PLANES TRIPLY TANGENT TO CURVES
WITH NONVANISHING TORSION

MICHAEL H. FREEDMAN*†

(Received 12 May 1978)

EXPERIMENTATION with a closed loop of wire and a desk top quickly leads to the conclusion that except for certain special configurations, only a finite number of planes are tangent to a given curve \( \alpha(t) \) at three places. The main results is that generically this number is even when the torsion \( \tau_\alpha(t) \) is nonvanishing.

Let \( A \) denote the space of \( C^\infty \) regular closed curves \( \alpha: [0, 1] \rightarrow \mathbb{R}^3 \) with nonvanishing curvature, \( k_\alpha \), in the Whitney \( C^\infty \)-topology. This is the topology generated by the open sets \( B_{\alpha, \epsilon} = \left\{ g \in A: \frac{d^3\mathbf{g}}{dt^3} - \frac{d^2\mathbf{f}}{dt^2} \right\} < \epsilon \}. \) Let \( A^+ \) denote the subspace of curves with nonvanishing torsion, \( \tau_\alpha \).

A plane \( P \) in \( \mathbb{R}^3 \) is tangent to \( \alpha \) at \( t \in [0, 1] \) if \( \alpha'(t), \alpha''(t), \alpha'''(t) \) are linearly dependent. Let \( \eta: A \rightarrow \mathbb{Z}^+ \cup \{0\} \) assign to each \( \alpha \) the number of planes which are tangent to \( \alpha \) at exactly three points, \( t_1, t_2 \) and \( t_3 \). Note that such a plane may meet \( \alpha(t) \) nontangentially at other points and thus avoid detection by the wire and table top method.

MAIN THEOREM: There is an open dense subset \( B^+ \) of \( A^+ \) satisfying: \( \eta(B^+) \subset \{ \text{even integers} \} \).

The proof will show that no open subset of \( A \) is mapped by \( \eta \) to \( \infty \), confirming our intuition. In §4 an experimental method is described for computing \( \eta \). To sharpen the main theorem examples may be constructed (the last word must be taken physically rather than mathematically) to persuade the reader that: (1) \( \eta(B^+) \subset \{ \text{even integers} \} \) and (2) for any integer \( s \geq 0 \) there is an open subset \( A_s \subset A \) with \( \eta(A_s) = s \). An analytic derivation of these examples appears to be technically difficult.

We outline the proof. Associated to a curve \( \alpha \in A \) is a mapping \( f_\alpha: T^2 \rightarrow \{ \text{planes in } \mathbb{R}^3 \} \equiv (RP^3 - \text{pt.}) \) with the property that triple points, that is points of \( (RP^3 - \text{pt.}) \) with three inverse images, are precisely triply tangent planes to \( \alpha \). An important step is to show that when \( \tau_\alpha(t) \) is nonvanishing \( f_\alpha \) is a locally flat topological immersion which is smooth away from a circle. This is accomplished by finding coordinates (given by \( x_1, y_1, z_1, w_1 \) in the notation below, \( i = 1, 2, 3 \) ) on any punctured affine space \( (R^3 - \text{pt.}) \subset (RP^3 - \text{pt.}) \) so that the image of \( f_\alpha \) meets this \( (R^3 - \text{pt.}) \) in the “cuspoidal” surface generated by the tangent lines to a related curve \( \gamma: [0, 1] \rightarrow (R^3 - \text{pt.}) \) also with nonvanishing torsion. This cuspoidal surface is topologically a flat immersion.

It is a topological fact that a smooth immersion of a torus into \( R^3 \) with normal crossings has an even number of triple points. Our immersion has three deficiencies (1) It is not smooth, (2) It does not necessarily have normal crossings and (3) The target is \( (RP^3 - \text{pt.}), \) not \( R^3 \). A technical section on genericity shows that for an open dense set of \( \alpha(=B^+) \) in \( A^+ \), \( f_\alpha \) will be a smooth normal immersion except at a cuspoidal circle and that all triple points will occur far away from this circle. Lemma 2 rounds the cusp over coming the first and second deficiencies while §3 handles the third. Thus the topological fact about immersions of a torus in \( R^3 \) is stretched to tell us that for \( \alpha \in B^+f_\alpha \) has an even number of triple points. The theorem follows.

†Partially supported by NSF grants.
The first three sections comprise the proof of this theorem.
Throughout we use the convention of [4] that a tangent vector to \( R^3 \), \( v_p \) is the vector at the origin, \( v = v_0 \), translated to the point \( p \).

§1. THE ASSOCIATED MAPPING \( f_\alpha \)

Let \( \alpha \in A \). Associate to \( \alpha \) the mapping \( f_\alpha : T^2 \to \{ \text{planes in } R^3 \} \) defined by \( f_\alpha(t, \theta) = \) the plane \( P \) containing \( T_\alpha(t) \) and \( (\cos \theta)N_\alpha(t) + (\sin \theta)B_\alpha(t) \) where

\[
T = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad N = \frac{T'}{||T'||}, \quad B = T \times N, \quad \text{and} \quad 0 \leq \theta < \pi.
\]

A plane \( P \) in \( R^3 \) is determined by its closest point to \( (0, 0, 0) \) unless it contains \( (0, 0, 0) \) in which case \( P \) is determined by its normal line. This establishes \( \{ \text{planes in } R^3 \} \equiv (R^3 \text{ with original blow up}) \equiv RP^3 - \text{pt.} \) Let \( b : \{ \text{planes in } R^3 \} \to R^3 \) be "blowing down", i.e. \( b \) (plane) = its closest point to \( (0, 0, 0) \).

If we express \( \alpha(t) \) as \( \alpha(t) - T + (\alpha(t) \cdot N)N + (\alpha(t) \cdot B)B \), we see that the closest point on the osculating plane to \( \alpha \) at \( t \) is \( (\alpha(t) \cdot B)B \). This means that \( b(f(t, 0)) = (\alpha(t) \cdot B)B \). We call this curve \( \beta(t) \). By the Frenet formulas:

\[
\beta'(t) = -\tau_\alpha(t)v_\alpha(t)((\alpha \cdot N)B + (\alpha \cdot B)N)
\]

where \( \tau_\alpha \) and \( v_\alpha \) denote torsion and speed.

Set \( c_t = \{ b(f(t, \theta)) \} \) \( t \) fixed, \( 0 \leq \theta < \pi \); \( c_t \) is the locus of points \( p \) in the \((N, B)\)-plane satisfying: \( p \) is the closest point to \( (0, 0, 0) \) on the line \((\alpha \cdot B)B + (\alpha \cdot N)N, p)\). See Fig. 1.

By plane geometry \( c_t \) is the circle through \( \{(0, 0, 0), (\alpha \cdot B)B, (\alpha \cdot N)N + (\alpha \cdot B)B\} \). By (*) the two shaded triangles are similar and an easy exercise in plane geometry establishes:

\[
** c_t \text{ and the curve } \beta \text{ are tangent at } \beta(t). \text{ So } b(\text{image } f_\alpha) \text{ is the union of all tangent circles to } \beta \text{ containing } (0, 0, 0).
\]

Except for values \( (t, \theta) \) satisfying \( (0, 0, 0) \in f_\alpha(t, \theta) \), the composition \( \text{inv} \circ (b(f_\alpha(t, \theta))) \) is defined where \( \text{inv} : R^3 - \{(0, 0, 0)\} \to R^3 - \{(0, 0, 0)\} \) is the inversion \( (\rho, \phi_1, \phi_2) \to (1/\rho, \phi_1, \phi_2) \) in polar coordinates. Tangencies are preserved by \( \text{inv} \) (since it is a conformal map) and circles through the origin are mapped to straight lines. Consequently \( \text{inv} \circ (b(\text{image } f_\alpha)) \) is the tangential developable (see [5]) \( S(t, x) = \gamma(t) + x\gamma'(t) \) where: \( \gamma(t) = (B_\alpha(t)/\alpha(t) \cdot B_\alpha(t)), t \in [0, 1], \) and \( x \in \text{Reals} \).

We need to give a local description of image \( f_\alpha \) as a smooth immersion away from an "unbranched family of cusps" along image \( f_\alpha(t, 0) \). It will be sufficient to give this description for image \( S(t, x) \) since \( \text{inv} \circ b \) is a diffeomorphism away from planes containing \( (0, 0, 0) \). Note that for any \( (t, \theta) \), the origin of \( R^3 \) may always be
PLANES TRIPLY TANGENT TO CURVES WITH NONVANISHING TORSION

Let $a$ be a regular curve with nonvanishing curvature and torsion. As long as $a^*$ is defined, i.e. $a(t) \cdot B_a(t) \neq 0$, $a^*$ is also a regular curve with nonvanishing curvature and torsion. Furthermore $(a^*)^* = a$.

Proof. It is well known (see [4] for example) that the hypothesis on $a$ is equivalent to saying: $\{a'(t), a''(t), a'''(t)\}$ are linearly independent for all $t$. For legibility, set $\gamma = a^*$. We have:

$$\alpha \cdot \gamma = 1 \quad \alpha' \cdot \gamma = 0$$

since $\gamma$ is a scalar multiple of $B_a$ and $\alpha'$ a scalar multiple $T_a$.

$$(***) \quad \alpha'' \cdot \gamma = 0$$

since $\alpha''$ is a linear combination of $T_a$ and $N_a$.

$$\alpha''' \cdot \gamma \neq 0.$$

By assumption $a(t) \cdot B_a(t) \neq 0$ so $B_a(t) \neq 0$ and $\gamma(t) \neq 0$. But $\gamma$ would be the zero functional if $\gamma$ annihilated the basis $\{a', a'', a'''\}$.

Differentiating the three equations (several times) yields:

$$\alpha' \cdot \gamma + \alpha \cdot \gamma' = 0 \quad \alpha'' \cdot \gamma + \alpha' \cdot \gamma' = 0 \quad \alpha''' \cdot \gamma + \alpha'' \cdot \gamma' = 0$$

$$\alpha'' \cdot \gamma + 2\alpha' \gamma' + \alpha \gamma'' = 0$$

$$\alpha''' \cdot \gamma + 3\alpha'' \gamma' + 3\alpha' \gamma'' + \alpha \gamma''' = 0.$$

So $\alpha \cdot \gamma' = 0$ and $\alpha' \cdot \gamma' = 0$ hence $\alpha \cdot \gamma'' = 0$. Differentiating the last two equalities: $\alpha'' \cdot \gamma' = -\alpha' \cdot \gamma''$, $\alpha' \cdot \gamma = -\alpha \cdot \gamma''$. We get

$$(******) \quad \alpha \cdot \gamma = 1$$

$$\alpha' \cdot \gamma = \alpha \cdot \gamma' = 0$$

$$\alpha'' \cdot \gamma = \alpha' \cdot \gamma' = \alpha \cdot \gamma'' = 0$$

$$\alpha''' \cdot \gamma = -\alpha'' \cdot \gamma' = \alpha' \cdot \gamma'' = -\alpha \cdot \gamma''' \neq 0.$$

Applying the functionals $\cdot a''$, $\cdot a'$ and $\cdot a$, we conclude that $\gamma' \neq 0, \{\gamma', \gamma''\}$, and finally $\{\gamma', \gamma'', \gamma'''\}$ are independent. This completes the proof of our first assertion.

Since $\alpha \cdot \gamma' = \alpha \cdot \gamma'' = 0$, $\alpha$ is perpendicular to the osculating plane of $\gamma(t)$ and is therefore a scalar multiple of the binormal to $\gamma$, $B_a(t) = x(t)B_b(t)$. Dotting with $\gamma(t)$ we find $1 = \gamma(t) \cdot B_a(t) = x(t)(\gamma(t) \cdot B_b(t))$, so $x = (1/(\gamma \cdot B_b))$ and $\alpha = (B_b(\gamma \cdot B_b)) = \gamma^* = (a^*)^*$. 

\[\square\]
Henceforth we will assume that $\gamma$ is regular with nonvanishing curvature and torsion. We have just seen that the same hypothesis now applies to $\gamma$.

For $x \neq 0$, $S(t, x)$ is smooth immersion, since the rows of the Jacobian, $(\partial S/\partial t) = \gamma'(t) + xy''(t)$ and $(\partial S/\partial x) = \gamma'(t)$ are linearly independent. The behavior of $S$ at $x = 0$ was thoroughly investigated more than a hundred years ago; a good account appears in pp. 66–72 of [5]. A "cuspidal edge" or "edge of regression" occurs at $\gamma$ on image $(S)$. Let the plane $P = (N, B, \gamma(0))$ with coordinate functions $N$ and $B$ be intersected with $S((0, -\epsilon, 0) \times \text{Reals})$.

This locus will be approximated at $(0, 0)$ by the cusp: $(B)^2 + (8/9)(\tau/K)(N)^3 = 0$ up to terms of higher degree. So for $\epsilon'$ sufficiently small this locus will meet the $\epsilon'$-circle in $P$ in exactly two distinct points which vary smoothly as functions of $t$. Since $\tau \neq 0$ the two sheets of the cusp are disjoint in a sufficiently small annular neighborhood of $(0, 0)$.

**LEMMA 2.** Let $\alpha$ have nonvanishing torsion and $f_\alpha(t, 0)$ be an imbedded circle. Choose a Riemannian metric on $\mathbb{RP}^3$. \( \forall \delta > 0, \exists \delta > \epsilon > 0 \) such that there is a smooth immersion $f_\alpha: T^2 \to \mathbb{RP}^3$ satisfying:

1. $f_\alpha(t, \theta) = f_\alpha(t, \theta)$ away from $X = \text{the connected component of } (t, 0)$ in $(\{t, \theta\} f_\alpha(t, \theta) \subseteq \text{a closed } \epsilon\text{-neighborhood of } f_\alpha(t, 0))$.

2. $f_\alpha/X$ is an imbedding into $\epsilon\text{-neighborhood } (f_\alpha(t, 0))$.

**Proof.** Given our description of the above locus when $\tau_\alpha \neq 0$ this lemma is an exercise in analysis. In each $(N, B)$-cross section we "round the cusp" by convolving with a smooth bump function. See Fig. 2 below.

Lemma 2 says that when $\tau_\alpha$ is nonvanishing, $f_\alpha$ is "as good as" a smooth immersion. On the other hand if $\tau_\alpha$ were to pass through zero, the $\cdot B_\gamma$-positive and $\cdot B_\gamma$-negative sheets of the cusp will cross yielding a topological branch point of $f_\alpha$ (and therefore of $S$). Such an $f_\alpha$ is not even a topological immersion.

We describe $J^3(I, R^3)$ as $\{t, x, b | t \in I, x \in R^1$ and $b = h_{ij} \leq i, j \leq 3 \} \subseteq R^{13}$.

§2. GENERICITY

We deal with the following technical question:

Normal immersions are certainly generic (see Chap. 4 of [2]). Yet how do we know there is an open, dense subset $B^* \subseteq A^*$ so that if $\alpha \in B^* f_\alpha$ can be rounded (as in Lemma 2) to a normal $f_\alpha$? Answering this question is an exercise in the use of the "multi-jet-transversality theorem". We will use the notation of [2].

Lojasiewicz[3] has shown that real analytic subsets of $R^n$ are smoothly triangulable and therefore a stratified union of manifolds. We will identify certain proper analytic subset $D_\alpha, E_\alpha$ and $F$ of the multi-jet bundle $J^3(I, R^3)$. Applying the above mentioned transversality theorem a strata at a time we conclude that $\{\alpha \in A^* | \gamma(\alpha) \in D_\alpha, E_\alpha$ and $F \}$ is open dense in $A^*$; this subset will be our $B^*$.

We describe $J^3(I, R^3)$ as $\{t, x, b | t \in I, x \in R^1$ and $b = h_{ij} \leq i, j \leq 3 \} \subseteq R^{13}$.
$J^3(I, R^3) = (β^*)^{-1}(R^{3s})$ where $β$ is the target jet map and $R^{3s}$ is the $s$-fold cartesian product minus the big diagonal. The data $(t_0, x_0, b)$ represent the germ $α(t_0 + t) = x + (b_{11}t + b_{22}t^2 + b_{33}t^3) U_1 + (b_2, t + b_{22}t + b_{33}t^3) U_2 + (b_{31}t + b_{33}t^3) U_3$. $J^3 \subset J^3$ will be the subset of data corresponding to $A^*$, i.e. $\{α', α'' , α'''\}$ should have rank 3.

For $s ≥ 2$ let $D_s = \{(t_1, . . . , t_s, x_1, . . . , x_s, b_1, . . . , b_s) \in J^3(I, R^3)\}$ the virtual osculating planes, $x_i + y(b_{11}, b_{22}, b_{33}) + z(b_{12}, b_{22}, b_{33}), y$ and $z$ variable, coincide for all $1 ≤ i ≤ s$. $D_s$ is an analytic subset of $J^3(I, R^3) \subset R^{3s}$ with codimension $= 3(s - 1)$. By transversality, there is an open dense subset $A^+ \subset A^*$ such that if $α \in A^+$, $J^3α$ is transverse to $D_s$, $∀s$. Since dimension (source $J^3α$) $= s < 3(s - 1)$ for $s = 2$, if $α \in A^+_2$ then $J^3α(1^*)$ is disjoint from $D_2$. It follows that for $α \in A^+_2$, $α^* = γ$ is imbedded.

Let $E_s = \{(t_1, . . . , t_s, x_1, . . . , x_s, b_1, . . . , b_s) \in J^3(I, R^3)\}$ the virtual tangent lines to $α^*(1^*)$. Recall that $α^*$ is defined from $α$, $α'$ and $α''$ so an analytic expression for tangent lines to $α^*$ may be written out from the 3-jet of $α$. Consequently $E_s$ is an analytic subset of $J^3(I, R^3)$. Since $∀(where defined) is an involution on $A^*$, $*$ maps data for $α^*$ to data for $α$ openly. So, codim $E_s =$ codimension of “s lines in $R^3$ meeting in a point” $= 2s - 3$. Let $A^+ \subset A^*$ be the open dense subset where $J^3α$ is transverse to $E_s$ $∀s$ (disjoint from $E_s$ for $s > 3$).

**LEMMA 3.** If $α \in A^+$ the associated mapping $f_α: \{(t, θ)|t \in I, θ ≠ 0\} \rightarrow RP^3$ is normal.

**Proof.** First we use $J^3α$ is traverse to $E_2$; let $ℓ_1$ and $ℓ_2$ be any two tangent lines based at different points on $α$ which meet at $p$. For at least one of the lines $ℓ_1$, the first variation of distance ($ℓ, θ$) must be positive as we move $ℓ_1$ in its ruled family, leaving $ℓ_2$ fixed. Consider the planes $P_1$ and $P_2$ generated by these first variations of $ℓ_1$ and $ℓ_2$. Their rulings cannot be parallel to each other by the uniqueness of “characteristic points” on a “characteristic line” of an envelope of planes (see p. 66 of [5]). It follows that $P_1$ and $P_2$ do not coincide. Thus the tangent planes of any two sheets at a self-intersection of $S_α/\{(t, x)|t \in I, x ≠ 0\}$ are distinct, so $S_α/\{\}$ and therefore $f_α/\{(t, θ)|t \in I, θ ≠ 0\}$ are 2-normal.

Since codim $E_3 \subset J^3$ is 3, $f_α/\{\}$ can have only isolated triple points. But a 2-normal immersion of a surface in a three manifold with isolated triple points is normal.

Let $F = \{(t_1, t_2, t_3, x_1, x_2, x_3, b_1, b_2, b_3)\}$ $2$ virtual tangent lines to $α^*$ meet at a point of $α^*$. $F \subset J^3(I, R^3)$ is of codimension $= 4$. By transversality there is an open dense subset $A^+ \subset J^3α$ with $J^3α(1^*)$ disjoint from $F$ for all $α \in A^+$.

Let $B^* = A^+_2 \cap A^+_2 \cap A^+_2 \subset A^*$. If $α \in B^*$ then: (1) $α^*$ is imbedded, (2) $f_α$ is a normal immersion on $\{(t, θ)|t \in I, θ ≠ 0\}$ and (3) the triple point set of $f_α/\{(t, θ)|t \in I, θ ≠ 0\}$ is disjoint from $α^*$. By (1) and (3) the triple point set of $f_α$ is disjoint from some $δ$-neighborhood of $f_α(t, 0)$. Applying Lemma 2, we see that passing from $f_α$ to $f_α$ leaves triple points in $(RP^3-δ$-neighborhood $(f_α(t, 0)))$ unaffected and cannot introduce triple points into $ε$-neighborhood $(f_α(t, 0))$. Since $f_α$ satisfies (2), we can require $f_α$ to be a normal immersion as well. In particular, we have proved:

**LEMMA 4.** There is an open, dense subset $B^* \subset A^*$ such that if $α \in B^*$ there is a normal immersion $f_α: T^2 \rightarrow RP^3$ with the same number of triple points as the map $f_α: T^2 \rightarrow RP^3$.

**§3. TOPOLOGICAL CONSIDERATIONS**

**LEMMA 5.** Given any normal immersion $f: T^2 \rightarrow RP^3$ there is a normal immersion $F: (T^2 × I; T^2 × 0, T^2 × 1) \rightarrow (W; RP^3, S^0)$ for some smooth manifold $W$ with $∂W = RP^3∪S^0$, satisfying $F((x, 0)) = f(x)$ for all $x \in T^2$. 
Proof. Let $W$ be a simply connected framed bordism between $(\mathbb{RP}^3, \text{some framing } K)$ and $(S^3, K')$. Let $c$ be a simple closed curve on $T'$ whose image $f(c)$ is null-homotopic in $\mathbb{RP}^3$. These exist since $\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(T') \cong \pi_1(\mathbb{RP}^3) \cong \mathbb{Z}$ must have indivisible elements in its kernel. $f$ extends over $T^2 \cup D^2 \cup D^2 \rightarrow \mathbb{RP}^3$. But $(T^2 \cup D^2 \cup D^2, T^2) = (S^1 \times D^2, T^2)$ so we have extended $f$ to a solid torus. Since $W$ is simply connected we may homotop the core circle $S^1 \times *$ across $W$ to obtain a map $h: (T^2 \times I, T^2 \times 0, T^2 \times 1) \rightarrow (W, \mathbb{RP}^3, S^3)$ with $h(x, 0) = f(x) \forall x \in T^2$.

The immersion $f$ together with the framing $K$ induces a framing of the stable tangent bundle $\tilde{\tau}(T^2 \times I)$. This framing extends to a framing of $(\tau^*(T^2 \times I): \tau^*(T^2 \times 0), \tau^*(T^2 \times I))$. According to Smale–Hirsch theory this is the data required to determine an extension of the immersion $f(x)$ to the desired immersion $F$. Since normal relative immersions are dense in immersions which are normal on a boundary component, we may take $F$ to be normal.

Corollary 1. The number of triple points of any normal immersion $f: T^2 \rightarrow \mathbb{RP}^3$ is even.

Proof. It is known that the number of triple points of a surface normally immersed in $S^3$ is congruent (mod 2) to its Euler Characteristic (see [1] for a proof). The triple point set of $F$ is an unoriented 1-manifold $T$ with boundary lying in $\mathbb{RP}^3$ and $S^3$. Since the number of boundary points is even: $\# \text{ points of } \partial_{\mathbb{RP}^3}(T) = \# \text{ points of } \partial_{S^3}(T) \equiv \chi(T^2) \equiv 0 \text{ (mod 2)}$. But $\# \text{ points of } \partial_{\mathbb{RP}^3}(T)$ is precisely the number of triple points of $f$.

Combining Corollary 1 and Lemma 4 we see that if $\alpha \in B^*$, then $f_{\alpha}: T^2 \rightarrow \mathbb{RP}^3$ has an even number of triple points. The main theorem now follows by observing that triple points of $f_{\alpha}$ correspond to planes in $\mathbb{R}^3$ tangent to $\alpha$ at three points.

§4. THE EXPERIMENTAL METHOD

Given a curve $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ the statement that there is a plane tangent to $\alpha$ at $t_1$, $t_2$ and $t_3$ is equivalent to saying that the following sets of vectors are all linearly dependent: $\{\alpha'(t_1), \alpha'(t_2), \alpha'(t_3)\}$, $\{\alpha(t_1) - \alpha(t_2), \alpha'(t_1), \alpha'(t_2)\}$ and $\{\alpha(t_1) - \alpha(t_2), \alpha'(t_1), \alpha'(t_2)\}$. So triple tangencies occur at the simultaneous zeros of three cubic polynomials in $\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_1'(t), \alpha_2'(t)$ and $\alpha_3'(t)$ with initial variables $t_1 = t_1$, $t_2$ and $t_3$. Even for simple parametrizations an analytic determination of the zeros seems too much to hope for. However if a closed curve is made of reasonably stiff wire (I recommend 16 gauge wire) and care is taken to make the model $C^1$ where the ends are joined, there is an efficient procedure for drawing the multiple point set: $M = \{(t_1, \theta_1) \in T^2 \exists \text{ (ties $\theta_2 \neq (t_1, \theta_1)$ with } f_{\alpha}(t_1, \theta_1) = f_{\alpha}(t_2, \theta_2)\}$ and from this set triple points may be counted. Doubly and triply tangent planes which bound the convex hull of $\alpha$ are best found by rolling the curve on a desk top; and general multiply tangent plane (tangent at $t_1$) can be found by closing one eye and sighting along $\alpha'(t_1)_{\alpha(t_1)}$, i.e. project $\alpha$ into span $(N_{\alpha}(t_1), B_{\alpha}(t_1))$. The result $\xi(t) = (\alpha - (\alpha \cdot \ell) I)(t)$, will usually be a smooth curve with a cusp at $t_1$. A doubly tangent plane now is seen in profile as a line through $\xi(t_1)$ tangent to $\xi$ at $\xi(t_2)(t_1 \neq t_2)$; the profile of a triply tangent plane would also be tangent at a third point $t_3$.

The dihedral angle $\theta$ between a multiply tangent plane and the osculating plane to $\alpha$ at $t_1$ may be read off the diagram. In the following examples we draw the multiple point set $M$ on a rectangle representing $T^2 = \{(t, \theta)|t \in I \text{ and } 0 \leq \theta < \pi\}$.
Example 1, a saddle curve $\alpha$. Let $\alpha$ be a parametrization of the intersection of the cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 2$, when $z > 0$. If we project $\alpha$ into the $x, z$-plane and the $y, z$-plane we see:

Here all doubly tangent planes bound the convex hull of $\alpha$ and correspond to the lines tangent to the interior points on the arcs in the above projections. There are no triply tangent planes. The diagram of $M$ is:

The four open circles correspond to the end points of the projections where families of double planes have died or were born. These are also the four points where the torsion of $\alpha$ passes through zero and, as we observed in §1, these 4 points are double branch points of $f_\alpha$.

Example 2, a "bent" saddle curve $\beta$. Reform the curve in the previous example to leave the $(x, z)$-projection unchanged and produce the following non-convex $(y, z)$ projection:
Application of our experimental method yields the following diagram for $M$:

![Diagram](image)

Fig. 7.

The three self intersections of $M$ in the above diagram reflect the unique plane tangent to $\beta$ at three points: $t_1, t_2, t_3$. This example is compatible with our theorem since $\pi_2$ passes through zero in six places.

Example 2 can be generalized to produce an example, in fact an open set of examples, $A_1 \subset A$ with $\pi(A_1) = s$ for all $s \in Z^+$.

Example 3, torus curves and their sums. On some standard torus in $R^3$ a curve $\alpha$ with nonvanishing torsion may be drawn representing the diagonal homology class. Apparently this curve may be chosen so that $\pi(\alpha) = 0$. Furthermore if the longitude and meridian are properly proportioned the linear $(2, 3)$-torus knot $\beta_1$ will have nonvanishing torsion. The experimental technique indicates that $\pi(\beta_1) = 2$.

To construct curves $\beta_n, n > 1$, with nonvanishing torsion and $\pi(\beta_n) = 2n$ we start by setting $\beta_2 = (\neq \beta^*)^s$ where the copies of $\beta^*$ are positioned so that the multiple point set of their tangential developables are disjoint. Connected sum, $\#$, must be taken carefully to preserve the assumption on torsion and the total number of triple points of the tangential developables. Now set $\beta_n = (\beta_2^{n-1} \# \beta^*)^s$.

§5. CONJECTURES

It seems possible that the method of this paper would extend to prove:

**CONJECTURE 1.** For $n \geq 3$, let $A_n^* = \{C^\infty$ closed curves $\alpha: I \rightarrow R^n$ with $\{\alpha^*, \alpha^{n-1}, \ldots, \alpha^1\}$ linearly independent at every point}. Then for every $\alpha$ belonging to some open, dense subset of $A_n^*$, the number of hyperplanes in $R^n$ tangent to $\alpha$ at exactly $n$ points is even.

With less conviction we also conjecture:

**CONJECTURE 2.** If $\alpha$ is a regular $C^\infty$ closed curve in $R^3$ and $\pi(\alpha) = 0$. Then $\alpha$ is unknotted.

REFERENCES


University of California
San Diego