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Borel reductions of profinite actions of $\mathrm{SL}_n(\mathbb{Z})$

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ABSTRACT

Greg Hjorth and Simon Thomas proved that the classification problem for torsion-free abelian groups of finite rank *strictly increases* in complexity with the rank. Subsequently, Thomas proved that the complexities of the classification problems for p-local torsion-free abelian groups of fixed rank n are *pairwise incomparable* as p varies. We prove that if $3 \leq m < n$ and p, q are distinct primes, then the complexity of the classification problem for p-local torsion-free abelian groups of rank m is again incomparable with that for q-local torsion-free abelian groups of rank n.

1. Introduction

This paper builds upon the methods introduced in [8,1], and further specialized in [15,12,3]. The theme of these papers is the intersection of two related pursuits:

- the study of the general structure of the countable Borel equivalence relations, and
- the particular case of the complexity of the classification problem for torsion-free abelian groups of finite rank.

At the heart of each is the use of powerful methods from ergodic theory and the superrigidity theory of Lie groups.

The study of Borel equivalence relations begins with the observation that many classification problems can be identified with an equivalence relation on a standard Borel space (*i.e.*, a Polish space equipped just with its σ-algebra of Borel sets). For instance, each group with domain $\mathbb{N}$ is determined by its group operation, a subset of $\mathbb{N}^3$. Hence, the space of countable groups may be identified with a subset $X_{\mathcal{G}} \subset \mathcal{P}(\mathbb{N}^3)$. Studying the classification problem for countable groups thus amounts to studying the isomorphism equivalence relation $\cong_{\mathcal{G}}$ on $X_{\mathcal{G}}$. The relation $\cong_{\mathcal{G}}$ is extremely complex in the intuitive sense that to check whether $(\mathbb{N}; \times_1) \cong_{\mathcal{G}} (\mathbb{N}; \times_2)$, one must conduct an unbounded search for a witnessing bijection $\phi : \mathbb{N} \to \mathbb{N}$. This intuition is reflected in descriptive set theory in part by the fact that $\cong_{\mathcal{G}}$ is not a Borel subset of $X_{\mathcal{G}} \times X_{\mathcal{G}}$.

However, there are many subcollections of the class of countable groups whose isomorphism equivalence relation *is* Borel. For instance, in this paper we will focus on the space of torsion-free abelian groups of finite rank. Since any torsion-free abelian group of rank n is isomorphic to a subgroup of $\mathbb{Q}^n$, the space of torsion-free abelian groups of rank n can be identified with a subset $R(n) \subset \mathcal{P}(\mathbb{Q}^n)$. Moreover, it is easily seen that for $A, B \leq \mathbb{Q}^n$, we have that $A \cong B$ iff there exists $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $B = g(A)$. It follows easily that the isomorphism equivalence relation $\cong_n$ on $R(n)$ is a Borel equivalence relation.

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The Borel/non-Borel dichotomy is a useful one, but we will shortly introduce a much finer notion of complexity which is specially tailored for equivalence relations. As a start, an equivalence relation E on the standard Borel space X is said to be *smooth*, or completely classifiable, if there exists a standard Borel space Y and a Borel function $f : X \to Y$ satisfying

$$x \mathrel{E} x' \iff f(x) = f(x').$$

In other words, Y is a space of complete invariants for the classification problem up to E. The condition that f is Borel amounts to the requirement that the invariants can be computed in a reasonably "explicit" manner. For instance, the classification problem for countable divisible groups is smooth. Indeed, any countable divisible group is decomposable into a product of Prüfer p-groups, and so any such group A is determined up to isomorphism by the sequence that lists the number of factors of each Prüfer group in a decomposition of A.

On the other hand, it follows from a 1937 result of Baer that even the classification problem for torsion-free abelian groups of rank 1 is not smooth. To explain this, however, we must first define the notion of *Borel reducibility*. If E, F are equivalence relations on the standard Borel spaces X, Y, then we say E is Borel reducible to F and write $E \leq_B F$ iff there exists a Borel function $f : X \to Y$ satisfying

$$x \mathrel{E} x' \iff f(x) \mathrel{F} f(x').$$

We then let $E \sim_B F$ iff $E \leq_B F$ and $F \leq_B E$, $E \perp_B F$ iff $E \nleq_B F$ and $F \nleq_B E$, and finally $E <_B F$ iff $E \leq_B F$ and $E \not\sim_B F$. In these terms, Baer's result implies that $\cong_1 \sim_B E_0$, where E_0 is the equivalence relation defined on $2^{\mathbb{N}}$ by $x \mathrel{E_0} y$ iff $x(n) = y(n)$ for all but finitely many n. It is an elementary fact that E_0 is nonsmooth (in fact it is the $\leq_B$-least nonsmooth Borel equivalence relation), and so it follows that $\cong_1$ is nonsmooth as well.

For a span of 60 years following Baer's result, the classification problem for torsion-free abelian groups of rank 2 and higher remained open. Although Kurosh and Malcev wrote down complete invariants for torsion-free abelian groups of rank 2, they were considered inadequate as a solution to the classification problem because it was as difficult to distinguish the invariants as it was to distinguish the groups themselves. In 1998, Hjorth proved in [8] that $E_0 <_B \cong_2$, and hence that the classification problem for torsion-free abelian groups of rank 2 is indeed strictly more complicated than that for rank 1. Hjorth's solution did not provide any method for dealing with the torsion-free abelian groups of rank greater than 2. In particular, it remained open whether $\cong_2$ is universal for all torsion-free abelian groups of finite rank, and if it's not, then whether $\cong_3$ is universal, and so on.

This question was of major interest since the $\cong_n$ are examples *countable Borel equivalence relations*, and it was unknown at the time whether there could be an infinite strictly ascending chain of countable Borel equivalence relations. Here, a Borel equivalence relation E is said to be *countable* iff every E-class is countable. For instance, let Γ be a countable group and suppose that Γ acts in a Borel fashion on the standard Borel space X. Then the induced *orbit equivalence relation* E_Γ, defined on X by

$$x \mathrel{E_\Gamma} y \iff \Gamma x = \Gamma y,$$

is clearly countable and easily seen to be Borel. For instance, by our earlier remarks concerning the space $R(n)$ of torsion-free abelian groups of rank n, we have that the isomorphism relation $\cong_n$ is exactly the orbit equivalence relation on $R(n)$ induced by the action of $\mathrm{GL}_n(\mathbb{Q})$. By an amazing result of Feldman and Moore [5], *every* countable Borel equivalence relation arises as the orbit equivalence relation induced by a Borel action of some countable group.

Returning to Hjorth's question of whether $\cong_3$ is more complex than $\cong_2$, the first progress was made by Adams and Kechris in [1], who answered the analogous question for the class of rigid groups. Here, a group A is said to be *rigid* iff its only automorphisms are $\pm Id$. Let $S(n) \subset R(n)$ denote the subset consisting of just the rigid torsion-free abelian groups of rank n, and let $\cong_n^*$ be the restriction of the isomorphism equivalence relation to $S(n)$. Adams and Kechris proved the following:

Theorem ([1, Theorem 6.1]). *For all n, we have* $\cong_n^* <_B \cong_{n+1}^*$.

This was one of the earliest results in the subject which separated two known equivalence relations; indeed, before this result there were only six known countable Borel equivalence relations up to Borel bireducibility. The proof made use of some powerful results from the ergodic theory of lattices in Lie groups, most notably, Zimmer's cocycle superrigidity theorem. The reader who is familiar with Zimmer's theorem may wonder exactly how it is relevant to this problem. But recall that $\cong_n^*$ is induced by the action of $\mathrm{GL}_n(\mathbb{Q})$ on $S(n)$, and note the following two facts:

- There exists an ergodic, $\mathrm{SL}_n(\mathbb{Z})$-invariant probability measure on $S(n)$ (see [8] or [13, Theorem 2.4]), and
- $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in the higher rank simple Lie group $\mathrm{SL}_n(\mathbb{R})$ (see [17, Theorem 3.1.7]).

Of course, more is necessary to meet the hypotheses of Zimmer's theorem, and even then Adams and Kechris expended a great deal of effort to extract information from its conclusion. Shortly after this was done, Thomas was able to refine in [14] the method of Adams and Kechris to fully answer the question on the complexity of the isomorphism problem for torsion-free abelian groups of rank 3 and higher.

Theorem ([13, Theorem 1.4]). *For all n, we have* $\cong_n <_B \cong_{n+1}$.

As a stepping stone towards this result, Thomas proved the analogous result for the quasi-isomorphism problem. Here, we say that subgroups $A, B \leq \mathbb{Q}^n$ are *quasi-isomorphic* iff B is commensurable with an isomorphic copy of A. Let $\sim_n$ denote the quasi-isomorphism equivalence relation on the space $R(n)$ of torsion-free abelian groups of rank n.

Theorem (*[13, Theorem 4.6]*)**.** *For all n, we have* $\sim_n <_B \sim_{n+1}$.

These results of Adams and Kechris and of Thomas provided the first examples of infinite chains of *naturally occurring* classification problems. The proofs again made use of Zimmer's cocycle superrigidity theorem for lattices in higher rank Lie groups. Very loosely speaking, at the heart of the proof that $\cong_{n+1} \not\leq_B \cong_n$ is the simple observation that the "dimension" of $\mathrm{SL}_{n+1}(\mathbb{Z})$ is larger than that of $\mathrm{SL}_n(\mathbb{Z})$ (or more precisely, the rank of the ambient Lie group $\mathrm{SL}_{n+1}(\mathbb{R})$ is larger than that of $\mathrm{SL}_n(\mathbb{R})$).

Thomas later gave an example of an infinite *antichain* of naturally occurring equivalence relations. Recall that a torsion-free abelian group A is said to be *p-local* iff it is q-divisible for every prime $q \neq p$. Let $\cong_{n,p}$ denote the isomorphism equivalence relation (and $\sim_{n,p}$ the quasi-isomorphism relation) on the space of p-local torsion-free abelian groups of rank n. Thomas proved the following:

Theorem (*[12, Theorem 1.2 and implicit]*)**.** *Let* p, q *be distinct primes and* $n \geq 3$. *Then we have:*

- $\cong_{n,p} \perp_B \cong_{n,q}$, *and*
- $\sim_{n,p} \perp_B \sim_{n,q}$.

Before this theorem, every Borel non-reducibility result in the area of torsion-free abelian groups had relied on some notion of the dimension of (the ambient Lie group of) the acting group as an invariant. The significance of this result is that this dimension is fixed, since of course both $\sim_{n,p}$ and $\sim_{n,q}$ are induced by actions of the same group.

This left open the question of whether the locality prime p could be used to distinguish between isomorphism relations when the dimension is *not* fixed.

Theorem. *Let* p, q *be distinct primes and* $m, n \geq 3$. *Then we have:*

A. $\cong_{m,p} \perp_B \cong_{n,q}$, *and*
B. $\sim_{m,p} \perp_B \sim_{n,q}$.

More generally, one might ask what role the dimension plays in deciding whether $E \leq_B F$. Theorems A and B shed some light on this question, since in these cases the dimension has no effect as long as it is greater than 2. Theorem A will be established in Corollary 4.2, and Theorem B in Corollary 4.4. These results unfortunately leave open a slightly more technical question, based on the following result from [3].

Theorem (*[3, Theorem B]*)**.** *If* $n \geq 3$, *then* $\cong_{n,p}$ *is Borel incomparable with* $\sim_{n,p}$.

It would be extremely interesting to know whether the isomorphism/quasi-isomorphism distinction is sufficient to establish Borel incomparability between the two classification problems, again even as the dimension increases.

Conjecture. *For* $m, n \geq 3$ *and* p, q *prime, we have* $\cong_{m,p} \perp_B \sim_{n,q}$.

The substantial case is when $q = p$, since if $q \neq p$ then this can easily be shown using the methods in this paper.

The remainder of this paper is organized as follows. In the next section, we discuss some properties of the action of a dense subgroup of a compact group K on homogeneous K-spaces. We then state and prove a result due to Furman which implies that these actions exhibit some intrinsic rigidity. We shall pay particular attention to the Grassmann space consisting of all linear subspaces of $\mathbb{Q}_p^n$ together with its $\mathrm{SL}_n(\mathbb{Z})$-action. In the third section we shall state a superrigidity result of Ioana, and use it to establish the Borel incomparability of some natural equivalence relations on Grassmann space. In the last section, we explain how the isomorphism and quasi-isomorphism equivalence relations can be viewed as equivalence relations on Grassmann spaces, and use this together with the results of Section 3 to prove Theorems A and B.

2. Homogeneous spaces of compact groups

In this section, we give an introduction to homogeneous spaces of compact groups and affine maps between them. We then give the definition of ergodicity of a general measure-preserving action, and a characterization of ergodicity in the case of homogeneous spaces. Finally, we present two lemmas (due to Gefter and Furman), which loosely speaking imply that if Γ, Λ act ergodically on homogeneous spaces, then any conjugacy between these actions comes from an affine map.

If K is a compact group, then a *homogeneous K-space* is a standard Borel space X together with a transitive Borel action of K on X. If X is a homogeneous K-space, then X is isomorphic as a K-space to the left coset space K/L, where $L \leq K$ is the stabilizer of an arbitrary point $x \in X$. Hence, X admits a K-invariant *Haar measure*, namely the push-forward to X of the usual Haar measure on K.

For instance, let $\mathrm{Gr}_k(\mathbb{Q}_p^m)$ denote the *Grassmann space* of all k-dimensional subspaces of $\mathbb{Q}_p^m$. By [15, Proposition 6.1], the compact group $\mathrm{SL}_m(\mathbb{Z}_p)$ acts transitively on $\mathrm{Gr}_k(\mathbb{Q}_p^m)$, and it follows that $\mathrm{Gr}_k(\mathbb{Q}_p^m)$ is a homogeneous $\mathrm{SL}_m(\mathbb{Z}_p)$-space. For purely æsthetic reasons, we sometimes write $\mathrm{Gr}_1(\mathbb{Q}_p^m)$ instead of $\mathbb{P}(\mathbb{Q}_p^m)$.

Definition 2.1. For $i = 0, 1$, let K_i be a compact group and L_i a closed subgroup. A map $f: K_0/L_0 \to K_1/L_1$ between homogeneous spaces is said to be *affine* iff there exists a homomorphism $\Phi: K_0 \to K_1$ and $t \in K_1$ such that $f(kL_0) = \Phi(k)tL_1$ for almost all $k \in K_0$.

Affine maps are the natural morphisms between homogeneous spaces of compact groups. It is trivial to see that any affine map $f(kL_0) = \Phi(k)tL_1$ has the property that the pair (Φ, f) is a *homomorphism of permutation groups*, in the sense that $f(kx) = \Phi(k)f(x)$ for all $x \in K_0/L_0$. Lemmas 2.2 and 2.3, taken together, provide a very strong converse to this observation. First, we shall need to introduce the notion of ergodicity of a measure-preserving action.

Let Γ be a countable group acting on the standard Borel space X (which we denote by $\Gamma \curvearrowright X$), and suppose the action preserves a probability measure on X. Then the action $\Gamma \curvearrowright X$ is said to be *ergodic* iff every Γ-invariant measurable subset $A \subset X$ has either $\mu(A) = 0$ or $\mu(A) = 1$. We shall have more use for the following equivalent formulation of this property: $\Gamma \curvearrowright X$ is ergodic iff for every standard Borel space Y and every Γ-invariant Borel function $\beta: X \to Y$, we have that β is constant on a conull set.

For instance, if X is a homogeneous K-space and Γ is a countable subgroup of K, then Γ acts on X and preserves the Haar measure. It is easily seen that in this case, $\Gamma \curvearrowright X$ is ergodic iff Γ is dense in K. The statement and proof of Lemma 2.2 were extracted from [7, Theorem 3.3].

Lemma 2.2. *For $i = 0, 1$ let K_i/L_i be a homogeneous space for the compact group K_i, let $\Gamma_i < K_i$ be a countable dense subgroup, and suppose that*

$$(\phi, f): \Gamma_0 \curvearrowright K_0/L_0 \longrightarrow \Gamma_1 \curvearrowright K_1/L_1$$

is a homomorphism of permutation groups. If ϕ extends to a homomorphism $\Phi: K_0 \to K_1$, then after adjusting f on a set of measure zero, f is an affine map.

Proof. Following Gefter's argument, define the map $\beta: K_0 \to K_1/L_1$ by

$$\beta(k) := \Phi(k)^{-1} f(kL_0).$$

We first observe that β is Γ_0-invariant. Indeed, for $\gamma \in \Gamma_0$, we compute that

$$\begin{aligned}\beta(\gamma k) &= \Phi(\gamma k)^{-1} f(\gamma k L_0)\\ &= \Phi(k)^{-1}\Phi(\gamma)^{-1}\phi(\gamma) f(kL_0)\\ &= \Phi(k)^{-1} f(kL_0)\\ &= \beta(k).\end{aligned}$$

Now, since Γ_0 is a dense subgroup of K_0, the action $\Gamma_0 \curvearrowright K_0$ is ergodic. Hence, there exists $t \in K_1$ such that for almost every $k \in K_0$, we have that $\beta(k) = tL_1$. In other words, there exists a conull subset $K_0^* \subset K_0$ such that for all $k \in K_0^*$ we have the identity $f(kL_0) = \Phi(k)tL_1$. Now, we will be done if we show that the function $f'(kL_0) := \Phi(k)tL_1$ is well-defined, for then f' is an affine map which is equal to f almost everywhere.

For this, a moment's pause reveals that f' is well-defined if and only if $\Phi(L_0) = tL_1t^{-1}$. Now, given $\ell \in L_0$, choose $k \in K_0^*$ such that also $k\ell \in K_0^*$. (This is possible: the right Haar measure has the same null sets as the left Haar measure, so $K_0^*\ell^{-1}$ is non-null.) We now have

$$\begin{aligned}\Phi(k)tL_1 &= f(kL_0)\\ &= f(k\ell L_0)\\ &= \Phi(k\ell)tL_1\\ &= \Phi(k)\Phi(\ell)tL_1.\end{aligned}$$

It follows that $tL_1 = \Phi(\ell)tL_1$ and so $\Phi(\ell) \in tL_1t^{-1}$, which completes the proof. □

Although the proof Lemma 2.2 was a key point in [3], it will not be explicitly needed in this paper. However, it clearly goes hand in hand with Lemma 2.3, which will be used crucially in the next section. The statement and proof of Lemma 2.3 were easily adapted from [6, Proposition 7.2].

Lemma 2.3. *For $i = 0, 1$, let $\Gamma_i \curvearrowright K_i/L_i$ be as in Lemma 2.2. Suppose additionally that the action $K_1 \curvearrowright K_1/L_1$ has trivial kernel. Suppose that $\phi: \Gamma_0 \to \Gamma_1$ is a surjective homomorphism and that*

$$(\phi, f): \Gamma_0 \curvearrowright K_0/L_0 \longrightarrow \Gamma_1 \curvearrowright K_1/L_1$$

is a homomorphism of permutation groups. Then ϕ extends to a homomorphism $\Phi: K_0 \to K_1$.

Proof. We first observe that f is measure-preserving. Indeed, letting μ_i denote the Haar measure on K_i/L_i, since ϕ is surjective we have that $f_*\mu_0$ is Γ_1-invariant. Now, it is well-known that since Γ_1 is a dense subgroup of K_1, we not only have that $\Gamma_1 \curvearrowright K_1/L_1$ is ergodic but also that it is *uniquely ergodic*. Here, an action $\Lambda \curvearrowright Y$ is said to be uniquely ergodic iff there exists a unique Λ-invariant probability measure on Y. Clearly, it follows from this property that $f_*\mu_0 = \mu_1$, and so f is measure-preserving.

Now, let ν be the lift of μ_0 to the measure on $K_0/L_0 \times K_1/L_1$ concentrating on the graph of f. In other words, for $A \subset K_0/L_0 \times K_1/L_1$, let

$$\nu(A) := \mu_0 \{x \in K_0/L_0 \mid (x, f(x)) \in A\}.$$

Next, we let

$$R := \{(k_0, k_1) \in K_0 \times K_1 \mid (k_0, k_1)_*\nu = \nu\}.$$

It is easy to see that R is a closed (and hence compact) subgroup of $K_0 \times K_1$. Moreover, it follows from the fact that (ϕ, f) is a homomorphism of permutation groups that R contains the graph of ϕ. Hence, by the density of Γ_i in K_i, we have $\pi_i(R) = K_i$, where π_i is the canonical projection onto K_i.

Now, consider the normal subgroups

$$R_0 := \{k_0 \in K_0 \mid (k_0, e) \in R\} \lhd K_0, \text{ and}$$
$$R_1 := \{k_1 \in K_1 \mid (e, k_1) \in R\} \lhd K_1.$$

Then loosely speaking, R_1 measures how far R is from being the graph of a function. And if R is the graph of a function, then R_0 is the kernel of that function.

Claim. $R_1 = 1$, *and thus R is the graph of a function.*

Proof of claim. Let $k_1 \in R_1$ be arbitrary, so that $(e, k_1)_*\nu = \nu$. This means that for all measurable sets $A \subset K_0/L_0 \times K_1/L_1$, we have that $\nu((e, k_1)^{-1}A) = \nu(A)$. Appealing to the definition of ν, we have that

$$\mu_0 \{x \in K_0/L_0 \mid (x, k_1 f(x)) \in A\} = \mu_0 \{x \in K_0/L_0 \mid (x, f(x)) \in A\}.$$

Applying this in the case where A is the graph of f, it follows that

$$\mu_0 \{x \in K_0/L_0 \mid k_1 f(x) = f(x)\} = 1.$$

Since f is measure-preserving, we can conclude that

$$\mu_1 \{y \in K_1/L_1 \mid k_1 y = y\} = 1.$$

We have shown that k_1 fixes almost every point of K_1/L_1, and hence that $k_1 \in L_1$. Thus, we have that R_1 is a normal subgroup of K_1 which is contained in L_1. It follows that R_1 is contained in the kernel of the action of K_1 on K_i/L_i, which we have assumed is trivial. $\dashv$

Hence, R is the graph of a homomorphism $\Phi : K_0 \to K_1$, and since R contains the graph of ϕ, we have that Φ extends ϕ. $\square$

Let us make some further observations that help to explain the hypotheses of the last result, and which will be useful later on when we apply it.

Remark 2.4. If in Lemma 2.3 we add the symmetric hypotheses that ϕ is injective and that $K_0 \curvearrowright K_0/L_0$ has trivial kernel, then we can repeat the argument given in the Claim to show that $R_0 = 1$ and thus that Φ is injective.

Remark 2.5. When we apply Lemma 2.3, we will unfortunately be interested in the case where ϕ is *not* surjective. To deal with this, consider the action of just $\phi(\Gamma_0)$ on K_1/L_1. Since the map $x \mapsto \overline{\phi(\Gamma_0)}f(x)$ is Γ_0-invariant, we can use the ergodicity of $\Gamma_0 \curvearrowright K_0/L_0$ to suppose that $f(X)$ is contained in some $\overline{\phi(\Gamma_0)}$-orbit, say $\overline{\phi(\Gamma_0)}z$. Now, $\overline{\phi(\Gamma_0)}z$ is naturally a homogeneous space for $\overline{\phi(\Gamma_0)}$, and we may replace $\Gamma_1 \curvearrowright K_1/L_1$ with the action $\phi(\Gamma_0) \curvearrowright \overline{\phi(\Gamma_0)}z$. We may then apply Lemma 2.3 to the latter action.

3. Superrigidity and Grassmann spaces

The first goal of this section is to state a version of a superrigidity theorem from ergodic theory due to Adrian Ioana. The conclusion of this theorem is slightly technical, and so we'll start with the necessary definitions. Afterwards, we shall use Ioana's theorem to establish a template Borel incomparability result for the actions $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_p)$ as n and p vary. We conclude the section with Theorem 3.6, which is the key result of the paper. Theorem 3.6 provides a strong form of Borel incomparability for actions of $\mathrm{GL}_n(\mathbb{Q})$ on the p-adic Grassmann spaces.

Suppose that X is a standard Borel space, μ is a Borel probability measure on X, and $\Gamma \curvearrowright X$ is ergodic with respect to μ. If $\Lambda < \Gamma$ is an arbitrary subgroup, then of course Λ need not act ergodically on X. However, if Λ is a subgroup of finite index in Γ, then it is not difficult to see that there exists a Λ-invariant subset $Z \subset X$ of positive measure such that $\Lambda \curvearrowright Z$ is ergodic with respect to the restriction (and renormalization) of μ to Z. Generally, we say that $\Lambda \curvearrowright Z$ is an *ergodic component* for $\Gamma \curvearrowright X$ iff $\Lambda \leq \Gamma$ is a subgroup of finite index, $Z \subset X$ is a Λ-invariant subset of positive measure, and $\Lambda \curvearrowright Z$ is ergodic.

For example, suppose that the ergodic action $\Gamma \curvearrowright X$ has a finite factor, that is, a finite Γ-space X_0 together with a Γ-invariant and measure-preserving function $\pi : X \to X_0$. Then the stabilizer Λ_0 in Γ of any $x_0 \in X_0$ is a subgroup of Γ of finite index, and it is easy to check that $\Lambda_0 \curvearrowright \pi^{-1}(x_0)$ is an ergodic component for $\Gamma \curvearrowright X$.

Ioana's theorem is about profinite group actions; these actions are built up from their finite factors, and hence have a rich structure of ergodic components. Somewhat more precisely, if $\Gamma \curvearrowright X$ is a probability measure-preserving action, then we say that $\Gamma \curvearrowright X$ is *profinite* iff as a Γ-space, X is the inverse limit of a directed system of finite measure-preserving Γ-spaces. For example, the action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_p)$ is profinite; in this case $\mathrm{SL}_n(\mathbb{Z}_p)$ is the inverse limit of the sequence of $\mathrm{SL}_n(\mathbb{Z})$-spaces given by $\mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})$. Similarly, since $\mathrm{Gr}_k(\mathbb{Q}_p^n)$ is a transitive $\mathrm{SL}_n(\mathbb{Z}_p)$-space, it is not hard to see that $\mathrm{Gr}_k(\mathbb{Q}_p^n)$ also carries the structure of a profinite $\mathrm{SL}_n(\mathbb{Z})$-space. (In general, if $\Gamma \curvearrowright K$ is the inverse limit of $\Gamma \curvearrowright K/K_n$ and K acts transitively on X, then $\Gamma \curvearrowright X$ is the inverse limit of $\Gamma \curvearrowright X_n$, where X_n is the set of K_n orbits on X.)

The consequence of Ioana's superrigidity theorem which we will state will give conditions under which any Borel homomorphism

$$f : \Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y$$

comes from a homomorphism of permutation groups

$$(\phi, f) : \Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y.$$

Here if E, F are equivalence relations on X, Y, then a function $f : X \to Y$ is called a *Borel homomorphism* from E to F iff for all $x, x' \in X$,

$$x \mathrel{E} x' \implies f(x) \mathrel{F} f(x').$$

We have abused notation, so that any reference to Borel homomorphism between actions will always refer to a Borel homomorphism between their corresponding orbit equivalence relations.

We must remark that Ioana's theorem makes use of property (T), which we shall not define. It is sufficient for our purposes to note that $\mathrm{SL}_n(\mathbb{Z})$ has property (T) for $n \geq 3$. See [11] for the definition as well as a discussion of this key property.

Theorem 3.1 (*[9, Theorem 4.1]*). *Suppose that Γ is a countable discrete group with property (T), and let $\Gamma \curvearrowright X$ be a free, ergodic and profinite action. Let Λ be a countable group, $\Lambda \curvearrowright Y$ a free action, and suppose that f is a Borel homomorphism from E_Γ to E_Λ. Then there exists an ergodic component $\Gamma_0 \curvearrowright X_0$ for $\Gamma \curvearrowright X$ and a homomorphism of permutation groups*

$$(\phi, f') : \Gamma_0 \curvearrowright X_0 \longrightarrow \Lambda \curvearrowright Y$$

such that for all $x \in X_0$, we have that $f'(x) \mathrel{E_\Lambda} f(x)$.

In other words, under the hypotheses of Ioana's theorem, the Borel homomorphism f can be replaced by one which is more or less equivalent to f, and which moreover comes from a homomorphism of permutation groups. Ioana's theorem is stated in a significantly higher generality in [10]; for a proof of Theorem 3.1 from his result, see [3, Corollary 3.3]. We will now combine Theorem 3.1 together with Lemma 2.3 to obtain the following result. Although the statement of Theorem 3.2 will not be needed later on, the argument will be expanded upon during the proof of Theorem 3.6.

Theorem 3.2. *Suppose that $m, n \geq 3$ are natural numbers, and p, q are distinct primes. Then the orbit equivalence relation induced by the action $\mathrm{PSL}_m(\mathbb{Z}) \curvearrowright \mathrm{PSL}_m(\mathbb{Z}_p)$ is Borel incomparable with that induced by $\mathrm{PSL}_n(\mathbb{Z}) \curvearrowright \mathrm{PSL}_n(\mathbb{Z}_q)$.*

The fact that the orbit equivalence relation induced by $\mathrm{PSL}_m(\mathbb{Z}) \curvearrowright \mathrm{PSL}_m(\mathbb{Z}_p)$ is not Borel reducible to that induced by $\mathrm{PSL}_n(\mathbb{Z}) \curvearrowright \mathrm{PSL}_n(\mathbb{Z}_q)$ was essentially established by Thomas for $n < m$ in [14, Theorem 2.4] and for $n = m$ in [15]. The arguments in this section are almost entirely built upon his, but also apply in the case where $m < n$.

Proof. Suppose, towards a contradiction, that f is a Borel reduction from $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$ to $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_q)$. Then the hypotheses of Theorem 3.1 are satisfied, so there exists an ergodic component $\Gamma \curvearrowright X$ for the action $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$ and a homomorphism $\phi : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$ such that

$$(\phi, f) : \Gamma \curvearrowright X \longrightarrow \mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_q)$$

is a permutation group homomorphism. We now wish to apply Lemma 2.3, but at the moment the hypothesis that ϕ is surjective isn't satisfied. However, recall that by the remarks following Lemma 2.3, we can suppose that $f(X)$ is contained in some $\phi(\Gamma)$-orbit, say $\overline{\phi(\Gamma)}z$. We may now apply Lemma 2.3 to the permutation group homomorphism

$$(\phi, f) : \Gamma \curvearrowright X \longrightarrow \phi(\Gamma) \curvearrowright \overline{\phi(\Gamma)}z$$

to conclude that ϕ lifts to a homomorphism $\Phi : \bar{\Gamma} \to \mathrm{SL}_n(\mathbb{Z}_q)$, where $\bar{\Gamma}$ denotes the closure of Γ in $\mathrm{SL}_m(\mathbb{Z}_p)$. It will now suffice to argue that Φ is injective, for this clearly contradicts Proposition 3.3, below.

Indeed, if Φ is not injective, then by Margulis's theorem on normal subgroups [17, Theorem 8.1.2], either $\ker(\Phi)$ lies in the center of $\mathrm{SL}_n(\mathbb{Z}_p)$ or it has finite index in $\bar{\Gamma}$. In the case when $\ker(\Phi)$ is central, Φ clearly induces an injective homomorphism $\bar{\Gamma}' \to \mathrm{PSL}_n(\mathbb{Z}_q)$, where $\bar{\Gamma}'$ denotes the image of $\bar{\Gamma}$ in $\mathrm{PSL}_m(\mathbb{Z}_p)$. Once again, this clearly contradicts Proposition 3.3. Hence, we may suppose that $\Phi(\bar{\Gamma})$ is a finite subgroup of $\mathrm{SL}_m(\mathbb{Z}_q)$. Now, replacing $\Gamma \curvearrowright X$ with an ergodic subcomponent if necessary, we can suppose without loss of generality that $\Phi = 1$. This implies that f is Γ-invariant and since $\bar{\Gamma} \curvearrowright X$ is ergodic, f is almost constant. Hence, in this case f maps a conull set into a single $\mathrm{SL}_n(\mathbb{Z})$-orbit, which is impossible since f is countable-to-one. □

Proposition 3.3. *Let $m, n \geq 2$ be arbitrary and p, q be distinct primes. Then for any subgroup $K \leq \mathrm{SL}_m(\mathbb{Z}_p)$ of finite index, K does not embed into $\mathrm{SL}_n(\mathbb{Z}_q)$. Similarly, any subgroup $K \leq \mathrm{PSL}_m(\mathbb{Z}_p)$ of finite index does not embed into $\mathrm{PSL}_n(\mathbb{Z}_q)$.*

For the proof, recall that $\mathrm{SL}_m(\mathbb{Z}_p)$ is the inverse limit of the system of maps

$$\mathrm{pr}_k\colon\ \mathrm{SL}_m(\mathbb{Z}_p) \to \mathrm{SL}_m(\mathbb{Z}/p^k\mathbb{Z}),$$

where pr_k always stands for the natural surjection. We shall also use the fact that any subgroup of $\mathrm{SL}_m(\mathbb{Z}_p)$ of finite index contains some *principal congruence subgroup*, that is, a subgroup of the form $\ker(\mathrm{pr}_k)$. (This is not as difficult as some instances of the congruence subgroup problem. Rather, it follows from elementary properties of profinite and pro-p groups. See [16] for the general properties of profinite groups, and [4, Exercise 1.9] for this particular fact.)

Proof of Proposition 3.3. Passing to a finite index subgroup of K if necessary, we may suppose without loss of generality that K is a principal congruence subgroup of $\mathrm{SL}_m(\mathbb{Z}_p)$. We shall use the well-known fact that for all k there exists an i such that the size of $\mathrm{SL}_m(\mathbb{Z}/p^k\mathbb{Z})$ divides bp^i, where b is some constant depending only on m and p. It follows that if K' is any principal congruence subgroup of K then $[K : K']$ also divides some bp^i. The same reasoning applies to $\mathrm{SL}_n(\mathbb{Z}_q)$, and so there exists some $c \in \mathbb{N}$ with the analogous properties.

Now, suppose towards a contradiction that $\Phi\colon K \to \mathrm{SL}_n(\mathbb{Z}_q)$ is an injective homomorphism. For each k, let $N_k \leq K$ denote the kernel of the composition:

$$K \xrightarrow{\Phi} \mathrm{SL}_n(\mathbb{Z}_q) \xrightarrow{\mathrm{pr}_k} \mathrm{SL}_n(\mathbb{Z}/q^k\mathbb{Z}).$$

Then for each k, we have that K/N_k embeds into $\mathrm{SL}_n(\mathbb{Z}/q^k\mathbb{Z})$, and so there exists a j such that $[K : N_k]$ divides cq^j. On the other hand, N_k also contains a principal congruence subgroup, and so there exists an i such that $[K : N_k]$ divides bp^i. Now each $[K : N_k]$ divides both some cq^j and some bp^i, and it follows that the sequence of indices $[K : N_k]$ must be bounded.

Now, to reach a contradiction, we shall argue that $\bigcap N_k = 1$ and hence $[K : N_k]$ tends to infinity. Indeed, if $\gamma \in \bigcap N_k$ then $\gamma \in \ker(\mathrm{pr}_k \circ \Phi)$ for all k. Since Φ is injective, $\gamma \in \ker(\mathrm{pr}_k)$ for all k. Since $\mathrm{SL}_m(\mathbb{Z}_q)$ is precisely the inverse limit corresponding to the maps pr_k, it follows that $\gamma = 1$, which completes the proof. □

Remark 3.4. The same argument can be used to show that $\mathrm{SL}_n(\mathbb{Z}_p)$ does not even embed into any quotient of a closed subgroup of $\mathrm{SL}_m(\mathbb{Z}_q)$. To see this, one may check that such a group can again be expressed as an inverse limit of groups whose cardinalities are essentially powers of q (that is, dividing cq^i for some fixed c). This is precisely the property that was required in the proof.

Next, we shall adapt the argument of Proposition 3.3 to establish our key result. In order to express the result in the greatest generality, we will use the following strengthening of the notion of ergodicity.

Definition 3.5. Let $\Gamma \curvearrowright X$ be a probability measure-preserving action, and let F be an arbitrary equivalence relation on the standard Borel space Y. Then $\Gamma \curvearrowright X$ is said to be F-*ergodic* iff whenever $f : X \to Y$ is a Borel homomorphism from E_Γ to F, there exists a conull $A \subset X$ such that $f(X)$ is contained in a single F-class.

Recall that if $\Gamma \curvearrowright X$ is ergodic, then E_Γ is nonsmooth. We have similarly that if $\Gamma \curvearrowright X$ is F-ergodic, then $E_\Gamma \not\leq_B F$. Moreover, in this case, if E is any countable Borel equivalence relation such that $E_\Gamma \subset E$, then also $E \not\leq_B F$.

Theorem 3.6. *Suppose that $m, n \geq 3$ and $k \leq n$, and that p, q are distinct primes. Then $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ is F-ergodic, where F is the orbit equivalence relation induced by the action $\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{Gr}_k(\mathbb{Q}_q^n)$.*

Once again, this has already been established by Thomas for $n < m$ in [14, Theorem 2.4] and for $n = m$ in [12, Theorem 4.7].

Proof. Suppose that f is a Borel homomorphism from the orbit equivalence relation induced by $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ to that induced by $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright (\mathbb{P}\mathbb{Q}_p^m)$. We cannot immediately apply Theorem 3.1, since neither action is free. By [15, Lemma 6.2], the action $\mathrm{PSL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ is *almost free*, meaning that there exists a conull subset of $\mathbb{P}(\mathbb{Q}_p^m)$ on which $\mathrm{PSL}_m(\mathbb{Z})$ acts freely. Hence, we may restrict f to this set to satisfy the freeness condition on the left-hand side. On the other hand, for the right-hand side we must consider the *free part*:

$$Y := \left\{ y \in \mathrm{Gr}_k(\mathbb{Q}_q^n) \mid 1 \neq \gamma \in \mathrm{PGL}_n(\mathbb{Q}) \implies \gamma y \neq y \right\}.$$

If there exists a conull subset $X \subset \mathbb{P}(\mathbb{Q}_p^m)$ such that $f(X) \subset Y$, then we may apply Theorem 3.1 and we are done after repeating the argument from Theorem 3.2. Hence, since the action $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ is ergodic, we may suppose instead that there exists an invariant conull subset $X \subset \mathbb{P}(\mathbb{Q}_p^m)$ such that $f(X) \subset \mathrm{Gr}_k(\mathbb{Q}_q^n) \setminus Y$.

In this case, we will follow the argument found in [12, Lemma 5.1] to replace the target action $\mathrm{PGL}_n(\mathbb{Q}) \curvearrowright \mathrm{Gr}_k(\mathbb{Q}_q^n)$ with a closely related free action. For this argument, it is helpful to think of elements of $\mathrm{Gr}_k(\mathbb{Q}_q^n)$ as one-dimensional subspaces of the exterior power $\bigwedge^k \mathbb{Q}_q^n$. Here, if $y \in \mathrm{Gr}_k(\mathbb{Q}_q^n)$ is a k-dimensional subspace of $\mathbb{Q}_q^n$ with basis $v_1, \dots, v_k$, then we identify V with the linear subspace of $\bigwedge^k \mathbb{Q}_q^n$ spanned by the simple tensor $v_1 \wedge \cdots \wedge v_k$. The relations which hold in the exterior algebra then ensure that this identification is well-defined.

Now, for each $x \in X$, since $f(x) \notin Y$, we must have that $f(x)$ is contained in a proper eigenspace of some element of $\mathrm{GL}_n(\mathbb{Q})$. Notice that such eigenspaces are $\bar{\mathbb{Q}}$-*subspaces* of $\bigwedge^k \mathbb{Q}_q^n$, where E is said to be a $\bar{\mathbb{Q}}$-subspace iff there exists a basis for E which consists only of vectors over the algebraic closure $\bar{\mathbb{Q}}$ of the rationals. Hence, for $x \in X$ we may let E_x denote a minimal $\bar{\mathbb{Q}}$-subspace of $\bigwedge^k \mathbb{Q}_q^n$ such that $f(x) \leq E_x$. Since there are only countably many possibilities for E_x, by the ergodicity of $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright X$ we may suppose that there exists a fixed $\bar{\mathbb{Q}}$-subspace V such that $E_x = V$ for all $x \in X$. Let H denote the group of projective linear transformations induced on V by elements of the setwise stabilizer $\mathrm{PGL}_n(\mathbb{Q})_{\{V\}}$ of V in $\mathrm{PGL}_n(\mathbb{Q})$. Then it is easily checked using the minimality of V that H acts freely on $\mathbb{P}(V)$.

Now, let d denote the dimension of V and regard V as the vector space $\mathbb{Q}_q^d$, so that H corresponds to a subgroup of $\mathrm{PGL}_d(\bar{\mathbb{Q}} \cap \mathbb{Q}_q)$. Then we may regard f as a Borel homomorphism from the orbit equivalence relation induced by $\mathrm{PSL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ to that induced by $H \curvearrowright \mathbb{P}(\mathbb{Q}_p^d)$. Since the action of H on $\mathbb{P}(\mathbb{Q}_q^d)$ is free, we may now apply Theorem 3.1. Hence, we may suppose that there exists an ergodic component $\Gamma \curvearrowright X$ for $\mathrm{PSL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m)$ and a homomorphism $\phi\colon \Gamma \to H$ such that

$$(\phi, f)\colon \Gamma \curvearrowright X \longrightarrow H \curvearrowright \mathbb{P}(\mathbb{Q}_q^d)$$

is a homomorphism of permutation groups.

Now, since Γ has property (T) (see [11, Theorem 1.5]), it is in particular finitely generated (see [11, Proposition 1.24]). Hence, $\phi(\Gamma)$ is finitely generated, and it follows that H is contained in some $\mathrm{PGL}_d(F)$, where $F \leq \bar{\mathbb{Q}} \cap \mathbb{Q}_q$ is a finite field extension of $\mathbb{Q}$. Moreover, the commutator subgroup $\Gamma' := [\Gamma, \Gamma]$ is a finite index subgroup of Γ (see [11, Corollary 1.29]). Since $\mathrm{PGL}_d(F)/\mathrm{PSL}_d(F) \cong F^\times$ is abelian, we have that

$$\phi(\Gamma') \leq [\mathrm{PGL}_d(F), \mathrm{PGL}_d(F)] \leq \mathrm{PSL}_d(F).$$

(Actually, the latter inequality is an equality.) Hence, replacing $\Gamma \curvearrowright X$ with an ergodic component for the action of Γ' if necessary, we may suppose without loss of generality that $\phi(\Gamma) \subset \mathrm{PSL}_d(F)$.

Claim. *We can suppose without loss of generality that $\phi(\Gamma) \subset \mathrm{PSL}_d(\mathcal{O}_F)$, where $\mathcal{O}_F$ denotes the ring of integers of F.*

Proof of claim. Recall that an element $x \in F$ lies in the ring of integers $\mathcal{O}_F$ if and only if $v(x) \geq 0$ for every nonarchimedian valuation v on F. More generally, if S is a set of valuations on F, then we say that $x \in F$ is an S-*integer* iff $v(x) \geq 0$ for all nonarchimedian valuations $v \notin S$. We denote the ring of S-integers of F by $F(S)$, so in particular the notation implies that $\mathcal{O}_F = F(\emptyset)$.

Now, note that F is the union of the rings $F(S)$ as S varies over all finite sets of valuations on F. Therefore, using the fact that $\phi(\Gamma)$ is finitely generated, there exists a finite set S of valuations on F such that

$$\phi(\Gamma) \subset \mathrm{SL}_d(F(S)). \tag{3.7}$$

Next, for any valuation v on F, let F_v denote the completion of F with respect to v, and $\mathcal{O}_v$ the ring of integers of F_v. It is clear from the definitions that we have

$$\mathrm{PSL}_d(\mathcal{O}_F) = \mathrm{PSL}_d(F(S)) \cap \bigcap_{v \in S} \mathrm{PSL}_d(\mathcal{O}_v). \tag{3.8}$$

By [2, Theorem VII.5.16], for each nonarchimedian valuation v on F, $\phi(\Gamma)$ is relatively compact in $\mathrm{SL}_d(F_v)$. (To see that the hypotheses of [2, Theorem VII.5.16] are satisfied, note that by [2, Theorem VIII.3.10], the Zariski closure in $\mathrm{PSL}_d(F_v)$ of $\phi(\Gamma)$ is semisimple.) Since $\mathrm{PSL}_d(\mathcal{O}_v)$ is an open subgroup of $\mathrm{PSL}_d(F_v)$, we have that $\phi(\Gamma) \cap \mathrm{PSL}_d(\mathcal{O}_v)$ is of finite index in $\phi(\Gamma)$. Since S is finite, it follows that

$$\phi(\Gamma) \cap \bigcap_{v \in S} \mathrm{PSL}_d(\mathcal{O}_v)$$

is also of finite index in $\phi(\Gamma)$. This, together with Eqs. (3.7) and (3.8), implies that $\phi(\Gamma) \cap \mathrm{PSL}_d(\mathcal{O}_F)$ has finite index in $\phi(\Gamma)$. Thus, replacing Γ with a subgroup of finite index establishes the claim. $\dashv$

Now, recall that $F \subset \mathbb{Q}_q$, and it follows that $\mathcal{O}_F \subset \mathbb{Z}_q$. (Indeed, F carries a q-adic valuation and so each $x \in \mathcal{O}_F$ has $v_q(x) \geq 0$.) Combining this with the Claim, we have that $\phi(\Gamma) \subset \mathrm{PSL}_d(\mathbb{Z}_q)$. For the remainder of the proof, let K_0 denote the closure of Γ in $\mathrm{SL}_m(\mathbb{Z}_p)$ and let K_1 denote the closure of $\phi(\Gamma)$ in $\mathrm{PSL}_d(\mathbb{Z}_q)$. Roughly speaking, we now wish to maneuver into a situation where we can apply Lemma 2.3 to the permutation group homomorphism

$$(\phi, f)\colon \Gamma \curvearrowright X \longrightarrow \phi(\Gamma) \curvearrowright \mathbb{P}(\mathbb{Q}_q^d)$$

to obtain an embedding of K_0 into K_1, which would be a contradiction. First, by the remarks following Lemma 2.3, we can suppose that $f(X)$ is contained in a single K_1-orbit, say K_1z. We would like to apply Lemma 2.3 to the permutation group homomorphism

$$(\phi, f)\colon \Gamma \curvearrowright X \longrightarrow \phi(\Gamma) \curvearrowright K_1 z,$$

but it is not necessarily the case that K_1 acts faithfully on K_1z. However, if there is a kernel $N \trianglelefteq K_1$ for this action, then K_1z is naturally a homogeneous K_1/N-space. Composing (ϕ, f) with the obvious factor map, we may now apply Lemma 2.3 to obtain a homomorphism $\Phi : K_0 \to K_1/N$. Arguing as in the proof of Theorem 3.2 we can suppose that Φ is injective, but this contradicts the remark following Proposition 3.3. □

4. Torsion-free abelian groups

In this section, we shall use Theorem 3.6 to prove Theorems A and B. In order to do so, we must first show that the isomorphism equivalence relations on spaces of local torsion-free abelian groups are in fact very closely related to orbit equivalence relations on Grassmann spaces. For this, we shall rely on some methods of Hjorth, Thomas and myself which ultimately make use of the Kurosh–Malcev invariants for torsion-free abelian groups of finite rank.

Recall that $\sim_{m,p}$ denotes the quasi-isomorphism relation on the space of p-local torsion-free abelian groups of rank m. The following result is a straightforward application of the Kurosh–Malcev p-adic localization technique.

Lemma 4.1 ([12, Theorem 4.3]). *The quasi-isomorphism relation $\sim_{m,p}$ is Borel bireducible with the orbit equivalence relation induced by the action of $\mathrm{GL}_n(\mathbb{Q})$ on the full Grassmann space $\mathrm{Gr}(\mathbb{Q}_p^m)$ of all vector subspaces of $\mathbb{Q}_p^m$.*

Of course, the full Grassmann space decomposes naturally into the invariant components $\mathrm{Gr}_k(\mathbb{Q}_p^m)$, for $k = 0, \dots, n$.

Corollary 4.2 (*Theorem A*). *If $m, n \geq 3$ and p, q are distinct primes, then $\sim_{m,p}$ is Borel incomparable with $\sim_{n,q}$.*

Proof. Suppose that there exists a Borel reduction from $\sim_{m,p}$ to $\sim_{n,q}$. Then by Lemma 4.1 there exists a Borel reduction

$$f\colon \mathrm{GL}_m(\mathbb{Q}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_p^m) \longrightarrow \mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_q^n).$$

Now, consider the restriction of f to $\mathbb{P}(\mathbb{Q}_p^m)$. Since each $Gr_k(\mathbb{Q}_q^n)$ is $\mathrm{GL}_n(\mathbb{Q})$-invariant, by the ergodicity of $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_q^n)$, we can adjust f on a null set to suppose that f takes values in $\mathrm{Gr}_k(\mathbb{Q}_q^n)$ for some fixed k. Therefore, f is a Borel homomorphism

$$f\colon \mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^m) \longrightarrow \mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{Gr}_k(\mathbb{Q}_q^n).$$

By Theorem 3.6, the image $f\left(\mathbb{P}(\mathbb{Q}_p^m)\right)$ is a countable set, which is impossible since f is a countable-to-one function. □

The proof of Theorem B is nearly identical, modulo the following rather technical piece of machinery.

Lemma 4.3 ([3, Lemma 4.1]). *The isomorphism relation $\cong_{m,p}$ is Borel bireducible with an equivalence relation $\cong'_{m,p}$ which, thought of as a set of pairs, lies properly between the orbit equivalence relations induced by the actions $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_p^m)$ and $\mathrm{GL}_m(\mathbb{Q}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_p^m)$.*

Corollary 4.4 (*Theorem B*). *If $m, n \geq 3$ and p, q are distinct primes, then $\cong_{m,p}$ is Borel incomparable with $\cong_{n,q}$.*

Proof. If there exists a Borel reduction from $\cong_{m,p}$ to $\cong_{n,q}$, then there exists a Borel reduction f from $\cong'_{m,p}$ to $\cong'_{n,q}$. It follows from the containments described in Lemma 4.3 that f is also a Borel homomorphism:

$$f\colon \mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_p^m) \longrightarrow \mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{Gr}(\mathbb{Q}_q^n).$$

Arguing as in the proof of Corollary 4.2, we again arrive at a contradiction. □

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