STRUCTURAL STABILITY ON TWO-DIMENSIONAL MANIFOLDS†

A FURTHER REMARK

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(*Received* 15 *April* 1963)

THE ABOVE paper appeared in Volume 1, pp. 101–120 of *Topology*, and the aim of it was to prove that the set of all structurally stable differential equations is open and dense in the space, with the C^1-topology, of all differential equations defined on a compact M^2. In this note we clarify a point concerning the proof of that theorem that was brought to our attention by E. Lima and S. Schwartzman.

In Lemma 4 we consider the segment $\sigma: x = -1$, $0 \geqslant y \geqslant -\frac{1}{2}$ on the co-ordinate square $R: |x| \leqslant 1, |y| \leqslant 1$, and call q_i the point of the trajectory γ of Y_1 through $p = (0, 0)$ where it hits σ for the i-th time and p_i the corresponding point on the y-axis, i.e. the point where the arc of γ beginning at q_i hits the y-axis for the first time. We then consider the trajectory $\gamma(u)$ of $X(u)$ through p. Assuming that γ does not pass through the top or bottom side of R, which can always be done, then there is a u_0 so small that for $u \leqslant u_0$, $\gamma(u)$ hits σ for the i-th time at a point $q_i(u)$ and if we call $p_i(u)$ the corresponding point on the y-axis then we pass continuously from the arc pq_i to the arc $pq_i(u)$; besides the point $q_i(u)$ varies continuously and monotonically with u.

Now in the proof of Lemma 4 we need to consider the extremity $q_i(u)$; of the arc $pq_i(u)$ of $\gamma(u)$, for values of u between u_0 and 1. But when u goes beyond u_0 the arc $pq_i(u)$ may hit suddenly the bottom of the segment σ for $u = \bar{u}$ and $q_i(u) \to q_{i+1}(\bar{u})$ as $u \to \bar{u}$ where $q_{i+1}(\bar{u})$ is the point where $\gamma(\bar{u})$ hits σ for the (1 + 1)-th time. So, to extend $q_i(u)$ for $u > u_0$ one has to consider i depending on u and the problem is to give a precise definition of the extension $q_{i(u)}(u)$, continuous and increasing on u. In Lemma 4 it is assumed implicitly that this can be done up to $u = 1$. No doubt $q_{i(u)}(u)$ may be defined on a certain interval $0 < u < u^* \leqslant 1$ but it might happen that $i(u) \to \infty$ as $u \to u^* < 1$ so that $u = 1$ would never be reached.

Actually this difficulty appears also in Lemmas 5 and 7. We now indicate how to remedy this situation.

We first remark that for the proof of these lemmas we need only to consider the case where M^2 is orientable; otherwise lifting the field Y_1 together with the co-ordinate square R

† This note was written with the partial support of the Air Force office of Scientific Research.

to the orientable covering M^2_* of M^2_* we get either a closed orbit or else a new connection between saddle points in M^2 and therefore the same happens in M^2. We now observe that if we drop in Lemma 3 the assumption that $Ta \neq d$, $Tb \neq c$ then the statement of this Lemma is to be changed by adding that c_0, d_0 may be isolated points of Γ and this has no effect whatsoever on what follows, so that we may use Lemma 3 with

$$Ta = d, Tb = c. \quad (1)$$

Now, to simplify matters we define q_i, $q_i(u)$ as the i-th intersection of γ and $\gamma(u)$, respectively, not with σ as before but with the side cd of R, and p_i, $p_i(u)$ as before; we recall that i is an integer determined so that q_i is closed enough to $(-1, 0)$.

When dealing with Lemmas 4 and 5 we consider a co-ordinate rectangle such that ca and db are arcs of the same trajectory of μ so that (1) is satisfied and $a = d_0$. Under these assumptions when $i(u)$ has a discontinuity at u_1 then $i(u') = i(u'') = i(u_1) - 1$ where u' and u'' are close enough to u_1, $u' < u_1 < u''$. So $i(u)$ is certainly bounded. Now a 'new' intersection between $\gamma(u)$ and cd which is introduced from the bottom can never go to the top and give rise to another since before that we get a closed orbit (Lemma 4) or else a new connection between saddle points (Lemma 5). Therefore if these things are avoided, after a certain value of u, $i(u)$ in fact remains constant and $q_{i(u)}(u)$ is defined up to $u = 1$. This settles the situation in case of Lemmas 4 and 5.

In case of Lemma 7, one takes the side db of R as an arc of δ in such a way that it corresponds to the first time δ meets R; then the function $i(u)$ can never increase and this is enough to settle the question in this case.

We end this note with two remarks unrelated to the situation raised above; (a) in Lemma 7 when considering the trajectory ξ it may be necessary to 'go back' so that ξ does not come from a saddle point; this is always possible but it may be that $\omega(\xi) \neq \alpha(\delta)$; (b) The problem raised on p. 113 about the existence of non-trivial minimal sets for fields of class C^2 on M^2 has been answered in the negative by A. Schwartz in a forthcoming paper. This does not imply that in the proof of the density theorem one may skip §5, since in order to cover the situation of Lemma 7 we still need the arguments of Lemmas 3, 4 and 5.

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