On the $C^1$ Stability Conjecture for Flows

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We give a proof of the $C^1$ stability conjecture of Palis and Smale for flows, which has reduced to proving that $C^1$ structural stability implies Axiom A. The proof is based on the fundamental work of Liao and Mañé, and on the recent powerful $C^1$ connecting lemma of Hayashi.

1. INTRODUCTION

We study the $C^1$ stability conjecture of Palis and Smale [PS] for flows in this paper.

Let $M$ be a compact Riemannian manifold without boundary. For each $r \geq 1$, let $\mathcal{X}^r(M)$ denote the set of $C^r$ vector fields of $M$, endowed with the $C^r$ topology. Every $S \in \mathcal{X}^r(M)$ generates a $C^r$ flow $\phi_t: M \times \mathbb{R} \to M$. We say $S$ is $C^r$ structurally stable if $S$ has a $C^r$ neighborhood $U$ in $\mathcal{X}^r(M)$ such that every $X \in U$ generates a flow $\phi_t$ that is topologically equivalent to $\phi^t$. As usual, two flows are topologically equivalent if there is a homeomorphism $h: M \to M$ that maps the orbits of one flow onto those of the other flow while preserving the orientation. Thus $C^r$ structural stability implies $C^{r+1}$ structural stability, when both make sense.

In [PS] Palis and Smale conjectured that $S$ is $C^r$ structurally stable if and only if $S$ satisfies Axiom A plus the strong transversality condition (see below for definitions). The “if” part was proved first for $r \geq 2$ by Robbin [R], and then for $r = 1$ by Robinson [Rs1]. The “only if” part was reduced by Robinson [Rs2] to proving that $C^r$ structural stability implies Axiom A, which has been known as the $C^r$ stability conjecture. Unlike the “if” part, the first target for studying the $C^r$ stability conjecture is the case of $r = 1$, i.e., the $C^1$ stability conjecture. The case of $r \geq 2$ is beyond our knowledge as discussed in [M2], and will not be considered in the present paper.

The $C^1$ stability conjecture has attracted attention for a long time. Many researchers made important contributions to this problem, among whom

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particularly notable are Ricardo Mañé and Shantao Liao. For the discrete case, i.e., for diffeomorphisms, the $C^1$ stability conjecture was proved by Mañé in a remarkable paper [M2]. For flows, a great deal of fundamental work has been done by Liao [L1–L6]. Recently, Hayashi [Ha2] proved a very important theorem on the $C^1$ connecting lemma. (See Theorem 3.5.) This type of result has long been desired but this strong a closing lemma result was unavailable until now. Using all these previous works we prove the $C^1$ stability conjecture for flows in this paper.

**Theorem A.** $C^1$ structurally stable flows of compact manifolds without boundary satisfy Axiom A.

For the case of dimension 2 Theorem A is contained in the classical work of Peixoto [Pe2]. For dimension 3 Theorem A was proved by Hu [Hu] and Liao [L6] independently. For general dimensions Theorem A was recently obtained by Hayashi [Ha2], whose paper contains the original proof of the $C^1$ connecting lemma. Our proof of the theorem is also for general dimensions but assumes the $C^1$ connecting lemma, which solves the major difficulty of the problem—to rule out accumulations of periodic orbits on singularities or on periodic orbits of different indices. Then we establish the ergodic closing lemma for flows and rule out explosions of periodic orbits of fixed indices. The latter again uses the $C^1$ connecting lemma. Also, by giving a unified treatment we show how the use of the $C^1$ connecting lemma can greatly simplify the whole approach of this problem in the literature. The $C^1$ connecting lemma of Hayashi is indeed the right key to this work.

As a corollary of Theorem A, we obtain the equivalence of two apparently different definitions of structural stability. The structural stability defined as above was due to Peixoto [Pe1] in the late '50s. There was another definition of structural stability introduced by Andronov and Pontryagin [AP] in the '30s. We say $S$ is $C^r$-structurally stable if for any $\varepsilon > 0$, there is a $C^r$ neighborhood $\mathcal{U}$ in $X(M)$ such that for any $X \in \mathcal{U}$, there is a homeomorphism $h: M \to M$ that maps the sensed orbits of $S$ onto that of $X$ such that $h$ is within $\varepsilon$ of the identity. Thus $\varepsilon$-structural stability implies the non-$\varepsilon$-structural stability. However, what Axiom A plus the strong transversality imply is actually the $C^1$ $\varepsilon$-structural stability [Rsl]. Thus Theorem A implies the following.

**Corollary.** The $C^1$ non-$\varepsilon$-structural stability is equivalent to the $C^1$ $\varepsilon$-structural stability for flows of closed manifolds.

It was Peixoto who first discovered this equivalence in dimension 2 [Pe1]. This equivalence is somewhat thought-provoking and, indeed, constitutes a major challenge in the proof of Theorem A.
Now we give the precise definitions of Axiom A and strong transversality. A point \( x \in M \) is nonwandering of \( S \) if for any neighborhood \( V \) of \( x \) in \( M \), there is \( t \geq 1 \) such that \( \phi_\tau(V) \cap V \neq \phi_\tau \). The set of nonwandering points of \( S \) is the nonwandering set of \( S \), and denoted by \( \Omega(S) \). Singularities and points of periodic orbits are all nonwandering. An \( S \)-invariant set \( A \) is hyperbolic of \( S \) if the restricted tangent bundle \( T_A \) has a continuous \( S \)-invariant splitting \( E_s \oplus E_\sigma \) such that for some constants \( \lambda > 0 \) and \( T > 0 \), the inequalities

\[
\|d\phi_t(x)\| E_s(x) \leq e^{-\lambda t} \quad \text{and} \quad m(d\phi_t(x) E_s(x)) \geq e^{\lambda t}
\]

hold for all \( x \in A \) and all \( t \geq T \), where \( m \) denotes the mini-norm, i.e.,

\[
m(A) = \min \{ \|Av\| \mid \|v\| = 1 \}
\]

for any linear map \( A \). Note that the hyperbolic splitting defined over a singularity is different from that over a non-singular point. Thus, in a hyperbolic set \( A \), singularities must be isolated from the rest of \( A \). Now \( S \) satisfies Axiom A if \( \Omega(S) \) is hyperbolic and the singularities and periodic orbits are dense in \( \Omega(S) \). It is well known [HPS] that if \( S \) satisfies Axiom A, then for any \( x \in M \), the stable manifold

\[
W^s(x) = \{ y \in M \mid \lim_{t \to +\infty} d(\phi_t(y), \phi_t(x)) = 0 \}
\]

of \( x \) and the unstable manifold

\[
W^u(x) = \{ y \in M \mid \lim_{t \to -\infty} d(\phi_t(y), \phi_t(x)) = 0 \}
\]

of \( x \) are each an injectively immersed \( C^r \) submanifold of \( M \), if \( S \) is \( C^r \). An Axiom A system \( S \) satisfies the strong transversality condition if \( W^s(x) \) is transverse to \( W^u(x) \) at all \( x \in M \). We see that Axiom A and the strong transversality condition are more concrete than the notion of structural stability, and are entirely visualizable in dimension 2 [Pe2]. Also, they are in terms of the unperturbed system only.

Still ahead is the \( C^1 \) \( \Omega \)-stability conjecture of Palis and Smale [PS], which asserts that \( C^1 \) \( \Omega \)-stability implies Axiom A. Recall that \( S \) is \( C^r \) \( \Omega \)-stable if \( S \) has a \( C^r \) neighborhood \( U \) in \( \mathcal{X}(M) \) such that every \( X \in U \) generates a flow \( \phi^X \) that is topologically equivalent to \( \phi^S \) when restricted to corresponding nonwandering sets. Examples show that \( \Omega \)-stability is strictly weaker than structural stability. Unfortunately, we are unable to prove the \( C^1 \) \( \Omega \)-stability conjecture in this paper. We will stay with the assumption of \( C^1 \) \( \Omega \)-stability as much as possible however, and single out the place where we need the whole weight of the assumption of \( C^1 \) structural stability in our approach of proving Theorem A. Recently Hayashi proved the \( C^1 \) \( \Omega \)-stability conjecture for flows.

An even weaker condition than \( C^1 \) \( \Omega \)-stability is the star condition. Recall that \( S \) is a star flow, denoted as \( S \in \mathcal{X}_s(M) \), if \( S \) has a \( C^1 \) neighborhood \( U \) in \( \mathcal{X}(M) \) such that every \( X \in U \) has, if any, only hyperbolic singularities...
and hyperbolic periodic orbits. It is well known that $C^1\Omega$-stability implies the star condition [F]. For diffeomorphisms the converse is also true. In fact, for diffeomorphisms, the star condition is equivalent to $C^1\Omega$-stability, and equivalent to Axiom A plus no cycle [Pa, Ha1]. However, for flows, the situations are quite different. For flows, the star condition does not imply $C^1\Omega$-stability, nor Axiom A, and even plus Axiom A still does not imply the no cycle condition. The famous Guckenheimer–Lorenz attractor [G] and two more elementary examples [D, LW] illustrate this. Nevertheless, the star condition still deserves attention, since it is easier to handle than the stabilities, yet still strong enough. In fact, many results of Section 3 below hold by assuming the star condition only. Moreover, the fact that the three examples just mentioned all have singularities singles out a particularly interesting class of flows—the class of nonsingular star flows. It is interesting to ask if they are closer to diffeomorphisms, or closer to these three examples, as far as these stability properties are concerned [L6]. That is to ask whether or not the class of nonsingular star flows is the same as the class of nonsingular $C^1\Omega$-stable flows, or as the class of nonsingular Axiom A plus no cycle flows. So far, what is known is from Liao [L4] that, in dimension three, these three classes are indeed the same.

It was S. T. Liao who introduced this problem to me several years ago, and gave me many invaluable advice ever since. Z. Xia and I had many benefitial discussions this year. Especially, he told me about the $C^1$ connecting lemma of Hayashi which plays a crucial role in this work, and provided me with his proof of the connecting lemma which simplifies that of Hayashi. C. Robinson gave me many good suggestions when I was visiting Northwestern University. My visit to Georgia Tech and Northwestern University this year have been very helpful to this work. I also thank the referees for critical comments and good suggestions to the first version of this paper.

2. AN INFORMAL OUTLINE FOR THE PROOF OF THEOREM A

In this section, we give an informal outline for the proof of Theorem A. We are assuming that $S \in \mathcal{X}^1(M)$ is $C^1$ structurally stable, and proceed to prove that $S$ satisfies Axiom A. It is well known that, for $C^1\Omega$-stable flows, the nonwandering set is the closure of the union of singularities and periodic orbits. Thus we are led to prove that $\text{Sing}(S) \cup \text{Per}(S)$ is hyperbolic, where $\text{Sing}(S)$ and $\text{Per}(S)$ denote the singularities of $S$ and the points of periodic orbits of $S$, respectively. It is also well known that, for $C^1\Omega$-stable flows, singularities and periodic orbits are each hyperbolic. Since $M$ is compact, there are only finitely many hyperbolic singularities. Thus $\text{Sing}(S) = \text{Sing}(S)$ is hyperbolic. It remains to prove that $\text{Per}(S)$ is
hyperbolic. Note that there could be infinitely many periodic orbits, since the periods may not bound. Write \( \text{Per}(S) \) as \( \bigcup_{i=0}^{\dim M} \text{Per}_i(S) \), where \( \text{Per}_i(S) \) denotes the points of periodic orbits of \( S \) whose index is \( i \), i.e., whose stable manifold has dimension \( i \). We are led to prove that each \( \text{Per}_i(S) \) is hyperbolic.

The proof goes by induction on \( i \). It is well known that in this case \( \text{Per}_0 \) consists of only finitely many (expanding) periodic orbits. Thus \( \text{Per}_0 = \text{Per}_0 \) is hyperbolic. Now assume \( \text{Per}_0, \ldots, \text{Per}_{j-1} \) are all hyperbolic, and hence \( \text{Per}_0 \cup \cdots \cup \text{Per}_{j-1} \) decomposes into its finitely many disjoint basic sets. From now on, \( j \) will be fixed through the end of the proof. Up to this stage, everything is well known. And the proof of Theorem A reduces to proving that \( \text{Per}_j \) is hyperbolic.

The proof that \( \text{Per}_j \) is hyperbolic constitutes the following four steps.

**Step 1 (Disjointness).** Prove that \( \text{Per}_j \) is disjoint from singularities, and from the closure of periodic orbits of lower indices as well. That is, to prove that \( \text{Per}_j \) is disjoint from \( \text{Sing} \cup \text{Per}_0 \cup \cdots \cup \text{Per}_{j-1} \). This disjointness must hold because a hyperbolic splitting needs to be continuous.

**Step 2 (Splitting).** Prove that there is a continuous splitting \( A_- \oplus A_+ \) of the normal bundle of \( S \) over \( \text{Per}_j \) that agrees with the hyperbolic splitting over each periodic orbit of \( \text{Per}_j \). (We are using an equivalent definition of hyperbolicity in terms of the normal bundle of \( S \) over nonsingular points.) In other words, to prove that the hyperbolic splittings over these individual periodic orbits of \( \text{Per}_j \) fit nicely so that they extend to a continuous splitting \( A_- \oplus A_+ \) over the closure \( \text{Per}_j \). This is clearly the unique candidate splitting for a possible hyperbolic structure over \( \text{Per}_j \).

**Step 3 (Contraction).** Prove \( A_- \) is contracting.

**Step 4 (Expansion).** Prove \( A_+ \) is expanding.

The statement of Step 1 does not mention the periodic orbits of higher indices because of the induction. Once the induction is complete, the conclusion reached will in fact include periodic orbits of all indices. That is, \( \text{Sing}(S), \text{Per}_0, \text{Per}_1, \ldots, \text{Per}_{\dim M-1} \) will be all hyperbolic and hence mutually disjoint. This is of course the way an Axiom A flow should be. The disjointness stated in Step 1 has been a major point of difficulty in studying the \( C^1 \) stability conjecture. For diffeomorphisms, this disjointness is proved by Mané [M2] via an indirect approach, which is very difficult and could be more difficult if we want to extend it to flows. The novelty of our approach of proving this disjointness, as mentioned in the introduction, will be the use of the long expected \( C^1 \) connecting lemma. In fact, according to the \( C^1 \) connecting lemma, if periodic orbits of \( P_j \) had accumulated on one of the basic sets \( A \) of \( \text{Sing} \cup \text{Per}_0 \cup \cdots \cup \text{Per}_{j-1} \), then some \( C^1 \) perturbation away
from $\text{Sing} \cup \overline{P_0} \cup \cdots \cup \overline{P_{j-1}}$ would simply create a homoclinic orbit associated with $A$, contradicting the $C^1$ $\Omega$-stability of $S$. This proves the disjointness directly, and completes Step 1. Thus the use of the $C^1$ connecting lemma is crucial to our approach. The original proof of the $C^1$ connecting lemma is in Hayashi [Ha2]. A simpler proof can be found in [WX].

The extendability formulated in Step 2 follows from a result of Liao [L1]. Results of this type for diffeomorphisms appear in Mañé [M1] and Pliss [Pl]. Among other things, this result says that the quotient of the contracting rate over the expanding rate is uniformly away from one for all (periodic) points of $P_j$. Together with an argument of certain uniqueness, this uniformity yields the extended splitting $A_- \oplus A_+$ of the normal bundle of $S$ over $\overline{P_j} - \text{Sing}$, which is called a dominated splitting. Now $\overline{P_j}$ has been proved in Step 1 disjoint from $\text{Sing}(S)$. Thus Step 2 is complete.

Step 3 is to prove that the domination summand $A_-$ is in fact contracting. The strategy is to prove that if $A_-$ were not contracting, then some $C^1$ perturbation would simply create a non-hyperbolic periodic orbit, contradicting the structural stability of $S$. This is done with the aid of the ergodic closing lemma. What the ergodic closing lemma asserts is that, respecting $S$-invariant measures almost every nonsingular point is strongly closable, a property meaning roughly that the orbit could be closed up nicely enough so that estimates along the original orbit transfer to the new periodic orbit. The role the ergodic closing lemma plays is the following. We are assuming for contradiction that $A_-$ is not contracting along the positive orbit of some point of $\overline{P_j}$. Via a sort of average, the ergodic closing lemma then guarantees that we may assume this point simply strongly closable. The noncontractingness of $A_-$ along the original orbit of this point therefore transfers to the corresponding summand of a dominated splitting of the newly created periodic orbit. Some further arguments about a uniform (respecting perturbations) separation of periodic orbits of different indices force this dominated splitting of the new periodic orbit to be its hyperbolic splitting, and hence reach a contradiction. We remark that this uniform separation is closely related to the structural stability mentioned in the introduction. We also remark that it is in arguing this uniform separation where we need the whole weight of the assumption of the $C^1$ structural stability in our approach to proving Theorem A. This is the way $A_-$ is proved to be contracting. These beautiful ideas, the ergodic closing lemma of diffeomorphisms in particular, are due to Mañé [M1]. Ideas of similar spirits also appear in Liao [L3, L4]. In the present paper, we need the ergodic closing lemma for flows, a proof of which will be given in Section 4 separately.

The last step is to prove that $A_+$ is expanding. Hence $\overline{P_j}$ is hyperbolic. This completes the induction process and hence proves Theorem A. The proof for expandingness of $A_+$ is simply a flow copy of a diffeomorphism
result of Mañé [M2]. We only remark that the proof is not by simply reversing the time. This is because we do not yet know if $P_{j+1} \cup P_j$ through $P_{\dim M - 1}$ are hyperbolic and disjoint from $P_j$.

3. THE PROOF OF THEOREM A

Let $S$ be a $C^1$ vector field of $M$. $S$ induces a $C^1$ flow $\phi_t: M \to M$, $t \in \mathbb{R}$, and a $C^0$ flow $d\phi_t: TM \to TM$, $t \in \mathbb{R}$, which is linear on fibers. Let $\text{Sing}(S)$ be the set of singularities of $S$, and $\mathcal{D}$ be the normal bundle of $S$ over $M - \text{Sing}(S)$. Thus, at every $x \in M - \text{Sing}(S)$, the fiber $\mathcal{D}(x)$ is the codimension 1 subspace of $T_x M$ that is perpendicular to $S(x)$. For any $u \in \mathcal{D}(x)$, let $\psi_t(u)$ be the orthogonal projection of $d\phi_t(u)$ onto $\mathcal{D}(\phi_t(x))$. Then

$$\psi_t: \mathcal{D} \to \mathcal{D}$$

is a $C^0$ flow, which is linear on fibers. The flow $\psi_t$ is convenient when dealing with nonsingular orbits. For instance, if $A$ is an $S$-invariant set which contains no singularities of $S$. Then $A$ is hyperbolic of $S$ if and only if $\mathcal{D}|_A$ has a continuous $\psi_t$-invariant splitting $D' \oplus D''$ such that for some constants $\lambda_1 > 0$ and $\lambda_2 > 0$, the inequalities

$$|\psi_t| D'(x)| \leq e^{-\lambda_1 t} \quad \text{and} \quad m(\psi_t)| D''(x)) \geq e^{\lambda_2 t}$$

hold for all $x \in A$ and all $t \geq T_1$. We will refer to these two inequalities briefly as that $D'$ is contracting, and $D''$ is expanding, respectively.

All these notations depend on $S$, and should have been written as $\phi_{t, S}$, $d\phi_{t, S}$, $\mathcal{D}(S)$, $\mathcal{D}(x, S)$, $\psi_{t, S}$, etc. To simplify notation, we will drop the vector field $S$ from the notations as we just did, if no confusion should occur. Unfortunately, sometimes we have to write something like $\psi_t| D'(x, X)$, when we consider perturbations as well.

Now we prove Theorem A. Assume $S$ is $C^1$ structurally stable. We prove $S$ satisfies Axiom A. As we saw in Section 2, the first part of the proof is well known in the literature. We just state these results as theorems.

**Theorem 3.1 [Pu].** If $S$ is $C^1$ $\Omega$-stable, then $\Omega(S) = \overline{\text{Sing}(S)} \cup \text{Per}(S)$.

Then we are led to prove that $\overline{\text{Sing}(S)} \cup \text{Per}(S)$ is hyperbolic.

**Theorem 3.2 [F].** If $S$ is $C^1$ $\Omega$-stable, then $S$ is in $\mathcal{A}^*(M)$.

Thus $S$ has only finitely many singularities, since $M$ is compact. Then $\overline{\text{Sing}(S)} = \text{Sing}(S)$ is hyperbolic. Therefore we need to prove that $\text{Per}(S)$ is hyperbolic. Note that unlike the singularities, there could be infinitely many periodic orbits, since the periods may not bound above. But we can group them according to their indices anyway. Thus we write

$$\text{Per}(S) = P_0 \cup P_1 \cup \cdots \cup P_{\dim M - 1},$$
where $P_i$ denotes the periodic points of index $i$. And we need to prove that each $P_i$ is hyperbolic.

The proof goes by induction on $i$. That $P_0$ is hyperbolic is a consequence of the following result of Liao [L3]. For diffeomorphisms this result was proved by Pliss [Pl].

**Theorem 3.3 [L3].** If $S$ is a star flow, then $P_0$ contains only finitely many periodic orbits.

Thus $P_0 = P_0$ is hyperbolic. Now assume $P_0, ..., P_{j-1}$ are all hyperbolic. Then $P_0 \cup \cdots \cup P_{j-1}$ is disjoint from the singularities, and decomposes into its finitely many disjoint basic sets $A_1, ..., A_s$. From now on the index $j$ will be fixed throughout the proof. We need to prove that $P_j$ is hyperbolic to complete our induction process.

Before going on, we insert a lemma here about homoclinic points. First we need a more flexible notion than that of basic sets, which is independent of some decomposition of a larger set. A compact invariant set $A$ is a prebasic set of $S$ if $A$ is isolated, transitive, and hyperbolic of $S$. Here, by isolated, we mean $A$ is the maximal invariant set in a neighborhood $U$ of $A$. Note that a prebasic set is either a singularity, or else without singularities. Periodic orbits are dense in any nonsingular prebasic set, and any prebasic set is in phase. As usual, $x \in M$ is a homoclinic point associated with an invariant set $K$ if $x \in W^u(K) \cap W^s(K) = K$. A basic fact used below is that any $C^r$ ($r \geq 1$) $\Omega$-stable flow $S$ is $\Omega$-equivalent to a Kupka–Smale flow $Y$ because Kupka–Smale flows are $C^r$ dense.

**Lemma 3.4.** Let $S$ be $C^1$ $\Omega$-stable. Then

1. $S$ has no homoclinic points associated with any singularity $\sigma$ of $S$, and has no homoclinic tangencies associated with any prebasic set $A$ of $S$.

2. If, moreover, $P_i(S)$ is hyperbolic for some $i$ and hence $P_i(S)$ is decomposed into its basic sets $C_1, ..., C_m$, then $S$ has no homoclinic points associated with any $C_k$. Besides, there is a $C^1$ neighborhood $U$ of $S$ in $\mathcal{H}(M)$ such that if $X \in \mathcal{H}$ and $X = S$ on a neighborhood $U$ of $C_1 \cup \cdots \cup C_m$, then each $C_k$ is a prebasic set of $X$, and $X$ has no homoclinic points associated with any $C_k$.

Note that the word tangency cannot be removed from (1). For instance, in a suspension of the horseshoe, any periodic orbit itself is a prebasic set with homoclinic points.

**Proof.** For the first statement of (1), suppose $S$ has a homoclinic point $x \in W^u(\sigma) \cap W^s(\sigma) = \{ \sigma \}$. Then $x$ is nonwandering of $S$ by a standard argument. Let $h : \Omega(S) \to \Omega(Y)$ be an $\Omega$-equivalence, where $Y$ is Kupka–
Smale. Then $h(x)$ is a homoclinic point associated with the singularity $h(p)$, respecting $Y$. This is a contradiction because Kupka–Smale flows do not have homoclinic points associated with singularities (they must be non-transverse).

For the second statement of (1), suppose $S$ has a homoclinic tangency $x \in W^r(A) \cup W^u(A) - A$. First we $C'$ perturb $S$ away from $A \cup \{x\}$ to an $X$ so that $x \in W^u(\gamma_1, X) \cap W^u(\gamma_2, X) - A$ is still a tangency, where $\gamma_1$ and $\gamma_2$ are periodic orbits in $A$. It is easy to see $x$ is nonwandering of $X$. Then we $C'$ perturb $X$ near $x$ to a $Z$ so that $W^s(\gamma_1, Z) \cap W^s(\gamma_2, Z) - A$ are in $\Omega(Z)$ but has dimension $\geq 2$. Moreover, we may assume $X$ and $Z$ are $C^1$ $\Omega$-stable. This contradicts that $Z$ should be $\Omega$-equivalent to a Kupka–Smale flow.

For the first statement of (2), note that $S$ has no transverse homoclinic point $x$ associated with any $C_k$. Otherwise $x$ would be accumulated by periodic orbits of $S$ of index $i$. Since $x \notin C_1 \cup \cdots \cup C_m$, this would contradict that $C_1, \ldots, C_m$ absorb all of $P_f(S)$. Then $S$ has no homoclinic points at all associated with any $C_k$ by (1).

For the second statement of (2), take $\epsilon > 0$ such that any $X \in B(S, \epsilon)$ is $C^1$ $\Omega$-stable, where $B(S, \epsilon)$ denotes the $\epsilon$-ball of $S$ in $\mathcal{F}^1(M)$. Let $\mathcal{U} = B(S, \epsilon/2)$. If $X = S$ on a neighborhood $U$ of $C_1 \cup \cdots \cup C_m$, then each $C_k$ is clearly a prebasic set of $X$ (it is not clear at this stage if $C_1, \ldots, C_m$ give a basic set decomposition for $P_f(S)$). Now suppose there is $X \in \mathcal{U}$ such that $X = S$ on a neighborhood $U$ of $C_1 \cup \cdots \cup C_m$, and such that $X$ has a homoclinic point $x$ associated with one of these sets, say $C_1$. Since $C_1$ is in phase, there are two orbits $\gamma_1$ and $\gamma_2$ in $C_1$ such that $x \in W^s(\gamma_1) \cap W^u(\gamma_2) - C_1$. This intersection at $x$ must be transverse by (1). Now $S$ and $X$ can be joint by a continuous arc $X_t$ in $\mathcal{U}$, $0 \leq t \leq 1$, such that $S = X_0$, $X = X_1$, and that $X_t = S$ on $U$ for all $t$. Note that there is $\delta > 0$ such that if for some $0 \leq t \leq 1$, $W^u(\gamma_1, X_t)$ and $W^u(\gamma_2, X_t)$ has a transverse intersection $y$ such that the angle $A$ between $T_y(W^u(\gamma_1, X_t))$ and $T_y(W^u(\gamma_2, X_t))$ is less than $\delta$, then there is a $Z \in B(S, \epsilon)$ such that $y \in W^s(\gamma_1, Z) \cap W^u(\gamma_2, Z) - C_1$ but the intersection at $y$ for $Z$ becomes nontransverse. On the other hand, there is $\delta > 0$ such that if the angle $A \geq \delta$, then $W^u(\gamma_1, X_{t+})$ and $W^u(\gamma_2, X_{t+})$ have transverse intersections for all $t$ in $(-\delta, \delta)$. Now $W^u(\gamma_1, X_0)$ does not intersect $W^u(\gamma_2, X_0)$ by the first statement of (2), but $W^u(\gamma_1, X_0)$ intersects $W^u(\gamma_2, X_0)$ transversely. Hence there must be $s \in (0, 1)$ such that $W^u(\gamma_1, X_s)$ and $W^u(\gamma_2, X_s)$ have an intersection $y$ with angle $< \delta$. Thus a $C'$ perturbation $Z$ in $B(S, \epsilon)$ would have homoclinic tangencies associated with a prebasic set of $Z$, which contradicts (1). This proves Lemma 3.4.

Now we return to proving that $P_f$ is hyperbolic to complete our induction process. As we have seen in Section 2, the proof consists of four steps.
Step 1. Prove $\overline{P}_j \cap (\text{Sing}(S) \cup \overline{P}_0 \cup \cdots \cup \overline{P}_{j-1}) = \emptyset$.

The key to Step 1 is the following $C^1$ connecting lemma of Hayashi, which can be obtained by applying Theorem F of [WX] twice.

**Theorem 3.5 (The $C^1$ Connecting Lemma).** Let $S \in \mathcal{X}^1(M)$ and $p, q \in M - \text{Sing}(S)$ be given. We assume that $p$ and $q$ are not periodic. We also assume that for any two neighborhoods $U$ of $p$ in $M$, and $V$ of $q$ in $M$, respectively, there is a point $x \in U$ and $t \geq 0$ such that $\phi^t(x) \in V$. Then for any $C^1$ neighborhood $\mathcal{U}$ of $S$ in $\mathcal{X}^1(M)$, there is a real number $L > 0$ such that for any $\delta > 0$, there is $X \in \mathcal{U}$ such that

(a) $X = S$ on $M - B(\phi^S_{[0,L]}(p), \delta) - B(\phi^S_{[-L,0]}(q), \delta)$

(b) $q$ is on the positive $X$-orbit of $p$.

Here $B(A, \delta)$ denotes the neighborhood of $A$ in $M$.

Intuitively, such two loosely connected points $p$ and $q$ can be made really connected by some $C^1$ perturbation supported on an arbitrarily small neighborhood of a finitely timed portion of the positive orbit of $p$ and the negative orbit of $q$. If we allow $C^0$ perturbation, this kind of connecting could be done trivially. But with $C^1$ perturbations, it is hard. This is very much like the situation of the $C^1$ closing lemma.

Now we adopt the $C^1$ connecting lemma to complete Step 1. Suppose for contradiction that $P_j$ accumulates on a singularity $\sigma$ of $S$. Then there is a point $p$ on the unit sphere of $W^s(\sigma)$, and a point $q$ on the unit sphere of $W^u(\sigma)$ that periodic orbits of $P_j$ accumulate on. Then $p$ and $q$ meet the condition of the $C^1$ connecting lemma. Take a $C^1$ neighborhood $\mathcal{U}$ of $S$ such that every $X \in \mathcal{U}$ is $C^1$ $\Omega$-stable. Let $L = L(p, q, \mathcal{U})$ be the constant given by the $C^1$ connecting lemma. Note that $S$ does not have homoclinic points associated with hyperbolic singularities by Lemma 3.4. Thus we may assume $p \notin W^s(\sigma)$ and $q \notin W^u(\sigma)$. Then there is $\varepsilon > 0$ such that $\text{Orb}^{-}(p, S)$ and $\text{Orb}^{+}(q, S)$ are disjoint from $B(\phi^S_{[0,L]}(p), \varepsilon)$ and $B(\phi^S_{[-L,0]}(q), \varepsilon)$. By the $C^1$ connecting lemma, there is $X \in \mathcal{U}$ such that $\text{Orb}^{-}(p, X) = \text{Orb}^{-}(p, S)$, $\text{Orb}^{+}(q, X) = \text{Orb}^{+}(q, S)$, and $q \in \text{Orb}^{+}(p, X)$. This means, $p$ is homoclinic of $X$ associated to the singularity $\sigma$. This contradicts (1) of Lemma 3.4.

For the case that $P_j$ accumulates on a basic set $A$ of $\overline{P}_1 \cup \cdots \cup \overline{P}_{j-1}$, the proof is similar using statement (2) of Lemma 3.4. This completes Step 1.

Now we come to the next step.

Step 2. Prove there is a continuous splitting $A_- \oplus A_+$ of $D_t | \overline{P}_j$ that agrees with the hyperbolic splitting over each periodic orbit of $P_j$.

The key to Step 2 is the following result of Liao [L4]. Results of this type for diffeomorphisms appear in Mañé [M1] and Pliss [Pl].
Theorem 3.6 [L4]. Let $S$ be a star flow. There is a $C^1$ neighborhood $\mathcal{U}$ of $S$ in $\mathcal{AX}(M)$, and two numbers $\lambda = \lambda(\mathcal{U}) > 0$ (usually small) and $T = T(\mathcal{U}) > 0$ (usually large) such that for any $X \in \mathcal{U}$, and any periodic point $x$ of $X$, the following two estimates hold.

(a) $\|\psi_t^X|D^t(x, X)/m(\psi_t^X|D^t(x, X))\| \leq e^{-\lambda t}$ for any $t \geq T$.

(b) If $\tau$ is the period of $x$, $m$ is any positive integer, and if $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then

$$\frac{1}{mt} \sum_{i=0}^{k-1} \log \|\psi^X_{t_{i+1} - t_i}|D^t(\psi^X_{t_i}(x), X)\| \leq -\lambda,$$

$$\frac{1}{mt} \sum_{i=0}^{k-1} \log(m(\psi^X_{t_{i+1} - t_i}|D^t(\psi^X_{t_i}(x), X))) \geq \lambda.$$

This theorem with $m = 1$ appears earlier in [L1]. We will need the case of general $m$ below.

Theorem 3.6 describes certain estimates that are close to but not the same as the hyperbolicity of $\text{Per}(X)$, which means by definition that for some $\lambda > 0$, $T > 0$, the two estimates $\|\psi_t^X|D^t(x, X)\| \leq e^{-\lambda t}$ and $m(\psi_t^X|D^t(x, X)) \geq e^{\lambda t}$ hold uniformly for all $x \in \text{Per}(X)$ and all $t \geq T$. The estimate in (a) of Theorem 3.6 is uniform for all $x \in \text{Per}(X)$ and all $t \geq T$, but is only for the quotient and not separate. The estimates in (b) of Theorem 3.6 are separate, and even better (because $\|AB\| \leq \|A\| \|B\|$ in general), and are uniform for all $x \in \text{Per}(X)$. But they hold only for some specific time, i.e., only for the multiples of the periods. Hence they may not hold for all $t \geq T$ if the periods are not bounded above. Thus Theorem 3.6 is close but not the same as the hyperbolicity of $\text{Per}(X)$. In fact, as discussed in Section 2, the Guckenheimer–Lorenz attractor shows that the star condition may not imply the hyperbolicity of the set of periodic orbits $\{G\}$. For our purpose of extending the splittings however, the estimate (a) is enough. This is because of a uniqueness of the so-called dominated splitting discussed below.

Let $x \in M = \text{Sing}(S)$. A splitting $A_- (x) \oplus A_+ (x)$ of $\mathcal{D}(x)$ is called a dominated splitting of index $i$ if $\dim A_- (x) = i$, and if there are two numbers $\lambda > 0$, $T > 0$ such that

$$\|\psi_t|A_-(x)\|/m(\psi_t|A_+(x)) \leq e^{-\lambda t},$$

and

$$\|\psi_t|A_+(x)\|/m(\psi_t|A_-(x)) \leq e^{-\lambda t},$$

for all $t \geq T$. 
Hyperbolic splittings are clearly dominated splittings. But, unlike hyperbolic splittings, a point $x$ may have more than one dominated splitting. However, if the index is specified, dominated splitting is unique. More precisely \([L4]\).

**Lemma 3.7.** Let $A \oplus B$ and $A' \oplus B'$ be two dominated splittings at $x$ of index $i$ and $i'$, respectively. If $i \leq i'$, then $A \subset A'$ and $B \supset B'$. In particular, if $i = i'$, then $A = A'$ and $B = B'$.

**Proof.** It follows from the definition of dominated splitting $A_-(x) \oplus A_+(x)$ that for any $u \in A_-(x)$, $\|u\| = 1$, and any $v \notin A_-(x)$, $\|v\| = 1$, the inequality $\|\psi^t(u)\| < \|\psi^t(v)\|$ holds for all sufficiently large $t > 0$. Therefore either $A \subset A'$ or $A \ni A'$ must hold, because otherwise there would be two vectors $u \in A - A'$, $\|u\| = 1$, and $v \in A' - A$, $\|v\| = 1$. Then both $\|\psi^t(u)\| < \|\psi^t(v)\|$ and $\|\psi^t(v)\| < \|\psi^t(u)\|$ would hold for all sufficiently large $t > 0$. This contradiction shows either $A \subset A'$ or $A \ni A'$. Reversing time shows $B \ni B'$.

As a consequence of Theorem 3.6 and Lemma 3.7, we have the following theorem.

**Theorem 3.8 (Dominated Splittings on $P_i(X) - \text{Sing}(X)$).** Let $S$ be a star flow. Then there is a $C^1$ neighborhood $\mathcal{U}$ of $S$ in $\mathcal{F}$, and two numbers $\lambda > 0$, $T > 0$ such that for any $X \in \mathcal{U}$, and any $i = 0, 1, ..., \dim M - 1$, there is a continuous $X$-invariant splitting $A_-(x) \oplus A_+$ on $P_i(X) - \text{Sing}(X)$ such that

(a) $A_-(x) = D^-(x)$, $A_+(x) = D^+(x)$, if $x \in P_i(X)$,

(b) $\|\psi^t(x)\| A_-(x) ||/\|A_+(x)\| \leq e^{-2\lambda t}$ for all $x \in P_i(X) - \text{Sing}(X)$, and all $t \geq T$.

(c) If $\tau$ is the period of any $x \in P_i(X)$, $m$ is any positive integer, and if $0 = t_0 < t_1 < ... < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log ||D^\tau(\psi^t(x), X)|| \leq -\lambda,$$

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log(m||D^\tau(\psi^t(x), X)||) \geq \lambda.$$

Note that (a) implies $\dim A_-(x) = i$ for each $x \in P_i(X) - \text{Sing}(X)$. Also note that (c) is the same as (b) of Theorem 3.6. We rewrite it here just for completeness of Theorem 3.8.

**Proof.** Take $\mathcal{U}$, $\lambda$, $T$ as Theorem 3.6 guarantees. Let $X \in \mathcal{U}$, and $0 \leq i \leq \dim M - 1$ be given, and let $x$ be any point of $P_i(X) - \text{Sing}(X)$. 

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Take any sequence \( \{ y_n \} \subset \mathcal{P}_j(X) \) that converges to \( x \). By taking subsequence if necessary, we assume \( D'(y_n) \) and \( D''(y_n) \) converge to the subspaces \( A_-(x) \) and \( A_+(x) \) of \( D(x) \), respectively. Note that \( \dim A_-(x) = i \), \( \dim A_+(x) = \dim M - 1 \). Since \( \lambda \) and \( T \) in the estimate (a) of Theorem 3.6 are uniform for all \( y \in \mathcal{P}_j(X) \) and all \( t \geq T \), passing to limit simply proves the inequality (b) of Theorem 3.8. This inequality then implies that \( A_-(x) + A_+(x) \) is a direct sum of \( D(x) \), hence \( A_-(x) \oplus A_+(x) \) is a dominated splitting of index \( i \) at \( x \). By the uniqueness of Lemma 3.7, \( A_-(x) \oplus A_+(x) \) is independent of the choice of \( \{ y_n \} \). Hence this gives a well defined bundle splitting \( A_- \oplus A_+ \) of \( D \) over \( \mathcal{P}_j(X) - \text{Sing}(X) \). Its \( S \)-invariance follows also from the uniqueness. Its continuity just follows from the way \( A_-(x) \) and \( A_+(x) \) are defined. Statement (a) of Theorem 3.8 is obvious.

Now we apply Theorem 3.8 to our system \( S \) and the index \( j \) in Step 2 to get a continuous splitting \( A_- \oplus A_+ \) of \( D \) over \( \mathcal{P}_j(S) - \text{Sing}(S) \) with all the properties described in Theorem 3.8. This completes Step 2 since \( \mathcal{P}_j(S) \) has been proved disjoint from \( \text{Sing}(S) \) in Step 1. Actually this is much more than what Step 2 claims because of the \( C^1 \) neighborhood \( \mathcal{U} \) of \( S \) and because of these detailed estimates of Theorem 3.8. We will need all of these below.

Now we come to the next step.

**Step 3.** Prove \( A_- \) over \( \mathcal{P}_j(S) \) is contracting under \( \psi_j \).

The key to Step 3 is the ergodic closing lemma below. Recall that the usual \( C^1 \) closing lemma claims that, passing near any nonwandering nonsingular point \( x \), one can create a periodic orbit \( \gamma \) by some \( C^1 \) small perturbation. Here the nearness means that there is a point \( z \in \gamma \) that is near \( x \). It does not tell if the new orbit of \( z \) is near the old orbit of \( x \) up to the whole period of \( \gamma \) even if \( x \) is recurrent. This is because the closer to \( x \) the point \( z \) is, the larger the period of \( \gamma \) will be. Therefore this is not a problem of the continuous dependence of initial values. Let us make this more precise. A point \( x \in M - \text{Sing}(S) \) is strongly closable of \( S \) if for any \( C^1 \) neighborhood \( \mathcal{U} \) of \( S \) in \( \mathcal{F}(M) \), and any \( \delta > 0 \), there are \( X \in \mathcal{U} \), \( z \in M \), \( \tau > 0 \), \( L > 0 \) such that the following three conditions hold.

(a) \( \phi^\tau_j(z) = z \).
(b) \( d(\phi^t_j(x), \phi^t_j(z)) < \delta, \) for any \( 0 \leq t \leq \tau \).
(c) \( X = S \) on \( M - B(\phi^\tau_j L_0, \delta) \).

The set of strongly closable points of \( S \) will be denoted by \( \Sigma(S) \). It is easy to see that strongly closable points must be recurrent. But the converse is unknown. The usual \( C^1 \) closing lemma does not tell this as...
mentioned above. The ergodic closing lemma then asserts that the set of strongly closable points is of full measure in \( M - \text{Sing}(S) \), respecting \( S \)-invariant measures.

**Theorem 3.9 (The Ergodic Closing Lemma for Flows).** For any \( S \in \mathcal{A}^1(M) \), \( \mu(\text{Sing}(S) \cup \Sigma(S)) = 1 \) for every \( S \)-invariant Borel probability measure \( \mu \).

It was Mañé [M1] who first proved this surprising ergodic closing lemma for diffeomorphisms. Our proof of Theorem 3.9 below mainly follows his approach.

Since the proof of Theorem 3.9 is long, we put it in Section 4.

Now we use the ergodic closing lemma to complete Step 3. Let \( \mathcal{H}, \lambda, T \) and the splitting \( A_+ \oplus A_- \) on \( \mathcal{P}_j(S) \) be guaranteed by Theorem 3.8 to our system \( S \) in Theorem A. We also assume \( \mathcal{H} \) is guaranteed by Lemma 3.4. We have to prove that \( A_- \) is contracting under \( \psi_t \).

It is easy to see that if for any \( x \in \mathcal{P}_j \), there is a \( t \geq 0 \) such that \( \log \| \psi_t \| A_-(x) \| < 0 \), then \( A_- \) is contracting. Thus for contradiction we suppose there is a point \( p \in \mathcal{P}_j \) such that

\[
\log \| \psi_t \| A_-(p) \| \geq 0 \tag{1}
\]

for all \( t \geq 0 \).

Consider \( \phi_T = \phi_T^\mathcal{H} \) as a discrete system. Let \( \eta: \mathcal{P}_j \to \mathbb{R} \) be defined as

\[
\eta(x) = \frac{1}{T} \log \| \psi_T \| A_-(x) \|
\]

Then (1) implies

\[
\frac{1}{n} \sum_{i=0}^{n-1} \eta(\phi_{iT}(p)) \geq 0 \tag{2}
\]

for any \( n \geq 1 \).

Let \( \mu_n = (1/n) \sum_{i=0}^{n-1} \delta(\phi_{iT}(p)) \), where \( \delta(x) \) denotes the atom measure at \( x \). Then (2) becomes

\[
\int \eta \, d\mu_n \geq 0
\]

for any \( n \geq 1 \).

Take a subsequence \( \{n_k\} \) such that

\[
\mu_{n_k} \to \mu
\]

for some \( \phi_T \) invariant Borel probability measure \( \mu \). Then

\[
\int \eta \, d\mu \geq 0.
\]
By the Birkhoff theorem,
\[ \int \eta^* \, d\mu \geq 0, \]  
where
\[ \eta^*(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta(\phi_j(x)), \]
which is defined on \( \overline{P}_j \) except for a set of \( \mu \)-measure zero.

It is easy to see that \( \mu(\Sigma(S)) = 1 \), where \( \Sigma(S) \) is the set of strongly closable points of \( S \). In fact, let \( \nu \) be the Borel measure defined so that
\[ \int f \, dv = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( f \circ \phi_t \right) \, dt \]
for any \( f \in C^0(\overline{P}_j) \). Then \( \nu \) is \( S \)-invariant, and \( \nu(A) = \mu(A) \) for any \( S \)-invariant Borel set \( A \). Since \( \Sigma(S) \) is an \( S \)-invariant Borel set, and \( \nu(\Sigma(S)) = 1 \) by the ergodic closing lemma, it follows that \( \mu(\Sigma(S)) = 1 \).

Therefore (3) implies
\[ \int_{\Sigma(S)} \eta^* \, d\mu \geq 0. \]
Hence there is a point \( q \in \overline{P}_j(S) \cap \Sigma(S) \) such that \( \eta^*(q) \geq 0 \), i.e.,
\[ \lim \frac{1}{nT} \sum_{j=0}^{n-1} \log \| \psi_T \mid A_j(\phi_j(q)) \| \geq 0. \]  
(4)

We claim that \( q \) is not periodic of \( S \). For otherwise \( \dim D^*(q, S) \geq f \) since \( \overline{P}_j \) is disjoint from the periodic orbits of \( S \) of lower indices. And then \( D^*(q) \Rightarrow A_j(\phi_j(q)) \) by Lemma 3.8. Talking \( m \) large in the first inequality of (3) of Theorem 3.8 would give a contradiction to (4). This shows that \( q \) is not periodic of \( S \).

Take \( 0 < \xi_1 < \xi_2 < \lambda \). Then there is \( n_0 \) such that
\[ \frac{1}{nT} \sum_{j=0}^{n-1} \log \| \psi_T \mid A_j(\phi_j(q)) \| \geq -\xi_1 \]  
(5)
for any \( n \geq n_0 \).

Take \( \delta > 0 \) small so that \( \text{B}(\overline{P}_j, \delta) \) is disjoint from \( \text{Sing} \cup \overline{P}_u \cup \cdots \cup \overline{P}_{j-1} \).

Since \( q \) is strongly closable of \( S \), there are \( X \in \mathcal{W}, z \in M, \tau > 0, L > 0 \) such that
(a) \( \phi^t_\gamma(z) = z \).
(b) \( d(\phi^t_\gamma(q), \phi^t_\gamma(z)) < \delta \), for all \( 0 \leq t \leq \tau \).
(c) \( X = S \) on \( M - B(\phi^\tau_{-L_0}(q), \delta) \).

Note that \( \tau \) can be made arbitrarily large by taking \( X \) sufficiently close to \( S \), because \( q \) is not \( S \)-periodic. Moreover, as shown in [M1], we may make the perturbation such that the following condition (d) holds as well.

(d) \( \mathcal{D} \) restricted to the \( X \)-orbit \( \gamma \) (which is periodic) has an \( X \)-invariant splitting \( G_- \oplus G_+ \) such that \( \dim G_-(x) = j \) for all \( x \in \gamma \), and that

\[
\| \psi^X_Y | G_-(\phi^X_{\tau Y}(z)) \| = \| \psi^X_Y | A_-(\phi^X_{\tau Y}(q)) \|
\]

and

\[
m(\psi^X_Y | G_-(\phi^X_{\tau Y}(z))) = m(\psi^X_Y | A_-(\phi^X_{\tau Y}(q)))
\]

for all \( 0 \leq i \leq [\tau/T] - 1 \). That is, we may deform the finite \( S \)-orbit \( \beta \) from \( q \) to \( \phi^X_{\tau Y}(q) \) to \( \gamma \) so that it carries the splitting \( A_- \oplus A_+ \) on \( \beta \) to an \( X \)-invariant splitting \( G_- \oplus G_+ \) on \( \gamma \) of the same dimension in a norm preserving way at iterates of \( \phi^Y_\tau \) up to \( [\tau/T] - 1 \). Because \( A_-(q) \oplus A_+(q) \) is dominated of \( S \); \( \gamma \) is periodic of \( X \); and because \( \tau \) can be arbitrarily large but \( T \) is fixed, it is easy to see that \( G_- \oplus G_+ \) must be a dominated splitting of \( X \) by the norm preserving property up to \([\tau/T] - 1 \) iterates. Then \( G_- \oplus G_+ \) is identical with the dominated splitting \( A_- \oplus A_+ \) on \( \gamma \) of index \( j \) guaranteed by Theorem 3.8, by the uniqueness of Lemma 3.7. And then the inequality (5) for \( S \) becomes, writing \([\tau/T] = k\),

\[
\frac{1}{kT} \sum_{j=0}^{k-1} \log \| \psi^X_Y | A_-^X(\phi^X_{\tau Y}(z)) \| \geq -\xi_1,
\]

as long as \( k \geq n_0 \). This yields

\[
\frac{1}{\tau} \left( \sum_{j=0}^{k-1} \log \| \psi^X_Y | A_-^X(\phi^X_{\tau Y}(z)) \| + \log \| \psi^X_Y | A_+^X(\phi^X_{\tau Y}(\tau z)) \| \right) \geq -\xi_2,
\]

(6)

because \( \tau \) can be arbitrarily large and \( T \) is fixed. The inequality (6) will contradict the inequality (c) of Theorem 3.8 for \( m = 1 \) as long as we can prove

\[
A_-^X = D^-(X), \quad A_+^X = D^+(X),
\]
on the $X$-periodic orbit $\gamma$. By Lemma 3.7, it suffices to prove that
\[
\dim D'(z, X) = j,
\]
or, what is the same, to prove that $z \in P_j(X)$. But (6) and the inequality of Theorem 3.8 together imply
\[
\log(m(\psi^X_\tau | A^X_+ (z))) \geq (2\lambda - \xi z) \tau > 0.
\]
and this implies (since $\gamma$ is periodic)
\[
A^X_+ (z) \subset D'(z, X),
\]
hence
\[
\dim D'(z, X) \leq j.
\]
Thus it remains to prove that
\[
\dim D'(z, X) \geq j.
\]
(7)

This is guaranteed by the following Lemma 3.10. We remark that this is the only lemma in this paper that assumes $C^1$ structural stability, and proving (7) is the only place in our approach to proving Theorem A where we need the whole weight of the $C^1$ structural stability assumption.

**Lemma 3.10.** Let $S$ be $C^1$ structurally stable, and let $P_i(S)$ be hyperbolic for some $i$. Then there is a $C^1$ neighborhood $\mathcal{U}$ of $S$ in $\mathcal{F}(M)$ such that if $X \in \mathcal{U}$ agrees with $S$ on a neighborhood $U$ of $P_i(S)$ in $M$, then $P_i(X) = P_i(S)$.

**Proof.** Let $\mathcal{U}$ be a $C^1$ neighborhood of $S$ in $\mathcal{F}(M)$ such that for every $X \in \mathcal{U}$, there is a homeomorphism $h: M \to M$ that maps the orbits of $S$ onto that of $X$. Then $h(P_i(S)) = P_i(X)$ because homeomorphism of $M$ preserves the dimensions of submanifolds of $M$. Then $h(P_i(S)) = P_i(X)$. In particular, $h$ preserves connected components of $P_i(S)$. Decompose $P_i(S)$ into its basic sets (since $P_i(S)$ is hyperbolic) $C_1, ..., C_N$, which are the same as the connected components of $P_i(S)$.

Now if $X = S$ on $U$, then $P_i(S) \subset P_i(X)$. Hence $P_i(S) \subset P_i(X)$. To prove $P_i(S) = P_i(X)$, or equivalently $P_i(S) = P_i(X)$, suppose for contradiction that $Q = P_i(X) - P_i(S) \neq \emptyset$. Then $Q \cap P_i(S) \neq \emptyset$, because otherwise $P_i(X)$ would have more than $N$ connected components. This means periodic orbits of $Q$ accumulate on some basic set $C_k$ of $P_i(S)$. Then as we did in Step 1, using the $C^1$ connecting lemma, an arbitrarily $C^1$ small perturbation $Y$ of $X$ would create a homoclinic point associated with a prebasic set of $Y$, contradicting (2) of Lemma 3.4. This proves Lemma 3.10.
Using Lemma 3.10, the inequality (7) becomes immediate since the periodic orbit $\gamma$ of $X$ through $z$ is clearly not contained in $P_{\psi}(S) \cup \cdots \cup P_{\psi}(S)$. This completes the proof that $\Delta_{\psi}^S$ on $P_{\psi}(S)$ is $\psi\Delta_{\psi}^S$ contracting. That is, Step 3 is complete.

**Step 4.** This step is a flow copy of a diffeomorphism result of Mañé [M2] and we refer the reader to that paper. This finishes the induction step that $P_{\psi}(S)$ is hyperbolic. This proves Theorem A.

### 4. THE ERGODIC CLOSING LEMMA FOR FLOWS

In this section we prove Theorem 3.9, the ergodic closing lemma for flows. The key to the proof is a ratio property demonstrated in the following version of the usual $C^1$ closing lemma formulated in [M]. Also see [W].

**Theorem 4.1 (The $C^1$ Closing Lemma, the Idealized Ratio Version).** Let $V_0, V_1, \ldots, V_n, \ldots$ be a sequence of $m$-dimensional inner product spaces, and $T_n: V_n \to V_0$ be a sequence of linear isomorphisms, and let $\epsilon > 0$. Then there are $\rho > 2$ (usually large) and an integer $L \geq 1$ (usually large) with the following properties: For any finite ordered set $\mathcal{P} = \{p_0, p_1, \ldots, p_T\}$ in $V_0$, there are two points $w$ and $y$ in $\mathcal{P} \cap B(p_0, \rho |p_0 - p_T|)$, where $w$ is before $y$ in $\mathcal{P}$, together with $L + 1$ points $c_0, c_1, \ldots, c_L$ in $B(p_0, \rho |p_0 - p_T|)$, not necessarily distinct, such that the following two conditions are satisfied.

(a) $c_0 = w, c_L = y$.

(b) $|T_n^{-1}(c_n) - T_n^{-1}(c_{n+1})| \leq \epsilon d(T_n^{-1}(c_{n+1}), T_n^{-1}(y))$

for $n = 0, 1, \ldots, L - 1$, where $T_0$ stands for the identity, $Y$ stands for the set $(\mathcal{P} - \{w, y\}) \cup B(p_0, \rho |p_0 - p_T|)$, and $d$ is the distance in $V_0$.

Via a linearization process, Theorem 4.1 applies to the manifold $M$ and the flow $\phi_t, V_n$ becomes $\mathcal{D}(\phi_{-t}(x))$ for some $x \in M - \text{Sing}(S)$, and $T_n$ becomes $\psi t$. The set $\mathcal{P}$ becomes the ordered intersections of the finite orbit from $p_0$ to $p_T$ with a local cross section at $x$. Thus Theorem 4.1 yields the following version of the $C^1$ closing lemma. For any $x \in M - \text{Sing}(S)$ and small $\alpha > 0$, we will denote by $\Pi(x, \alpha)$ the local cross section $\exp B(\alpha, \mathcal{D}(x))$, where $B(0, \alpha \mathcal{D}(x))$ denotes the $\alpha$-ball of center 0 in $\mathcal{D}(x)$.

**Theorem 4.2 (The $C^1$ Closing Lemma, the Ratio Version).** Let $S \in \mathcal{X}(M)$ and $x \in M - \text{Sing}(S)$. Given any $C^1$ neighborhood $\mathcal{U}$ of $S$ in $\mathcal{X}(M)$ and any $\delta > 0$, there are $r > 0$ (usually small), $\rho > 2$ (usually large) and $L \geq 1$ (usually large) with the following properties: Whenever $p$ and $\psi_t(p)$ are both in $\Pi(x, b)$ for some $0 < b \leq r$ and some $T > 2h$, then there are
0 \leq T_1 < T_2 \leq T \text{ and } X \in \mathcal{U} \text{ such that } \phi_{T_2}(p) \text{ and } \phi_{T_2}(p) \text{ are both in } \Pi(x, pb), \text{ and that for any } z \in \phi_{T_2}^{-1}([T_1-b, T_1+b]) \text{, we have}

(a) \phi_{T_2-T_1}^X(z) = z,

(b) d(\phi_{T_2}^X(z), \phi_{T_2}(z)) < \delta \text{ for all } 0 \leq t \leq T_2 - T_1.

(c) X = S \text{ on } M - B(\phi_{T_2}^X([-L,0]), \delta).

We remark that a refined formulation for Theorem 4.2 could be this:

Given \mathcal{U}, there are \rho > 2 \text{ and } L \geq 1 \text{ such that for any } \delta > 0, \text{ there is } r > 0 \text{ with the following properties, etc. We will not need these details but stay with Theorem 4.2 formulated as above.}

The formulation of Theorems 4.1 and 4.2 are a lot more complicated than the usual formulation of the C^1 closing lemma. This is because they describe how the detailed closing process is carried out, especially how a ratio \rho \text{ is involved. This will play a crucial role in proving the ergodic closing lemma below. During the proof we will try to illustrate Theorem 4.2 further as well.}

Now we prove the ergodic closing lemma for flows. Let \mu \text{ be any } S\text{-invariant Borel probability measure. We need to prove}

\mu(M - \text{Sing}(S) - \Sigma(S)) = 0.

The proof goes through a series of reductions as follows.

Let \mathcal{U} \text{ be a } C^1 \text{ neighborhood of } S \text{ in } \mathcal{A}(M), \text{ and let } \delta > 0. \text{ A point } x \in M - \text{Sing}(S) \text{ is } (\mathcal{U}, \delta)\text{-strongly closable of } S \text{ if there are } X \in \mathcal{U}, z \in M, \text{ and } T > 0, L > 0 \text{ such that}

(1) \phi_{T_2}^X(z) = z.

(2) d(\phi_T^X(x), \phi_T(z)) < \delta \text{ for all } 0 \leq t \leq T.

(3) X = S \text{ on } M - B(\phi_{T_2}^X([-L,0]), \delta).

Denote by \Sigma(\mathcal{U}, \delta) \text{ the set of } (\mathcal{U}, \delta)\text{-strongly closable points of } S. \text{ It is a Borel set. Clearly, if } \mathcal{U}_n \text{ is a basis of } \mathcal{A}(M) \text{ at } S, \text{ and } \delta_n \to 0, \text{ then}

\Sigma(S) = \bigcap_n \Sigma(\mathcal{U}, \delta_n).

Thus the ergodic closing lemma reduces to proving

\mu(M - \text{Sing}(S) - \Sigma(\mathcal{U}, \delta)) = 0 \quad \text{(E1)}

for any \mathcal{U}, \delta.

Now we make more reductions. Let \rho > 0, \rho > 2. \text{ A point } x \in M - \text{Sing}(S) \text{ is } (\mathcal{U}, \delta, r, \rho)\text{-responsible of } S \text{ if there is } L > 0 \text{ such that whenever } p \text{ and } \phi_{T_2}(p) \text{ are both in } \Pi(x, b) \text{ for some } 0 < b \leq r \text{ and some } T > 2pb, \text{ then there}
are \(0 < T_1 < T_2 \leq T\) and \(X \in \Psi\) such that \(\phi_{T_1}(p)\) and \(\phi_{T_2}(p)\) are both in \(\Pi(x, \rho b)\), and for any \(z \in \phi_{\tau_{T_1 - b, T_1 + b}} \Pi(x, \rho b)\), we have

1. \(\phi_{T_2 - T_1}(z) = z\).
2. \(d(\phi_{\tau_{T_1 - b, T_1 + b}}(z), \phi_{\tau_{T_2 - b, T_2 + b}}(z)) < \delta\) for all \(0 \leq t \leq T_2 - T_1\).
3. \(X = S\) on \(M - B(\phi_{\tau_{T_1 - L_0}}(x), \delta)\).

Note that \(z\) is \((\Psi, \delta)\)-strongly closable. Thus whenever an orbit hits a sufficiently small \(b\)-box of a \((\Psi, \delta, r, \rho)\)-responsible point \(x\) twice, there will be a \((\Psi, \delta)\)-strongly closable segment that hits the enlarged \(\rho b\)-box of \(x\) at some time in between. Roughly, \(x\) responds with \((\Psi, \delta)\)-strongly closable segments within the ratio \(\rho\). Note that Theorem 4.2 simply claims that, for every \(\Psi, \delta\), any non-singular point is \((\Psi, \delta, r, \rho)\)-responsible for some \(r, \rho\).

Denote \(\mathcal{R}(x, \delta, r, \rho)\) the set of \((\Psi, \delta, r, \rho)\)-responsible points of \(S\). This is a Borel set. Clearly, if \(r_n \to 0, \rho_m \to 0\), then

\[
M - \text{Sing}(S) = \bigcup_{n=0}^{\infty} \mathcal{R}(\Psi, \delta, r_n, \rho_m)
\]

for any \(\Psi, \delta\). Thus the ergodic closing lemma reduces again to proving

\[
\mu(\mathcal{R}(\Psi, \delta, r, \rho) \setminus \Sigma(\Psi, \delta)) = 0 \quad (E2)
\]

for any \(\Psi, \delta, r, \rho\).

The following lemma illustrates the notion of \((\Psi, \delta, r, \rho)\)-responsible points in terms of ergodic measures.

**Lemma 4.3.** If \(x\) is \((\Psi, \delta, r, \rho)\)-responsible, then for any \(0 < b \leq r\),

\[
\mu(\Sigma(\Psi, \delta) \cap \phi_{\tau_{1-b, 1+b}} \Pi(x, \rho b)) \geq \mu(\phi_{\tau_{1-b, 1+b}} \Pi(x, b))
\]

Proof. Let us abbreviate the two sets \(\Sigma(\Psi, \delta) \cap \phi_{\tau_{1-b, 1+b}} \Pi(x, \rho b)\) and \(\phi_{\tau_{1-b, 1+b}} \Pi(x, b)\) as \(A\) and \(B\), respectively. Since \(\mu\) is ergodic, there is a point \(q \in M\) such that

\[
\mu(A) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \chi_A(\phi(t)(q)) \, dt
\]

and

\[
\mu(B) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \chi_B(\phi(t)(q)) \, dt
\]

But \(B\) is a flow box of time length \(2b\), so

\[
\int_0^T \chi_B(\phi(t)(q)) \, dt \approx 2b \# \{\phi_{\tau_{T_1, T_2}}(q) \cap \Pi(x, b)\}
\]
where the error is no bigger than $2b$, which is fixed when $T \to +\infty$. Now since \( x \) is \((\mathcal{U}, \delta, r, \rho)\)-responsible, between any two hits at the \( b \)-box, there is a \((\mathcal{U}, \delta)\)-strongly closable hit at the \( \rho \)-box. Thus

\[
\int_{\mathbb{S}} \mathcal{Z}_d(\phi(t)) \, dt \approx 2b(\# \{ \phi_{t_0, T}(q) \cap \Pi(x, b) \} - 1),
\]

where the error is also no bigger than $2b$. This proves Lemma 4.3.

Roughly, Lemma 4.3 says that, about \((\mathcal{U}, \delta, r, \rho)\)-responsible points, the \((\mathcal{U}, \delta)\)-strongly closable points contained in the \( pb \)-box has \( \mu \)-measure no less than the \( \mu \)-measure of the \( b \)-box. If \( \mu \) is the volume, we may in turn say that this is no less than a constant percentage of the measure of the \( pb \)-box itself. Now our \( \mu \) is somewhat arbitrary. Our next goal is to estimate the \( \mu \)-measure of the set of points around which a percentage \( \sigma \) of such a compatibility holds. First we make an easy \( \sigma \)-compactness reduction. For each positive integer \( i \), denote

\[
G_i = \left\{ x \in M \mid d(x, \text{Sing}(S)) > \frac{1}{i} \right\}.
\]

Then \( G_i \) is open and \( G_i \) is disjoint from \( \text{Sing}(S) \). Since

\[
M - \text{Sing}(S) = \bigcup_i G_i,
\]

the ergodic closing lemma reduces again to proving

\[
\mu(R(\mathcal{U}, \delta, r, \rho) \cap G_i - \Sigma(\mathcal{U}, \delta)) = 0 \quad \text{(E3)}
\]

for all \( \mathcal{U}, \delta, r, \rho, i \).

Now we proceed to the last reduction. Let \( 0 < \sigma < 1 \). A point \( x \in M - \text{Sing}(S) \) is \((\rho, \sigma)\)-compatible if there are arbitrarily small \( b > 0 \) such that

\[
\mu(\phi_{\{ -b, b \}}(x, b)) \geq \sigma \mu(\phi_{\{ -b, b \}}(x, pb)).
\]

Denote by \( C(\rho, \sigma) \) the set of \((\rho, \sigma)\)-compatible points.

**Lemma 4.4.** For any \( \rho \) and \( i \), there is \( K = K(\rho, i) \) such that

\[
\mu(G_i \cap C(\rho, \sigma)) \geq \mu(G_i) - \sigma K
\]

for any \( \sigma \).
Proof. Isometrically embed $M$ into $T'$ for some large $s$. For each positive integer $k$, let $\mathcal{P}_1^k \subseteq \cdots \subseteq \mathcal{P}_n^k \subseteq \cdots$ be a nested sequence of partitions of $T'$ that partition $T'$ into cubes of equal sides of $1/k^n$. Associated with $\mathcal{P}_n^k$ is a covering $\mathcal{E}_n^k$ of $T'$ by cubes of the same centers but of equal sides of $1/k^n-1$. Let $\mathcal{P}_n^k(x)$ denote the cube of $\mathcal{P}_n^k$ containing $x$. Similarly define $\mathcal{E}_n^k$. Extend $\mu$ to a Borel measure on $T'$ by defining $\mu(A) = \mu(A \cap M)$ for any Borel set $A$ of $T'$. Let

$$A(k, n, \sigma) = \{ x \in T' \mid \mu(\mathcal{E}_n^k(x)) < \sigma \mu(\mathcal{P}_n^k(x)) \}.$$ 

As shown in [M1], for $k$ odd,

$$\mu(A(k, n, \sigma)) < \sigma^k \quad (\ast)$$

for any $k, n, \sigma$.

Let $\rho, i$ be given. Since $\overline{G_{i+1}}$ is disjoint from $\text{Sing}(S)$, sufficiently small $S$-flow boxes contained in $G_{i+1}$ and about points of $\overline{G_i}$ are distorted not too badly. Now $M$ is isometrically embedded in $T'$, so there is $m = m(\rho, i)$ such that for any $x \in G_i$, and any $n \geq m$, there is $b > 0$ such that

$$\mathcal{P}_n^k(x) \subset N(\phi_{[-h, h]} \Pi(x, \rho b)) \subset N(\phi_{[-h, h]} \Pi(x, \rho b)) \subset \mathcal{E}_n^k(x),$$

where $N$ (box) denotes the tubular neighborhood of the box in $T'$. Thus for any $\sigma > 0$,

$$G_i - A(m, n, \sigma) \subset C(\rho, \sigma).$$

Applying the inequality $(\ast)$, we get

$$\mu(G_i \cap C(\rho, \sigma)) \geq \mu(G_i) - \sigma^m.$$

Note that $m$ and $s$ are independent of $\sigma$. This proves Lemma 4.4. □

By Lemma 4.4, the ergodic closing lemma reduces finally to proving

$$\mu(R(\mathcal{U}, \delta, r \rho) \cap G_i \cap C(\rho, \sigma) - \Sigma(\mathcal{U}, \delta)) = 0 \quad (E4)$$

for any $\mathcal{U}$, $\delta$, $r$, $\rho$, $i, \sigma$.

Note that by Lemma 4.3 and the definition of $(\rho, \sigma)$-compatible points, any point $x \in R(\mathcal{U}, \delta, r \rho) \cap G_i \cap C(\rho, \sigma)$ has arbitrarily small $0 < b \leq r$ such that

$$\mu(\Sigma(\mathcal{U}, \delta) \cap \phi_{[-h, h]} \Pi(x, \rho b)) \geq \sigma \mu(\phi_{[-h, h]} \Pi(x, \rho b)).$$

Or equivalently,

$$\mu((\phi_{[-h, h]} \Pi(x, \rho b)) - \Sigma(\mathcal{U}, \delta)) < (1 - \sigma) \mu(\phi_{[-h, h]} \Pi(x, \rho b)). \quad (\ast\ast)$$


Roughly, the set \( R(\mathcal{U}, \delta, r, \rho) \cap G \cap C(\rho, \sigma) - \Sigma(\mathcal{U}, \delta) \) has the property that it misses a constant percentage \( \sigma \) of measure in arbitrarily small boxes about each of its points. The following argument shows such a set must be of measure zero.

**Proof of (E4).** Given \( \mathcal{U}, \delta, r, \rho, i, \sigma \). Abbreviate the two sets \( R(\mathcal{U}, \delta, r, \rho) \cap G \cap C(\rho, \sigma) \) and \( \Sigma(\mathcal{U}, \delta) \) as \( E \) and \( \Sigma \), respectively. Take any open set \( U \) containing \( E \) and \( \Sigma \). As \((***)\) claimed, any \( x \in E - \Sigma \) has arbitrarily small \( 0 < b \leq r \) such that

\[
\phi_{(-b, b)}(x, pb) \subset U,
\]

and

\[
\mu((\phi_{(-b, b)}(x, pb)) - \Sigma) < (1 - \sigma) \mu(\phi_{(-b, b)}(x, pb)).
\]

All these boxes \( \phi_{(-b, b)}(x, pb) \) form a Vitali covering of \( E - \Sigma \). By a variation of Vitali’s covering lemma \([M1]\), there are countably many of the boxes

\[
B_1, B_2, ..., B_k, ...
\]

that cover \( E - \Sigma \) except for a set of \( \mu \)-measure zero. Then

\[
\mu(E - \Sigma) = \mu \left( \bigcup_{k} B_k - \Sigma \right) = \sum_{k} \mu(B_k - \Sigma) < (1 - \sigma) \sum_{k} \mu(B_k) < (1 - \sigma) \mu(U).
\]

But, on the other hand, if \( \mu(E - \Sigma) \) is not zero, we can take \( U \) so small that

\[
\mu(U) < \frac{1}{1 - \sigma} \mu(E - \Sigma),
\]

contradicting the last inequality. This proves \((E4)\), and hence proves the ergodic closing lemma.

**REFERENCES**

C^1 STABILITY OF FLOWS


