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On a problem of Eidelheit from `The Scottish Book` concerning absolutely continuous functions

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ABSTRACT

A negative solution of Problem 188 posed by Max Eidelheit in *the Scottish Book* concerning superpositions of separately absolutely continuous functions is presented. We discuss here this and some related problems which have also negative solutions. Finally, we give an explanation of such negative answers from the "embeddings of Banach spaces" point of view.

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1. Introduction

There are several equivalent definitions of the concept of absolute continuity. The notion and the term of *absolutely continuous* was introduced in 1905 by G. Vitali [27]. Let $I = [a, b] \subset \mathbb{R}$ and $f : I \to \mathbb{R}$. The function f is called *absolutely continuous on I* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $a \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n \leqslant b$ the condition $\sum_{k=1}^{n}(b_k - a_k) < \delta$ implies that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ (cf. I.P. Natanson [14, p. 243]). Also we can say that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite collection of mutually disjoint intervals $I_k = (a_k, b_k) \subset I$ ($k = 1, 2, \ldots, n$) we have that $\sum_{k=1}^{n} |I_k| < \delta$ implies $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$. The requirement that the open intervals I_k are disjoint is sometimes stated by saying that the corresponding closed intervals $\{[a_k, b_k]\}_{k=1}^{n}$ must be *nonoverlapping*, that is, their interiors are disjoint. Note that since the number $n \in \mathbb{N}$ is arbitrary, we can also take $n = \infty$, that is, replace finite sums by series.

It is obvious that an absolutely continuous function is continuous and it is easy to show that it is also of bounded variation. The classical Banach–Zareckiĭ theorem states that a function $f : I \to \mathbb{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the *Luzin (N) property*, that is, maps null sets into null sets (cf. [4, Theorem 7.11], [14, p. 250], [23, p. 146] and [29]). Of course, every *Lipschitz function* on I, that is, any function $f : I \to \mathbb{R}$ satisfying the condition that there exists a constant $C > 0$ such that $|f(x) - f(y)| \leqslant C|x - y|$ for all $x, y \in I$, is absolutely continuous on I. The set of all Lipschitz functions on I we denote by $\mathrm{Lip}_1(I)$. One of the equivalent norms in $\mathrm{Lip}_1(I)$ is defined by

$$\|f\| = |m_f| + \sup_{x, y \in I, \; x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad \text{where } m_f = \int_I f(x)\, dx.$$

The fundamental theorem of calculus for absolutely continuous functions (cf. [14, pp. 253–255], [21, pp. 197–198] and [23, pp. 148–149]) gives that a function $f : I \to \mathbb{R}$ is absolutely continuous if and only if f is differentiable almost everywhere on I, the derivative $f' \in L_1(I)$, i.e. is Lebesgue integrable, and $f(x) = f(a) + \int_a^x f'(t)\, dt$ for every $x \in I$. On the other hand, if we put instead of $f' \in L_1(I)$ the stronger assumption $f' \in L_\infty(I)$ we obtain a characterization of Lipschitz functions on I. Therefore, for an absolutely continuous function f, the condition $\int_a^b |f'(x)|^p\, dx < \infty$ for each $p > 1$ is a natural weakening of the Lipschitz condition.

In what follows we consider only the segment $I = [a, b] = [0, 1]$ and the square $Q = I^2$.

Max Eidelheit on October 27, 1940 wrote in *The Scottish Book* the following problem concerning superposition of absolutely continuous functions (cf. [12, Problem 188.1, p. 261[2]]):

Problem *(Eidelheit).* Let a function $f : Q \to \mathbb{R}$ be absolutely continuous on every straight line parallel to the axes of the coordinate system and let $g_1, g_2 : I \to I$ be absolutely continuous functions. Is the function $f(g_1(t), g_2(t))$ also absolutely continuous? If not, then perhaps this holds under the additional assumptions that $\iint_Q |f'_x|^p\, dx\, dy < \infty$ and $\iint_Q |f'_y|^p\, dx\, dy < \infty$, where $p > 1$?

There is no any comment about this problem in the book [12] on page 261.

Note that there are several different meanings of the conditions in the problem: f can be absolutely continuous on every straight line parallel to the axes or on almost every straight line parallel to the axes, the integrals can be bounded for some p or for every p and the derivatives can exist everywhere or almost everywhere.

Our intention here is to give some short historical comments to the Eidelheit problem (as we will show in Theorem 1, the answer has been known to a great extent even before the problem was posed), and present some variations and generalizations of known results connected with this problem.

It easy to see that the first part of Eidelheit's question has a negative answer. Consider the Schwarz function

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & \text{if } x^2 + y^2 > 0, \\ 0, & \text{if } x = y = 0. \end{cases}$$

The function f is absolutely continuous in each variable since for any fixed $y \in (0, 1]$ we have that $|f(x, y) - f(u, y)| \leqslant \frac{2}{y}|x - u|$ for all $x, u \in I$ and $f(x, 0) - f(u, 0) = 0$. Similarly with fixed x. If we take the functions $g_1(t) = g_2(t) = t$, then the superposition $f(g_1(t), g_2(t)) = f(t, t)$ is 2 for $t \neq 0$ and 0 for $t = 0$ and, hence, it is discontinuous at $t = 0$, and therefore not absolutely continuous on I. Note that the integrals in the Eidelheit problem are unbounded for the Schwartz function if $p > 1$.

Also the second part of Eidelheit's problem has a negative answer, which we will present in the next section. We even give a negative answer to the diagonal case, that is, when $g_1(t) = g_2(t) = t$.

The paper is organized as follows: In Section 2 we show how to obtain the answer to Eidelheit's question using a well-known theorem of Fichtenholz. Then we obtain two variable Fichtenholz theorems concerning superposition of absolutely continuous functions as a corollary of a general theorem on superpositions in Banach spaces. Section 3 contains a counterexample to the diagonal version of Eidelheit's problem. Finally, in Section 4 we give the "embeddings of Banach spaces" approach to this problem.

2. Superposition of absolutely continuous functions

We start with the question about the superposition of one variable functions. As is well known, the functions $f(x) = x^{1/2}$ and $g(x) = x^2 \sin^2(\frac{1}{x})$, if $x > 0$ and $= 0$, if $x = 0$ are absolutely continuous on I but their superposition $f \circ g$ is not since it has infinite variation. We even have that $g \in \mathrm{Lip}_1(I)$ since $|g'(x)| \leqslant 4$ for all $x \in I$.

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[2] In the original handwritten *Scottish Book* in Polish language this problem has number 188 and there was double numeration of Problem 185 (one written by Saks and the other one by Banach). In the English translation done by Ulam in 1957 appeared instead double numeration of Problem 188 (one by Sobolev, which originally had number 187, and the other one by Eidelheit). This was probably the reason why in the Mauldin edition of *The Scottish Book* [12] we have the numbers 188 on the Sobolev problem and 188.1 on the Eidelheit problem. One can suppose that the integral conditions of the Eidelheit problem are connected with Sobolev's visit to the *Scottish Café* after which Eidelheit became more familiar with Sobolev spaces.

For the first time the existence of superposition of absolutely continuous functions which is not absolutely continuous was noticed by W. Wilkosz [28, pp. 479–480] (cf. also [20, Example 1]) and D. Jackson. More exactly, it was noted in [26, p. 462] that the proposition from [25, p. 280] on absolute continuity of the superposition of absolutely continuous functions is not true and that Jackson informed de la Vallée Poussin about this fact. In the Carathéodory book [5, pp. 554–555] there is a construction of $g : I \to I$ such that its variation is equal to 1, but the variation of $\sqrt{g}$ is infinite. The absolute continuity of the superposition of absolutely continuous functions was investigated in detail by G. Fichtenholz. Already in 1922 (cf. [7, pp. 436–439] and [8, pp. 289–291]) Fichtenholz proved the following result, where we can see big difference between nonoverlapping and overlapping intervals:

Theorem A *(Fichtenholz). Let $f : I \to \mathbb{R}$ be a function. Then the following conditions are equivalent*:

 (i) *f is Lipschitz on I.*
 (ii) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $0 \leqslant a_k < b_k \leqslant 1$ ($k = 1, 2, \ldots, n$) the condition $\sum_{k=1}^{n}(b_k - a_k) < \delta$ implies $\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon$.*
(iii) *For every absolutely continuous function $g : I \to I$ the superposition $f \circ g$ is absolutely continuous on I.*
(iv) *For every Lipschitz function $g : I \to I$ the superposition $f \circ g$ is absolutely continuous on I.*

The Banach–Zareckiĭ theorem indicates that a superposition of two absolutely continuous functions can fail to be absolutely continuous if and only if it is not of bounded variation since both continuity and Luzin's condition (N) are preserved under superposition. Therefore the question about superposition of functions of bounded variation has the same answer as that about superposition of absolutely continuous functions. From the Fichtenholz characterization in Theorem A we can get a similar characterization for functions of bounded variation (BV), which was done in 1981 by M. Josephy [9, Theorem 2]: for $f : I \to I$ the superposition $f \circ g \in$ BV for all $g \in$ BV if and only if f is a Lipschitz function on I.

It is well known since a long time that the absolutely continuous function $f(x) = \int_0^x \ln t \, dt$ is not Lipschitz, because its derivative $f'(x) = \ln x$ is unbounded, and that $\int_0^1 |\ln x|^p \, dx < \infty$ for every $p > 1$. This fact together with Theorem A answers the one variable version of Eidelheit's question. A negative answer to one variable Eidelheit's question gives automatically the negative answer to the corresponding two (and n) variable question. More exactly, we have:

Theorem 1. *There exists a function $\varphi : Q \to \mathbb{R}$, absolutely continuous in each variable, such that $\iint_Q |\varphi_x'|^p \, dx \, dy < \infty$, $\iint_Q |\varphi_y'|^p \, dx \, dy < \infty$ for every $p > 1$, and Lipschitz functions g_1, g_2 in I such that the superposition $\varphi(g_1(t), g_2(t))$ is not absolutely continuous.*

Proof. Indeed, let $f(x) = \int_0^x \ln t \, dt$ and g be the corresponding Lipschitz function from Theorem A. Put $\varphi(x, y) = f(x)$ (the function φ depends on y only formally) and $g_1 = g_2 = g$. Then φ is absolutely continuous in each variable and for each $p > 0$ we have that

$$\iint_Q |\varphi_x'|^p \, dx \, dy = \int_I |f'|^p \, dx < \infty, \qquad \iint_Q |\varphi_y'|^p \, dx \, dy = 0,$$

and the superposition $\varphi(g_1(t), g_2(t)) = f(g(t))$ is not absolutely continuous. $\quad \square$

Naturally there arises a question on the validity of the two variables (and n variables) Fichtenholz theorem. In fact, we will derive this generalization as a corollary of a more general theorem on superpositions in normed spaces.

Let us recall that a mapping $f : X \to Y$ defined on a metric space (X, d) with values in a metric space (Y, ρ) is called *Lipschitz on X* if there exists a constant $C > 0$ such that

$$\rho\big(f(x), f(y)\big) \leqslant C \, d(x, y) \quad \text{for all } x, y \in X.$$

A mapping $g : I \to X$ is called *absolutely continuous on I* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n \leqslant 1$ the condition $\sum_{k=1}^{n}(b_k - a_k) < \delta$ implies that $\sum_{k=1}^{n} d(g(b_k), g(a_k)) < \varepsilon$.

Theorem 2. *Let P be a convex set in a normed space X and $f : P \to \mathbb{R}$ be a function. Then the following conditions are equivalent*:

 (i) *For every compact set $K \subset P$ the restriction $f|_K$ is Lipschitz on K.*
 (ii) *For every absolutely continuous mapping $g : I \to P$ the superposition $h = f \circ g$ is absolutely continuous on I.*
(iii) *For every Lipschitz mapping $g : I \to P$ the superposition $h = f \circ g$ is absolutely continuous on I.*

Proof. (i) $\Rightarrow$ (ii). Let $g : I \to P$ be absolutely continuous. Since the set $K = g(I)$ is compact as a continuous image of the compact set I, the function $f \circ g$ is absolutely continuous on I as a superposition of Lipschitz and absolutely continuous mappings.

The implication (ii) ⇒ (iii) is obvious.

(iii) ⇒ (i). Suppose that there exists a compact set $K \subset P$ for which the restriction $f|_K$ is not Lipschitz. Then there are points $x_n, y_n \in P$, $n = 1, 2, \ldots$, such that for every $n \in \mathbb{N}$ we have

$$\frac{|f(x_n) - f(y_n)|}{d_n} \geqslant 2n^3 \max_{x \in K} |f(x)|,$$

where $d_n = \|x_n - y_n\|$. Hence, $d_n \leqslant \frac{1}{n^3}$ for every n. Since K is compact, without loss of generality, one can assume that $\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$. Then, in particular, there exists $x_0 = \lim_{n \to \infty} x_n$.

For every $n \in \mathbb{N}$ put $k_n = [\frac{1}{n^2 d_n}]$. Note that $\frac{1}{n^2} - d_n < k_n d_n \leqslant \frac{1}{n^2}$. Hence, $k_n d_n \sim \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} k_n d_n < \infty$.

Define a sequence of segments $[a_n, b_n]$ recursively as follows: put $a_1 = 0$ and for each $n > 0$,

$$b_n = a_n + 2k_n d_n, \qquad a_{n+1} = b_n + \|x_n - x_{n+1}\|.$$

Let $b = \lim_n b_n$. Then $b < \infty$, since $\sum_{n=1}^{\infty} k_n d_n < \infty$ and $\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$. By definition,

$$0 = a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots.$$

Moreover,

$$[0, b] = \bigcup_{n=1}^{\infty} \left([a_n, b_n] \cup [b_n, a_{n+1}]\right) \cup \{b\}.$$

Let us construct a Lipschitz mapping $g : [0, b] \to P$ such that the composition function $h : [0, b] \to \mathbb{R}$ given by $h(x) = f(g(x))$ is not absolutely continuous.[3]

We define the mapping g, on each segment $[a_n, b_n]$, as follows:

(1) $g(a_n + 2i d_n) = x_n$ for $0 \leqslant i \leqslant k_n$ and $g(a_n + (2i - 1)d_n) = y_n$ for $1 \leqslant i \leqslant k_n$;
(2) g is linear on every segment $[a_n + (j - 1)d_n, a_n + j d_n]$ for $1 \leqslant j \leqslant 2k_n$.

The length of each segment $[a_n + (j - 1)d_n, a_n + j d_n]$ is equal to $d_n = \|x_n - y_n\|$, so g is Lipschitz with the constant $C = 1$ on each such segment (hence on the whole interval $[a_n, b_n]$).

Define g to be linear on each segment $[b_n, a_{n+1}]$. Since, by (1), $g(b_n) = x_n$, $g(a_{n+1}) = x_{n+1}$, and $\|x_n - x_{n+1}\| = a_{n+1} - b_n$, the mapping g is Lipschitz with the constant $C = 1$ on the segment $[b_n, a_{n+1}]$. Finally, $g(b) = x_0$.

Since g is Lipschitz with the constant $C = 1$ on $[0, b)$ and is continuous at b, it is Lipschitz on the segment $[0, b]$. However, the variation of h between a_n and b_n is

$$\bigvee_{a_n}^{b_n}(h) = \sum_{i=1}^{2k_n} \left|h(a_n + i d_n) - h\left(a_n + (i - 1)d_n\right)\right|$$

$$= \sum_{i=1}^{2k_n} \left|f(x_n) - f(y_n)\right| \geqslant n \sum_{i=1}^{2k_n} d_n = 2n d_n k_n \sim \frac{2}{n},$$

and, therefore, $\bigvee_0^b(h) \geqslant \sum_{n=1}^{\infty} \bigvee_{a_n}^{b_n}(h) = \infty$. Thus h is not absolutely continuous on $I = [0, b]$. □

Remark 1. Theorem 1 is false for convex sets P in linear metric spaces. Let, for example, $0 < p < 1$ and $P = [0, 1] \subset (\mathbb{R}, |\cdot|_p)$, with the distance $|x - y|_p := |x - y|^p$. It is not difficult to verify that every absolutely continuous function $g : I \to P$ is constant. Therefore, the set P and an arbitrary mapping f satisfy the conditions (ii) and (iii). But, for example, the function $f : P \to \mathbb{R}$, defined by $f(x) = x^{p/2}$, is not Lipschitz because

$$\frac{|x^{p/2} - 0|}{|x - 0|_p} = x^{-p/2} \to \infty \quad \text{as } x \to 0^+.$$

Corollary 1. *Let $f : Q \to \mathbb{R}$ be a function. The following conditions are equivalent*:

(i) *f is Lipschitz on Q.*
(ii) *For every absolutely continuous functions $g_1, g_2 : I \to I$ the superposition $f(g_1(x), g_2(x))$ is absolutely continuous.*
(iii) *For every Lipschitz functions $g_1, g_2 : I \to I$ the superposition $f(g_1(x), g_2(x))$ is absolutely continuous.*

Proof. Since the mapping $g = (g_1, g_2) : I \to Q$ is Lipschitz (absolutely continuous) if and only if the functions g_1, g_2 are Lipschitz (absolutely continuous), Theorem 1 implies Corollary 1. □

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[3] In this construction we use the interval $I = [0, b]$ instead of $I = [0, 1]$, but this is not essential.

3. The diagonal case

By putting $g_1(x) = g_2(x) = x$ in Eidelheit's problem, we obtain the question on absolute continuity for the diagonal of a separately absolutely continuous function. In this section we give a negative answer to a stronger version of this question.

Theorem 3. *Let $(u_n)_{n=1}^{\infty}$ be a sequence of reals $u_n > 0$ such that $\sum_{n=1}^{\infty} u_n < \infty$. Then there exists a separately Lipschitz function $f : Q \rightarrow \mathbb{R}$ such that the Lebesgue measure*

$$\lambda\left(\left\{z \in Q : f'_x(z) \neq 0 \text{ or } f'_y(z) \neq 0\right\}\right) \leqslant \sum_{n=1}^{\infty} u_n, \tag{1}$$

$$\lambda\left(\left\{z \in Q : \left|f'_x(z)\right| \geqslant 2^n\right\}\right) \leqslant \sum_{k=n}^{\infty} u_k \tag{2}$$

and

$$\lambda\left(\left\{z \in Q : \left|f'_y(z)\right| \geqslant 2^n\right\}\right) \leqslant \sum_{k=n}^{\infty} u_k \tag{3}$$

for every $n \in \mathbb{N}$, and the function $h(x) = f(x, x)$ has unbounded variation on I.

Proof. For every $n \in \mathbb{N}$ let

$$I_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right], \qquad k_n = \left[\frac{1}{4^n u_n}\right] + 1 \quad \text{and} \quad d_n = \frac{1}{2^n k_n}.$$

Note that $k_n d_n^2 \leqslant u_n$. Moreover, for every $n \in \mathbb{N}$ and $1 \leqslant i \leqslant k_n$ let

$$I_{n,i} = [a_{n,i}, b_{n,i}] = \left[\frac{1}{2^n} + (i-1)d_n, \frac{1}{2^n} + id_n\right], \qquad c_{n,i} = \frac{a_{n,i} + b_{n,i}}{2}, \qquad Q_{n,i} = I_{n,i} \times I_{n,i}$$

and choose a continuous separately Lipschitz function $\varphi_{n,i} : Q_{n,i} \rightarrow \mathbb{R}$ so that

(1) $\varphi_{n,i}(x, y) = 0$ if $(x, y) \notin \text{int } Q_{n,i}$,
(2) $\varphi_{n,i}(c_{n,i}, c_{n,i}) = \frac{1}{2k_n}$,
(3) $\varphi_{n,i}$ is linear on every segment connecting a boundary point of $Q_{n,i}$ with $(c_{n,i}, c_{n,i})$.

Next, put

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \varphi_{n,i}(x, y).$$

Since all segments $(a_{n,i}, b_{n,i})$ are disjoint, for every $x_0, y_0 \in (a_{n,i}, b_{n,i})$ and $x, y \in I$ we have that $f(x_0, y) = \varphi_{n,i}(x_0, y)$ and $f(x, y_0) = \varphi_{n,i}(x, y_0)$. Therefore, f is separately Lipschitz.

For every $n \in \mathbb{N}$ consider

$$A_n = \left\{z \in Q : \left|f'_x(z)\right| \geqslant 2^n\right\} \quad \text{and} \quad B_n = \left\{z \in Q : \left|f'_y(z)\right| \geqslant 2^n\right\}.$$

Note that $|(\varphi_{n,i})'_x(z)| \leqslant \frac{1}{d_n k_n} = 2^n$ almost everywhere on $Q_{n,i}$. Therefore

$$A_n \subseteq \bigcup_{m=n}^{\infty} \bigcup_{i=1}^{k_m} Q_{m,i},$$

and, thus,

$$\lambda(A_n) \leqslant \sum_{m=n}^{\infty} \sum_{i=1}^{k_m} \lambda(Q_{m,i}) = \sum_{m=n}^{\infty} k_m d_m^2 \leqslant \sum_{m=n}^{\infty} u_m.$$

Similarly we find that $\lambda(B_n) \leqslant \sum_{m=n}^{\infty} u_m$. Since

$$C := \left\{z \in Q : f'_x(z) \neq 0 \text{ or } f'_y(z) \neq 0\right\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} Q_{n,i}$$

it follows that

$$\lambda(C) \leqslant \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \lambda(Q_{n,i}) = \sum_{n=1}^{\infty} u_n.$$

We only need to show that the function $h(x) = f(x, x)$ has unbounded variation on I. We have that

$$\bigvee_{0}^{1}(h) \geqslant \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \left(\left| h\left(c_{n,i} - h(a_{n,i})\right) \right| + \left| h\left(b_{n,i} - h(c_{n,i})\right) \right| \right)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \left(\frac{1}{2k_n} + \frac{1}{2k_n} \right) = \infty.$$

The proof is complete. $\square$

A Banach space E of classes of measurable functions $f : Q \to \mathbb{R}$ is called *rearrangement invariant* (r.i.) *Banach function space* or *symmetric space* (over Q with the Lebesgue measure λ such that $\lambda(Q) = 1$), if it satisfies the following conditions (see e.g. [2, p. 59], [10, p. 90] and [11, pp. 114–119]):

(1) if $|f(z)| \leqslant |g(z)|$ for λ-almost every $z \in Q$, f measurable on Q and $g \in E$, then $g \in E$ and $\|f\|_E \leqslant \|g\|_E$;
(2) if f and g are equimeasurable, that is, $\lambda(\{z \in Q : |f(z)| > \alpha\}) = \lambda(\{z \in Q : |g(z)| > \alpha\})$ for every $\alpha \geqslant 0$ and $g \in E$, then $f \in E$ and $\|f\|_E = \|g\|_E$.

Note that for any r.i. space E on Q we have continuous embeddings

$$L_\infty(Q) \subset E \subset L_1(Q) \quad \text{with } \|f\|_{L_1(Q)} \leqslant \frac{\|f\|_E}{\|\chi_Q\|_E} \leqslant \|f\|_{L_\infty(Q)} \text{ for all } f \in L_\infty(Q).$$

Moreover, since $\lambda(Q) = 1$ we can have as the definition of equimeasurability of f and g in (2) the equality $\lambda(\{z \in Q : |f(z)| \geqslant \alpha\}) = \lambda(\{z \in Q : |g(z)| \geqslant \alpha\})$ for every $\alpha > 0$.

The following lemma is well known (see e.g. [3, p. 2], [10, p. 98]):

Lemma 1. *Let E be a r.i. Banach function space on Q, $g \in E$ and $f : Q \to \mathbb{R}$ be a measurable function such that*

$$\lambda\left(\left\{z \in Q : \left|f(z)\right| \geqslant \alpha\right\}\right) \leqslant \lambda\left(\left\{z \in Q : \left|g(z)\right| \geqslant \alpha\right\}\right)$$

for every $\alpha \geqslant 0$. Then $f \in E$ and $\|f\|_E \leqslant \|g\|_E$.

Lemma 2. *Let $(v_n)_{n=1}^{\infty}$ be a decreasing sequence of reals $v_n > 0$. Then there exists a sequence $(u_n)_{n=1}^{\infty}$ of reals $u_n > 0$ such that $\sum_{k \geqslant n} u_k \leqslant v_n$ for every n.*

Proof. It is sufficient to take $u_n = \frac{v_n}{n} - \frac{v_{n+1}}{n+1}$ for every n. $\square$

Corollary 2. *Let $\{E_s\}_{s \in S}$ be a family of r.i. Banach function spaces E_s on Q such that $(\bigcap_{s \in S} E_s) \setminus L_\infty(Q) \neq \emptyset$. Then there exists a separately Lipschitz function $f : Q \to \mathbb{R}$ such that $f'_x, f'_y \in \bigcap_{s \in S} E_s$ and the function $h(x) = f(x, x)$ has unbounded variation.*

Proof. First, let us take a function $g \in (\bigcap_{s \in S} E_s) \setminus L_\infty(Q)$ and for every $n \in \mathbb{N}$ put

$$v_n = \lambda\left(\left\{z \in Q : \left|g(z)\right| \geqslant 2^{n+1}\right\}\right).$$

By using Lemma 2 we can choose a sequence $(u_n)_{n=1}^{\infty}$ of reals $u_n > 0$ such that $\sum_{k=n}^{\infty} u_k \leqslant v_n$ for every $n \in \mathbb{N}$.

Theorem 3 implies that there exists a separately Lipschitz function $f : Q \to \mathbb{R}$ such that the conditions (1)–(3) from Theorem 3 are satisfied for every $n \in \mathbb{N}$, and the function $h(x) = f(x, x)$ has unbounded variation.

We only need to show that $f'_x, f'_y \in \bigcap_{s \in S} E_s$. For fixed $\alpha > 0$ let

$$A_\alpha = \left\{z \in Q : \left|f'_x(z)\right| \geqslant \alpha\right\} \quad \text{and} \quad B_\alpha = \left\{z \in Q : \left|g(z)\right| \geqslant \alpha\right\}.$$

If $\alpha \in (0, 4]$, then

$$\lambda(A_\alpha) \leqslant \lambda\left(\left\{z \in Q : f'_x(z) \neq 0\right\}\right) \leqslant \sum_{n=1}^{\infty} u_n \leqslant v_1$$

$$= \lambda\left(\left\{z \in Q : \left|g(z)\right| \geqslant 4\right\}\right) \leqslant \lambda(B_\alpha).$$

If $\alpha > 4$, then choosing $n \in \mathbb{N}$ with $2^n < \alpha \leqslant 2^{n+1}$ we have that

$$\lambda(A_\alpha) \leqslant \lambda\big(\big\{z \in Q \colon \big|f_x'(z)\big| \geqslant 2^n\big\}\big) \leqslant \sum_{k=n}^{\infty} u_k \leqslant v_n$$

$$= \lambda\big(\big\{z \in Q \colon \big|g(z)\big| \geqslant 2^{n+1}\big\}\big) \leqslant \lambda(B_\alpha).$$

Thus $\lambda(A_\alpha) \leqslant \lambda(B_\alpha)$ and $f_x' \in \bigcap_{s \in S} E_s$ by Lemma 1. Similarly we can prove that $f_y' \in \bigcap_{s \in S} E_s$. The proof is complete. $\quad\square$

Corollary 3. *There exists a separately Lipschitz function $f : Q \to \mathbb{R}$ such that $\iint_Q |f_x'|^p \, dx\, dy < \infty$ and $\iint_Q |f_y'|^p \, dx\, dy < \infty$ for every $p > 1$, and the function $h(x) = f(x, x)$ has unbounded variation.*

Proof. Note that for every $p > 1$ the space $L_p(Q)$ is r.i. space and the function $g(x, y) = \ln x$ belongs to $\bigcap_{p>1} L_p(Q) \setminus L_\infty(Q)$. Hence, by Corollary 2, we obtain the existence of such a function. $\quad\square$

Remark 2. Note that $g(x_2) - g(x_1) = \int_{x_1}^{x_2} g'(x) \, dx$ for every absolutely continuous function $g : I \to \mathbb{R}$. If the partial derivatives f_x' and f_y' of a separately absolutely continuous function $f : Q \to \mathbb{R}$ are such that $|f_x'| \leqslant C$ and $|f_x'| \leqslant C$ almost everywhere on Q, then the Fubini theorem implies that f is separately Lipschitz with the constant C and, therefore, f is jointly Lipschitz with the constant C with respect to the sum-distance on Q. In particular, this means that the restriction of f on any straight line is a Lipschitz function.

4. "Embeddings of Banach spaces" approach

We show how, using Theorems A and 2, one can give an answer (in a classical Banach style) to the Eidelheit question. Moreover, we obtain, as a byproduct, stronger results, which are not evident under the function theory approach.

Our approach is based on the following well-known notion: A bounded linear operator T from a topological vector space X into a topological vector space Y is called *strictly singular* if there exists no infinite-dimensional subspace $Z \subset X$ such that $T|_Z$ is an isomorphism. The operator T from a Banach space X into a Banach space Y is called *superstrictly singular* (*SSS* for short) if there does not exist a number $c > 0$ and a sequence of subspaces $E_n \subset X$, $\dim E_n = n$, such that $\|Tx\| \geqslant c\|x\|$ for all x in $\bigcup_n E_n$. Obviously, each compact operator is SSS, each SSS operator is strictly singular and T is SSS if it is SSS on a finite-codimensional closed subspace (cf. [18]).

A Rudin's version [22, Theorem 5.2] of the well-known Grothendieck's result says that the natural (noncompact) embedding $\mathcal{I} : L_\infty \hookrightarrow L_p$, $p \geqslant 1$, is SSS. On the other hand, generalizing the Grothendieck's result, S.Ya. Novikov [16, Theorem 1] has proved that the natural embedding $\mathcal{I} : L_\infty \hookrightarrow E$ is strictly singular for every rearrangement invariant (r.i.) space $E \neq L_\infty$. The following result contains both the Rudin and Novikov theorems.

Theorem 4. *Let E be a r.i. Banach function space on $I = [0, 1]$ different from $L_\infty(I)$. Then the natural embedding $\mathcal{I} : L_\infty \hookrightarrow E$ is SSS.*

In the proof we will use the following well-known lemma (see e.g. [19, Lemma 3.3]).

Lemma 3. *Let $b > 0$. Then for every $k \in \mathbb{N}$, $k \geqslant 2$ there exists $n = n(b, k) \in \mathbb{N}$ such that for any collection of measurable subsets $A_i \subset I$, $i = 1, \ldots, n$ with the Lebesgue measure $\lambda(A_i) > b$ there is a subcollection $(A_{i_j})_{j=1}^k$ with*

$$\lambda\left(\bigcap_{j=1}^{k} A_{i_j}\right) > 0. \tag{4}$$

Proof of Theorem 4. Let us on the contrary suppose that there exist $\varepsilon > 0$ and n-dimensional subspaces $E_n \subset L_\infty$ such that for every n and $f \in E_n$

$$\varepsilon \|f\|_{L_\infty} \leqslant \|f\|_E. \tag{5}$$

Since $E \neq L_\infty$, there are a and b such that for every $f \in L_\infty$ with $\|f\|_{L_\infty} = 1$ and $\|f\|_E \geqslant \varepsilon$,

$$\lambda\big(\big\{x \colon \big|f(x)\big| > a\big\}\big) > b.$$

Then, by Lemma 3, there exists $c > 0$ such that for every k with the property (4) there is n so that for any elements $(f_i)_1^k$ with the property (5) we have that

$$\frac{1}{k}\left\|\sum_{i=1}^{k} f_i\right\|_E \geqslant c.$$

Take an orthogonal (with respect to natural inner product) basis $(f_i^n)_{i=1}^n$ of E_n with $\|f_i^n\|_{L_\infty} = 1$. Then

$$\left\| \sum_{i=1}^n f_i^n \right\|_{L_2} \leqslant n^{1/2}.$$

Put

$$\sigma(n, \delta) = \left\{ x \in (0, 1): \frac{1}{n} \left| \sum_{i=1}^n f_i^n(t) \right| > \delta \right\}.$$

Hence, for every $\delta > 0$ the measure $\lambda(\sigma(n, \delta)) \to 0$ as $n \to \infty$ not depending on the form of (f_i) (see e.g. [11, p. 160]). But

$$\frac{1}{n} \left\| \sum_{i=1}^n f_i^n \right\|_E \leqslant \delta + \|\chi_{\sigma(n,\delta)}\|_E.$$

Since E is different from L_∞ we have $\|\chi_{A_n}\|_E \to 0$ provided $\lambda(A_n) \to 0$ (see e.g. [11, p. 118]). Thus,

$$\frac{1}{n} \left\| \sum_{i=1}^n f_i^n \right\|_E \to 0 \quad \text{as } n \to \infty,$$

and we have a contradiction which complete the proof. $\quad\square$

Remark 3. Of course, Theorem 4 is valid for r.i. Banach function spaces on any subset $A \subset \mathbb{R}^n$ of positive finite Lebesgue measure.

Let us look at the integral condition in the one variable version of Eidelheit's problem. It seems that he means the existence of the derivative almost everywhere on I, i.e. Eidelheit had in mind generalized derivatives. The corresponding one variable space was considered as far back as by Banach. Namely, in [1, pp. 134, 167] he introduced, in particular, a space of absolutely continuous functions on I with derivative in $L^p(I)$. On this space one can introduce the norm $\|f\| = |m_f| + \|f'\|_{L_p}$, where $m_f = \int_I f(x)\,dx$ (this is just one of equivalent forms). Banach noted that this space is in fact complete.

Given an arbitrary r.i. Banach function space E on I, one can define the Beppo Levi space $BL_E^1(I)$ (why this space is named after Beppo Levi, we will explain below) of absolutely continuous functions f for which $f' \in E$ with the natural norm (this is just one of the equivalent norms):
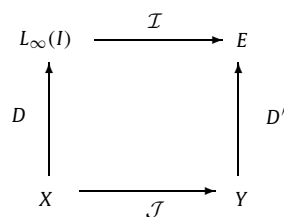
$$\|f\| = |m_f| + \|f'\|_E.$$

Remark 4. Let Y be the one-codimensional subspace of $BL_E^1(I)$ consisting of all f with $m_f = 0$ and let $D': Y \to E$ be the generalized derivative operator. Obviously, this operator is an onto isometry, which means that $BL_E^1(I)$ is complete.

Corollary 4. *Let E be a r.i. Banach function space on I different from L_∞. Then the natural embedding $\mathcal{J} : \mathrm{Lip}_1(I) \hookrightarrow BL_E^1(I)$ is noncompact but SSS.*

Proof. Take the Rademacher functions (r_n) and put $f_n(x) = \int_0^x r_n(t)\,dt$, $x \in I$ (the Schauder functions). Then $f_n \in \mathrm{Lip}_1(I)$ and $\|f_n\| = 1$ for every n. On the other hand, $\|\mathcal{J}f_n - \mathcal{J}f_m\| \geqslant 1$ for any $n \neq m$. Thus, the mapping $\mathcal{J}$ is not compact.

Denote by X the subspace of $\mathrm{Lip}_1(I)$ consisting of all f with $m_f = 0$. Then the generalized derivation operator $D: X \to L_\infty$ defined by $Df = f'$ is bounded. Let Y and D' be from Remark 4. Then from commutativity of the following operator diagram

$$
\begin{array}{ccc}
L_\infty(I) & \xrightarrow{\;\mathcal{I}\;} & E \\
{\scriptstyle D} \uparrow & & \uparrow {\scriptstyle D'} \\
X & \xrightarrow[\;\mathcal{J}\;]{} & Y
\end{array}
$$

we obtain that $\mathcal{J} = (D')^{-1}\mathcal{I}D$. Since, by Theorem 4, $\mathcal{I}$ is SSS, it follows that the restriction $\mathcal{J}|_X$ is SSS, and we conclude that $\mathcal{J}$ is SSS. The proof is complete. $\quad\square$

Let us consider a sequence of r.i. Banach function spaces (E_k) different from $L_\infty(I)$ and the topological vector space $F = \bigcap_k BL^1_{E_k}(I)$ with the fundamental neighborhood of 0 formed by the unit balls of the spaces $BL^1_{E_k}(I)$, $k = 1, 2, \ldots$. It is easy to verify that F is a Fréchet space.

Corollary 5. *The natural embedding* $\mathcal{J} : \mathrm{Lip}_1(I) \hookrightarrow F$ *is strictly singular.*

Proof. Let $Z \subset \mathrm{Lip}_1(I)$ be an infinite-dimensional subspace and B_Z be its unit ball. If $\mathcal{J}|_Z$ is an isomorphism, then $\mathcal{J}(B_Z)$ is a neighborhood of 0 in $\mathcal{J}(Z) \subset F$. Hence there exists k such that $\mathcal{J}(B_Z)$ is a neighborhood of 0 in $\mathcal{J}(Z) \subset BL^1_{E_k}(I)$. This contradicts Corollary 4 and the proof follows. $\square$

The next corollary generalizes the solution of the one variable version of Eidelheit's problem.

Corollary 6. *Every infinite-dimensional closed subspace of F contains an absolutely continuous function f for which there is a Lipschitz function g with the non-absolutely continuous superposition $f \circ g$.*

Proof. Indeed, by Corollary 5, every infinite-dimensional closed subspace of F contains an (absolutely continuous) function f which does not belong to $\mathrm{Lip}_1(I)$. Moreover, by Theorem A, there is a Lipschitz function g with the non-absolutely continuous superposition $f \circ g$. $\square$

Before presenting the abstract versions of the two variable results and the diagonal Eidelheit problems let us start with a short historical excursion. The conditions in the Eidelheit problem mean that $f \in BL^1_p(Q)$, where the (Beppo Levi) space $BL^1_p(Q)$ consists of all functions $f : Q \to \mathbb{R}$, which are absolutely continuous in each variable and whose (classical generalized) first order partial derivatives belong to $L_p(Q)$. For $p = 2$ and three variables a similar space was considered by O. Nikodym [15]. This space is called *Beppo Levi space* since functions in this class were studied as far back as 1906 by Beppo Levi (for $p = 2$), and later by Tonelli (for $p = 1$ and $p \geqslant 2$) in the problem of minimization of variational integrals. The name Beppo Levi space was introduced by O. Nikodym [15] for $p = 2$ in 1933 and in general by J. Deny and J.L. Lions [6] in 1953. On the other hand, in 1936 S.L. Sobolev [24] developed the so-called *Sobolev space* $W^1_p(Q)$ as a space of all $f \in L_p(Q)$ whose generalized (distributional) derivatives belong to $L_p(Q)$. Surprisingly, we have that $BL^1_p(\Omega) = W^1_p(\Omega)$ even for subsets $\Omega \subset \mathbb{R}^n$ (cf. [13, Theorem 1, p. 8] and [30, Theorem 2.1.4]) but we must explain in which sense since $BL_p(Q)$ is a space of functions and $W^1_p(Q)$ a space of equivalence classes of functions which differ on sets of measure zero. More correctly, we mean that any function in $BL_p(Q)$ belongs to (an equivalence class in) $W^1_p(Q)$, while every element of $W^1_p(Q)$ has a representative in $BL_p(Q)$.

Denote by $\mathrm{Lip}_1(Q)$ the space of Lipschitz functions on Q with the norm

$$\|f\| = |m_f| + \sup_{z, z' \in Q, \, z \neq z'} \frac{|f(z) - f(z')|}{d(z, z')},$$

where $m_f = \iint_Q f(x, y) \, dx \, dy$ and d denotes the Euclidean distance in $\mathbb{R}^2$.

Let E be a r.i. Banach function space on Q. Denote by $BP^1_E(Q)$ the space of functions f on Q, which are absolutely continuous with respect to each variable for almost all other variables, and whose classical generalized partial derivatives $f'_x, f'_y \in E$ with the natural norm

$$\|f\| = |m_f| + \|f'_x\|_E + \|f'_y\|_E.$$

Similar spaces were considered by J. Deny and J.L. Lions [6]. They mean the derivatives in the sense of generalized functions. Deny and Lions have proved that these spaces are complete.

Remark 5. Let Y be the one-codimensional subspace of E consisting of all f with $m_f = 0$. Let $D' : Y \to E \times E$ be defined by $D'(f) = (f'_x, f'_y)$. This operator is an into isomorphism. Thus, $BL^1_E(Q)$ is isomorphic to a subspace of E. Is it isomorphic to E? For the spaces $BL^1_p(Q)$ the answer is positive [17, pp. 1373–1377].

Corollary 7. *Let E be a Banach r.i. space on Q different from $L_\infty(Q)$. Then the natural embedding* $\mathcal{J} : \mathrm{Lip}_1(Q) \hookrightarrow BL^1_E(Q)$ *is SSS.*

Proof. Denote by X the subspace of $\mathrm{Lip}_1(Q)$ consisting of all f with $m_f = 0$. Then the generalized derivation operator $D : X \to L_\infty(Q) \times L_\infty(Q)$, $Df = (f'_x, f'_y)$ is bounded. Let $\mathcal{I} : L_\infty(Q) \times L_\infty(Q) \hookrightarrow E \times E$ be the natural embedding. By Remark 1, it is SSS.

Let Y and D' be from Remark 4. Then from the commutativity of the following diagram

$$
\begin{array}{ccc}
L_\infty(Q) \times L_\infty(Q) & \xrightarrow{\ \mathcal{I}\ } & E \times E \\
\Big\uparrow{\scriptstyle D} & & \Big\uparrow{\scriptstyle D'} \\
X & \xrightarrow[\ \mathcal{J}\ ]{} & Y
\end{array}
$$

we have that $\mathcal{J} = (D')^{-1}\mathcal{I}D$. Hence, $\mathcal{J}|_X$ is SSS, and so $\mathcal{J}$ is SSS. □

Let us consider a sequence of r.i. Banach function spaces (E_k) on Q, different from $L_\infty(Q)$, and the topological vector space $F = \bigcap_k BL^1_{E_k}(Q)$ with the fundamental neighborhood of 0 formed by unit balls of the spaces $BL^1_{E_k}(Q)$, $k = 1, 2, \ldots$. It is easy to verify that F is a Fréchet space. The proof of the following corollary is the same as that of Corollary 5.

Corollary 8. *The natural embedding $\mathcal{J} : \mathrm{Lip}_1(Q) \hookrightarrow F$ is strictly singular.*

The next corollary generalizes the solution of Eidelheit's problem.

Corollary 9. *Every infinite-dimensional closed subspace of F contains an (absolutely continuous) function f for which there are Lipschitz functions g_1, g_2 with the non-absolutely continuous superposition $f(g_1(t), g_2(t))$.*

Proof. Indeed, by Corollary 8, every infinite-dimensional closed subspace of F contains an (absolutely continuous) function f which does not belong to $\mathrm{Lip}_1(Q)$. By Corollary 1, there are Lipschitz functions g_1, g_2 with the non-absolutely continuous superposition $f(g_1(t), g_2(t))$. □

Let us now consider the functional analytic meaning of Theorem 3. Denote by Z the "diagonal" subspace of $BL^1_E(Q)$ consisting of functions $f(x, y) \in BL^1_E(Q)$ for which $f(x + \lambda x, x - \lambda x) = f(x, x)$, $x \in I$, $\lambda \in \mathbb{R}$. Then, from Corollary 7 we have immediately that:

Corollary 10. *There exists $f \in Z$ such that $f \notin \mathrm{Lip}_1(Q)$.*

Note that Corollary 3 is stronger than Corollary 10 since in Corollary 3 f is a separately Lipschitz function.

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