Riesz potentials and integral geometry in the space of rectangular matrices

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Abstract

Riesz potentials on the space of rectangular $n \times m$ matrices arise in diverse “higher rank” problems of harmonic analysis, representation theory, and integral geometry. In the rank-one case $m=1$ they coincide with the classical operators of Marcel Riesz. We develop new tools and obtain a number of new results for Riesz potentials of functions of matrix argument. The main topics are the Fourier transform technique, representation of Riesz potentials by convolutions with a positive measure supported by submanifolds of matrices of rank $< m$, the behavior on smooth and $L^p$ functions. The results are applied to investigation of Radon transforms on the space of real rectangular matrices.

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1. Introduction

Problems of harmonic analysis and integral geometry on $\mathbb{R}^n$ get a new flavor if we replace $n$ by the product $nm$, and regard $\mathbb{R}^{nm}$ as the space $\mathbb{M}_{n,m}$ of $n \times m$ matrices $x = (x_{i,j})$. Once we accept this point of view, a number of new “higher rank” phenomena come into play. In the present paper we focus on analytic tools which allow us to investigate the properly defined Radon transform $f(x) \rightarrow \hat{f}(\tau)$ on $\mathbb{M}_{n,m}$. This transform assigns to a function $f(x)$ a collection of integrals of $f$ over the so-called matrix $k$-planes $\tau$ in $\mathbb{M}_{n,m}$. Each such plane represents a certain linear manifold defined by the corresponding matrix equation, and is, in fact, an ordinary $km$-dimensional plane in $\mathbb{R}^{nm}$. An important feature is that the manifold $\mathcal{F}$ of all matrix $k$-planes is “much smaller” than the manifold of all $km$-dimensional planes in $\mathbb{R}^{nm}$. The main problem is to reconstruct $f(x)$ from known data $\hat{f}(\tau)$, $\tau \in \mathcal{F}$, for possibly large class of functions $f$.

In the rank-one case $m = 1$ this problem is well investigated; see, e.g., [E,G,GG,H,Hi,Ru2], and references therein. The standard tools are the Fourier analysis and the corresponding real-variable techniques, including approximation to the identity, singular integral operators, and especially, Riesz potentials and fractional integrals. In the higher rank case $m \geq 2$, all these tools still require a further development.
Let us pass to details. Suppose that $f(x)$ is a Schwartz function on $\mathfrak{M}_{n,m}$, $n \geq m$. The Fourier transform of $f$ is defined by

$$\mathcal{F} f(y) = \int_{\mathfrak{M}_{n,m}} \exp(\text{tr}(iy'x)) f(x) \, dx, \quad y = (y_{i,j}) \in \mathfrak{M}_{n,m},$$

where $y'$ denotes the transpose of $y$, $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$. This is the ordinary Fourier transform on $\mathbb{R}^{nm}$. Given a complex number $\alpha$, we denote

$$Z(f, \alpha - n) = \int_{\mathfrak{M}_{n,m}} f(x) |x|_{m}^{\alpha - n} \, dx,$$

where $|x|_m = \det(x'x)^{1/2}$ is the volume of the parallelepiped spanned by the column-vectors of the matrix $x$. For $m = 1$, $|x|_m$ is the usual euclidean norm on $\mathbb{R}^n$. Integrals (1.2) converge absolutely for $\Re \alpha > m - 1$ and extend meromorphically to other $\alpha$. Such integrals are known as zeta integrals, the $S'$-distribution defined by analytic continuation of (1.2) is called a zeta distribution, and the corresponding function $\alpha \to Z(f, \alpha - n)$ is sometimes called a zeta function. The Riesz potential associated to the zeta integral (1.2) is defined by

$$(I^{\alpha} f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int_{\mathfrak{M}_{n,m}} f(x - y) |y|^{\alpha - n} \, dy,$$

where $\gamma_{n,m}(\alpha) = \frac{2^{nm} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n - \alpha)/2)}$, $\alpha \neq n - m + 1, n - m + 2, \ldots$. (1.3)

These operators play a vital role in integral geometry and analysis on matrix spaces [Kh,OR1,P,Sh]. If $m = 1$ then $I^{\alpha} f$ coincides with the classical potential of Riesz [Ri,L,Ru1,SKM].

We begin our study with the functional equation

$$\frac{Z(f, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} \gamma_m(\alpha - n) \frac{Z(\mathcal{F} f, \alpha)}{\Gamma_m((\alpha - n)/2)},$$

where the normalizing denominators are Siegel gamma functions associated to the cone of positive definite symmetric $m \times m$ matrices. Both sides of (1.4) are understood in the sense of analytic continuation. Equality (1.4) implies

$$\mathcal{F}[I^{\alpha} f](y) = |y|_{m}^{-\alpha}(\mathcal{F} f)(y)$$

and is of fundamental importance. Different modifications of (1.4) can be found in [Cl,Far,FK,Ge,Ra,Sh]. Unfortunately, the argument in these publications is not “self-contained enough” and some crucial details are skipped. These details play, in fact, a
decisive role, by taking into account an essential difference between the case \(2m < n+2\) and \(2m \geq n + 2\) (this difference is not indicated in some papers). The crux is that the distribution on the left-hand side of (1.4) is regular if and only if \(\text{Re} \, x > m - 1\), whereas the right-hand side is regular if and only if \(\text{Re} \, x < n - m + 1\). For \(2m \geq n + 2\) these two sets on the complex plane are separated one from another! In the case \(2m < n + 2\) the proof is elementary [OR1]. For \(2m \geq n + 2\), justification of (1.4) represents a difficult problem.

The phenomenon of lack of common domain of regularity was investigated by Stein [St1] for Riesz distributions on the space of complex \(n \times n\) matrices. For more information on zeta distributions, see [Ig,SS,Shin], and references therein.

For convenience of the readers and our future work in the area (see, e.g., [OR2]), we present a relatively simple proof of (1.4). Our argument contains important new elements which make the proof essentially self-contained and transparent. The core of the proof is the so-called Bernstein identity

\[
\Delta |x|^\lambda_m = B(\lambda)|x|^\lambda_{m-2},
\]

\[
B(\lambda) = (-1)^m \prod_{i=0}^{m-1} (\lambda + i)(2 - n - \lambda + i),
\]

where \(\Delta\) is the Cayley–Laplace operator defined by \(\Delta = \det(\partial_i \partial_j)\), \(\partial = (\partial_{i,j})_{n \times m}\). The idea of such an identity amounts to pioneering papers of Bernstein; see, e.g., [B,Ig].

The novelty of our approach is that we first obtain an explicit formula for the radial part of \(\Delta\). This formula makes derivation of (1.5) completely elementary. It is worth noting that the proof of the Bernstein identity in [FK,Cl] employs the heat kernel and integration by parts over the corresponding cone. Justification of this integration by parts was unfortunately skipped, and we could not reconstruct it without imposing artificial restrictions on \(n\) and \(m\). That was one of the reasons for us why we tried to find an alternative approach and present it in full detail. We hope that our method can be extended to more general settings similar to those in [FK,Cl].

The second concern of the paper is connected with the so-called Wallach set \(\mathcal{W}\) of the normalized zeta distribution

\[
(\zeta_x, f) = a.c. \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathbb{R}^n_{a,m}} f(x) |x|^\alpha_n \, dx, \quad f \in \mathcal{S}(\mathbb{R}^n_{a,m}),
\]

where “a.c.” abbreviates analytic continuation. This notion was introduced in [FK] for another class of distributions associated to symmetric cones. The expression \((\zeta_x, f)\) is an entire function of \(x\). The Wallach set \(\mathcal{W}\) is constituted by all \(x\) for which \(\zeta_x\) is a positive measure. Clearly, each \(x > m - 1\) belongs to \(\mathcal{W}\). We show that the integers \(x = 0, 1, \ldots, m - 1\) (outside of the domain of regularity) also belong to \(\mathcal{W}\). Note that in the rank-one case, the discrete part of \(\mathcal{W}\) is trivial, namely, we have only one value \(x = 0\) corresponding to the Dirac mass \(\zeta_0\) at the origin. We obtain explicit representation
of $\zeta_\alpha$ for all $\alpha = 0, 1, \ldots, m-1$, and investigate convolutions $(\zeta_\alpha \ast f)(x)$ assuming $f \in L^p(\mathcal{M}_{n,m})$. In particular, we prove that if $Re \alpha > m-1$ or $\alpha = 1, 2, \ldots, m-1$, then these convolutions are well defined almost everywhere on $\mathcal{M}_{n,m}$ provided

$$1 \leq p < \frac{n}{Re \alpha + m - 1}. \quad (1.7)$$

This result agrees with the case $m = 1$ in [St2]. The question whether condition (1.7) is sharp remains open.

We conjecture that the values $\alpha = 0, 1, \ldots, m-1$ and $\alpha > m-1$ exhaust the Wallach set $\mathcal{W}$, i.e., for no other $\alpha$, the distribution $\zeta_\alpha$ is a positive measure. We plan to investigate this question in forthcoming publications.

The operator $I^\alpha$ can be regarded as the $(-\alpha/2)$th power of $(-1)^m \Delta$ where $\Delta$ is the Cayley–Laplace operator. Our investigation of the Riesz potentials $I^\alpha f$ employs matrix modification of the Gauss–Weierstrass integral (see [SW,Ta] for $m = 1$) defined by

$$(W_tf)(x) = \int_{\mathcal{M}_{n,m}} h_t(x - y) f(y) \, dy, \quad t \in \mathcal{P}_m, \quad (1.8)$$

where $\mathcal{P}_m$ is the cone of positive definite $m \times m$ matrices, and

$$h_t(x) = (4\pi)^{-nm/2} \det(t)^{-n/2} \exp(-\text{tr}(t^{-1} x'x)/4) \quad (1.9)$$

is the corresponding heat kernel on $\mathcal{M}_{n,m}$. This approach allows us to circumvent essential technical difficulties. The idea is to represent the Riesz potential through the lower-dimensional Gårding–Gindikin fractional integral by a simple formula

$$(W_t[I^\alpha f])(x) = (I^\alpha_{-\alpha/2} g_x)(t), \quad g_x(t) = (W_tf)(x), \quad (1.10)$$

provided $m - 1 < Re \alpha < n - m + 1$. We recall that the Gårding–Gindikin fractional integral on $\mathcal{P}_m$ has the form

$$(I^\lambda_{-\lambda} g)(t) = \frac{1}{\Gamma(m(\lambda))} \int_t^\infty g(\tau) \det(\tau - t)^{\lambda-d} \, d\tau, \quad (1.11)$$

where $d = (m+1)/2$, $Re \lambda > d - 1$, and integration is performed over all $\tau \in \mathcal{P}_m$ so that $\tau - t \in \mathcal{P}_m$. These integrals were introduced by Lars Gårding [Gå1] who wrote [Gå2]:

‘... Actually the origin of my integral was a statistics paper (samples of mean, variance etc. from a multivariate Gaussian distribution). The idea to use it in analysis came from my many meetings with my beloved teacher Marcel Riesz’.
Integrals (1.11) were substantially generalized by Gindikin [Gi]. They have proved to be an important tool in PDE [VG,Rab], and in the theory of Radon transforms on Grassmann manifolds and matrix spaces [GR,OR2,Ru3]. If \( t = 0 \), then \( W_t \) turns into the identity operator and (1.10) reads

\[
(I^2 f)(x) = \frac{1}{\Gamma_m(z/2)} \int_{P_m} |t|^{z/2-d} (W_t f)(x) \, dt.
\] (1.12)

This representation is well known in the case \( m = 1 \); see, e.g., [Ru1].

The third topic of the paper is application of zeta distributions to integral geometry. It is based on intimate connection between the corresponding Riesz potentials (1.3) and the Radon transform \( f \to \hat{f} \) on \( M_{n,m} \). This connection is realized through the generalized Fuglede formula

\[
(\hat{f})^\vee(x) = \text{const} \ (I_k f)(x)
\] (1.13)

which is well known for \( m = 1 \) [Fu,Hel]. Here the left-hand side is the mean value of \( \hat{f}(\tau) \) over all matrix \( k \)-planes \( \tau \) “passing through \( x \”). For sufficiently good functions \( f \), this formula was established in [OR1]. We justify it for all \( f \in L^p(\mathfrak{M}_{n,m}), 1 \leq p < n/(k + m - 1) \), and apply to the inversion problem for the Radon transform \( f \to \hat{f} \). Solution to this problem is presented in the framework of the relevant theory of distributions. For \( m = 1 \), such a theory was developed by Semyanistyi [Se].

The paper is organized as follows. In Section 2 we establish our notation, recall some basic facts, and make necessary preparations. In Section 3 we derive an explicit formula for the radial part of the Cayley–Laplace operator and prove the Bernstein identity (1.5). In Section 4 we study the zeta integral (1.2), prove (1.4), and obtain different explicit representations of the normalized zeta distribution (1.6) for \( z = 0, 1, \ldots, m - 1 \). Convolutions with zeta distributions and Riesz potentials of \( L^p \) functions are investigated in Section 5. Section 6 is devoted to application of the results of preceding sections to integral geometry. We define the Radon transform over matrix \( k \)-planes and justify the generalized Fuglede formula (1.13) for \( f \in L^p(\mathfrak{M}_{n,m}), 1 \leq p < n/(k + m - 1) \). Then we introduce the space \( \Phi' \) of distributions of the Semyanistyi type. This space and the relevant space of test functions have proved to be useful in further developments [Kh4]. Two \( \Phi' \)-distributions coincide if and only if their Fourier transforms differ by a tempered distribution supported by the singular set \( \{ y : y \in \mathfrak{M}_{n,m}, \text{rank}(y) < m \} \). An inversion formula for the Radon transform \( f \to \hat{f} \) for \( f \in L^p(\mathfrak{M}_{n,m}) \) is then obtained in the \( \Phi' \)-sense. For convenience of the reader, evaluation of some useful integrals is presented in Appendix A.

Some open problems that might be of interest are stated in Remarks 5.5, 5.14, and 6.9.

After the manuscript was submitted, a series of new related results, containing further progress, has been obtained; see [BSZ,Kh1,Kh2,Kh3,Kh4,OOR,OR3].
2. Preliminaries

2.1. Matrix spaces

Let \( \mathcal{M}_{n,m} \) be the space of real matrices having \( n \) rows and \( m \) columns. We identify \( \mathcal{M}_{n,m} \) with the real Euclidean space \( \mathbb{R}^{nm} \). The letters \( x, y, r, s \) etc. stand for both the matrices and points since it is always clear from the context which is meant. If \( x = (x_{i,j}) \in \mathcal{M}_{n,m} \), we write \( dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j} \) for the elementary volume in \( \mathcal{M}_{n,m} \). In the following \( x' \) denotes the transpose of \( x \), \( I_m \) is the identity \( m \times m \) matrix, \( 0 \) stands for zero entries. Given a square matrix \( a \), we denote by \( |a| \) the absolute value of the determinant of \( a \); \( \text{tr}(a) \) stands for the trace of \( a \).

Let \( S_m \) be the space of \( m \times m \) real symmetric matrices \( s = (s_{i,j}), s_{i,j} = s_{j,i} \). It is a measure space isomorphic to \( \mathbb{R}^{m(m+1)/2} \) with the volume element \( ds = \prod_{i \leq j} ds_{i,j} \). We denote by \( \mathcal{P}_m \) the open convex cone of positive definite matrices in \( S_m \); \( \overline{\mathcal{P}}_m \) is the closure of \( \mathcal{P}_m \) that consists of positive semi-definite matrices. For \( r \in \mathcal{P}_m \) we write \( r > 0 \). The inequality \( r_1 > r_2 \) means \( r_1 - r_2 \in \mathcal{P}_m \). If \( a \) and \( b \) are positive semi-definite matrices, the symbol \( \int_a^b f(s)ds \) denotes integration over the set

\[ \{s : s \in \mathcal{P}_m, a < s < b\} = \{s : s - a \in \mathcal{P}_m, b - s \in \mathcal{P}_m\}. \]

The group \( G = GL(m, \mathbb{R}) \) of real non-singular \( m \times m \) matrices \( g \) acts on \( \mathcal{P}_m \) transitively by the rule \( r \rightarrow grg' \). The corresponding \( G \)-invariant measure on \( \mathcal{P}_m \) is

\[ d_\star r = |r|^{-d} dr, \quad |r| = \det(r), \quad d = (m + 1)/2 \] (2.1)
[Te, p. 18]. The cone \( \mathcal{P}_m \) is a \( G \)-orbit in \( S_m \) of the identity matrix \( I_m \). The boundary \( \partial \mathcal{P}_m \) of \( \mathcal{P}_m \) is a union of \( G \)-orbits of \( m \times m \) matrices

\[ e_k = \begin{bmatrix} I_k & 0 \\
0 & 0 \end{bmatrix}, \quad k = 0, 1, \ldots, m - 1. \]

More information about the boundary structure of \( \mathcal{P}_m \) can be found in [FK, p. 72, Bar, p. 78].

We denote by \( T_m \) a group of upper triangular matrices

\[ t = \begin{bmatrix} t_{1,1} & \ddots & t_* \\
& \ddots & \ddots \\
0 & \ddots & t_{m,m} \end{bmatrix}, \quad t_{i,i} > 0, \]

\[ t_* = \{t_{i,j} : i < j\} \in \mathbb{R}^{m(m-1)/2}. \] (2.2)
Each \( r \in \mathcal{P}_m \) has a unique representation \( r = t' t, \ t \in T_m \), so that
\[
\int_{\mathcal{P}_m} f(r) \, dr = \int_0^\infty t_{1,1}^m \, dt_{1,1} \int_0^\infty t_{2,2}^{m-1} \, dt_{2,2} \ldots \\
\times \int_0^\infty t_{m,m} \, \tilde{f}(t_{1,1}, \ldots, t_{m,m}) \, dt_{m,m},
\]
\[
\tilde{f}(t_{1,1}, \ldots, t_{m,m}) = 2^m \int_{\mathbb{R}^{m(m-1)/2}} f(t' t) \, dt_\ast, \quad dt_\ast = \prod_{i<j} dt_{i,j}, \tag{2.3}
\]

[Te, p. 22, Mu, p. 592]. In the last integration, the diagonal entries of the matrix \( t \) are given by the arguments of \( \tilde{f} \), and the strictly upper triangular entries of \( t \) are variables of integration.

Let us recall some useful formulas for Jacobians.

**Lemma 2.1** (see, e.g., Muirhead [Mu, pp. 57–59]).

(i) If \( x = ayb \), where \( y \in \mathbb{M}_{n,m}, \ a \in GL(n, \mathbb{R}), \ b \in GL(m, \mathbb{R}) \), then \( dx = |a|^m |b|^n \, dy \).

(ii) If \( r = qsq' \), where \( s \in S_m, \ q \in GL(m, \mathbb{R}) \), then \( dr = |q|^{m+1} \, ds \).

(iii) If \( r = s^{-1} \), where \( s \in \mathcal{P}_m \), then \( r \in \mathcal{P}_m \), and \( dr = |s|^{-m-1} \, ds \).

In the following \((\lambda, m) = \lambda(\lambda+1) \ldots (\lambda+m-1)\) is the Pochhammer symbol, \( \delta_{ij} \) is the Kronecker delta; “a.c.” abbreviates analytic continuation. To facilitate presentation, we shall use the symbols “\( \simeq \)” and “\( \ll \)”, instead of “\( = \)” and “\( \leq \)”, respectively, to indicate that the corresponding relation holds up to a constant multiple. All standard spaces of functions of matrix argument, say, \( x = (x_{i,j}) \), are identified with the corresponding spaces of functions of \( nm \) variables \( x_{1,1}, x_{1,2}, \ldots, x_{n,m} \). The Fourier transform of a function \( f \in L^1(\mathbb{M}_{n,m}) \) is defined by
\[
(\mathcal{F} f)(y) = \int_{\mathbb{M}_{n,m}} \exp(\text{tr}(iy'x)) f(x) \, dx, \quad y \in \mathbb{M}_{n,m}. \tag{2.4}
\]

### 2.2. Gamma functions, beta functions, and Bernstein polynomials

The Siegel gamma function associated to the cone \( \mathcal{P}_m \) is defined by
\[
\Gamma_m(x) = \int_{\mathcal{P}_m} \exp(-\text{tr}(r)) |r|^{x-d} \, dr, \quad d = (m+1)/2. \tag{2.5}
\]
It is easy to check [Mu, p. 62], that integral (2.5) converges absolutely if and only if \( \text{Re } \beta > d - 1 \), and represents a product of ordinary gamma functions:

\[
\Gamma_m(\beta) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\beta - j/2). \tag{2.6}
\]

This implies

\[
(-1)^m \frac{\Gamma_m(1 - \beta/2)}{\Gamma_m(-\beta/2)} = 2^{-m} \frac{\Gamma(\beta + m)}{\Gamma(\beta)} = 2^{-m}(\beta, m), \tag{2.7}
\]

\((\beta, m) = \beta(\beta + 1) \cdots (\beta + m - 1)\) being the Pochhammer symbol. If \( 1 \leq k < m, k \in \mathbb{N} \), then

\[
\Gamma_m(\beta) = \pi^{k(m-k)/2} \Gamma_k(\beta) \Gamma_{m-k}(\beta - k/2), \tag{2.8}
\]

\[
\frac{\Gamma_m(\beta)}{\Gamma_m(\beta + k/2)} = \frac{\Gamma(\beta + (k - m)/2)}{\Gamma_k(\beta + k/2)}. \tag{2.9}
\]

The beta function of the cone \( \mathcal{P}_m \) is defined by

\[
B_m(\alpha, \beta) = \int_0^1 |r|^{\alpha-d} |I_m - r|^{\beta-d} dr, \quad d = (m + 1)/2. \tag{2.10}
\]

This integral converges absolutely if and only if \( \text{Re } \alpha, \text{Re } \beta > d - 1 \), and obeys the classical relation [FK, p. 130]

\[
B_m(\alpha, \beta) = \frac{\Gamma_m(\alpha) \Gamma_m(\beta)}{\Gamma_m(\alpha + \beta)}. \tag{2.11}
\]

Let \( r = (r_{i,j}) \in \mathcal{P}_m \). We define the following differential operator acting in the \( r \)-variable:

\[
D \equiv D_r = \det \left( \eta_{i,j} \frac{\partial}{\partial r_{i,j}} \right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j. \end{cases} \tag{2.12}
\]

**Lemma 2.2.**

(i) For \( r \in \mathcal{P}_m \) and \( z \in \mathcal{S}^C_m \) (the complexification of \( \mathcal{S}_m \)),

\[
D_r[\exp(-\text{tr}(rz))] = (-1)^m \det(z) \exp(-\text{tr}(rz)). \tag{2.13}
\]
(ii) For \( r \in \mathcal{P}_m \) and \( z \in \mathbb{C} \),

\[
D(|r|^{2-d}/\Gamma_m(z)) = |r|^{z-1-d}/\Gamma_m(z-1), \quad d = (m+1)/2.
\] (2.14)

The proof of this statement is simple and can be found in [Gå1, p. 813, FK, p. 125].

By changing notation and using (2.7), one can write (2.14) as

\[
D|r|^z = b(z)|r|^{z-1},
\] (2.15)

where

\[
b(z) = z(z + 1/2) \cdots (z + d - 1)
\] (2.16)

is the so-called Bernstein polynomial of the determinant [FK].

2.3. Bessel functions of matrix argument

We recall some facts from [Herz,FK].

2.3.1. \( J \)-Bessel functions

The \( J \)-Bessel function \( J_\nu(r) \), \( r \in \mathcal{P}_m \), can be defined in terms of the Laplace transform by the property

\[
\int_{\mathcal{P}_m} \exp(-\text{tr}(zr))J_\nu(r)|r|^{\nu-d} \, dr = \Gamma_m(\nu) \exp(-\text{tr}(z^{-1})) \det(z)^{-\nu},
\]

\[
d = (m+1)/2, \quad z \in \mathcal{T}_m = \mathcal{P}_m + iS_m
\] (2.17)

(the branch of the multi-valued function \( \det(z)^{-\nu} \) is chosen so that for \( z \in \mathcal{P}_m \), the argument of \( \det(z)^{-\nu} \) is zero). This gives

\[
J_\nu(r)|r|^{\nu-d} = \frac{\Gamma_m(\nu)}{(2\pi i)^N} \int_{Rez=\sigma_0} \exp(\text{tr}(rz - z^{-1})) \det(z)^{-\nu} \, dz,
\] (2.18)

\( \sigma_0 \in \mathcal{P}_m, \ N = m(m+1)/2 \), or, by changing variable,

\[
J_\nu(r) = \frac{\Gamma_m(\nu)}{(2\pi i)^N} \int_{Rez=\sigma_0} \exp(\text{tr}(z - rz^{-1})) \det(z)^{-\nu} \, dz.
\] (2.19)
Both integrals are absolutely convergent for $\Re \nu > m$. Applying operator (2.12), we obtain the following recurrence formulae:

$$
D[\mathcal{J}_\nu(r)|r|^{\nu-d}] = \frac{\Gamma_m(\nu)}{\Gamma_m(\nu-1)} \mathcal{J}_{\nu-1}(r)|r|^{\nu-1-d},
$$
(2.20)

$$
D[\mathcal{J}_\nu(r)] = (-1)^m \frac{\Gamma_m(\nu)}{\Gamma_m(\nu+1)} \mathcal{J}_{\nu+1}(r).
$$
(2.21)

Eq. (2.20) allows to extend $\mathcal{J}_\nu(r)$ analytically to all $\nu \in \mathbb{C}$.

There is an intimate connection between the Fourier transform and $J$-Bessel functions.

**Theorem 2.3** (Herz [Herz, p. 492], Faraut and Korányi [FK, p. 355]). Let $f(x)$ be an integrable function on $\mathfrak{M}_{n,m}$ of the form $f(x) = f_0(x^\prime x)$, where $f_0$ is a function on $\mathcal{P}_m$. Then

$$(\mathcal{F}f)(y) = \int_{\mathfrak{M}_{n,m}} \exp(\text{tr}(iy^\prime x)) f_0(x^\prime x) \, dx = \frac{\pi^{mn/2}}{\Gamma_m(n/2)} \tilde{f}_0 \left( \frac{y^\prime y}{4} \right), \quad (2.22)$$

where

$$
\tilde{f}_0(s) = \int_{\mathcal{P}_m} J_{n/2}(rs)|r|^{n/2-d} f_0(r) \, dr, \quad d = (m+1)/2, \quad (2.23)
$$

is the Hankel transform of $f_0$.

This statement is a matrix generalization of the classical result of Bochner, see, e.g., [SW, Chapter IV, Theorem 3.3].

One can meet another notation for the $J$-Bessel function in the literature. We follow the notation from [FK]. It relates to the notation $A_\delta(r)$ in [Herz] by the formula $J_{\nu}(r) = \Gamma_m(\nu)A_\delta(r)$, $\delta = \nu - d$. For $m = 1$, the classical Bessel function $J_{\nu}(r)$ expresses through $J_{\nu}(r)$ as

$$
J_{\nu}(r) = \frac{1}{\Gamma(v+1)} \left( \frac{r}{2} \right)^v J_{v+1} \left( \frac{r^2}{4} \right).
$$

2.3.2. $K$-Bessel functions

Let, as above, $r \in \mathcal{P}_m$, $d = (m+1)/2$. The $K$-Bessel function $K_{\nu}(r)$ is defined by

$$
K_{\nu}(r) = \int_{\mathcal{P}_m} \exp(\text{tr}(-s - rs^{-1}))|s|^{v-d} \, ds, \quad (2.24)
$$
or (replace $s$ by $s^{-1}$)

$$K_v(r) = \int_{\mathcal{P}_m} \exp(\text{tr}(-s^{-1} - rs)) |s|^{-v-d} \, ds, \quad (2.25)$$

see [Te,FK]. For $m = 1$,

$$K_v(r) = \int_0^\infty \exp(-s - r/s) s^{v-1} \, ds = 2r^{v/2} K_v(2\sqrt{r}), \quad (2.26)$$

$K_v$ being the classical Macdonald function.

**Lemma 2.4.** Let $r \in \mathcal{P}_m$, $d = (m + 1)/2$.

(i) Integral (2.24) (or (2.25)) converges absolutely for all $v \in \mathbb{C}$, and is an entire function of $v$.

(ii) The following estimates hold:

(a) For $\text{Re} \, v > d - 1$:

$$|K_v(r)| \leq \Gamma_m(\text{Re} \, v). \quad (2.27)$$

(b) For $\text{Re} \, v < 1 - d$:

$$|K_v(r)| \leq \Gamma_m(-\text{Re} \, v) |r|^\text{Re} \, v. \quad (2.28)$$

(c) For $1 - d \leq \text{Re} \, v \leq d - 1$:

$$|K_v(r)| \leq \Gamma_m(d) + |r|^{1-d} \Gamma_m(d - 1 + \varepsilon) \quad \forall \varepsilon > 0. \quad (2.29)$$

(iii) If $\text{Re} \, v < 1 - d$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{-mv} K_v(\varepsilon r) = \Gamma_m(-v) |r|^v. \quad (2.30)$$

**Proof.** All statements, with probable exception of (2.29), are known [FK]. Estimate (2.27) follows from (2.24), (2.28) and (2.30) are consequences of (2.25). To prove (2.29), we write

$$K_v(r) = \left( \int_{|s| > 1} + \int_{|s| < 1} \right) \exp(\text{tr}(-s - rs^{-1})) |s|^{v-d} \, ds = I_1 + I_2.$$

For $I_1$ we have

$$|I_1| < \int_{|s| > 1} \exp(\text{tr}(-s)) \, ds < \Gamma_m(d)$$
provided \( \text{Re} \, \psi \leq d \). For \( I_2 \), by changing variable \( s \to s^{-1} \), we obtain

\[
I_2 = \int_{|s|>1} \exp(\text{tr}(-s^{-1} - rs)) |s|^{-\psi - d} \, ds.
\]

If \( \text{Re} \, \psi \geq 1 - d \), then \( |s|^{-\text{Re} \, \psi - d} \leq |s|^\varepsilon - 1 \) \( \forall \varepsilon > 0 \), and

\[
|I_2| \leq \int_{|s|>1} \exp(\text{tr}(-s^{-1} - rs)) |s|^\varepsilon - 1 \, ds
\]

\[
< \int_{\mathcal{P}_m} \exp(\text{tr}(-rs)) |s|^\varepsilon - 1 \, ds
\]

\[
= \Gamma_m(d - 1 + \varepsilon) |r|^{1-d-\varepsilon}.
\]

This gives (2.29). \( \Box \)

### 2.4. Stiefel manifolds

For \( n \geq m \), let \( V_{n,m} = \{ v \in \mathfrak{W}_{n,m} : v'v = I_m \} \) be the Stiefel manifold of orthonormal \( m \)-frames in \( \mathbb{R}^n \). If \( n = m \), then \( V_{n,n} = O(n) \) is the orthogonal group in \( \mathbb{R}^n \). The group \( O(n) \) acts on \( V_{n,m} \) transitively by the rule \( g : v \to gv, \quad g \in O(n) \), in the sense of matrix multiplication. The same is true for the special orthogonal group \( SO(n) \) provided \( n > m \). We fix the corresponding invariant measure \( dv \) on \( V_{n,m} \) normalized by

\[
\sigma_{n,m} \equiv \int_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}.
\]  

**Lemma 2.5** (Muirhead [Mu, p. 589]). Let \( x \) and \( y \) be real matrices such that \( x \) is \( n \times k \) and \( y \) is \( m \times k \), \( n \geq m \). Then \( x'x = y'y \) if and only if there exists \( v \in V_{n,m} \) such that \( x = vy \). In particular, if \( x \) is \( n \times m \), \( n \geq m \), then there exists \( v \in V_{n,m} \) such that \( x = vy \), where \( y = (x'x)^{1/2} \).

**Lemma 2.6** (Polar decomposition). Let \( x \in \mathfrak{W}_{n,m}, \ n \geq m \). If \( \text{rank}(x) = m \), then

\[
x = vr^{1/2}, \quad v \in V_{n,m}, \quad r = x'x \in \mathcal{P}_m
\]

and \( dx = 2^{-m} |r|^{(n-m-1)/2} \, dr \, dv \).

This statement and its generalizations can be found in different places, see, e.g., [Herz, p. 482, Mu, pp. 66, 591, FT, p. 130]. A modification of Lemma 2.6 in terms of upper triangular matrices \( t \in T_m \) (see (2.2)) reads as follows.
Lemma 2.7. Let \( x \in \mathfrak{M}_{n,m} \), \( n \geq m \). If \( \text{rank}(x) = m \), then

\[
x = vt, \quad v \in V_{n,m}, \quad t \in T_m,
\]

so that

\[
dx = \prod_{j=1}^{m} t_{j,j}^{n-j} dt_{j,j} dt_s dv, \quad dt_s = \prod_{i<j} dt_{i,j}.
\]

**Proof.** This statement is also well known. It can be easily derived from Lemma 2.6 and (2.3). Indeed, if \( \text{rank}(x) = m \), then \( x'x = t \in P_m \) and there exists \( t \in T_m \) such that \( x'x = t't \). We set \( v = xt^{-1} \). Then \( v'v = I_m \) and, therefore, \( v \in V_{n,m} \). This proves the representation \( x = vt \), where \( v \in V_{n,m} \) and \( t \in T_m \). Furthermore, by Lemma 2.6 and (2.3),

\[
\int_{\mathfrak{M}_{n,m}} f(x) dx = 2^{-m} \int_{V_{n,m}} dv \int_{P_m} |r|^{(n-m-1)/2} f(vr^{1/2}) dr \quad \text{ (2.3)}
\]

\[
\overset{\text{(2.3)}}{=} \int_{V_{n,m}} dv \int_{T_m} f(v(t't)^{1/2}) \prod_{j=1}^{m} t_{j,j}^{n-j} dt_{j,j} dt_s dv.
\]

Now we denote \( \lambda = (t't)^{1/2} t^{-1} \in O(m) \), then change the order of integration, and set \( v\lambda = u \). This gives

\[
\int_{\mathfrak{M}_{n,m}} f(x) dx = \int_{T_m} \prod_{j=1}^{m} t_{j,j}^{n-j} dt_{j,j} dt_s \int_{V_{n,m}} f(ut) du
\]

and we are done. \( \square \)

3. Radial functions and the Cayley–Laplace operator

A function \( f(x) \) on \( \mathfrak{M}_{n,m} \) is called radial, if there exists a function \( f_0(r) \) on \( P_m \) such that \( f(x) = f_0(x'x) \) for all (or almost all) matrices \( x \in \mathfrak{M}_{n,m} \). One can readily check that \( f \) is radial if and only if it is \( O(n) \) left-invariant, i.e., \( f(\gamma x) = f(x) \) for all \( \gamma \in O(n) \).

The **Cayley–Laplace operator** \( \Delta \) on the space \( \mathfrak{M}_{n,m} \) of matrices \( x = (x_{i,j}) \) is defined by

\[
\Delta = \det(\partial^\gamma \partial^\gamma).
\]

(3.1)
Here $\hat{\partial}$ is an $n \times m$ matrix whose entries are partial derivatives $\hat{\partial}/\hat{\partial} x_{i,j}$. In the Fourier transform terms, the action of $\Delta$ represents a multiplication by the polynomial $(-1)^m P(y)$, where $y = (y_{i,j}) \in \mathfrak{N}_{n,m}$,

$$P(y) = |y'y| = \det \begin{bmatrix} y_1 \cdot y_1 \cdots y_1 \cdot y_m \\ \vdots \cdots \cdots \\ y_m \cdot y_1 \cdots y_m \cdot y_m \end{bmatrix},$$

$y_1, \ldots, y_m$ are column-vectors of the matrix $y$, and "\cdot" stands for the usual inner product in $\mathbb{R}^n$. Clearly, $P(y)$ is a homogeneous polynomial of degree $2m$ of $nm$ variables $y_{i,j}$, and $\Delta$ is a homogeneous differential operator of order $2m$. For $m = 1$, it coincides with the Laplace operator on $\mathbb{R}^n$.

The Cayley–Laplace operator (3.1) and its generalizations were studied by Khekalo [Kh]. For $m > 1$, the operator $\Delta$ is not elliptic because $P(y) = 0$ for all non-zero matrices $y$ of rank $< m$. Moreover, $\Delta$ is not hyperbolic, although, for some $n, m$ and $\ell$, its power $\Delta^\ell$ enjoys the strengthened Huygens’ principle; see [Kh] for details.

Our nearest goal is to find a radial part of $\Delta$ corresponding to the polar decomposition $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x'x \in \mathcal{P}_m$. For $m = 1$, the classical result states that the radial part of the Laplace operator on $\mathbb{R}^n$ is

$$L = \rho^{1-n} \frac{\hat{\partial}}{\hat{\partial} \rho} \rho^{n-1} \frac{\hat{\partial}}{\hat{\partial} \rho}, \quad \rho = |x|, \quad x \in \mathbb{R}^n.$$  

By changing variable $r = \rho^2$, we get

$$L = 4r^{1-n/2} \frac{\hat{\partial}}{\hat{\partial} r} r^{n/2} \frac{\hat{\partial}}{\hat{\partial} r}. \quad \text{(3.2)}$$

The following statement is one of the main results of the paper. It extends (3.2) to the higher rank case.

**Theorem 3.1.** Let $\Omega \subset \mathfrak{N}_{n,m}$ be an open set consisting of matrices of rank $m$; $n \geq m \geq 1$. If $f(x) = f_0(x'x)$, $f_0(r) \in C^{2m}(\mathcal{P}_m)$, then for $x \in \Omega$,

$$(\Delta f)(x) = (Lf_0)(x'x), \quad \text{(3.3)}$$

where

$$L = 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D, \quad d = (m + 1)/2, \quad \text{(3.4)}$$

$D = \det[((1 + \delta_{ij})/2)\hat{\partial}/\hat{\partial} r_{i,j}]$ being operator (2.12).
Proof. For $m = 1$, (3.4) coincides with (3.2). To prove the theorem, we first note that without loss of generality, one can assume $f$ to be compactly supported away from the surface $\{ x : \det(x'x) = 0 \}$. Otherwise, $f(x)$ can be replaced by $f_1(x) = \phi(x)\psi(x)f(x)$, where $\phi$ and $\psi$ are radial cut-off functions of the form

$$
\phi(x) = \phi_0(\text{tr}(x'x)), \quad \psi(x) = \psi_0(\det(x'x)), \quad \phi_0, \psi_0 \in C^\infty(\mathbb{R}_+),
$$

and

$$
\phi_0(\rho) = \begin{cases} 
1 & \text{if } 0 \leq \rho < N, \\
0 & \text{if } \rho \geq N + 1,
\end{cases} \quad \psi_0(\rho) = \begin{cases} 
0 & \text{if } 0 \leq \rho \leq \varepsilon, \\
1 & \text{if } \rho \geq 2\varepsilon.
\end{cases}
$$

The positive numbers $N$ and $\varepsilon$ should be chosen sufficiently large and small, respectively.

By the generalized Bochner formula (2.22),

$$
\mathcal{F}[\Delta f](y) = (-1)^m |y'|y| (\mathcal{F} f)(y) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} h \left( \frac{y'y}{4} \right),
$$

where

$$
h(s) = (-4)^m \int_{\mathcal{P}_m} |rs| J_{n/2}(rs) |r|^{n/2-d-1} f_0(r) \, dr.
$$

Let us transform $|rs| J_{n/2}(rs)$. By (2.20) and (2.21),

$$
D[|r|^{n/2+1-d} D J_{n/2}(r)] = (-1)^m \frac{\Gamma_m(n/2)}{\Gamma_m(n/2 + 1)} D[J_{n/2+1}(r) |r|^{n/2+1-d}]
$$

$$
= (-1)^m J_{n/2}(r) |r|^{n/2-d}. 
$$

Hence,

$$
|r| J_{n/2}(r) = (-1)^m (\tilde{L} J_{n/2})(r), \quad \tilde{L} = (|r|^{d+1-n/2} D) |r|^{n/2-d}(|r| D).
$$

Since $|r|^{d+1-n/2} D |r|^{n/2-d}$ and $|r| D$ are invariant differential operators with respect to the transformation $r \rightarrow gr g'$, $g \in GL(m, \mathbb{R})$, [FK, p. 294], then so is $\tilde{L}$. Thus for any $s \in \mathcal{P}_m$,

$$
|rs| J_{n/2}(rs) = (-1)^m (\tilde{L} J_{n/2})(sr) = (-1)^m \tilde{L}_r [J_{n/2}(sr)],
$$
where \( \tilde{L}_r \) stands for the operator \( \tilde{L} \) acting in the \( r \)-variable (here we use the symmetry property \( J_\rho(rs) = J_\rho(s^{1/2}r^{1/2}) \)). It follows that

\[
\text{Owing to remark at the beginning of the proof, one can integrate by parts and get}

h(s) = (-4)^m \int_{P_m} D_r |r|^{n/2-d+1} (D_f_0)(r) dr
\]

where \( L = 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D \). Thus

\[
\mathcal{F}[\Delta f](y) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} \int_{P_m} J_{n/2}(sr) |r|^{n/2-d} (L_f_0)(r) dr,
\]

which implies (3.3). \( \square \)

**Example 3.2.** Let \( f(x) = |x|^{\lambda}_m, \ |x|_m = \det(x'x)^{1/2} \). By Theorem 3.1 and (2.15), \((\Delta f)(x) = \varphi(x'x)\), where

\[
\varphi(r) = 4^m |r|^{d-n/2} D |r|^{n/2-d+1} D |r|^{\lambda/2}
\]

\[
= 4^m b(\lambda/2) |r|^{d-n/2} D |r|^{(n+\lambda)/2-d}
\]

\[
= 4^m b(\lambda/2) b((n + \lambda)/2 - d)|r|^{\lambda/2-1}.
\]

Thus, we have arrived at the following identity of the Bernstein type

\[
\Delta |x|^{\lambda}_m = B(\lambda) |x|^{\lambda-2}_m,
\]

where, owing to (2.16), the polynomial \( B(\lambda) \) has the form

\[
B(\lambda) = (-1)^m \prod_{i=0}^{m-1} (\lambda + i)(2 - n - \lambda + i).
\]

An obvious consequence of (3.5) in a slightly different notation reads

\[
\Delta^k |x|^{\lambda+2k-n}_m = B_k(\lambda) |x|^{\lambda-n}_m,
\]

(3.7)
\[ B_k(x) = \prod_{i=0}^{m-1} \prod_{j=0}^{k-1} (x - i + 2j)(x - n + 2 + 2j + i) \]
\[ = B_k(n - x - 2k). \quad (3.8) \]

4. Zeta integrals

4.1. Definition and example

Let us consider the zeta integral

\[ Z(f, x - n) = \int_{\mathfrak{M}_{n,m}} f(x)|x|^{x-n} dx, \quad (4.1) \]

where \( f(x) \) is a Schwartz function on \( \mathfrak{M}_{n,m}, n \geq m, |x|_m = \det(x'x)^{1/2} \). The following example gives a flavor of basic properties of \( Z(f, x - n) \).

**Example 4.1.** Let \( e(x) = \exp(-\text{tr}(x'x)) \) be the Gaussian function. By Lemma 2.6, for \( \Re x > m - 1 \) we have

\[ Z(e, x - n) = 2^{-m} \sigma_{n,m} \int_{P_m} |r|^{2-d} \exp(-\text{tr}(r)) dr \]
\[ = c_{n,m} \Gamma_m(x/2), \quad c_{n,m} = \frac{\pi^{nm/2}}{\Gamma_m(n/2)^{2}}, \quad (4.2) \]

\( d = (m + 1)/2 \). On the other hand, the well known formula for the Fourier transform yields

\[ (F e)(y) = \pi^{nm/2} e(y/2) \quad (4.3) \]

and for \( \Re x < n - m + 1 \) we obtain

\[ Z(F e, -x) = c_{n,m} \pi^{nm/2} \int_{P_m} |r|^{(n-x)/2-d} \exp(-\text{tr}(r/4)) dr \]
\[ = d_{n,m} \Gamma_m((n - x)/2), \quad d_{n,m} = c_{n,m} \pi^{nm/2} 2^{m(n-x)}. \quad (4.4) \]

Thus, after analytic continuation we obtain the following meromorphic functions:

\[ Z(e, x - n) = c_{n,m} \Gamma_m(x/2), \quad x \neq m - 1, m - 2, \ldots, \quad (4.5) \]
\[ \mathcal{Z}(\mathcal{F}e, -\alpha) = d_{n,m} \Gamma_m((n - \alpha)/2), \quad \alpha \neq n - m + 1, n - m + 2, \ldots. \quad (4.6) \]

These equalities imply the functional relation
\[ \frac{\mathcal{Z}(e, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} 2^{m(\alpha-n)} \frac{\mathcal{Z}(\mathcal{F}e, -\alpha)}{\Gamma_m((n - \alpha)/2)}, \quad (4.7) \]

which is a prototype of similar formulas for much more general zeta functions. Since each side of (4.7) equals \( c_{n,m} \), this formula extends to all complex \( \alpha \). Note that excluded values of \( \alpha \) in (4.5) and (4.6) correspond to \( m \geq 2 \). If \( m = 1 \), they proceed with step 2, namely, \( \alpha \neq 0, -2, -4, \ldots \), and \( \alpha \neq n, n + 2, \ldots \), respectively.

In the following, throughout the paper, we assume \( m \geq 2 \).

**Lemma 4.2.** Let \( f \in S(\mathbb{M}_{n,m}) \). For \( \Re \alpha > m - 1 \), integral (4.1) is absolutely convergent, and for \( \Re \alpha \leq m - 1 \), it extends as a meromorphic function of \( \alpha \) with the only poles \( m - 1, m - 2, \ldots \). These poles and their orders are exactly the same as of the gamma function \( \Gamma_m(\alpha/2) \). The normalized zeta integral \( \mathcal{Z}(f, \alpha - n)/\Gamma_m(\alpha/2) \) is an entire function of \( \alpha \).

**Proof.** This statement is known; see, e.g., [Kh,Sh]. We present the proof for the sake of completeness. Equality (4.5) says that the function \( \alpha \to \mathcal{Z}(f, \alpha - n) \) has poles at least at the same points and at least of the same order as the gamma function \( \Gamma_m(\alpha/2) \). Our aim is to show that no other poles occur, and the orders cannot exceed those of \( \Gamma_m(\alpha/2) \). Let us transform (4.1) by passing to upper triangular matrices \( t \in T_m \) according to Lemma 2.7. We have

\[ \mathcal{Z}(f, \alpha - n) = \int_{\mathbb{R}^m_+} F(t_{1,1}, \ldots, t_{m,m}) \prod_{i=1}^{m} t_{i,i}^{\alpha-i} dt_{i,i}, \quad (4.8) \]

\[ F(t_{1,1}, \ldots, t_{m,m}) = \int_{\mathbb{R}^{m(m-1)/2}} dt_* \int_{V_{n,m}} f(vt) dv, \quad dt_* = \prod_{i<j} dt_{i,j} . \]

Since \( F \) extends as an even Schwartz function in each argument, it can be written as

\[ F(t_{1,1}, \ldots, t_{m,m}) = F_0(t_{1,1}^2, \ldots, t_{m,m}^2), \]

where \( F_0 \in S(\mathbb{R}^m) \) (use, e.g., Lemma 5.4 from [Tr, p. 56]. Replacing \( t_{i,i}^2 \) by \( s_{i,i} \), we represent (4.8) as a direct product of one-dimensional distributions

\[ \mathcal{Z}(f, \alpha - n) = 2^{-m} \left( \prod_{i=1}^{m} (s_{i,i})^{(\alpha-i)/2} \right) F_0(s_{1,1}, \ldots, s_{m,m}) , \quad (4.9) \]
which is a meromorphic function of $\alpha$ with the poles $m - 1, m - 2, \ldots$, see [GSh1]. These poles and their orders coincide with those of the gamma function $\Gamma_m(\alpha/2)$, cf. (4.5). To normalize function (4.9), following [GSh1], we divide it by the product

$$\prod_{i=1}^{m} \Gamma((\alpha - i + 1)/2) = \prod_{i=0}^{m-1} \Gamma((\alpha - i)/2) = \Gamma_m(\alpha/2)/\pi^{m(m-1)/4}.$$ 

As a result we obtain an entire function. $\square$

### 4.2. A functional equation for the zeta integral

Let us prove another main result of the paper.

**Theorem 4.3.** If $f \in S(\mathcal{H}_{n,m})$, $n \geq m$, then

$$\frac{Z(f, \alpha - n)}{\Gamma_m(\alpha/2)} = \pi^{-nm/2} 2^{m(\alpha - n)} \frac{Z(F f, -\alpha)}{\Gamma_m((n - \alpha)/2)}. \quad (4.10)$$

**Proof.** We recall that both sides of this equality are understood in the sense of analytic continuation. Furthermore, the distribution on the left-hand side of (4.10) is regular if and only if $\text{Re} \alpha > m - 1$, whereas the right-hand side is regular if and only if $\text{Re} \alpha < n - m + 1$. For $2m < n + 2$ these two sets have no common points. We consider the cases $2m < n + 2$ and $2m \geq n + 2$ separately.

(i) The case $2m < n + 2$. We start with the following equality:

$$\int_{\mathcal{H}_{n,m}} (F f)(y)|\varepsilon I_m + y'y|^{-\alpha/2} dy = \frac{\pi^{-nm/2} \varepsilon^{m(n-\alpha)/2}}{\Gamma_m(\alpha/2)} \int_{\mathcal{H}_{n,m}} f(x) \mathcal{K}_{(x-n)/2} \left(\frac{\varepsilon x'}{4} \right) dx, \quad \varepsilon > 0. \quad (4.11)$$

Here $\mathcal{K}_{(x-n)/2}$ is the $\mathcal{K}$-Bessel function (see Section 2.3.2), and

$$m - 1 < \text{Re} \alpha < n - m + 1 \quad (4.12)$$

(since $2m < n + 2$, domain (4.12) is not void). Suppose for a moment, that (4.11) is true. Then the result follows if we pass to the limit as $\varepsilon \to 0$. Indeed, by (2.28) for all $\varepsilon > 0$ we have

$$\left|\varepsilon^{m(n-\alpha)/2} \mathcal{K}_{(x-n)/2} \left(\frac{\varepsilon x'}{4} \right) \right| \leq \Gamma_m \left(\frac{n - \text{Re} \alpha}{2} \right) \left|\frac{x'}{4} \right|^{(\text{Re} \alpha - n)/2}. \quad (4.13)$$
Furthermore, by (2.30),
\[
\lim_{\varepsilon \to 0} \varepsilon^{m(n-z)/2} \mathcal{K}_{(z-n)/2} \left( \frac{\varepsilon}{4} x' x \right) = 2^{m(n-z)} \Gamma_m \left( \frac{n-z}{2} \right) |x|^{z-n}. \tag{4.14}
\]

Hence, by the Lebesgue theorem on dominated convergence, we are done.

To prove (4.11), let \( e_s(x) = \exp(-\text{tr}(xsx')/4\pi), \ s \in \mathcal{P}_m \). By the Plancherel formula,
\[
|s|^{-n/2} \int_{\mathbb{R}^m} (\mathcal{F} f)(y) \exp(-\text{tr}(\pi y s^{-1} y')) dy = \int_{\mathbb{R}^m} f(x) e_s(x) dx. \tag{4.15}
\]

Now we multiply (4.15) by \( |s|^{(n-z)/2-d} \exp(-\text{tr}(\varepsilon ps^{-1})), \ d = (m+1)/2 \), then integrate in \( s \), and change the order of integration. We obtain
\[
\int_{\mathbb{R}^m} (\mathcal{F} f)(y) a(y) dy = \int_{\mathbb{R}^m} f(x) b(x) dx,
\]
where
\[
a(y) = \int_{\mathcal{P}_m} |s|^{-z/2-d} \exp[-\text{tr}(\pi s^{-1} (y'y + \varepsilon I_m))] ds \quad (s = t^{-1})
\]
\[
= \int_{\mathcal{P}_m} |t|^{-z/2-d} \exp[-\text{tr}(\pi t (y'y + \varepsilon I_m))] dt
\]
\[
= \Gamma_m (z/2) \pi^{-zm/2} |y'y + \varepsilon I_m|^{-z/2}, \quad \text{Re} z > m - 1
\]
and
\[
b(x) = \int_{\mathcal{P}_m} |s|^{(n-z)/2-d} \exp[-\text{tr}(sx'x/4\pi + \varepsilon ps^{-1})]] ds
\]
\[
= (\varepsilon \pi)^{m(n-z)/2} \int_{\mathcal{P}_m} |u|^{(n-z)/2-2-d} \exp[-\text{tr}(u \varepsilon x'x/4 + u^{-1})] du
\]
\[
= (\varepsilon \pi)^{m(n-z)/2} \mathcal{K}_{(z-n)/2} \left( \frac{\varepsilon}{4} x' x \right).
\]

This gives (4.11). Application of the Fubini theorem in this argument is justified because both integrals in (4.11) converge absolutely. Indeed, for the left-hand side the absolute convergence is obvious, and for the right-hand side it follows from (4.14).

(ii) The case \( 2m \geq n + 2 \). In this case interval (4.12) is void. To circumvent this difficulty, we replace \( f(x) \) in (4.15) by \( |x|^{2k} \Delta^k f(x) \) and proceed as above, assuming
Formally we obtain
\[\int_{\mathbb{R}^{n,m}} \Delta^k[|y|^{2k}_m(\mathcal{F} f)(y)] |\varepsilon I_m + y'y|^{-\varkappa/2} dy = \frac{\pi^{nm/2} \Gamma(m/2)}{\Gamma_m(\varkappa/2)} \int_{\mathbb{R}^{n,m}} |x|^{2k}_m \Delta^k f(x) K_{(\varkappa-n)/2} \left( \frac{\varepsilon x' x}{4} \right) dx. \]  
Equality (4.16) will become meaningful if we justify application of the Fubini theorem and specify a suitable interval for \( \text{Re} \varkappa \). Clearly, the left-hand side of (4.16) absolutely converges for all \( \varkappa \in \mathbb{C} \), and one should take care of the right-hand side only. By Lemma 2.4(ii), the integral on the right-hand side absolutely converges for \( \text{Re} \varkappa > m - 1 - 2k \) provided \( 2k > 2m - 2 - n \). Indeed, for small \( |x|_m \) we have

\[ |x|^{2k}_m \Delta^k f(x) K_{(\varkappa-n)/2} \left( \frac{\varepsilon x' x}{4} \right) = O(|x|^{\hat{\varkappa}-n}_m), \]

where

\[ \hat{\varkappa} = \begin{cases} 
2k + n & \text{if } \text{Re} \varkappa > n + m - 1, \\
2k + n + 1 - m - \varepsilon_0, \forall \varepsilon_0 > 0 & \text{if } n - m + 1 \leq \text{Re} \varkappa \leq n + m - 1, \\
2k + \text{Re} \varkappa & \text{if } \text{Re} \varkappa < n - m + 1.
\] 

If \( \text{Re} \varkappa > m - 1 - 2k \) and \( \varepsilon_0 \) is small enough \( \varepsilon_0 < 2k - 2m + n + 2 \) then \( \hat{\varkappa} > m - 1 \) and the desired convergence follows. Thus, (4.16) is justified for \( \text{Re} \varkappa > m - 1 \). Since both sides of (4.16) are analytic functions of \( \varkappa \) in a larger domain \( \text{Re} \varkappa > m - 1 - 2k \) containing a non-void strip

\[ m - 1 - 2k < \text{Re} \varkappa < n - m + 1, \]

(4.16) also holds in this strip, and one can utilize (4.13) and (4.14) in order to pass to the limit as \( \varepsilon \to 0 \). This gives

\[ \mathcal{Z} (\Delta^k[|y|^{2k}_m(\mathcal{F} f)(y)], -\varkappa) = c_{\varkappa} \mathcal{Z} (\Delta^k f, \varkappa + 2k - n), \]

\[ c_{\varkappa} = \frac{\pi^{nm/2} \Gamma(2m-n-\varkappa)}{2 \Gamma_m \left( \frac{n-\varkappa}{2} \right)} \Gamma_m \left( \frac{\varkappa}{2} \right). \]

To transform (4.18), we use the equality

\[ \Delta^k |x|^{\varkappa+2k-n}_m = B_k(n) |x|^{\varkappa-n}_m, \]

\[ B_k(n) = \prod_{i=0}^{\frac{m-1}{2}} \prod_{j=0}^{k-1} (\varkappa - i + 2j)(\varkappa - n + 2 + 2j + i) = B_k(n - \varkappa - 2k), \]
see (3.7). Then the right-hand side of (4.18) becomes \( c_\beta B_k(z) \mathcal{Z}(f, \alpha - n) \). For the left-hand side we obtain (set \( \varphi(y) = |y|^{2k}_m (\mathcal{F}f)(y), \ \alpha = n - \beta - 2k \))

\[
\mathcal{Z}(\Delta^k [ |y|^{2k}_m (\mathcal{F}f)(y)], -\alpha) = \mathcal{Z}(\Delta^k \varphi, \beta + 2k - n) = B_k(\beta) \mathcal{Z}(\varphi, \beta - n) = B_k(n - \alpha - 2k) \mathcal{Z}(\mathcal{F}f, -\alpha) = B_k(\alpha) \mathcal{Z}(\mathcal{F}f, -\alpha).
\]

Finally, (4.18) reads

\[
\mathcal{Z}(\mathcal{F}f, -\alpha) = c_\alpha \mathcal{Z}(f, \alpha - n), \hspace{1em} m - 1 - 2k < \text{Re} \alpha < n - m + 1
\]

and the desired equality (4.10) follows by analytic continuation. \( \square \)

4.3. Normalized zeta distributions of integral order

It is convenient to introduce a special notation for the normalized zeta integral \( \mathcal{Z}(f, \alpha - n)/\Gamma_m(\alpha/2) \) which is an entire function of \( \alpha \). We denote

\[
\zeta_\alpha(x) = \frac{|x|_{m}^{\alpha-n}}{\Gamma_m(\alpha/2)}, \hspace{1em} (\zeta_{\alpha}, f) = a.c. \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathfrak{g}_{n,m}} f(x)|x|_{m}^{\alpha-n} dx, \hspace{1em} (4.21)
\]

where “a.c.” abbreviates analytic continuation. We call \( \zeta_\alpha \) a normalized zeta distribution of order \( \alpha \).

In view of forthcoming applications in Section 6, normalized zeta distributions of integral order deserve special treatment. A striking feature of the distribution \( \zeta_\alpha \) is that for \( \alpha = 0, 1, 2, \ldots, m - 1 \), (outside of the domain of absolute convergence !) it is a positive measure. This measure is supported by a lower-dimensional manifold (in the rank-one case \( m = 1 \) we have only one point \( \alpha = 0 \) corresponding to the delta function at the origin). Below we obtain a number of explicit representations of \( \zeta_\alpha \) for \( \alpha = 1, 2, \ldots, n \).

**Theorem 4.4.** Let \( f \in S(\mathfrak{g}_{n,m}) \). For \( \alpha = k, k = 1, 2, \ldots, n \),

\[
(\zeta_k, f) = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int_{SO(n)} d\gamma \int_{\mathfrak{g}_{k,m}} f(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}) d\omega. \hspace{1em} (4.22)
\]

Furthermore, in the case \( \alpha = 0 \) we have

\[
(\zeta_0, f) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} f(0). \hspace{1em} (4.23)
\]
Proof.

Step 1: Let first \( k > m - 1 \). In polar coordinates we have

\[
Z(f, k - n) = \int_{\mathcal{M}_{n,m}} f(x) |x|^{k-n} dx
\]

\[
= 2^{-m} \sigma_{n,m} \int_{\mathcal{P}_{m}} |r|^{k/2-d} dr \int_{SO(n)} f \left( \gamma \left[ \begin{array}{c} r^{1/2} \\ 0 \end{array} \right] \right) d\gamma.
\]

Now we replace \( \gamma \) by \( \beta = \begin{pmatrix} 0 & \beta \\ I_{n-k} & 0 \end{pmatrix} \), \( \beta \in SO(k) \), then integrate in \( \beta \in SO(k) \), and replace the integration over \( SO(k) \) by that over \( V_{k,m} \). We get

\[
Z(f, k - n) = 2^{-m} \sigma_{n,m} \int_{SO(n)} d\gamma \int_{\mathcal{P}_{m}} |r|^{k/2-d} dr \int_{V_{k,m}} f \left( \gamma \left[ \begin{array}{c} vr^{1/2} \\ 0 \end{array} \right] \right) dv
\]

(set \( \omega = vr^{1/2} \in \mathcal{M}_{k,m} \))

\[
= \sigma_{n,m} \int_{\mathcal{M}_{k,m}} d\omega \int_{SO(n)} f \left( \gamma \left[ \begin{array}{c} \omega \\ 0 \end{array} \right] \right) d\gamma.
\]

This coincides with (4.22).

Step 2: Our next task is to prove that analytic continuation of \((\zeta_{\alpha}, f)\) at the point \( \alpha = k (\leq m - 1) \) has the form (4.22). We first note that for \( \alpha = 0 \), (4.23) is an immediate consequence of (4.10). Let \( k > 0 \). We split \( x \in \mathcal{M}_{n,m} \) in two blocks \( x = [y; b] \) where \( y \in \mathcal{M}_{n,k} \) and \( b \in \mathcal{M}_{n,m-k} \). Then for \( Re \alpha > m - 1 \),

\[
(\zeta_{\alpha}, f) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{M}_{n,k}} dy \int_{\mathcal{M}_{n,m-k}} f([y; b]) \left| \begin{array}{cc} y' & y' \\ y' & b' \end{array} \right|^{(\alpha-n)/2} db,
\]

where \( \left| \begin{array}{cc} * & * \\ * & * \end{array} \right| \) denotes the determinant of the respective matrix \( \left| \begin{array}{cc} * & * \\ * & * \end{array} \right| \). By passing to polar coordinates (see Lemma 2.6) \( y = vr^{1/2}, \ v \in V_{n,k}, \ r \in \mathcal{P}_{k} \), we have

\[
(\zeta_{\alpha}, f) = \frac{2^{-k}}{\Gamma_m(\alpha/2)} \int_{V_{n,k}} dv \int_{\mathcal{P}_{k}} |r|^{(n-k-1)/2} dr
\]

\[
\times \int_{\mathcal{M}_{n,m-k}} f([vr^{1/2}; b]) \left| \begin{array}{cc} r & r^{1/2}v'b' \\ r^{1/2}v'b' & b'b' \end{array} \right|^{(\alpha-n)/2} db
\]

\[
= \frac{2^{-k}}{\Gamma_m(\alpha/2)} \int_{SO(n)} d\gamma \int_{\mathcal{P}_{k}} |r|^{(n-k-1)/2} dr
\]

\[
\times \int_{\mathcal{M}_{n,m-k}} f_{\gamma}([\zeta_{\alpha}r^{1/2}; b]) \left| \begin{array}{cc} r & r^{1/2}b' \zeta_{\alpha} \\ r^{1/2}b' \zeta_{\alpha} & b'b' \end{array} \right|^{(\alpha-n)/2} db.
\]
Here

$$\lambda_0 = \begin{bmatrix} I_k & 0 \end{bmatrix} \in \mathbb{V}_{n,k}, \quad f_\gamma(x) = f(\gamma;x).$$

We write

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad b_1 \in \mathfrak{M}_{k,m-k}, \quad b_2 \in \mathfrak{M}_{n-k,m-k}.$$ 

Since $\lambda'_0 b = b_1$, then

$$\langle \zeta_\gamma, f \rangle = \frac{2^{-k} \sigma_{n,k}}{\Gamma_m(\alpha/2)} \int_{\mathbb{S}\mathbb{O}(n)} d\gamma \int_{\mathbb{P}_k} |r|^{(n-k-1)/2} dr \int_{\mathfrak{M}_{k,m-k}} db_1 \times \int_{\mathfrak{M}_{n-k,m-k}} f_\gamma \left( \begin{bmatrix} r^{1/2} b_1 \\ 0 \\ b_1' r^{1/2} b_2 \end{bmatrix} \right) \left| r b_1' r^{1/2} b_2 \right|^{(\alpha-n)/2} db_2.$$ 

Note that

$$\begin{bmatrix} r & r^{1/2} b_1 \\ b_1' r^{1/2} & b_1' b_1 + b_2' b_2 \end{bmatrix} = \begin{bmatrix} r & 0 \\ b_1' r^{1/2} & I_{m-k} \end{bmatrix} \begin{bmatrix} I_k & r^{-1/2} b_1' \\ 0 & b_2' \end{bmatrix}$$ 

and

$$\det \begin{bmatrix} r & r^{1/2} b_1 \\ b_1' r^{1/2} & b_1' b_1 + b_2' b_2 \end{bmatrix} = \det(r) \det(b_2' b_2),$$

see, e.g., [Mu, p. 577]. Therefore,

$$\langle \zeta_\gamma, f \rangle = c_\alpha \int_{\mathbb{S}\mathbb{O}(n)} d\gamma \int_{\mathbb{P}_k} |r|^{(\alpha-k-1)/2} dr \int_{\mathfrak{M}_{k,m-k}} \psi_{\alpha-k}(\gamma, r, b_1) db_1,$$ 

where

$$c_\alpha = \frac{2^{-k} \sigma_{n,k} \Gamma_{m-k}(\alpha-k)/2}{\Gamma_m(\alpha/2)},$$

$$\psi_{\alpha-k}(\gamma, r, b_1) = \frac{1}{\Gamma_{m-k}((\alpha-k)/2)} \int_{\mathfrak{M}_{n-k,m-k}} f_\gamma \left( \begin{bmatrix} r^{1/2} b_1 \\ 0 \\ b_2 \end{bmatrix} \right) \times |b_2' b_2|^{(\alpha-k)/2-(n-k)/2} db_2.$$
The last expression represents the normalized zeta distribution of order \( \alpha - k \) in the \( b_2 \)-variable. Owing to (4.23), analytic continuation of (4.24) at \( \alpha = k \) reads

\[
(\zeta_k, f) = c_k \int_{SO(n)} d\gamma \int_{\mathcal{P}_k} |r|^{-1/2} dr \int_{\mathfrak{g}_{k,m-k}} f_{\gamma} \left( \begin{bmatrix} r^{1/2} b_1 \\ 0 \\ 0 \end{bmatrix} \right) db_1,
\]

\[c_k = [\pi^{(n-k)(m-k)/2}/\Gamma_{m-k}(n-k/2)] \lim_{\alpha \to k} c_\alpha.\]

To transform this expression, we replace \( \gamma \) by \( \gamma \left[ \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix} \right] \), \( \beta \in SO(k) \), and integrate in \( \beta \). This gives

\[
(\zeta_k, f) = c_k \int_{\mathcal{P}_k} |r|^{-1/2} dr \int_{SO(k)} d\beta \int_{\mathfrak{g}_{k,m-k}} db_1 \times \int_{SO(n)} f_{\gamma} \left( \begin{bmatrix} \beta r^{1/2} b_1 \\ 0 \\ 0 \end{bmatrix} \right) d\gamma
\]

(set \( \zeta = \beta b_1 \), \( \eta = b|r|^{1/2} \) and use Lemma 2.6)

\[
= \frac{2}{\sigma_{k,k}} \int_{\mathfrak{g}_{k,k}} d\eta \int_{\mathfrak{g}_{k,m-k}} d\zeta \int_{SO(n)} f_{\gamma} \left( \begin{bmatrix} \eta & \zeta \\ 0 & 0 \end{bmatrix} \right) d\gamma
\]

\[
= c \int_{SO(n)} d\gamma \int_{\mathfrak{g}_{k,m}} f \left( \gamma \left[ \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right] \right) d\omega,
\]

\[
c = \frac{\pi^{(n-k)(m-k)/2}/\sigma_{n,k} \Gamma_{m-k}((\alpha - k)/2)}{\Gamma_{m-k}((\alpha - k)/2)} \lim_{\alpha \to k} \frac{\Gamma_{m-k}((\alpha - k)/2)}{\Gamma_m(\alpha/2)} = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)}.
\]

(here we used formulae (2.8) and (2.31)). \( \square \)

The following formulas are consequences of (4.22).

**Corollary 4.5.** For all \( k = 1, 2, \ldots, n \),

\[
(\zeta_k, f) = c_1 \int_{V_{n,k}} dv \int_{\mathfrak{g}_{k,m}} f(v\omega) d\omega,
\]

\[
c_1 = 2^{-k} \pi^{(nm-km-nk)/2} \Gamma_k(n/2)/\Gamma_m(n/2).
\]

Moreover, if \( k = 1, 2, \ldots, m - 1 \), then

\[
(\zeta_k, f) = c_1 \int_{V_{m,k}} du \int_{\mathfrak{g}_{n,k}} f(yu')|y|^{m-n}_k dy
\]

(4.27)
and

\[
(\zeta_k, f) = c_2 \int_{\mathfrak{M}_{n,k}} \frac{dy}{|y|^{n-m}} \int_{\mathfrak{M}_{k,m-k}} f([y; yz]) \, dz, \quad (4.28)
\]

\[
c_2 = 2^{(m-k)(n/2-k)/2} \Gamma_k(k/2) \Gamma_m((n-k)/2). \quad (4.29)
\]

**Proof.** From (4.22) we have

\[
(\zeta_k, f) = \frac{\pi (n-k)m/2}{\Gamma_m(n/2)} \int_{\mathfrak{P}_k} \int_{\mathfrak{V}_{n,k}} f(v \lambda_0 \omega) \, dv \quad (\lambda_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in \mathfrak{V}_{n,k})
\]

\[
= \frac{\pi (n-k)m/2}{\sigma_n \Gamma_m(n/2)} \int_{\mathfrak{P}_k} \int_{\mathfrak{V}_{n,k}} f(v \omega) \, dv,
\]

which coincides with (4.25). To prove (4.27), we pass to polar coordinates in (4.25) by setting \( u' = ur^{1/2}, \ u \in \mathfrak{V}_{m,k}, \ r \in \mathfrak{P}_k \). This gives

\[
(\zeta_k, f) = 2^{-k} c_1 \int_{\mathfrak{V}_{n,k}} \int_{\mathfrak{P}_k} |r|^{(m-k-1)/2} dr \int_{\mathfrak{V}_{n,k}} f(vr^{1/2}u') \, du
\]

\[
= c_1 \int_{\mathfrak{V}_{m,k}} \int_{\mathfrak{V}_{n,k}} f(yu') |y|^{m-n} \, dy.
\]

To prove (4.28), we represent \( \omega \) in (4.25) in the block form \( u = [\eta; \zeta], \ \eta \in \mathfrak{M}_{k,k}, \ \zeta \in \mathfrak{M}_{k,m-k} \), and change the variable \( \zeta = \eta z \). This gives

\[
(\zeta_k, f) = c_1 \int_{\mathfrak{V}_{k,k}} |\eta|^{m-k} \, d\eta \int_{\mathfrak{V}_{k,m-k}} d\zeta \int_{\mathfrak{V}_{n,k}} f(v[\eta; \eta z]) \, dv.
\]

Using Lemma 2.6 repeatedly, and changing variables, we obtain

\[
(\zeta_k, f) = 2^{-k} c_1 \sigma_{k,k} \int_{\mathfrak{P}_k} |r|^{(m-k-1)/2} dr \int_{\mathfrak{V}_{k,m-k}} dz \int_{\mathfrak{V}_{n,k}} f(v[r^{1/2}, r^{1/2}z]) \, dv
\]

\[
= c_1 \sigma_{k,k} \int_{\mathfrak{V}_{k,k}} \frac{dy}{|y|^{n-m}} \int_{\mathfrak{V}_{k,m-k}} f([y; yz]) \, dz.
\]

By (4.26), (2.31) and (2.9), this coincides with (4.28). \( \Box \)

The representation (4.28) was obtained in [Sh,Kh] in a different way. An idea of (4.27) is due to Ournycheva.
Remark 4.6. One can also write \((\zeta_k, f)\) as

\[
(\zeta_k, f) = \int_{\mathfrak{M}_{n,m}} f(x) \, dv_k(x), \quad f \in S(\mathfrak{M}_{n,m}),
\]

(4.30)

where \(v_k\) is a positive locally finite measure defined by

\[
(v_k, \varphi) = c_1 \int_{V_{n,k}} dv \int_{\mathfrak{M}_{k,m}} \varphi(v_\omega) \, d\omega, \quad \varphi \in C_c(\mathfrak{M}_{n,m}),
\]

(4.31)

\(C_c(\mathfrak{M}_{n,m})\) being the space of compactly supported continuous functions on \(\mathfrak{M}_{n,m}\); cf. (4.25). In order to characterize the support of \(v_k\), we denote

\[
\mathfrak{M}_{n,m}(k) = \{ x : x \in \mathfrak{M}_{n,m}, \, \text{rank}(x) = k \},
\]

(4.32)

\[
\bar{\mathfrak{M}}_{n,m}(k) = \bigcup_{j=0}^{k} \mathfrak{M}_{n,m}(j) \quad \text{(the closure of } \mathfrak{M}_{n,m}(k)\text{)}.
\]

(4.33)

Lemma 4.7. The following statements hold:

(i) \(\text{supp} v_k = \bar{\mathfrak{M}}_{n,m}(k)\).

(ii) The manifold \(\mathfrak{M}_{n,m}(k)\) is an orbit of \(e_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}\) under the group of transformations

\[
x \rightarrow g_1 x g_2, \quad g_1 \in GL(n, \mathbb{R}), \quad g_2 \in GL(m, \mathbb{R}).
\]

(iii) The manifold \(\bar{\mathfrak{M}}_{n,m}(k)\) is a collection of all matrices \(x \in \mathfrak{M}_{n,m}\) of the form

\[
x = \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}, \quad \gamma \in SO(n), \quad \omega \in \mathfrak{M}_{k,m},
\]

(4.34)

or

\[
x = v_\omega, \quad v \in V_{n,k}, \quad \omega \in \mathfrak{M}_{k,m}.
\]

(4.35)

Proof. (i) Let us consider (4.31). Since \(\text{rank}(v_\omega) \leq k\), then \((v_k, \varphi) = 0\) for all \(\varphi \in C_c(\mathfrak{M}_{n,m})\) supported away from \(\mathfrak{M}_{n,m}(k)\). This means that \(\text{supp} v_k = \bar{\mathfrak{M}}_{n,m}(k)\).

(ii) Let us show that each \(x \in \mathfrak{M}_{n,m}(k)\) is represented in the form \(x = g_1 e_k g_2\) for some \(g_1 \in GL(n, \mathbb{R})\) and \(g_2 \in GL(m, \mathbb{R})\). By Lemma 2.5, each matrix \(x\) can be written as \(x = u \rho\), where \(u \in V_{n,m}\) and \(\rho = (x'x)^{1/2}\) is a positive semi-definite \(m \times m\) matrix
of rank $k$. By taking into account that $\rho = g_2^T \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} m$ for some $g_2 \in GL(m, \mathbb{R})$, and $u = \gamma \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ for some $\gamma \in O(n)$, we obtain $x = g_1 e_k g_2$ with

$$g_1 = \gamma \begin{bmatrix} g_2' & 0 \\ 0 & I_{n-m} \end{bmatrix} \in GL(n, \mathbb{R}).$$

(iii) Clearly, each matrix of the form (4.34) or (4.35) has rank $\leq k$. Conversely, if $\text{rank}(x) \leq k$ then, as above, $x = u \rho = \gamma \begin{bmatrix} \rho \\ 0 \end{bmatrix}$ where $\gamma \in O(n)$ and $\rho = (x'x)^{1/2}$ is a positive semi-definite $m \times m$ matrix of rank $\leq k$. The latter can be written as

$$\rho = g \lambda g', \quad g \in O(m), \quad \lambda = \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$$

and therefore,

$$x = \gamma \begin{bmatrix} g \lambda \\ 0 \end{bmatrix} g' = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \end{bmatrix} g' = \gamma_1 \begin{bmatrix} \omega \\ 0 \end{bmatrix},$$

where $\gamma_1 = \gamma \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix}$ and $\omega \in \mathcal{M}_{k,m}$. If $\text{det}(\gamma_1) = 1$, we are done. If $\text{det}(\gamma_1) = -1$, one should replace $\gamma_1$ by $\gamma_1 e$, $e = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$ and change $\omega$ appropriately. Representation (4.35) follows from (4.34). □

**Corollary 4.8.** Integral (4.30) can be written as

$$(\zeta_k^*, f) = \int_{\text{rank}(x) \leq k} f(x) dv_k(x) = \int_{\text{rank}(x) = k} f(x) dv_k(x). \quad (4.36)$$

Indeed, the first equality follows from Lemma 4.7(i). The second one is clear from the observation that if $\text{rank}(x) \leq k - 1$ then by (4.35), $x = v \omega$, $v \in V_{n,k-1}$, $\omega \in \mathcal{M}_{k-1,m}$. The set of all such pairs $(v, \omega)$ has measure 0 in $V_{n,k} \times \mathcal{M}_{k,m}$.

5. Convolutions with zeta distributions and Riesz potentials

5.1. Definitions

Let us consider the convolution operator $\zeta_2^*$ defined by

$$(\zeta_2^* f)(x) = \frac{1}{\Gamma_m(\xi/2)} \int_{\mathcal{M}_{n,m}} f(x - y) |y|^{\xi-n} dy \quad (5.1)$$
if $\Re z > m - 1$, and
\[
(\zeta_k * f)(x) = c_1 \int_{V_n,k} dv \int_{\Omega_{k,m}} f(x - v\omega) d\omega,
\]
(5.2)
\[
c_1 = 2^{-k} \pi^{(nm-km-nk)/2} \Gamma_k(n/2)/\Gamma_m(n/2).
\]

if $z = k$, $k = 1, 2, \ldots, n$; cf. Corollary 4.5. Note that for $m - 1 < k \leq n$, both representations are applicable to $\zeta_k * f$.

Another important normalization of the zeta distribution and the corresponding convolutions (5.1) and (5.2) leads to the Riesz distributions and Riesz potentials. We observe that the functional equation (4.10) for the zeta function can be written in the form
\[
\frac{1}{\gamma_{n,m}(z)} \zeta(f, z - n) = (2\pi)^{-nm} \zeta(\mathcal{F}f, -z),
\]
(5.3)
\[
\gamma_{n,m}(z) = \frac{2^{zn} \pi^{zn/2} \Gamma_m(z/2)}{\Gamma_m((n - z)/2)}, \quad z \neq n - m + 1, n - m + 2, \ldots.
\]
(5.4)

Now we have excluded the values $z = n - m + 1, n - m + 2, \ldots$, because these are poles of the gamma function $\Gamma_m((n - z)/2)$ sitting in the numerator in (5.4) (we recall that $m \geq 2$). The corresponding Riesz distribution $h_x$ is defined by
\[
(h_x, f) = \frac{1}{\gamma_{n,m}(x)} \zeta(f, x - n) = a.c. \frac{1}{\gamma_{n,m}(x)} \int_{\Omega_{n,m}} |x|^x_m f(x) dx,
\]
(5.5)

where $f \in \mathcal{S}(\Omega_{n,m})$. For $\Re z > m - 1$, the distribution $h_x$ is regular and agrees with the ordinary function $h_x(x) = |x|^x_m/\gamma_{n,m}(x)$.

The Riesz potential of a function $f \in \mathcal{S}(\Omega_{n,m})$ is defined by
\[
(I^z f)(x) = (h_x, f_x) = \frac{1}{\gamma_{n,m}(x)} \zeta(f_x, x - n), \quad f_x(\cdot) = f(x - \cdot).
\]
(5.6)

For $\Re z > m - 1, x \neq n - m + 1, n - m + 2, \ldots$, (5.6) is represented in the classical form by the absolutely convergent integral
\[
(I^z f)(x) = \frac{1}{\gamma_{n,m}(x)} \int_{\Omega_{n,m}} f(x - y)|y|^x_m dy.
\]
(5.7)

This integral operator is well known in the rank-one case $m = 1$.

Riesz potentials of integral order deserve special mentioning. The following representations are inherited from those for the normalized zeta function; see Theorem 4.4.
Theorem 5.1. Let $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Suppose that $\alpha = k$ is a positive integer. If $k \neq n - m + 1, n - m + 2, \ldots$, then

\[ (I^k f)(x) = \gamma_1 \int_{SO(n)} d\gamma \int_{\mathbb{R}^n_{k,m}} f \left( x - \gamma \left[ \begin{array}{c} \omega \\ 0 \end{array} \right] \right) d\omega \]

\[ = \gamma_2 \int_{V_{n,k}} dv \int_{\mathbb{R}^n_{k,m}} f(x - v\omega) d\omega, \]

where

\[ \gamma_1 = 2^{-km} \pi^{-km/2} \Gamma_m \left( \frac{n - k}{2} \right) / \Gamma_m \left( \frac{n}{2} \right), \]

\[ \gamma_2 = 2^{-k(m+1)} \pi^{-k(m+n)/2} \Gamma_k \left( \frac{n - m}{2} \right). \]

The constant $\gamma_2$ above was evaluated by making use of (2.9).

The following theorem resumes basic properties of Riesz distributions and Riesz potentials.

Theorem 5.2. Let $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, $\alpha \in \mathbb{C}$, $\alpha \neq n - m + 1, n - m + 2, \ldots$.

(i) The Fourier transform of the Riesz distribution $h_\alpha$ is evaluated by the formula

\[ (\mathcal{F}h_\alpha)(y) = |y|^{-\alpha} \mathcal{L}(\mathcal{F}f)(y) = (2\pi)^{-nm} |y|^{-\alpha} \mathcal{L}(\mathcal{F}f, -\alpha). \]

(ii) If $k = 0, 1, \ldots$, and $\Delta$ is the Cayley–Laplace operator, then

\[ (-1)^{mk} \Delta^k h_{\alpha + 2k} = h_\alpha, \text{ i.e. } (-1)^{mk} (h_{\alpha + 2k}, \Delta^k f) = (h_\alpha, f) \]

and, therefore,

\[ (I^{-2k} f)(x) = (-1)^{mk} (\Delta^k f)(x). \]

In particular,

\[ (I^0 f)(x) = f(x). \]
**Proof.** (i) follows immediately from definition (5.5) and the functional equation (5.3). To prove (5.13), for sufficiently large $\lambda$ we have

$$
\Delta^k h_{\lambda+2k}(x) = \frac{1}{\gamma_{n,m}(\lambda + 2k)} \Delta^k |x|^{\lambda+2k-n} = \frac{B_k(\lambda)}{\gamma_{n,m}(\lambda + 2k)} \Delta^k |x|^{\lambda-n} = c h_{\lambda}(x),
$$

where by (3.8) and (2.7),

$$
c = \frac{B_k(\lambda)\gamma_{n,m}(\lambda)}{\gamma_{n,m}(\lambda + 2k)} = \frac{B_k(\lambda)\Gamma_m(\lambda/2)\Gamma_m((n-\lambda)/2-k)}{4^m k \Gamma_m(\lambda/2+k)\Gamma_m((n-\lambda)/2)} = 1.
$$

For all admissible $\lambda \in \mathbb{C}$, (5.13) follows by analytic continuation. Equality (5.14) is a consequence of (5.12) and (5.13). Indeed, by (5.6),

$$(I^{-2k} f)(x) = (h_{-2k}, f_x) \overset{(5.13)}{=} (-1)^m k (h_0, \Delta^k f_x) \overset{(5.12)}{=} (-1)^m k (2\pi)^{-nm} \mathcal{Z}(\mathcal{F}(\Delta^k f_x), 0) = (-1)^m k (\Delta^k f_x)(0) = (-1)^m k (\Delta^k f)(x). \quad \square
$$

### 5.2. Riesz potentials and heat kernels

It is convenient to study Riesz potentials by making use of the heat kernels and the corresponding Gauss–Weierstrass integrals. In the rank-one case $m = 1$ this approach is described in [Ru1, Section 16]. In the higher rank case it was implicitly indicated in [Cl, FK]. The key idea is to represent the Riesz potential as a lower-dimensional fractional integral of the corresponding Gauss–Weierstrass integral which is easy to handle.

**Definition 5.3.** For $x \in \mathbb{M}_{n,m}$, $n \geq m$, and $t \in \mathcal{P}_m$, we define the (generalized) heat kernel $h_t(x)$ by the formula

$$
h_t(x) = (4\pi)^{-nm/2} |t|^{-n/2} \exp(-\text{tr}(t^{-1} x' x)/4),
$$

where $|t| = \det(t)$. The corresponding Gauss–Weierstrass integral $(W_t f)(x)$ of a function $f(x)$ on $\mathbb{M}_{n,m}$ is defined by

$$
(W_t f)(x) = \int_{\mathbb{M}_{n,m}} h_t(x - y) f(y) dy = \int_{\mathbb{M}_{n,m}} h_{t^1}(y) f(x - yt^{1/2}) dy.
$$


Lemma 5.4.

(i) For each \( t \in \mathcal{P}_m \),

\[
\int_{\mathfrak{H}_{n,m}} h_t(x) \, dx = 1. \tag{5.18}
\]

(ii) The Fourier transform of \( h_t(x) \) has the form

\[
(\mathcal{F} h_t)(y) = \exp(-\text{tr}(ty'y)), \tag{5.19}
\]

which implies the semi-group property

\[
h_t \ast h_\tau = h_{t+\tau}, \quad t, \tau \in \mathcal{P}_m. \tag{5.20}
\]

(iii) If \( f \in L^p(\mathfrak{H}_{n,m}) \), \( 1 \leq p \leq \infty \), then

\[
\|W_t f\|_p \leq \|f\|_p, \quad W_t W_\tau f = W_{t+\tau} f \tag{5.21}
\]

and

\[
\lim_{t \to 0} (W_t f)(x) = f(x) \tag{5.22}
\]
in the \( L^p \)-norm. If \( f \) is a continuous function vanishing at infinity, then (5.22) holds in the sup-norm.

Proof. To prove (i), we pass to polar coordinates. Owing to (2.5) and (2.31), we obtain

\[
\int_{\mathfrak{H}_{n,m}} h_t(x) \, dx = \frac{2^{-m} \sigma_{n,m}}{(4\pi)^{nm/2}} \int_{\mathcal{P}_m} \exp(-\text{tr}(t^{-1}r)/4) |r|^{n/2-d} \, dr
\]

\[
= \frac{2^{-m} \sigma_{n,m} \Gamma_m(n/2)}{(4\pi)^{nm/2}} |t^{-1}/4|^{-n/2} = 1.
\]

Formula (5.19) is well known (see, e.g., [Herz]), and the proof is elementary. Indeed, by changing variable \( x \to 2xt^{1/2} \) and setting \( \eta = 2yt^{1/2} \), we have

\[
(\mathcal{F} h_t)(y) = \pi^{-nm/2} \int_{\mathfrak{H}_{n,m}} \exp(-\text{tr}(tx'x)) \exp(i\text{tr}(\eta'x)) \, dx.
\]

This splits into a product of the one-dimensional Fourier transforms of Gaussian functions. Statement (iii) follows from (5.18) and (5.20). Indeed, the relations in (5.21) are
clear. Furthermore,

\[(W_t f)(x) - f(x) = \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) [f(x - yt^{1/2}) - f(x)] dy.\]

Hence, by the generalized Minkowski inequality,

\[\|W_t f - f\|_p \leq \int_{\mathfrak{M}_{n,m}} h_{I_m}(y) \|f(\cdot - yt^{1/2}) - f(\cdot)\|_p dy.\]

If \(t\) tends to the 0-matrix, then \(yt^{1/2} \to 0\) in \(\mathbb{R}^{nm}\). Since the integrand above does not exceed \(2\|f\|_p h_{I_m}(y)\), we can pass to the limit under the sign of integration, and the desired result follows. For continuous functions vanishing at infinity, the argument is similar. □

**Remark 5.5.** A challenging open problem is whether (5.22) holds for almost all \(x \in \mathfrak{M}_{n,m}\). This fact is well known in the case \(m = 1\) [St2, SW]. It follows from the estimate

\[
\sup_{t>0} |(W_t f)(x)| \leq (M^* f)(x),
\]

where \((M^* f)(x)\) is the Hardy–Littlewood maximal function. It would be desirable to extend this theory to the matrix case when the positive parameter \(t\) is replaced by a positive definite matrix.

Let us consider the Gårding–Gindikin fractional integrals on \(\mathcal{P}_m\) defined by

\[(I^\lambda g)(t) = \frac{1}{\Gamma_m(\lambda)} \int_\mathcal{P}_m |\tau - t|^{\lambda - d} d\tau, \quad (5.24)\]

where \(d = (m+1)/2\), \(Re \lambda > d - 1\), and integration is performed over all \(\tau \in \mathcal{P}_m\) so that \(\tau - t \in \mathcal{P}_m\). The following theorem establishes connection between the Riesz potentials, the Gauss–Weierstrass integrals, and the Gårding–Gindikin fractional integrals.

**Theorem 5.6.** Let \(m - 1 < Re \alpha < n - m + 1\), \(d = (m+1)/2\). Then

\[(I^\alpha f)(x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2 - d} (W_t f)(x) dt, \quad (5.25)\]

\[W_t[I^\alpha f](x) = I^{\alpha/2}[((W(\cdot) f)(x))(t)\], \quad (5.26)\]

provided that integrals on either side of the corresponding equality exist in the Lebesgue sense.
Proof. The right-hand side of (5.25) transforms as follows:

\[
\text{r.h.s.} = \frac{(4\pi)^{-nm/2}}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{(\alpha-n)/2-d} \exp(-\text{tr}(t^{-1}z)) dt \int f(y) dy
\]

where \( z = (x - y)'(x - y)/4 \). Now we change the order of integration and replace \( t^{-1} \) by \( t \). Since

\[
\int |t|^{(n-\alpha)/2-d} \exp(-\text{tr}(t z)) dt = \Gamma_m \left( \frac{n-\alpha}{2} \right) |z|^{(\alpha-n)/2}, \quad |z| \neq 0,
\]

(5.25) follows. The validity of (5.26) is a simple consequence of the semi-group property of the Gauss–Weierstrass integral. Indeed,

\[
W_t[I^\alpha f](x) = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |\tau|^{\alpha/2-d} (W_{t+\tau} f)(x) d\tau
\]

Theorem 5.6 has a number of remarkable consequences. Firstly, it enables us to invert the Riesz potential by inverting \( I^{\alpha/2} \) and applying the approximation property \( \lim_{t \to 0} W_t f = f \). We shall study this question in subsequent publications. Secondly, Theorem 5.6 provides a simple proof of the semigroup property of the Riesz potentials under mild assumptions for \( f \).

Corollary 5.7. Let

\[
\text{Re} \alpha > m - 1, \quad \text{Re} \beta > m - 1, \quad \text{Re} (\alpha + \beta) < n - m + 1.
\]

Then

\[
(I^\alpha I^\beta f)(x) = (I^{\alpha+\beta} f)(x)
\]

provided that the integral \( (I^{\alpha+\beta} f)(x) \) absolutely converges.

Proof. Applying Theorem 5.6 and changing the order of integration, we obtain

\[
I^\alpha I^\beta f = \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} |t|^{\alpha/2-d} W_t I^\beta f dt
\]
\[
\begin{align*}
&= \frac{1}{\Gamma_m(\alpha/2)} \int_{\mathcal{P}_m} \left| t \right|^{\alpha/2 - d} (I_\alpha f) (t) \, dt \\
&= \frac{1}{\Gamma_m(\alpha/2) \Gamma_m(\beta/2)} \int_{\mathcal{P}_m} W \tau f d\tau \int_0^\tau \left| t \right|^{\alpha/2 - d} \left| \tau - t \right|^{\beta/2 - d} \, dt \\
&= \frac{1}{\Gamma_m((\alpha + \beta)/2)} \int_{\mathcal{P}_m} \left| \tau \right|^{(\alpha + \beta)/2 - d} W \tau f d\tau = I^{\alpha + \beta} f. \quad \Box
\end{align*}
\]

**Remark 5.8.** For \( m = 1 \), conditions (5.27) have a well-known form

\[
Re \; \alpha > 0, \quad Re \; \beta > 0, \quad Re \; (\alpha + \beta) < n.
\]

Note that in the higher rank case, (5.27) is possible only if \( 3m < n + 3 \). It is also worth noting that the “usual” way to prove the semi-group property (5.28) for “rough” functions \( f \) via the Fourier transform formula and evaluation of the corresponding beta integral (cf. [SKM,St2]) becomes much more difficult in the higher rank case.

**Remark 5.9.** Since the “symbol” (i.e., the Fourier transform of the kernel) of the Riesz potential \( I^\alpha f \) is \( |y|_m^{-\alpha} \), then, for sufficiently good functions \( f \), (5.28) holds for arbitrary complex \( \alpha \) and \( \beta \). We will be concerned with this topic in Section 6.

### 5.3. \( L^p \)-convolutions

Another useful application of Theorem 5.6 is the following. By Lemma 4.2 and Theorem 4.4, convolutions with zeta distributions and Riesz potentials are well defined on test functions belonging to the Schwartz space. Are they meaningful for \( f \in L^p \)? Let us study this question.

**Theorem 5.10.** For \( f \in L^p(\mathcal{M}_{n,m}) \), the Riesz potential \( (I^\alpha f)(x) \) absolutely converges almost everywhere on \( \mathcal{M}_{n,m} \) provided

\[
Re \; \alpha > m - 1, \quad 1 \leq p < \frac{n}{Re \; \alpha + m - 1}. \tag{5.29}
\]

**Proof.** Note that (5.29) is possible if \( m \leq [(n + 1)/2] \) where \( [(n + 1)/2] \) is the integral part of \( (n + 1)/2 \). Conditions (5.29) are well known if \( m = 1 \) [St1] when they have the form \( 1 \leq p < n/Re \; \alpha \). To prove our theorem, it suffices to consider \( f \geq 0 \) and justify the following inequality:

\[
I \equiv \int_{\mathcal{M}_{n,m}} \exp(-\text{tr}(x^t x)) (I^\alpha f)(x) \, dx \leq c \| f \|_p, \quad c = \text{const}. \tag{5.30}
\]
Regarding $I$ as the Gauss–Weierstrass integral $(W_{I_m} |I^2 f|) (0)$ owing to (5.26), we have

$$I \simeq \int_{I_m}^\infty |\tau - I_m|^{\frac{\alpha}{2} - d} (W_{\tau} f) (0) d\tau$$

$$\simeq \int_{I_m}^\infty |\tau - I_m|^{\frac{\alpha}{2} - d} |\tau|^{-n/2} d\tau \int_{\mathbb{R}_{n,m}} \exp(-\text{tr}(\tau^{-1} x' x)/4) f(x) dx,$$

$d = (m + 1)/2$. By Hölder’s inequality, $I \lesssim A \|f\|_p$ where

$$A^p' = \int_{I_m}^\infty |\tau - I_m|^{\frac{\alpha}{2} - d} |\tau|^{-n/2} \left( \int_{\mathbb{R}_{n,m}} \exp(-p' \text{tr}(\tau^{-1} x' x)/4) dx \right)^{1/p'} d\tau,$$

$1/p + 1/p' = 1$. This integral can be easily estimated. Indeed, by setting $x = y \tau^{1/2}$, $dx = |\tau|^{n/2} dy$, we obtain

$$A^p' \simeq \int_{I_m}^\infty |\tau - I_m|^{\frac{\alpha}{2} - d} |\tau|^{-n/p} d\tau \quad (\tau^{-1} = r)$$

$$\simeq \int_0^\infty |I_m - r|^{\frac{\alpha}{2} - d} |r|^{(n/p - \alpha)/2} dr.$$

The last integral is finite if $p$ obeys (5.29). □

More precise information can be obtained for convolutions with zeta distributions (and therefore, for Riesz potentials) of integral order. For $k = 1, 2, \ldots, n - 1$, we denote

$$v_{n-k} = \begin{bmatrix} 0 & \; & \; \\ I_{n-k} & \; & \; \\ \; & \; & \; \end{bmatrix} \in V_{n,n-k}, \quad \text{Pr}_{I_{n-k}^*} v_{n-k} = v_{n-k} v_{n-k}^* = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix}, \quad (5.31)$$

$$\tilde{f}(x) = \int_{SO(n)} f(\gamma x) d\gamma. \quad (5.32)$$

**Lemma 5.11.** Let $n \geq m \geq 1; \; k = 1, 2, \ldots, n$. For any $\lambda > k + m - 1$,

$$\int_{\mathbb{R}_{n,m}} \frac{|(z_k \ast f)(x)|}{|I_m + x' x|^{\lambda/2}} dx = c_\lambda \int_{\mathbb{R}_{n,m}} \frac{\tilde{f}(x)}{|I_m + x' \text{Pr}_{I_{n-k}^*} x|^{(\lambda-k)/2}} dx, \quad (5.33)$$

$$c_\lambda = \frac{\pi^{nm/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(n/2) \Gamma_m(\lambda/2)}. \quad (5.34)$$
Proof. Suppose first that $k < n$, and denote the left-hand side of (5.33) by $I(f)$. By (5.2),

$$I(f) = c_1 \int_{V_{n,k}} dv \int_{\mathbb{R}^n} d\omega \int_{\mathbb{R}^n} f(x - v\omega) \frac{dx}{|I_m + x'x|^{\lambda/2}}$$

(set $v = \gamma v_0$, $v_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, $x = \gamma y$, $\gamma \in SO(n)$)

$$= c_1 \sigma_{n,k} \int_{\mathbb{R}^n} d\omega \int_{\mathbb{R}^n} \tilde{f}(x - v_0\omega) \frac{dx}{|I_m + y'y|^{\lambda/2}}$$

$$= c_1 \sigma_{n,k} \int_{\mathbb{R}^{n-k,m}} db \int_{\mathbb{R}^n} \tilde{f} \left( \begin{bmatrix} \omega \\ b \end{bmatrix} \right) d\omega \int_{\mathbb{R}^n} \frac{da}{|I_m + b'b + a'a|^{\lambda/2}}.$$}

For $\lambda > k + m - 1$, the last integral can be explicitly evaluated by the formula

$$\int_{\mathbb{R}^n} \frac{da}{|q + a'a|^{\lambda/2}} = \frac{\pi^{km/2}}{\Gamma_m((\lambda - k)/2)} \frac{\Gamma_m(\lambda/2)}{|q|^{(k-\lambda)/2}}, \quad q \in \mathcal{P}_m$$

(see (A.3)). This gives (5.33). For $k = n$, the proof is simpler and based on (5.35). In this case $Pr_{\mathbb{P}^{n-k}}$ should be replaced by the zero matrix. □

Corollary 5.12. If $n \geq m \geq 1$, $k = 1, 2, \ldots, n$, and $\lambda > k + m - 1$, then

$$\int_{\mathbb{R}^n} \frac{|(\zeta_k * f)(x)|}{|I_m + x'x|^{\lambda/2}} dx \leq c_\lambda \|f\|_1.$$  \hspace{1cm} (5.36)

If $k = n$ and $f \geq 0$, then (5.36) is a strict equality.

Let us extend (5.36) to $f \in L^p$ with $p \geq 1$. Now we have to impose extra restrictions because of the projection operator $Pr_{\mathbb{P}^{n-k}}$ on the right-hand side of (5.33).

Theorem 5.13. Let $n > m$; $k = 1, 2, \ldots, n - m$. If $f \in L^p$ and

$$\lambda > k + \max \left( m - 1, \frac{n + m - 1}{p'} \right), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

then

$$\int_{\mathbb{R}^n} \frac{|(\zeta_k * f)(x)|}{|I_m + x'x|^{\lambda/2}} dx \leq c \|f\|_p.$$  \hspace{1cm} (5.37)
provided

\[ 1 \leq p < \frac{n}{k + m - 1}. \]  

(5.38)

**Proof.** Note that (5.38) agrees with (5.29) in Theorem 5.10. It suffices to prove the statement for non-negative radial functions \( f(x) \equiv f_0(x'x) \). We remind the notation \( d = (m + 1)/2 \). Denote by \( I(f) \) the left-hand side of (5.37) and make use of (5.33). Splitting the integral in the right-hand side of (5.33) in \( \int_{\mathfrak{G}_{n-k,m}} \times \int_{\mathfrak{G}_{k,m}} \), and passing in \( \int_{\mathfrak{G}_{n-k,m}} \) to polar coordinates, we have

\[ I(f) \simeq \int_{\mathcal{P}_m} \frac{|r|^{(n-k)/2-d}}{|\lambda |^{(\lambda-k)/2}} d\lambda \int_{\mathfrak{G}_{k,m}} f_0(\omega'\omega + r) d\omega \]

\[ = \int_{\mathfrak{G}_{k,m}} d\omega \int_{s_0 < \omega} \frac{|s - \omega'\omega|^{(n-k)/2-d}}{|\lambda | + s - \omega'\omega|^{(\lambda-k)/2}} ds \]

\[ = \int_{\mathcal{P}_m} f_0(s) ds \int_{s_0 < \omega} \frac{|s - \omega'\omega|^{(n-k)/2-d}}{|\lambda | + s - \omega'\omega|^{(\lambda-k)/2}} d\omega \]

(set \( \omega = \eta s^{1/2}, \quad \eta \in \mathfrak{G}_{k,m}, \quad d\omega = |s|^{k/2} d\eta \))

\[ = \int_{\mathcal{P}_m} f_0(s)|s|^{n/2-d} ds \int_{\eta s < \lambda} \frac{|\lambda - \eta'|^{(n-k)/2-d}}{|\lambda | + s^{1/2}(\lambda - \eta')s^{1/2}|^{(\lambda-k)/2}} d\eta. \]

By taking into account that

\[ \| f \| \simeq \left( \int_{\mathcal{P}_m} |f_0(s)|^p |s|^{n/2-d} ds \right)^{1/p}, \]  

(5.39)

owing to the Hölder and the generalized Minkowski inequality, we obtain \( I(f) \lesssim c \| f \|_p \) where

\[ c = \int_{\eta s < \lambda} |\lambda - \eta'|^{(n-k)/2-d} A_{\mu'}^{1/p'}(\lambda - \eta') d\eta, \]

\[ A_{\mu}(r) = \int_{\mathcal{P}_m} \frac{|s|^{n/2-d}}{|\lambda | + s^{1/2}r s^{1/2}|^{\mu/2}} ds, \quad \mu = (\mu - k)p', \quad r = \lambda - \eta'. \]

Since \( |\lambda | + s^{1/2}r s^{1/2} = |\lambda | + r^{1/2}sr^{1/2} | \), by changing variable \( s = r^{-1/2}\rho r^{-1/2} \), we have

\[ A_{\mu}(r) = |r|^{-n/2} \int_{\mathcal{P}_m} \frac{|\rho|^{n/2-d}}{|\lambda | + \rho|^{\mu/2}} d\rho. \]
The last integral is finite for \( \mu > n + m - 1 \) or \((\lambda - k)p' > n + m - 1\) (moreover, it can be explicitly evaluated by (A.2)). Thus for \( \lambda > k + (n + m - 1)/p' \) we have
\[
c \simeq \int_{\eta' \eta < I_m} |I_m - \eta'\eta|^{(n-k)/2-n/2p'-d} d\eta
\]
\[
\simeq \int_0^1 |I_m - r|^{(n/p-k)/2-d}|r|^{-1/2} dr < \infty
\]
provided \( n/p - k > m - 1 \), i.e., \( 1 \leq p < n/(k + m - 1) \). This completes the proof. \( \square \)

**Remark 5.14.** We do not know whether conditions (5.38) and (5.29) are sharp. Moreover, it would be interesting to study boundedness properties of the Riesz potential operator \( I^\alpha \) on functions \( f \in L^p(M_{n,m}) \) in different norms.

6. Radon transforms

6.1. Definitions and basic properties

Let \( k, n, \) and \( m \) be positive integers, \( 0 < k < n \), \( V_{n,n-k} \) be the Stiefel manifold of orthonormal \( (n-k)\)-frames in \( \mathbb{R}^n \). For \( \xi \in V_{n,n-k} \), and \( t \in \mathcal{M}_{n-k,m} \), the linear manifold
\[
\tau = \tau(\xi, t) = \{ x : x \in \mathcal{M}_{n,m} ; \xi^t x = t \}
\]
will be called a matrix k-plane in \( \mathcal{M}_{n,m} \). We denote by \( \mathcal{I} \) the variety of all such planes. For \( m = 1 \), \( \mathcal{I} \) is an affine Grassmann manifold of \( k \)-dimensional planes \( \mathbb{R}^n \). The matrix k-plane Radon transform \( \hat{f}(\tau) \) of a function \( f(x) \) on \( \mathcal{M}_{n,m} \) assigns to \( f \) a collection of integrals of \( f \) over all matrix planes \( \tau \in \mathcal{I} \). Namely,
\[
\hat{f}(\tau) = \int_{x \in \tau} f(x), \quad \tau \in \mathcal{I}.
\]
The dual Radon transform \( \check{\varphi}(x) \) is the mean value of a function \( \varphi(\tau) \) on \( \mathcal{I} \) over all matrix planes \( \tau \) through \( x \):
\[
\check{\varphi}(x) = \int_{\tau \ni x} \varphi(\tau), \quad x \in \mathcal{M}_{n,m}.
\]
Precise meaning of these integrals is the following. Let \( \tau \) be the plane (6.1), \( \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k} \), and \( g_\xi \in SO(n) \) be a rotation satisfying \( g_\xi \xi_0 = \xi \). Then
\[
\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int_{\mathcal{M}_{k,m}} f \left( g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix} \right) d\omega
\]
and
\[ \tilde{\phi}(x) = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \varphi(\tilde{\xi}, \tilde{\xi}'x) \, d\tilde{\xi} = \int_{SO(n)} \varphi(\gamma \tilde{\xi}_0, \gamma' \tilde{\xi}_0'x) \, d\gamma. \] (6.3)

**Lemma 6.1.**

(i) The duality relation
\[ \int_{\mathcal{M}_{n,m}} f(x) \tilde{\phi}(x) \, dx = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\tilde{\xi} \int_{\mathcal{M}_{n-k,m}} \varphi(\tilde{\xi}, t) \hat{f}(\tilde{\xi}, t) \, dt \] (6.4)
holds provided that either side of (6.4) is finite for \( f \) and \( \varphi \) replaced by \( |f| \) and \( |\varphi| \), respectively.

(ii) If \( f \in L^1(\mathcal{M}_{n,m}) \), then the Radon transform \( \hat{f}(\xi, t) \) exists for all \( \xi \in V_{n,n-k} \) and almost all \( t \in \mathcal{M}_{n-k,m} \). Furthermore,
\[ \int_{\mathcal{M}_{n-k,m}} \hat{f}(\xi, t) \, dt = \int_{\mathcal{M}_{n,m}} f(x) \, dx \quad \forall \, \xi \in V_{n,n-k}. \] (6.5)

**Proof.** By (6.2), the right-hand side of (6.4) has the form
\[ \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\tilde{\xi} \int_{\mathcal{M}_{n-k,m}} \varphi(\tilde{\xi}, t) \hat{f}(\tilde{\xi}, t) \, dt \int_{\mathcal{M}_{k,m}} f(\tilde{g}_\xi \left[ \begin{array}{c} \omega \\ t \end{array} \right]) \, d\omega. \] (6.6)
Changing variables \( x = \tilde{g}_\xi \left[ \begin{array}{c} \omega \\ t \end{array} \right] \), we have
\[ \tilde{\xi}'x = (g_\xi \tilde{\xi}_0)' g_\xi \left[ \begin{array}{c} \omega \\ t \end{array} \right] = \tilde{\xi}' = \left[ \begin{array}{c} \omega \\ t \end{array} \right] = t. \]
Hence, by the Fubini theorem, (6.6) reads
\[ \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} d\tilde{\xi} \int_{\mathcal{M}_{n,m}} \varphi(\tilde{\xi}, \tilde{\xi}'x) f(x) \, dx = \int_{\mathcal{M}_{n,m}} \tilde{\phi}(x) f(x) \, dx. \]

Statement (ii) is a consequence of the Fubini theorem:
\[ \int_{\mathcal{M}_{n-k,m}} \hat{f}(\xi, t) \, dt = \int_{\mathcal{M}_{n-k,m}} dt \int_{\mathcal{M}_{k,m}} f(\tilde{g}_\xi \left[ \begin{array}{c} \omega \\ t \end{array} \right]) \, d\omega \]
\[ = \int_{\mathcal{M}_{n,m}} f(g_\xi x) \, dx = \int_{\mathcal{M}_{n,m}} f(x) \, dx. \] \( \square \)
The following properties of the Radon transform can be easily checked.

**Lemma 6.2.** Suppose that the Radon transform

\[ f(x) \rightarrow \hat{f}(\xi, t), \quad x \in \mathcal{M}_{n,m}, \quad (\xi, t) \in V_{n,n-k} \times \mathcal{M}_{n-k,m}, \]

exists (at least almost everywhere). Then

\[ \hat{f}(\xi', t) = \hat{f}(\xi, t) \quad \forall \xi' \in O(n-k). \] (6.7)

If \( g(x) = \gamma x + y \) where \( \gamma \in O(n), \beta \in O(m), \ y \in \mathcal{M}_{n,m} \), then

\[ (f \circ g)(\xi, t) = \hat{f}(\gamma \xi, \beta \xi + \xi'y). \] (6.8)

In particular, if \( f_y(x) = f(x + y) \), then

\[ \hat{f}_y(\xi, t) = \hat{f}(\xi, \xi'y + t). \] (6.9)

Equality (6.7) is a matrix analog of the “evenness property” of the classical Radon transform, cf. [Hel].

It is known [OR1] that the Radon transform \( f \rightarrow \hat{f} \) is injective on the Schwartz space \( \mathcal{S} = \mathcal{S}(\mathcal{M}_{n,m}) \) if and only if \( k \leq n-m \). The classical problem is how to reconstruct a function \( f \) from the integrals \( \hat{f}(\tau), \tau \in \mathfrak{T} \). One of the ways to do this lies via Riesz potentials \( I^k f \) of integral order. By Theorem 5.1, for \( 1 \leq k \leq n-m \) and \( f \in \mathcal{S} \) we have

\[ (I^k f)(x) = \gamma_1 \int_{SO(n)} d\gamma \int_{\mathcal{M}_{n,m}} f \left( x - \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega, \] (6.10)

\[ \gamma_1 = 2^{-km} \pi^{-km/2} \Gamma_m \left( \frac{n-k}{2} \right) / \Gamma_m \left( \frac{n}{2} \right). \] (6.11)

By Theorem 5.13, this expression is well defined a.e. on \( \mathcal{M}_{n,m} \) as an absolutely convergent integral for any \( f \in L^p \) provided \( 1 \leq p < n/(k+m-1) \). The following important statement establishes connection between Riesz potentials and Radon transforms. It generalizes the classical result of Fuglede; see [Fu,Hel] for \( m = 1 \).

**Theorem 6.3.** Let \( 1 \leq k \leq n-m \). Then

\[ \gamma_1 (\hat{f})^\vee(x) = (I^k f)(x), \] (6.12)

provided the Riesz potential on the right-hand side absolutely converges, e.g., for \( f \in L^p, \ 1 \leq p < n/(k+m-1) \).
Proof. Let \( f_x(y) = f(x + y) \). Then (6.9) yields \( \hat{f}_x(\xi, t) = \hat{f}(\xi, \xi'x + t) \). Owing to (6.3) and (6.2), by changing the order of integration, we have

\[
(\hat{f})^\vee(x) = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \hat{f}(\xi, \xi'x) d\xi = \frac{1}{\sigma_{n,n-k}} \int_{V_{n,n-k}} \hat{f}_x(\xi, 0) d\xi
\]

This coincides with \( \gamma^{-1}_1(I^k f)(x) \). \( \square \)

Corollary 6.4. If \( f \in L^p \), \( 1 \leq p < n/(k + m - 1) \), then \( \hat{f}(\tau) \) is finite for almost all \( \tau \in \mathbb{R} \).

Remark 6.5. The condition \( 1 \leq p < n/(k + m - 1) \) in Corollary 6.4 can be replaced by the weaker one, namely,

\[
1 \leq p < \frac{n + m - 1}{k + m - 1},
\]  

see [OR2]. The proof of this fact relies on completely different ideas related to representation of the Radon transform of radial functions by the Gårding–Gindikin fractional integral (5.24). For \( m = 1 \), both conditions coincide and cannot be improved. We are not sure that for \( m > 1 \), (6.13) is sufficient for the existence of the Riesz potential \( I^k f \) (in contrast to the case \( m = 1 \)); cf. Remark 5.14.

6.2. The inversion problem

Theorem 6.3 reduces the inversion problem for the Radon transform to that for the Riesz potentials (as in the rank-one case [Hel,Ru2]). Below we show how the unknown function \( f \) can be recovered in the framework of the theory of distributions.

Let us consider the Riesz distribution

\[
(h_\mathcal{S}, f) = a.c. \frac{1}{\gamma_{n,m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} |x|^{2-n} f(x) dx,
\]

where \( f \in S = S(\mathbb{R}^m) \). From the Fourier transform formula

\[
(h_\mathcal{S}, f) = (2\pi)^{-nm} (|y|^{-2}_m, (\mathcal{F} f)(y)),
\]

it is evident that the Schwartz class \( S \) does not suit well enough because it is not invariant under multiplication by \( |y|^{-2}_m \). To get around this difficulty, we choose another space of test functions. Let \( \Psi = \Psi(\mathbb{H}_{n,m}) \) be the collection of all functions
\( \psi(y) \in S(\mathcal{M}_{n,m}) \) vanishing on the manifold

\[
V = \{ y : y \in \mathcal{M}_{n,m}, \ \text{rank}(y) < m \} = \{ y : y \in \mathcal{M}_{n,m}, \ |y'| = 0 \} \tag{6.15}
\]

with all derivatives (the coincidence of both sets in (6.15) is clear because \( \text{rank}(y) = \text{rank}(y') \)). The manifold \( V \) is a cone in \( \mathbb{R}^{nm} \) with vertex 0. The set \( \Psi \) is a closed linear subspace of \( S \). Therefore, it can be regarded as a linear topological space with the induced topology of \( S \). Let \( \Phi = \Phi(\mathcal{M}_{n,m}) \) be the Fourier image of \( \Psi \). Since the Fourier transform \( \mathcal{F} \) is an automorphism of \( S \) (i.e., a topological isomorphism of \( S \) onto itself), then \( \Phi \) is a closed linear subspace of \( S \). Having been equipped with the induced topology of \( S \), the space \( \Phi \) becomes a linear topological space isomorphic to \( \Psi \) under the Fourier transform.

The spaces \( \Phi \) and \( \Psi \) were introduced by Semyanistyi [Se] in the rank-one case \( m = 1 \). They have proved to be very useful in integral geometry, fractional calculus, and the theory of function spaces. Further generalizations and applications can be found in [Li,Ru1,Sa,SKM].

In our case the following characterization of the space \( \Phi \) is a consequence of a more general result due to Samko [Sa].

**Theorem 6.6.** The Schwartz function \( \phi(x) \) on \( \mathcal{M}_{n,m} \) belongs to the space \( \Phi \) if and only if it is orthogonal to all polynomials \( p(x) \) on any hyperplane \( \tau \) in \( \mathbb{R}^{nm} \) having the form \( \tau = \{ x : \text{tr}(a'x) = c \} \), \( a \in V \):

\[
\int_{\tau} p(x)\phi(x) \, d\mu(x) = 0, \tag{6.16}
\]

\( d\mu(x) \) being the induced Lebesgue measure on \( \tau \).

Note that for \( m = 1 \), the space \( \Phi \) consists of Schwartz functions which are orthogonal to all polynomials on \( \mathbb{R}^n \).

We denote by \( \Phi' \) the space of all linear continuous functionals (generalized functions) on \( \Phi \). It is clear that two \( S' \)-distributions that coincide in the \( \Phi' \)-sense, differ from each other by an arbitrary \( S' \)-distribution with the Fourier transform supported by \( V \). Since for any complex \( z \), multiplication by \( |y|^{-z} \) is an automorphism of \( \Psi \), then, according to the general theory [GSh2], \( I^z \) is an automorphism of \( \Phi \), and we have

\[
\mathcal{F}[I^z\phi](y) = |y|^{-z} \mathcal{F}[\phi](y), \quad \phi \in \Phi. \tag{6.17}
\]

This gives

\[
I^z I^\beta \phi = I^{z+\beta} \phi \quad \forall \alpha, \beta \in \mathbb{C} \tag{6.18}
\]
(cf. Remark 5.9), and

$$\mathcal{F}[I^2 f](y) = |y|^{-2m}\mathcal{F}[f](y)$$  \hfill (6.19)

for all $\Phi'$-distributions $f$. The last formula implies the following

**Theorem 6.7.** The function $f \in L^p(\mathcal{M}_{n,m})$, $1 \leq p < n/(k + m - 1)$, can be recovered from the Radon transform $g = \hat{f}$ in the sense of $\Phi'$-distributions by the formula

$$(f, \phi) = \gamma_1(\hat{g}, I^{-k}\phi), \quad \phi \in \Phi,$$  \hfill (6.20)

where

$$(I^{-k}\phi)(x) = (\mathcal{F}^{-1}|y|^{k/m}\mathcal{F}\phi)(x),$$

$\gamma_1$ being the constant (6.11).

**Proof.** We have

$$\begin{align*}
(f, \phi) &= (2\pi)^{-nm}(\mathcal{F}[f], \mathcal{F}[\phi]) \\
&= (2\pi)^{-nm}(|y|^{-k/m}\mathcal{F}[f](y), |y|^{k/m}\mathcal{F}[\phi](y)) \\
&= (2\pi)^{-nm}((\mathcal{F}[I^k f])(y), |y|^{k/m}\mathcal{F}[\phi](y)) \\
&= (2\pi)^{-nm}\gamma_1((\mathcal{F}[\hat{g}](y), |y|^{k/m}\mathcal{F}[\phi](y)) \\
&= \gamma_1(\hat{g}, I^{-k}\phi). \quad \square
\end{align*}$$

**Remark 6.8.** For $k$ even, the Riesz potential $I^k f$ can be inverted (in the sense of $\Phi'$-distributions) by repeated application of the Cayley–Laplace operator $\Delta_m = \det(\partial^2\partial)$. This operator agrees with multiplication by $(-1)^m|y|^{2/m}$ in the Fourier terms, and therefore, $(-1)^km\Delta_m^{-1}I^k f = f$ in the $\Phi'$-sense.

**Remark 6.9.** It would be desirable to obtain pointwise inversion formulas for $I^k f$ and $\hat{f}$ (not in the $\Phi'$-sense). In the rank-one case such formulas can be found in [Ru1,Ru2].

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Appendix A

The formulas in this section are not new. Since it is not so easy to find simple proofs of them in the literature, we present such proofs for convenience of the reader. We recall that $\mathcal{P}_m$ denotes the cone of positive definite $m \times m$ matrices, $\overline{\mathcal{P}}_m$ is the closure of $\mathcal{P}_m$, $B_m$ is the beta function (2.10), $d = (m + 1)/2$. The following formulas hold:

\[ \int_s^\infty |r|^{-\gamma} |r - s|^{2 - d} \, dr = |s|^{2-\gamma} B_m(\alpha, \gamma - \alpha), \]
\[ s \in \mathcal{P}_m, \quad Re \alpha > d - 1, \quad Re (\gamma - \alpha) > d - 1, \tag{6.21} \]

\[ \int_s^\infty |I_m + r|^{-\gamma} |r - s|^{2 - d} \, dr = |I_m + s|^{2-\gamma} B_m(\alpha, \gamma - \alpha), \]
\[ s \in \overline{\mathcal{P}}_m, \quad Re \alpha > d - 1, \quad Re (\gamma - \alpha) > d - 1, \tag{6.22} \]

\[ \int_{\mathcal{M}_{k,m}} |b + y'y|^{-\lambda/2} \, dy = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)} |b|^{(k-\lambda)/2}, \]
\[ b \in \mathcal{P}_m, \quad Re \lambda > k + m - 1, \tag{6.23} \]

\[ \int_{\{y \in \mathcal{M}_{k,m}: y'y < b\}} |b - y'y|^{(\lambda-k)/2 - d} \, dy = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)} |b|^\lambda/2 - d, \]
\[ b \in \mathcal{P}_m, \quad Re \lambda > k + m - 1. \tag{6.24} \]

**Proof.** (A.1), (A.2). By setting $r = q^{-1}$, $dr = |q|^{-m-1} \, dq$, one can write the left-hand side of (A.1) as

\[ |s|^{2-\gamma} \int_0^{s^{-1}} |q|^{-\gamma-\alpha} |s^{-1} - q|^{2 - d} \, dq = |s|^{2-\gamma} B_m(\alpha, \gamma - \alpha) \]

and we are done. Equality (A.2) follows from (A.1) if we replace $s$ and $r$ by $I_m + s$ and $I_m + r$, respectively.

(A.3), (A.4). By changing variable $y \to y^{1/2}$, we obtain

\[ \int_{\mathcal{M}_{k,m}} |b + y'y|^{-\lambda/2} \, dy = |b|^{(k-\lambda)/2} J_1, \]

\[ \int_{\{y \in \mathcal{M}_{k,m}: y'y < b\}} |b - y'y|^{(\lambda-k)/2 - d} \, dy = |b|^\lambda/2 - d J_2, \]
where

\[ J_1 = \int_{\mathcal{M}_{k,m}} |I_m + y'y|^{-\lambda/2} dy, \]

\[ J_2 = \int_{\{y \in \mathcal{M}_{k,m} : y'y < I_m\}} |I_m - y'y|^{(\lambda-k)/2-d} dy. \]

Thus we have to show that

\[ J_1 = J_2 = \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)}. \]

**The case** \( k \geq m \). We write both integrals in the polar coordinates according to Lemma 2.6. For \( J_1 \) we have

\[ J_1 = 2^{-m} \sigma_{k,m} \int_{\mathcal{P}_m} |r|^{k/2-d} |I_m + r|^{-\lambda/2} dr \]

\[ = 2^{-m} \sigma_{k,m} B_m \left( \frac{k}{2}, \frac{\lambda - k}{2} \right) \]

(the second equality holds by (A.2) (with \( s = 0, \alpha = k/2, \gamma = \lambda/2 \)). Similarly,

\[ J_2 = 2^{-m} \sigma_{k,m} \int_0^{I_m} |r|^{k/2-d} |I_m - r|^{(\lambda-k)/2-d} dr \]

\[ = 2^{-m} \sigma_{k,m} B_m \left( \frac{k}{2}, \frac{\lambda - k}{2} \right). \]

Now the result follows by (2.10) and (2.31).

**The case** \( k < m \). We replace \( y \) by \( y' \) and pass to the polar coordinates. This yields

\[ J_1 = \int_{\mathcal{M}_{m,k}} |I_m + yy'|^{-\lambda/2} dy \]

\[ = 2^{-k} \int_{\mathcal{P}_k} dv \int_{\mathcal{V}_{m,k}} |I_m + vqv'|^{-\lambda/2} |q|^{(m-k-1)/2} dq, \]

\[ (|I_m + vqv'| = |I_k + q|) \]

\[ = 2^{-k} \sigma_{m,k} \int_{\mathcal{P}_k} |q|^{(m-k-1)/2} |I_k + q|^{-\lambda/2} dq. \]
By (A.2) (with $s = 0$, $m = k$, $\gamma = \lambda/2$) and (2.9),

$$J_1 = 2^{-k} \sigma_{m,k} B_k \left( \frac{m}{2}, \frac{\lambda - m}{2} \right)$$

$$= \frac{\pi^{km/2} \Gamma_k((\lambda - m)/2)}{\Gamma_k(\lambda/2)}$$

$$= \frac{\pi^{km/2} \Gamma_m((\lambda - k)/2)}{\Gamma_m(\lambda/2)}.$$

Similarly,

$$J_2 = \int_{\{y \in \mathbf{R}^m, k: yy' < I_m\}} |I_m - yy'|^{(\lambda - k)/2 - d} dy$$

$$= 2^{-k} \int_{V_{m,k}} dv \int_{\{q \in P_k: vq' < I_m\}} |I_m - vq v'|^{(\lambda - k)/2 - d} |q|^{(m - k - 1)/2} dq$$

$$= 2^{-k} \sigma_{m,k} \int_0^I |I_k - q|^{(\lambda - m - k - 1)/2} |q|^{(m - k - 1)/2} dq$$

$$= 2^{-k} \sigma_{m,k} B_k \left( \frac{m}{2}, \frac{\lambda - m}{2} \right)$$

and we get the same. $\square$

References


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