_4m_4$. *Then*

(1) $\Phi_2(w)(a) = (\mathrm{tr}(\phi_3(w)^2))(a) = 72(a_0a_4 - \frac{1}{4}a_1a_3 + \frac{1}{12}a_2^2)$,

(2) $F_w(a) = (\det \phi_3(w))(a)$
$= 72a_0a_2a_4 + 9a_1a_2a_3 - 2a_2^3 - 27a_0a_3^2 - 27a_1^2a_4.$

By these considerations, Φ_1, Φ_2 are non-trivial maps. By (3.16), the discriminant $\Delta(x)$ of $\Phi_1(x)$ is a non-zero polynomial. By Lemma (2.7), $\Delta(x)$ is a non-constant polynomial. Therefore, it is a relative invariant polynomial. So we reproved that $w \in V_k^{\mathrm{ss}}$. Since our case is known to be a regular prehomogeneous vector space, $V_{\bar{k}}^{\mathrm{ss}}$ is a single $G_{\bar{k}}$-orbit if k is algebraically closed. Therefore, V^{ss} consists of x's such that $\Phi_1(x)$ is non-degenerate.

Since $\Phi_2(w)$ is non-degenerate, $\Phi_2(x)$ is non-degenerate for all $x \in V^{\mathrm{ss}}$.

4. THE ORBIT SPACE $G_K \backslash V_K^{\mathrm{SS}}$

In this section, we prove that $G_k \backslash V_k^{\mathrm{ss}}$ corresponds bijectively with $\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k$-equivalence classes of ternary quadratic forms over k.

We first recall the relation between the orbit space $G_k \backslash V_k^{\mathrm{ss}}$ and the Galois cohomology set.

For any algebraic group G over k, let $\mathrm{H}^1(k, G)$ be the first Galois cohomology set. We choose the definition so that trivial classes are those of the form $\{g^{-1}g^\sigma\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ $(g \in G_{\bar{k}})$ and the cocycle condition is $h_{\sigma\tau} = h_\tau h_\sigma^\tau$ for a continuous map $\{h_\sigma\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ from $\mathrm{Gal}(\bar{k}/k)$ to $G_{\bar{k}}$.

Let (G, V) be an arbitrary regular prehomogeneous vector space, and $w \in V_k^{\mathrm{ss}}$. Then for any $x \in V_k^{\mathrm{ss}}$, there exists $g_x \in G_{\bar{k}}$ such that $x = g_xw$. Then

$c_x = \{g_x^{-1} g_x^{\sigma}\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ determines a cohomology class in $\mathrm{H}^1(k, G_w)$ and does not depend on the choice of g_x. The following theorem is due to Igusa [8].

THEOREM (4.1) (Igusa). *The correspondence*

$$G_k \backslash V_x^{\mathrm{ss}} \ni x \to c_x \in \mathrm{Ker}(\mathrm{H}^1(k, G_w) \to \mathrm{H}^1(k, G))$$

is bijective.

Note that $\mathrm{Ker}(\mathrm{H}^1(k, G_w) \to \mathrm{H}^1(k, G))$ is the set of elements $c \in \mathrm{H}^1(k, G_w)$ which map to the trivial class in $\mathrm{H}^1(k, G)$. In our case, $\mathrm{H}^1(k, G)$ is trivial. Therefore, $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, G_w)$.

We recall the correspondence between $\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k$-equivalence classes of ternary quadratic forms and quarternion algebras. Let $V_3 = \mathrm{Sym}^2 V_2^*$. Then $\mathrm{GL}(V_2) \cong \mathrm{GL}(3)$ acts on V_3 in the usual manner. We let $\mathrm{GL}(1)$ act on V_3 by the usual multiplication. Then $\mathrm{GL}(1) \times \mathrm{GL}(3)$ acts on V_3. Let $\{l_0, l_1, l_2\}$ and $\{p_0, p_1, p_2\}$ be as before. Let $\bar{w} = p_1^2 - 4p_0p_2$, and Q the corresponding quadratic form. It is well known (and is easy to verify) that the stabilizer of $\bar{w}$ is isomorphic to $\mathrm{SO}(Q) \times \mathrm{GL}(1)$ and $\mathrm{SO}(Q) \cong \mathrm{PGL}(2)$.

Therefore, $(\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k) \backslash V_{3k}^{\mathrm{ss}}$ corresponds bijectively with $\mathrm{H}^1(k, \mathrm{PGL}(2))$. Since $\mathrm{PGL}(2)$ is isomorphic to the automorphism group of the associative algebra $\mathrm{M}(2, 2)$, $\mathrm{H}^1(k, \mathrm{PGL}(2))$ corresponds bijectively with isomorphism classes of quarternion algebras. Given a ternary quadratic form, the corresponding quarternion algebra is the Clifford algebra associated with the quadratic form.

Now we go back to our situation. Let G, H, V be as before. We consider the element $w \in V_k^{\mathrm{ss}}$ which we defined in (3.10). We pointed out in (3.12) that $G_w^{\circ} \cong \mathrm{PGL}(2) \times \mathrm{GL}(1)$.

PROPOSITION (4.2). *The group G_w is connected.*

Proof. We may assume that k is algebraically closed. Suppose $g \in G_k/\widetilde{T}_k$. The identity component of the stabilizer of w in $G/\widetilde{T}$ is isomorphic to $\mathrm{PGL}(2)$. The conjugation by g induces an automorphism of $\mathrm{PGL}(2)$. Since there is no outer automorphism of $\mathrm{PGL}(2)$, by changing g if necessary, we may assume that g commutes with elements of $\mathrm{PGL}(2)$. Since V_1, V_2 are irreducible representations, by Schur's lemma, g is represented by an element of the form $(t_1 I_5, t_2 I_3)$. This element fixes w if and only if $t_1^2 t_2 = 1$. So $g = 1$ (in $G/\widetilde{T}$). ∎

PROPOSITION (4.3). (1) *The map $\Phi_1 \colon V \to V_3 = \mathrm{Sym}^2 V_1^*$ induces a bijection $G_k \backslash V_k^{\mathrm{ss}} \cong (\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k) \backslash V_{3k}^{\mathrm{ss}}$.*

(2) *If* $x \in V_k^{\mathrm{ss}}$, *the projections of* H_x *to* G_1, G_2 *induce isomorphisms to the images. In particular,* $H_x \cong \mathrm{SO}(\Phi_1(x))$.

Proof. Let $c \in \mathrm{H}^1(k, G_w)$. Then c becomes trivial in $\mathrm{H}^1(k, H)$ also. Let $g = (g_1, g_2) \in H_{\bar{k}}$ be the element such that c is represented by $\{g^{-1}g^{\sigma}\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$. Then the orbit in V_k^{ss} corresponding to c is gw. By Lemma (2.7), $\Phi_1(gw) = g_2 \Phi_1(w)$.

Since $\mathrm{H}^1(k, G_w) \cong \mathrm{H}^1(k, \mathrm{PGL}(2))$ and the projection of $\mathrm{PGL}(2)$ to G_2 is an isomorphism to its image,

$$\mathrm{H}^1(k, G_w) \ni \{g^{-1}g^{\sigma}\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)} \to \{g_2^{-1}g_2^{\sigma}\}_{\sigma \in \mathrm{Gal}(\bar{k}/k)} \in \mathrm{H}^1(k, \mathrm{PGL}(2))$$

is a bijection. Note that we are considering $\mathrm{PGL}(2) \subset G$ for the first element and $\mathrm{PGL}(2) \subset G_2$ for the second element. Since $(\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k) \backslash V_{3k}^{\mathrm{ss}} \cong \mathrm{H}^1(k, \mathrm{PGL}(2))$, this proves (1).

Note that $(\mathrm{PGL}(2) \times \tilde{T}) \cap H = \mathrm{PGL}(2)$. So $H_w \cong \mathrm{PGL}(2)$. We already pointed out that statement (2) holds for w in Section 2. Let $x \in V_k^{\mathrm{ss}}$. By Lemma (2.7), the projection of H_x to G_2 is contained in $\mathrm{SO}(\Phi_1(x))$. So it is enough to prove (2) when k is algebraically closed. But then x is in the orbit of w and (2) follows easily. ∎

Remark (4.4). The map Φ_1 induces a map $G_k \backslash V_k^{\mathrm{ss}} \to \mathrm{GL}(3)_k \backslash V_{3k}^{\mathrm{ss}}$ also, but this may not be surjective. This may be regarded as the section $(\mathrm{GL}(1)_k \times \mathrm{GL}(3)_k) \backslash V_{3k}^{\mathrm{ss}} \to \mathrm{GL}(3)_k \backslash V_{3k}^{\mathrm{ss}}$ defined by $x \to (\det x)^{-1} x$.

5. INTERMEDIATE GROUPS

Let $x \in V_{\mathbb{R}}^{\mathrm{ss}}$. By Proposition (4.3), $H_{x\mathbb{R}}$ is connected in classical topology. So $H_{x\mathbb{R}+}^{\circ} = H_{x\mathbb{R}}$. If $\Phi_1(x)$ is definite, $H_{x\mathbb{R}}$ is compact by Proposition (4.3) also. Then $H_{x\mathbb{R}} H_{\mathbb{Z}} \subset H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a compact set. Therefore, an analogue of the Oppenheim conjecture is not applicable to such points. The set of real indefinite non-degenerate ternary quadratic forms is a single $\mathrm{GL}(1)_{\mathbb{R}} \times \mathrm{GL}(3)_{\mathbb{R}}$-orbit. Therefore, we only consider $x \in G_{\mathbb{R}} w$.

We determine all the closed connected subgroups between $H_{x\mathbb{R}+}^{\circ}$ and $H_{\mathbb{R}}$ for all $x \in G_{\mathbb{R}} w$ for the rest of this section. This reduces to the consideration of Lie algebras. We consider an arbitrary ground field k of characteristic zero and specialize to $k = \mathbb{R}$ in (5.10).

We first describe possible candidates for such subgroups. By Lemmas (2.7), (2.13), Φ_1, Φ_2 are H-equivariant maps. As we pointed out at the end of Section 3, $\Phi_1(x) \in \mathrm{Sym}^2 V_1^*$, $\Phi_2(x) \in \mathrm{Sym}^2 V_2^*$ are non-degenerate for $x \in G_k w$. So let $\mathrm{SO}(\Phi_1(x))$, $\mathrm{SO}(\Phi_2(x))$ be the corresponding special orthogonal groups.

In the following definition, $x \in G_k w$.

DEFINITION (5.1). (1) $H_{x1} \subset \mathrm{GL}(V_1)$, $H_{x2} \subset \mathrm{GL}(V_2)$ are the images of the projections of H_x to G_1, G_2 respectively.

(2) $H_{x3} = \mathrm{SO}(\Phi_2(x)) \subset \mathrm{GL}(V_1)$.

Note that both H_{x1}, H_{x2} are isomorphic to H_x, and $H_x \cong \mathrm{PGL}(2)$.

Let $\mathfrak{h}$ be the Lie algebra of PGL(2) as before. Let $\mathfrak{h}_1 = \mathrm{sl}(5)$, $\mathfrak{h}_2 = \mathrm{sl}(3)$ (Lie algebras of SL(5), SL(3)). If $\mathfrak{f}$ is a Lie algebra between $\mathfrak{h}$ and $\mathfrak{h}_1 \times \mathfrak{h}_2$, it is an $\mathfrak{h}$-module. So we first decompose $\mathfrak{h}_1$, $\mathfrak{h}_2$ to direct sums of irreducible $\mathfrak{h}$-modules.

Let

$$B = B(b_0, ..., b_4) = \begin{pmatrix} 2b_2 & -3b_1 & b_0 & 0 & 0 \\ 12b_3 & -b_2 & -2b_1 & 3b_0 & 0 \\ 6b_4 & 3b_3 & -2b_2 & 3b_1 & 6b_0 \\ 0 & 3b_4 & -2b_3 & -b_2 & 12b_1 \\ 0 & 0 & b_4 & -3b_3 & 2b_2 \end{pmatrix},$$

$$C = C(c_0, ..., c_6) = \begin{pmatrix} c_3 & 3c_2 & -c_1 & c_0 & 0 \\ 12c_4 & -2c_3 & -4c_2 & 0 & 4c_0 \\ 6c_5 & -6c_4 & 0 & -6c_2 & 6c_1 \\ 4c_6 & 0 & -4c_4 & 2c_3 & 12c_2 \\ 0 & c_6 & -c_5 & 3c_4 & -c_3 \end{pmatrix}, \tag{5.2}$$

$$D = D(d_0, ..., d_8) = \begin{pmatrix} d_4 & -d_3 & d_2 & -d_1 & d_0 \\ 4d_5 & -4d_4 & 4d_3 & -4d_2 & 4d_1 \\ 6d_6 & -6d_5 & 6d_4 & -6d_3 & 6d_2 \\ 4d_7 & -4d_6 & 4d_5 & -4d_4 & 4d_3 \\ d_8 & -d_7 & d_6 & -d_5 & d_4 \end{pmatrix},$$

$$B' = B'(b'_0, ..., b'_4) = \begin{pmatrix} b'_2 & -b'_1 & b'_0 \\ 2b'_3 & -2b'_2 & 2b'_1 \\ b'_4 & -b'_3 & b'_2 \end{pmatrix},$$

where $b_0 \cdots \in k$.

We define

$$\begin{aligned} U_2 &= \{B(b_0, ..., b_4) \mid b_0, ..., b_4 \in k\}, \\ U_3 &= \{C(c_0, ..., c_6) \mid c_0, ..., c_6 \in k\}, \\ U_4 &= \{D(d_0, ..., d_8) \mid d_0, ..., d_8 \in k\}, \\ V_2 &= \{B'(b'_0, ..., b'_4) \mid b'_0, ..., b'_4 \in k\}. \end{aligned} \tag{5.3}$$

Let U_1, V_1 be the images of $\mathfrak{h}$ in $\mathfrak{h}_1, \mathfrak{h}_2$. U_1, V_1 are clearly, sub $\mathfrak{h}$-modules.

LEMMA (5.4). *The subspaces* U_2, U_3, U_4, V_2 *are irreducible sub* $\mathfrak{h}$*-modules with highest weights* $4\Lambda, 6\Lambda, 8\Lambda, 4\Lambda$ *respectively.*

Proof. By straightforward computations,

$$\begin{aligned}
&[A_0, B(1, 0, ..., 0)] = [A_0, C(1, 0, ..., 0)] = [A_0, D(1, 0, ..., 0)] = 0,\\
&[A'_0, B'(1, 0, ..., 0)] = 0,\\
&[A_1, B(1, 0, ..., 0)] = 4B(1, 0, ..., 0),\\
&[A_1, C(1, 0, ..., 0)] = 6C(1, 0, ..., 0),\\
&[A_1, D(1, 0, ..., 0)] = 8D(1, 0, ..., 0),\\
&[A'_1, B'(1, 0, ..., 0)] = 4B'(1, 0, ..., 0).
\end{aligned} \tag{5.5}$$

Also

$$\begin{aligned}
&[A_2, B(b_0, ..., b_4)] = B(0, b_0, 6b_1, b_2, 4b_3),\\
&[A_2, C(c_0, ..., c_6)] = C(0, 2c_0, c_1, -12c_2, c_3, 10c_4, 3c_5),\\
&[A_2, D(d_0, ..., d_8)] = D(0, d_0, 2d_1, 3d_2, ..., 8d_7),\\
&[A'_2, B'(b'_0, ..., b'_4)] = B'(0, b'_0, 2b'_1, 3b'_2, 4b'_3).
\end{aligned} \tag{5.6}$$

The author used MAPLE [1] to find U_2, U_3, U_4 but computed (5.5), (5.6) manually. So these computations can be managed manually in principle, but we checked (5.5) (5.6) by MAPLE also.

By (5.6), U_2 is spanned by elements of the form $\mathrm{ad}(A_2)^i\, B(1, 0, 0, 0, 0)$ ($\mathrm{ad}(*)$ is the adjoint representation). Since

$$\mathrm{ad}(A_0)\, B(1, 0, 0, 0, 0) = 0, \qquad \mathrm{ad}(A_1)\, B(1, 0, 0, 0, 0) = 4B(1, 0, 0, 0, 0),$$

U_2 is an irreducible sub $\mathfrak{h}$-module with highest weight 4Λ.

Other cases are similar. ∎

PROPOSITION (5.7). (1) $[U_2, U_2] = U_1 \oplus U_3$.

(2) $[U_2, U_3] = U_2 \oplus U_4$.

(3) $[U_2, U_4] = U_3$.

(4) $[U_3, U_3] = U_1 \oplus U_3$.

(5) $[U_3, U_4] = U_2 \oplus U_4$.

(6) $[U_4, U_4] = U_1 \oplus U_3$.

(7) $[V_2, V_2] = V_1$.

Proof. We first consider (1). Since U_2 is irreducible, for any non-zero element $X \in U_2$, U_2 is generated by X as an $\mathfrak{h}$-module. So $[U_2, U_2]$ is generated by $[X, U_2]$ as an $\mathfrak{h}$-module also.

By straightforward computations,

$$[B(0,0,1,0,0), B(b_0, ..., b_4)] \\ = -\frac{21b_1}{5} A_0 - \frac{21b_3}{5} A_2 + C\left(0, -4b_0, -\frac{8b_1}{5}, 0, -\frac{8b_3}{5}, -4b_4, 0\right). \tag{5.8}$$

We chose $B(0,0,1,0,0)$ because it is diagonal.

By (5.8), $[U_2, U_2] \subset U_1 \oplus U_3$. By choosing $b_1 = b_3 = 0$ in (5.8), $[U_2, U_2]$ contains a non-zero element of U_3. This implies $[U_2, U_2]$ contains U_3. By choosing $b_1 \neq 0$ in (5.8), $[U_2, U_2]$ contains an element of the form $X + X'$ where $X \in U_1$ is non-zero and $X' \in U_3$. So $X \in [U_2, U_2]$. This implies $[U_2, U_2]$ contains U_1 also. This proves (1).

Other cases follow from the following relations and by similar arguments. We found these relations manually. However, it can be checked by a routine program in MAPLE (which we did).

$$\begin{aligned}
&[B(0,0,1,0,0), C(c_0, ..., c_6)] \\
&\quad = B\left(-\frac{16c_1}{7}, -\frac{16c_2}{7}, 0, -\frac{16c_4}{7}, -\frac{16c_5}{7}\right) \\
&\qquad + D\left(0, -3c_0, -\frac{12c_1}{7}, -\frac{15c_2}{7}, 0, -\frac{15c_4}{7}, -\frac{12c_5}{7}, -3c_6, 0\right), \\
&[B(0,0,1,0,0), D(d_0, ..., d_8)] \\
&\quad = C(-3d_1, -4d_2, -d_3, 0, -d_5, -4d_6, -3d_7), \\
&[C(0,0,0,1,0,0,0), C(c_0, ..., c_6)], \\
&\quad = 6c_2 A_0 - 6c_4 A_2 + C(-c_0, c_1, c_2, 0, -c_4, -c_5, c_6), \\
&[C(0,0,0,1,0,0,0), D(d_0, ..., d_8)], \\
&\quad = B\left(\frac{20d_2}{7}, \frac{10d_3}{7}, 0, -\frac{10d_5}{7}, -\frac{20d_6}{7}\right) \\
&\qquad + D\left(2d_0, -d_1, -\frac{13d_2}{7}, -\frac{9d_3}{7}, 0, \frac{9d_5}{7}, \frac{13d_6}{7}, d_7, -2d_8\right), \\
&[D(0,0,0,0,1,0,0,0,0), D(d_0, ..., d_8)] \\
&\quad = -14d_3 A_0 - 14d_5 A_2 + C(-5d_1, 5d_2, 3d_3, 0, 3d_5, 5d_6, -5d_7), \\
&[B'(0,0,1,0,0), B'(b'_0, ..., b'_4)] \\
&\quad = -3b_1 A'_0 - 3b_3 A'_2. \quad \blacksquare
\end{aligned}$$

Note that the Lie algebras of H_{w1}, H_{w2} are isomorphic to sl(2). We denote the Lie algebra of H_{w3} by so(5) (more precisely so(3, 2)). Since $\dim \mathrm{so}(5) = 10$, $\mathrm{so}(5) = U_1 \oplus U_3$ by counting the dimension.

PROPOSITION (5.9). *If* $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{h}_1 \times \mathfrak{h}_2$ *is a Lie subalgebra,* $\mathfrak{k}$ *is one of the following subalgebras.*

$$\mathfrak{h},\ \mathrm{sl}(2)\times\mathrm{sl}(2),\ \mathrm{sl}(2)\times\mathrm{sl}(3),\ \mathrm{so}(5)\times\mathrm{sl}(2),$$

$$\mathrm{so}(5)\times\mathrm{sl}(3),\ \mathrm{sl}(5)\times\mathrm{sl}(2),\ \mathrm{sl}(5)\times\mathrm{sl}(3).$$

Proof. Let $\mathfrak{k}$ be as above. Note that sl(3) does not contain any $\mathfrak{h}$-module which is isomorphic to U_3 or U_4. Suppose $\mathfrak{k} \supset U_4$. Then $\mathfrak{k} \supset U_1 \oplus U_3$ by Lemma (5.7)(1). So $\mathfrak{k} \supset U_2$ by Lemma (5.7)(5). Since $\mathfrak{k} \supset U_1$, $\mathfrak{k} \supset U_1 \oplus V_1$. Therefore, $\mathfrak{k} \supset \mathrm{sl}(5)\times\mathrm{sl}(2)$. So $\mathfrak{k} = \mathrm{sl}(5)\times\mathrm{sl}(2)$ or $\mathrm{sl}(5)\times\mathrm{sl}(3)$.

Suppose the projection of $\mathfrak{k}$ to the first factor contains U_2. Then there exists an $\mathfrak{h}$-homomorphism $\alpha\colon U_2 \to V_2$ such that $(x, \alpha(x)) \in \mathfrak{k}$ for all $x \in U_2$. By Lemma (5.7)(1), the projection of $\mathfrak{k}$ to the first factor contains U_3. Since U_3 is not equivalent to any other factor, $\mathfrak{k}$ contains U_3. By Lemma (5.7)(4), $\mathfrak{k} \supset U_1$. Since $\mathfrak{k} \supset \mathfrak{h}$, $\mathfrak{k} \supset U_1 \oplus V_1$. If $x \in U_2$, $y \in U_1$, $(y, 0) \in \mathfrak{k}$. So $[(y, 0), (x, \alpha(x))] = ([y, x], 0) \in \mathfrak{k}$. Since $[U_1, U_2] = U_2$, $\mathfrak{k} \supset U_2$. By Lemma (5.7)(2), $\mathfrak{k} \supset U_4$ and it reduces to the previous case.

Suppose $\mathfrak{k}$ does not contain U_4 and the projection to the first factor does not contain U_2. Suppose $\mathfrak{k} \supset U_3$. By Lemma (5.7)(4), $\mathfrak{k} \supset U_1$. Therefore, $\mathfrak{k}$ has so(5) as the first factor. This implies $\mathfrak{k} = \mathrm{so}(5)\times\mathrm{sl}(2)$ or $\mathrm{so}(5)\times\mathrm{sl}(3)$.

Suppose the projection of $\mathfrak{k}$ to the first factor is U_1. If $\mathfrak{k} \supset V_2$, $\mathfrak{k} \supset V_1$ also. Therefore, $\mathfrak{k} = \mathrm{sl}(2)\times\mathrm{sl}(3)$. Otherwise the projection of $\mathfrak{k}$ to both factors are sl(2). Since there is no sub $\mathfrak{h}$-module between $\mathfrak{h}$ and $U_1 \times V_1$, $\mathfrak{k}$ is $\mathfrak{h}$ or $\mathrm{sl}(2)\times\mathrm{sl}(2)$. ∎

Now we specialize to the field $k = \mathbb{R}$.

PROPOSITION (5.10). *Let* $x \in G_{\mathbb{R}} w$ *and* $H_{x\mathbb{R}} \subset F \subset H_{\mathbb{R}}$ *be a closed connected subgroup. Then* F *is one of the following subgroups.*

$$H_{x\mathbb{R}},\ H_{x1\mathbb{R}}\times H_{x2\mathbb{R}},\ H_{x1\mathbb{R}}\times \mathrm{SL}(3)_{\mathbb{R}},$$

$$H_{x3\mathbb{R}}\times H_{x2\mathbb{R}},\ H_{x3\mathbb{R}}\times \mathrm{SL}(3)_{\mathbb{R}},\ \mathrm{SL}(5)_{\mathbb{R}}\times H_{x2\mathbb{R}},\ \mathrm{SL}(5)_{\mathbb{R}}\times \mathrm{SL}(3)_{\mathbb{R}}.$$

Proof. If $x = gw$ for $g \in G_{\mathbb{R}}$, $H_{x\mathbb{R}} = gH_{w\mathbb{R}}g^{-1}$, etc. So we may assume that $x = w$. Then this proposition follows from the previous proposition. ∎

6. AN ANALOGUE OF THE OPPENHEIM CONJECTURE

In this section, we prove an analogue of the Oppenheim conjecture.

In the following lemma, $x \in V_{\mathbb{C}}^{ss}$. We define $H_{x1\mathbb{C}}$, etc. as in Definition (5.1).

LEMMA (6.1). (1) *If $y \in V_{\mathbb{C}}$ is fixed by $H_{x\mathbb{C}}$, y is a scalar multiple of x.*

(2) *If $y \in \mathrm{Sym}^2 V_2^*$ is fixed by $H_{x2\mathbb{C}}$, y is a scalar multiple of $\Phi_1(x)$.*

(3) *If $y \in \mathrm{Sym}^2 V_1^*$ is fixed by $H_{x1\mathbb{C}}$ or $H_{x3\mathbb{C}}$, y is a scalar multiple of $\Phi_2(x)$.*

Proof. Consider (1). Let $x = gw$ with $g \in G_{\mathbb{C}}$. Then $H_{x\mathbb{C}} = gH_{w\mathbb{C}}g^{-1}$, and $g^{-1}y$ is fixed by $H_{w\mathbb{C}}$. So we may assume $x = w$. By (3.1), V contains the trivial representation of $\mathfrak{h}$ precisely once. Therefore, the set of fixed points of $H_{w\mathbb{C}}$ is of dimension one. This proves (1).

Consider the first part of (3). As in (1), we may assume $x = w$. Since $H_{w1\mathbb{C}} \cong \mathrm{PGL}(2)_{\mathbb{C}}$, it is enough to show that $\mathrm{Sym}^2 V_{2\mathbb{C}}^*$ contains the trivial representation of the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathrm{PGL}(2)_{\mathbb{C}}$ precisely once. Let Λ be the fundamental dominant weight of $\mathfrak{h}$ as before. Since $V_1 \cong V_1^* \cong 4\Lambda$, by considering weights, it is easy to see that

$$\mathrm{Sym}^2 V_{2\mathbb{C}}^* \cong (8\Lambda)_{\mathbb{C}} \oplus (4\Lambda)_{\mathbb{C}} \oplus \mathbb{C}.$$

The second part of (3) and (4) are well known and were used in the proof of the Oppenheim conjecture for quadratic forms. ∎

In the following theorem, let $x \in G_{\mathbb{R}}w$. Then $H_{x\mathbb{R}+}^{\circ}$ is generated by unipotent elements. Let $H_{x\mathbb{R}} \subset F \subset H_{\mathbb{R}}$ be the closed connected subgroup such that $\overline{H_{x\mathbb{R}}H_{\mathbb{Z}}} = FH_{\mathbb{Z}}$. By Ratner's theorem (Theorem (0.2)), such F exists.

THEOREM (6.2). (1) *If $\Phi_2(x) \notin \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$, $F = \mathrm{SL}(5)_{\mathbb{R}} \times H_{x2\mathbb{R}}$ or $F = \mathrm{SL}(5)_{\mathbb{R}} \times \mathrm{SL}(3)_{\mathbb{R}}$.*

(2) *If $\Phi_1(x) \notin \mathbb{P}(\mathrm{Sym}^2 V_2^*)_{\mathbb{Q}}$ and $\Phi_2(x) \notin \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$, $F = \mathrm{SL}(5)_{\mathbb{R}} \times \mathrm{SL}(3)_{\mathbb{R}}$.*

Proof. Suppose $F = H_{x\mathbb{R}}$. Then F is defined over $\mathbb{Q}$. Therefore, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, $H_{x\mathbb{C}}^{\sigma} = H_{x\mathbb{C}}$. Since $H_{x\mathbb{C}}^{\sigma} = H_{x^{\sigma}\mathbb{C}}$, x^{σ} is fixed by $H_{x\mathbb{C}}$. So x^{σ} is a scalar multiple of x by Lemma (6.1). Since this is the case for all σ, $[x] \in \mathbb{P}(V)_{\mathbb{Q}}$. Since Φ_1, Φ_2 are defined over $\mathbb{Q}$, $\Phi_1(x)$, $\Phi_2(x)$ are $\mathbb{Q}$-rational points.

We show that $\Phi_2(x) \in \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$ if $F = H_{x1\mathbb{R}} \times H_{x2\mathbb{R}}$ or $H_{x1\mathbb{R}} \times \mathrm{SL}(3)_{\mathbb{R}}$. Since the argument is similar, we only consider the first case.

For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

$$H^{\sigma}_{x1\mathbb{C}} \times H^{\sigma}_{x2\mathbb{C}} = H_{x^{\sigma}1\mathbb{C}} \times H_{x^{\sigma}2\mathbb{C}} = H_{x1\mathbb{C}} \times H_{x2\mathbb{C}}.$$

Since $H_{x\mathbb{C}} \subset H_{x3\mathbb{C}}$, $\Phi_2(x^{\sigma})$ is fixed by $H_{x1\mathbb{C}}$. This implies $\Phi_2(x^{\sigma}) = \Phi_2(x)$. Since Φ_2 is defined over $\mathbb{Q}$, $\Phi_2(x^{\sigma}) = \Phi_2(x)^{\sigma} = \Phi_2(x)$ by Lemma (6.1). Therefore, $\Phi_2(x) \in \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$.

By a similar argument, if $F = H_{x1\mathbb{R}} \times H_{x2\mathbb{R}}$, $H_{3\mathbb{R}} \times H_{x2\mathbb{R}}$, or $\mathrm{SL}(5)_{\mathbb{R}} \times H_{x2\mathbb{R}}$, $[\Phi_1(x)] \in \mathbb{P}(\mathrm{Sym}^2 V_2^*)_{\mathbb{Q}}$. Also if $F = H_{x3\mathbb{R}} \times H_{x2\mathbb{R}}$ or $H_{x3\mathbb{R}} \times \mathrm{SL}(3)_{\mathbb{R}}$, $[\Phi_2(x)] \in \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$.

By these considerations, conditions in (1), (2) force F to become the given subgroups. ▮

LEMMA (6.3). *Let $x \in G_{\mathbb{R}}w$. Then for any non-zero real number r, there exists $h \in H_{\mathbb{R}}$ and a primitive integer point $a \in V_{1\mathbb{Z}}$ such that $F_{h^{-1}x}(a) = r$.*

Proof. We may assume $x = \lambda w$ where $\lambda \in \mathbb{R}\setminus\{0\}$. Since $F_{\lambda h^{-1}w}(a) = \lambda^{60} F_w(ha)$, the above condition is equivalent to $F_w(ha) = \lambda^{-60} r$. Put $t = -\lambda^{-20}(r/2)^{1/3}$. Then

$$h = \left(\begin{pmatrix} t^{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, I_3 \right)$$

and $a = {}^t(0\,0\,1\,0\,0)$ satisfy the condition. ▮

In the following theorem, $x \in G_{\mathbb{R}}w$.

THEOREM (6.4). *If $[\Phi_2(x)] \notin \mathbb{P}(\mathrm{Sym}^2 V_1^*)_{\mathbb{Q}}$, the set of values of the cubic form $F_x(a)$ at primitive integer points is dense in $\mathbb{R}$.*

Proof. Let r be a non-zero real number. We choose $h = (h', h'') \in H_{\mathbb{R}}$ and $a \in V_{1\mathbb{Z}}$ as in Lemma (6.3). By Theorem (6.2), there exist $h_1 = (h_1', h_1'') \in H_{x\mathbb{R}}$ and $h_2 = (h_2', h_2'') \in H_{\mathbb{Z}}$ such that $h_1' h_2'$ is close to h'. Then

$$F_x(h_2' a) = F_{h_2^{-1}x}(a) = F_{h_2^{-1}h_1^{-1}x}(a) = F_{(h_2'^{-1}h_1',\, 1)\, x}(a)$$

is close to

$$F_{(h'^{-1},\, 1)\, x}(a) = F_{h^{-1}x}(a) = r.$$

Note that $F_{h^{-1}x}$ does not depend on the second component of h.

Since $h_2' a \in V_{1\mathbb{Z}}$ is primitive, this proves the theorem. ▮

Note that if $x = gw$ with $g = (g_1, g_2) \in G_{\mathbb{R}}$,

$$[\Phi_2(x)] = g_1[\Phi_2(w)], F_x(a) = (\det g_1)^{15} (\det g_2)^3 F_w(g_1^{-1}a).$$

Therefore, writing down F_w, etc. explicitly, we get the statement of Theorem (0.3).

Remark (6.5). We proved Theorem (6.4) as a consequence of Theorem (6.2). But we don't need our prehomogeneous vector space if we just want to prove Theorem (6.4). For that purpose, we only have to consider the situation $\mathrm{PGL}(2) \subset \mathrm{SL}(5)$ and apply Ratner's theorem using the computations in Section 5. We discuss this issue in [28].

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