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Algebraic independence of arithmetic gamma values and Carlitz zeta values ☆

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Abstract

We consider the values at proper fractions of the arithmetic gamma function and the values at positive integers of the zeta function for $\mathbb{F}_q[\theta]$ and provide complete algebraic independence results for them.

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1. Introduction

In this paper, we determine all of the algebraic relations among special values of two important functions in function field arithmetic, namely the arithmetic gamma function and the zeta function associated to $\mathbb{F}_q[\theta]$. For more background on function field arithmetic and on the properties of these functions, we refer to [14,20] and references there. We give only a brief introduction that is relevant here.

Let us write $A = \mathbb{F}_q[\theta]$, $k = \mathbb{F}_q(\theta)$, $k_\infty = \mathbb{F}_q((1/\theta))$ for the completion of k at its usual infinite place, and $\mathbb{C}_\infty$ for the completion of an algebraic closure of k_∞. It is well known that these are good analogues of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ respectively. Let A_+ denote the set of monic polynomials in A. This is considered to be an analogue of the set $\mathbb{N}$ of positive integers.

By (Carlitz) zeta values, we mean

$$\zeta_C(s) = \sum_{a \in A_+} \frac{1}{a^s} \in k_\infty, \quad s \in \mathbb{N}.$$

These values, first considered by Carlitz, are the special values at positive integers of the zeta function studied by Goss.

Let

$$D_n := \prod_{i=0}^{n-1} \bigl(\theta^{q^n} - \theta^{q^i}\bigr), \qquad \overline{D_n} := D_n / \bigl(\theta^{\deg D_n}\bigr).$$

The Carlitz factorial of n is defined to be $\prod D_i^{n_i} \in \mathbb{F}_q[\theta]$ for $n = \sum n_i q^i \in \mathbb{N}$, $0 \leqslant n_i < q$, and the interpolation of its unit part for $n \in \mathbb{Z}_p$, due to Goss [12], is

$$n! := \prod \overline{D_i}^{n_i} \in k_\infty \quad \text{for } n = \sum n_i q^i,\ 0 \leqslant n_i < q.$$

By (arithmetic or Carlitz–Goss) gamma values, we mean values at proper fractions of this function.

Our goal in this paper is to determine completely the algebraic relations among gamma and zeta values. We mention that there are many parallels (see [14] and [20]) between known facts about classical zeta and gamma values and their function field counterparts, such as Euler's theorem, prime factorization, interpolations at all finite primes, functional equations, Gross–Koblitz formulas, and Chowla–Selberg formulas. In fact, for $\mathbb{F}_q[\theta]$-arithmetic, there are two types of gamma functions, the arithmetic one above dealing with the cyclotomic theory of usual roots of unity corresponding to constant field extensions and the geometric one dealing with the cyclotomic theory of Carlitz–Drinfeld cyclotomic extensions. For a unified treatment of these two together with the classical gamma function, see [20, Sec. 4.12].

Very briefly, the development of the special value theory is the following.

For the classical gamma function the only gamma values (at proper fractions) known to be transcendental have denominators 2, 4 or 6, with $\Gamma(1/2) = \sqrt{\pi}$, and with the ones with denominators 4 and 6 related to periods of elliptic curves with complex multiplications by Gaussian or Eisenstein integers, via the Chowla–Selberg formula. There is an algebraic independence result concerning these values due to Chudnovsky. (The beta value theory is much better developed by results of Wolfart and Wüstholz.)

The third author (in his 1987 thesis, see [18]) proved analogues of the first formula (with $2\pi i$ replaced by $\tilde{\pi}$, the period of Carlitz module) and of the Chowla–Selberg formula for the arithmetic gamma function, resulting in a parallel transcendence and independence statement by results of the fourth author and Thiery on transcendence of periods. In [19], using automata methods, the transcendence of gamma values with any denominator, but with some restrictions on numerators, was established. These restrictions were then removed by Allouche [1]. Mendès France and Yao [15] gave an easier automata proof and finally the third author completely determined all transcendental monomials in gamma values. (See [20, Sec. 11.3]; we also take this opportunity to correct 'turns out to be a trivial monomial' in the last but one paragraph of p. 349, by adding 'after translating by integers appropriately to apply Theorem 4.6.4' as explained on p. 351.) But the more general algebraic dependence question remained.

As for the geometric gamma function, in [18], some very special values were related to the periods of Drinfeld modules, thus establishing their transcendence by the results of Wade and the fourth author. Anderson, Sinha, Brownawell and the second author [3,5,16,17] connected all values at proper fractions of geometric gamma to periods and quasi-periods of certain t-motives of Anderson [2] (generalizations of Drinfeld modules) and at the same time provided strong transcendence tools to completely determine [3] all algebraic relations among them. This paper does the same for the arithmetic gamma function, by using these new tools and connecting these values to the periods and quasi-periods of appropriate t-motives (see Section 3.3), thus bypassing the automata theory.

The main result of the present paper (Theorem 4.2.2) considers arithmetic gamma values and zeta values simultaneously and furthermore determines all algebraic relations among them. The earlier results of the fourth author [21] on transcendence of zeta values, already surpassing the parallel classical results, have recently been further improved [8,10] to complete algebraic independence results for zeta values. Here we show that these techniques generalize to give algebraic independence of both arithmetic gamma and zeta values together.

We briefly mention some additional avenues of research one can now pursue in light of Theorem 4.2.2. In [8], specific techniques inspired from [6] are introduced to deal with varying q, i.e. to obtain algebraic independence results for zeta values at positive integers with varying constant fields. This method certainly can also work for gamma values, in particular to determine all algebraic relations among special arithmetic gamma values as the constant fields vary in the same characteristic. This more complicated question will not be treated here however. We note that in another paper [9], the first, second, and fourth author have also established algebraic independence of geometric gamma and zeta values taken together. We leave the question of algebraic independence of arithmetic gamma and geometric gamma values taken together to a later work. Note that for the special case $q = 2$, the geometric gamma values in question are algebraic multiples of $\tilde{\pi}$ and the zeta value $\zeta_C(n)$ is a rational multiple of $\tilde{\pi}^n$ for each $n \in \mathbb{N}$, and so the present paper covers the algebraic relations of all three together completely in this case.

Overall the present results bring the special value theory of the arithmetic gamma function and zeta function for $A = \mathbb{F}_q[\theta]$ to a very satisfactory state; however, similar questions for (i) v-adic interpolations (see [20, Sec. 11.3] for very partial results about v-adic gamma values using automata methods and [21] for transcendence of v-adic zeta values), (ii) generalizations to other rings 'A' in the setting of Drinfeld modules [20, 4.5, 8.3], (iii) values of the two variable gamma function of Goss [20, 4.12], [18, Sec. 8], [13], are still open.

A main tool here for proving algebraic independence is a theorem of the second author [16], which is a function field version of Grothendieck's conjecture on periods of abelian varieties. The t-motives related to special arithmetic gamma values have "complex multiplication" by constant field extensions. This fact enables us to show that the associated Galois group by Tannakian duality is the Weil restriction of scalars of $\mathbb{G}_m$ from the constant field extension in question, hence a torus. On the other hand, according to Chang and Yu [10], Galois groups of the t-motives related to Carlitz zeta values are always extensions of $\mathbb{G}_m$ by vector groups. The Galois group of the direct sum of these two types of t-motives can be shown to be an extension of a torus by a vector group. Adding the dimension of the torus with that of the vector group in question proves the desired algebraic independence result. Thus, the story of arithmetic gamma values unfolds through t-motives having arithmetic (cyclotomic) CM in this paper, just as t-motives having geometric (cyclotomic) CM provide the proper setting for special geometric gamma values [3,9].

For the rest of the paper, we will only consider arithmetic gamma values and Carlitz zeta values as defined above.

2. t-Motives and periods

2.1. *Notations*

2.1.1. *Table of symbols*

$\mathbb{F}_q :=$ the finite field of q elements, q a power of a prime number p.
$\theta, t :=$ independent variables.
$A := \mathbb{F}_q[\theta]$, the polynomial ring in the variable θ over $\mathbb{F}_q$.
$A_+ :=$ the set of monic polynomials of A.
$k := \mathbb{F}_q(\theta)$, the fraction field of A.
$k_\infty := \mathbb{F}_q((\frac{1}{\theta}))$, the completion of k with respect to the place at infinity.
$\overline{k_\infty} :=$ a fixed algebraic closure of k_∞.
$\bar{k} :=$ the algebraic closure of k in $\overline{k_\infty}$.
$\mathbb{C}_\infty :=$ the completion of $\overline{k_\infty}$ with respect to canonical extension of the place at infinity.
$|\cdot|_\infty :=$ a fixed absolute value for the completed field $\mathbb{C}_\infty$.
$\mathbb{T} := \{f \in \mathbb{C}_\infty[\![t]\!]\colon\ f$ converges for $|t|_\infty \leqslant 1\}$, the Tate algebra.
$\mathbb{L} :=$ the fraction field of $\mathbb{T}$.
$\mathbb{G}_a :=$ the additive group.
$\mathrm{GL}_r/F :=$ for a field F, the F-group scheme of invertible rank r matrices.
$\mathbb{G}_m := \mathrm{GL}_1$, the multiplicative group.

2.1.2. *Twisting*

For $n \in \mathbb{Z}$, given $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$, define the twist of f by $\sigma^n(f) := f^{(-n)} = \sum_i a_i^{q^{-n}} t^i$. The twisting operation is an automorphism of the Laurent series field $\mathbb{C}_\infty((t))$ that stabilizes several subrings, e.g., $\bar{k}[\![t]\!]$, $\bar{k}[t]$, and $\mathbb{T}$. More generally, for any matrix B with entries in $\mathbb{C}_\infty((t))$ we define $B^{(-n)}$ by the rule ${B^{(-n)}}_{ij} = {B_{ij}}^{(-n)}$.

2.1.3. *Entire power series*

A power series $f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[\![t]\!]$ that satisfies

$$\lim_{i\to\infty} \sqrt[i]{|a_i|_\infty} = 0$$

and

$$\left[k_\infty(a_0, a_1, a_2, \dots)\colon k_\infty\right] < \infty$$

is called an entire power series. As a function of t, such a power series f converges on all $\mathbb{C}_\infty$ and, when restricted to $\overline{k_\infty}$, f takes values in $\overline{k_\infty}$. The ring of the entire power series is denoted by $\mathbb{E}$.

2.2. Tannakian categories and difference Galois groups

We follow [16] in working with the Tannakian category of t-motives and the Galois theory of Frobenius difference equations, which is analogous to classical differential Galois theory. In this subsection, we fix a positive integer ℓ and let $\bar{\sigma} := \sigma^\ell$.

Let $\bar{k}(t)[\bar{\sigma}, \bar{\sigma}^{-1}]$ be the noncommutative ring of Laurent polynomials in $\bar{\sigma}$ with coefficients in $\bar{k}(t)$, subject to the relation

$$\bar{\sigma} f := f^{(-\ell)}\bar{\sigma} \quad \text{for all } f \in \bar{k}(t).$$

A pre-t-motive M is a left $\bar{k}(t)[\bar{\sigma}, \bar{\sigma}^{-1}]$-module which is finite-dimensional over $\bar{k}(t)$. Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ be a $\bar{k}(t)$-basis of M. Multiplication by $\bar{\sigma}$ on M is represented by $\bar{\sigma}(\mathbf{m}) = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$. Furthermore, M is called rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_r(\mathbb{L})$ such that $\bar{\sigma}(\Psi) = \Phi\Psi$. Such a matrix Ψ is called a rigid analytic trivialization of the matrix Φ.

The category of pre-t-motives forms an abelian $\mathbb{F}_{q^\ell}(t)$-linear tensor category. Moreover, the category $\mathcal{R}$ of rigid analytically trivial pre-t-motives forms a neutral Tannakian category over $\mathbb{F}_{q^\ell}(t)$ (for the definition of neutral Tannakian categories, see [11, Chap. II]). For each $M \in \mathcal{R}$, Tannakian duality asserts that the smallest Tannakian subcategory of $\mathcal{R}$ containing M is equivalent to the finite-dimensional representations of some affine algebraic group scheme Γ_M over $\mathbb{F}_{q^\ell}(t)$.

The group Γ_M is called the Galois group of M and it can be described explicitly as follows. Let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ be the matrix providing multiplication by $\bar{\sigma}$ on a rigid analytically trivial pre-t-motive M with rigid analytic trivialization $\Psi \in \mathrm{GL}_r(\mathbb{L})$. Let $X := (X_{ij})$ be an $r \times r$ matrix whose entries are independent variables X_{ij}, and define a $\bar{k}(t)$-algebra homomorphism $\nu : \bar{k}(t)[X, 1/\det X] \to \mathbb{L}$ so that $\nu(X_{ij}) = \Psi_{ij}$. We let

$$\Sigma_\Psi := \operatorname{im}\nu = \bar{k}(t)[\Psi, 1/\det\Psi] \subseteq \mathbb{L},$$
$$Z_\Psi := \operatorname{Spec}\Sigma_\Psi.$$

Then Z_Ψ is a closed $\bar{k}(t)$-sub-scheme of $\mathrm{GL}_r/\bar{k}(t)$. Let

$$\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$$

be the matrices satisfying $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2$. Now define an $\mathbb{F}_{q^\ell}(t)$-algebra homomorphism

$$\mu : \mathbb{F}_{q^\ell}(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$$

so that $\mu(X_{ij}) = \widetilde{\Psi}_{ij}$. Let

$$\begin{aligned} \Delta &:= \operatorname{im}\mu, \\ \Gamma_\Psi &:= \operatorname{Spec}\Delta. \end{aligned} \tag{1}$$

By Theorems 4.2.11, 4.3.1, and 4.5.10 of [16], we obtain the following theorem.

Theorem 2.2.1. *(See Papanikolas [16].) The scheme Γ_Ψ is a closed $\mathbb{F}_{q^\ell}(t)$-subgroup scheme of $\mathrm{GL}_r/\mathbb{F}_{q^\ell}(t)$, which is isomorphic to the Galois group Γ_M over $\mathbb{F}_{q^\ell}(t)$. Moreover Γ_Ψ has the following properties:*

(a) *Γ_Ψ is smooth over $\overline{\mathbb{F}_q(t)}$ and geometrically connected.*
(b) *$\dim \Gamma_\Psi = \operatorname{tr.deg}_{\bar{k}(t)} \bar{k}(t)(\Psi)$.*
(c) *Z_Ψ is a Γ_Ψ-torsor over $\bar{k}(t)$.*

We call Γ_Ψ the Galois group of the functional equation $\bar{\sigma}\Psi = \Phi\Psi$. Here we note that Γ_Ψ can be regarded as a linear algebraic group over $\mathbb{F}_{q^\ell}(t)$ because of Theorem 2.2.1(a).

Finally, we review the definition of t-motives and the main theorem of [16]. Let $\bar{k}[t,\bar{\sigma}]$ be the noncommutative subring of $\bar{k}(t)[\bar{\sigma},\bar{\sigma}^{-1}]$ generated by t and $\bar{\sigma}$ over $\bar{k}$. An Anderson t-motive (cf. [2,3]) is a left $\bar{k}[t,\bar{\sigma}]$-module $\mathcal{M}$ which is free and finitely generated both as left $\bar{k}[t]$-module and left $\bar{k}[\bar{\sigma}]$-module and which satisfies

$$(t-\theta)^N \mathcal{M} \subseteq \bar{\sigma}\mathcal{M},$$

for $N \in \mathbb{N}$ sufficiently large. Let $\mathbf{m}$ be a $\bar{k}[t]$-basis of $\mathcal{M}$. Multiplication by $\bar{\sigma}$ on $\mathcal{M}$ is represented by $\bar{\sigma}\mathbf{m} = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{Mat}_r(\bar{k}[t]) \cap \mathrm{GL}_r(\bar{k}(t))$. By tensoring $\mathcal{M}$ with $\bar{k}(t)$ over $\bar{k}[t]$, $\mathcal{M}$ corresponds to a pre-t-motive $\bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}$ given by

$$\bar{\sigma}(f\otimes m) := f^{(-\ell)} \otimes \bar{\sigma}m, \quad f \in \bar{k}(t),\ m \in \mathcal{M}.$$

Furthermore, $\mathcal{M}$ is called rigid analytically trivial if there exists $\Psi \in \mathrm{Mat}_r(\mathbb{T}) \cap \mathrm{GL}_r(\mathbb{L})$ so that $\bar{\sigma}\Psi := \Psi^{(-\ell)} = \Phi\Psi$. Rigid analytically trivial pre-t-motives that can be constructed from Anderson t-motives using direct sums, subquotients, tensor products, duals and internal Hom's, are called t-motives. These t-motives form a neutral Tannakian category over $\mathbb{F}_{q^\ell}(t)$. For a t-motive M with rigid analytic trivialization Ψ, $\Psi(\theta)^{-1}$ is called a period matrix of M. The fundamental theorem of [16] can be stated as follows:

Theorem 2.2.2. *(See Papanikolas [16].) Let M be a t-motive with Galois group Γ_M. Suppose that $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$ represents multiplication by $\bar{\sigma}$ on M and that $\det\Phi = c(t-\theta)^s$, $c \in \bar{k}^\times$. Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{L}) \cap \mathrm{Mat}_r(\mathbb{E})$ and let $\bar{k}(\Psi(\theta))$ be the field generated by the entries of $\Psi(\theta)$ over $\bar{k}$. Then*

$$\dim \Gamma_M = \operatorname{tr.deg}_{\bar{k}} \bar{k}\big(\Psi(\theta)\big).$$

3. Arithmetic gamma values and t-motives

3.1. Basic properties of arithmetic gamma values

We are interested in the special values $r! \in k_\infty$ for $r \in \mathbb{Q} \cap \mathbb{Z}_p$ (see Section 1). We see from the definition that $r! \in k$ for a nonnegative integer r. For r a negative integer, $r!$ is a $k^\times$-multiple of $\tilde{\pi} := \tilde{\pi}_1$ (defined in Section 3.2) [18, p. 34], and it is thus transcendental over k. Moreover, for $r \in \mathbb{Q} \cap (\mathbb{Z}_p \setminus \mathbb{Z})$, $r!$ depends up to multiplication by a factor in $\bar{k}$ only on r modulo $\mathbb{Z}$ [18]. Hence, without loss of generality we focus on those $r!$ with $-1 < r < 0$.

Given such an r, write $r = \frac{a}{b}$, where a and b are integers and b is not divisible by p. By Fermat's little theorem we see that b divides $q^\ell - 1$ for some $\ell \in \mathbb{N}$. Hence r can be written in the form

$$r = \frac{c}{1-q^\ell} \quad \text{for some } 0 < c < q^\ell - 1.$$

Write $c = \sum_{i=0}^{\ell-1} c_i q^i$, with $0 \leqslant c_i < q$. By the definition of $r!$ we see that

$$r! = \prod_{i=0}^{\ell-1} \left(\frac{q^i}{1-q^\ell}\right)!^{c_i}. \tag{2}$$

Hence, to determine all the algebraic relations among

$$\left\{\left(\frac{1}{1-q^\ell}\right)!, \left(\frac{2}{1-q^\ell}\right)!, \dots, \left(\frac{q^\ell-2}{1-q^\ell}\right)!\right\},$$

we need only to concentrate on these ℓ values

$$\left\{\left(\frac{1}{1-q^\ell}\right)!, \left(\frac{q}{1-q^\ell}\right)!, \dots, \left(\frac{q^{\ell-1}}{1-q^\ell}\right)!\right\}.$$

3.2. The Carlitz $\mathbb{F}_{q^\ell}[t]$-module and its Galois group

For a fixed positive integer ℓ, we recall the Carlitz $\mathbb{F}_{q^\ell}[t]$-module, denoted by C_ℓ, which is given by the $\mathbb{F}_{q^\ell}$-linear ring homomorphism

$$\begin{aligned} C_\ell : \mathbb{F}_{q^\ell}[t] &\to \operatorname{End}_{\mathbb{F}_{q^\ell}}(\mathbb{G}_a), \\ t &\mapsto \left(x \mapsto \theta x + x^{q^\ell}\right). \end{aligned}$$

One has the Carlitz exponential,

$$\exp_{C_\ell}(z) = \sum_{h=0}^{\infty} \frac{z^{q^h}}{D_h} = z \prod_{0 \neq a \in \mathbb{F}_{q^\ell}[\theta]} \left(1 - \frac{z}{a\tilde{\pi}_\ell}\right),$$

where

$$\tilde{\pi}_\ell := \theta(-\theta)^{\frac{1}{q^\ell-1}} \prod_{i=1}^{\infty}\left(1-\frac{\theta}{\theta^{q^{\ell i}}}\right)^{-1} \tag{3}$$

is a fundamental period of C_ℓ over $\mathbb{F}_{q^\ell}[t]$. Throughout this paper we fix a choice of $(-\theta)^{\frac{1}{q^\ell-1}}$ so that $\tilde{\pi}_\ell$ is a well-defined element in $\overline{k_\infty}$. We also choose these roots in a compatible way so that when $\ell|\ell'$ the number $(-\theta)^{\frac{1}{q^\ell-1}}$ is a power of $(-\theta)^{\frac{1}{q^{\ell'}-1}}$.

We can regard C_ℓ also as a Drinfeld $\mathbb{F}_q[t]$-module, and then it is of rank ℓ with complex multiplication by $\mathbb{F}_{q^\ell}[t]$ (see [14] and [20]). There is a canonical t-motive associated to this Drinfeld $\mathbb{F}_q[t]$-module C_ℓ, which we denote by M_ℓ. Its construction is given below (cf. [7, Sec. 2.4]).

Define $\Phi_\ell := (t-\theta)$ if $\ell = 1$, and otherwise let

$$\Phi_\ell := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{GL}_\ell\big(\bar{k}(t)\big) \cap \mathrm{Mat}_\ell\big(\bar{k}[t]\big).$$

Let ξ_ℓ be a primitive element of $\mathbb{F}_{q^\ell}$ and define $\Psi_\ell := \Omega_\ell$ if $\ell = 1$, and otherwise let

$$\Psi_\ell := \begin{bmatrix} \Omega_\ell & \xi_\ell\Omega_\ell & \cdots & \xi_\ell^{\ell-1}\Omega_\ell \\ \Omega_\ell^{(-1)} & (\xi_\ell\Omega_\ell)^{(-1)} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_\ell^{(-(\ell-1))} & (\xi_\ell\Omega_\ell)^{(-1)} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-(\ell-1))} \end{bmatrix} \in \mathrm{Mat}_\ell(\mathbb{T}),$$

where

$$\Omega_\ell(t) := (-\theta)^{\frac{-q^\ell}{q^\ell-1}} \prod_{i=1}^{\infty}\left(1-\frac{t}{\theta^{q^{\ell i}}}\right) \in \overline{k_\infty}[\![t]\!] \subseteq \mathbb{C}_\infty(\!(t)\!).$$

Observe that Ω_ℓ is an entire function and that $\Omega_\ell(\theta) = \frac{-1}{\tilde{\pi}_\ell}$. Moreover, one has the following functional equations

$$\Omega_\ell^{(-\ell)} = (t-\theta)\Omega_\ell, \tag{4}$$

$$\Psi_\ell^{(-1)} = \Phi_\ell\Psi_\ell. \tag{5}$$

Since $\{1, \xi_\ell, \ldots, \xi_\ell^{\ell-1}\}$ is a basis of $\mathbb{F}_{q^\ell}$ over $\mathbb{F}_q$, we have that $\Psi_\ell \in \mathrm{GL}_\ell(\mathbb{L})$.

Let M_ℓ be an ℓ-dimensional vector space over $\bar{k}(t)$ with a basis $\mathbf{m} \in \mathrm{Mat}_{\ell\times 1}(M_\ell)$. We give M_ℓ the structure of a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module by defining $\sigma\mathbf{m} := \Phi_\ell\mathbf{m}$, thus making M_ℓ a pre-t-motive. One directly checks that this pre-t-motive M_ℓ is in fact a t-motive with rigid analytic trivialization provided by Ψ_ℓ. Working out its Galois group, we have the following lemma.

Lemma 3.2.1. *Let M_ℓ be the t-motive defined above. Then its Galois group $\Gamma_{M_\ell} \subseteq \mathrm{GL}_\ell/\mathbb{F}_q(t)$ is an ℓ-dimensional torus over $\mathbb{F}_q(t)$.*

Proof. Let $\bar{\sigma} := \sigma^\ell$. Since M_ℓ is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module, it also can be regarded as a left $\bar{k}(t)[\bar{\sigma}, \bar{\sigma}^{-1}]$-module given by $\bar{\sigma}\mathbf{m} = \widetilde{\Phi_\ell}\mathbf{m}$, where

$$\widetilde{\Phi_\ell} := \Phi_\ell^{(-(\ell-1))} \cdots \Phi_\ell^{(-1)} \Phi_\ell.$$

To distinguish the two different roles of M_ℓ, we let $\mathbf{M}_\ell$ have the same underlying space as M_ℓ, but we regard $\mathbf{M}_\ell$ to be a left $\bar{k}(t)[\bar{\sigma}, \bar{\sigma}^{-1}]$-module. Note that $\mathbf{M}_\ell$ is a rigid analytically trivial pre-t-motive because

$$\bar{\sigma}\Psi_\ell = \widetilde{\Phi_\ell}\Psi_\ell. \tag{6}$$

Since by Theorem 2.2.1 the Galois group $\Gamma_{\mathbf{M}_\ell}/\mathbb{F}_{q^\ell}(t)$ of $\mathbf{M}_\ell$ is isomorphic to the Galois group of the functional equation (6), by (1) we see that

$$\Gamma_{\mathbf{M}_\ell} \cong \Gamma_{M_\ell} \times_{\mathbb{F}_q(t)} \mathbb{F}_{q^\ell}(t) \quad \text{over } \mathbb{F}_{q^\ell}(t). \tag{7}$$

Now let $\widetilde{\Psi_\ell}$ be the diagonal matrix with entries

$$\Omega_\ell, \Omega_\ell^{(-1)}, \ldots, \Omega_\ell^{(-(\ell-1))}.$$

Since $\widetilde{\Phi_\ell}$ is equal to the diagonal matrix with entries

$$(t-\theta), (t-\theta)^{(-1)}, \ldots, (t-\theta)^{(-(\ell-1))},$$

using (4) we have $\bar{\sigma}\widetilde{\Psi_\ell} = \widetilde{\Phi_\ell}\widetilde{\Psi_\ell}$. That is, $\widetilde{\Psi_\ell}$ is also a rigid analytic trivialization of $\widetilde{\Phi_\ell}$ with respect to the operator $\bar{\sigma}$ and hence

$$\Gamma_{\mathbf{M}_\ell} \cong \Gamma_{\widetilde{\Psi_\ell}} \quad \text{over } \mathbb{F}_{q^\ell}(t).$$

On the other hand, since $\widetilde{\Psi_\ell}$ is a diagonal matrix, by (1) $\Gamma_{\widetilde{\Psi_\ell}}$ is a split torus inside $\mathrm{GL}_\ell/\mathbb{F}_{q^\ell}(t)$. Therefore, from (7) we see that Γ_{M_ℓ} is a torus over $\mathbb{F}_q(t)$.

To prove $\dim \Gamma_{M_\ell} = \ell$, it suffices by Theorem 2.2.1 to show that the transcendence degree of the field $\bar{k}(t)(\Psi_\ell) = \bar{k}(t)(\Omega_\ell, \Omega_\ell^{(-1)}, \ldots, \Omega_\ell^{(-(\ell-1))})$ over $\bar{k}(t)$ is ℓ. Now, we let $X_1, \ldots, X_\ell$ be the coordinates of the ℓ-dimensional split torus T in $\mathrm{GL}_\ell/\mathbb{F}_{q^\ell}(t)$. Suppose that $\Gamma_{\widetilde{\Psi_\ell}}$ is a proper subtorus of T. Then $\Gamma_{\widetilde{\Psi_\ell}}$ is the kernel of some characters of T, i.e., canonical generators of the defining ideal of $\Gamma_{\widetilde{\Psi_\ell}}$ can be chosen of the form ${X_1}^{m_1} \cdots {X_\ell}^{m_\ell} - 1$ for integers $m_1, \ldots, m_\ell$, not all zero. By (1), replacing each X_i by $(1/\Omega_\ell^{(-i+1)}) \otimes \Omega_\ell^{(-i+1)} \in \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$, we obtain

$$\prod_{i=1}^{\ell} \left(\Omega_\ell^{(-i+1)}\right)^{m_i} =: \beta \in \bar{k}(t). \tag{8}$$

Note that for each $1 \leqslant i \leqslant \ell$, the set of all zeros of $\Omega_\ell^{(-i+1)}$ is $\{\theta^{q^{\ell j - i+1}}\}_{j=1}^{\infty}$. Since β has only finitely many zeros and poles, comparing the order of vanishing at $t = \theta^{q^{\ell j-i+1}}$ on both sides of (8), for each $1 \leqslant i \leqslant \ell$ and for sufficiently large j, forces $m_i = 0$, and hence we obtain a contradiction. □

Combining Lemma 3.2.1 and Theorem 2.2.2, we also have the following corollary.

Corollary 3.2.2.

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\big(\Omega_\ell(\theta), \Omega_\ell^{(-1)}(\theta), \ldots, \Omega_\ell^{(-(\ell-1))}(\theta)\big) = \ell. \tag{9}$$

3.3. Determining algebraic relations for arithmetic gamma values

For nonzero elements $x, y \in \mathbb{C}_\infty$, we write $x \sim y$ when $x/y \in \bar{k}$.

Theorem 3.3.1. *(See Thakur [18].) For each positive integer ℓ, we have*

$$\frac{\left(\frac{1}{1-q^\ell}\right)!}{\left(\frac{q^{\ell-1}}{1-q^\ell}\right)!^q} \sim \Omega_\ell(\theta). \tag{10}$$

Theorem 3.3.2. *Fix an integer $\ell \geqslant 2$. For each j, $1 \leqslant j \leqslant \ell - 1$, we have*

$$\frac{\left(\frac{q^j}{1-q^\ell}\right)!}{\left(\frac{q^{j-1}}{1-q^\ell}\right)!^q} \sim \Omega_\ell^{(-(\ell-j))}(\theta). \tag{11}$$

The first theorem is an analogue (see [20, Sec. 4.12]) of the Chowla–Selberg formula and the second theorem is its quasi-periods counterpart. Proofs for both follow in exactly the same fashion by straightforward manipulation. Use $D_i/D_{i-1}^q = (\theta^{q^i} - \theta)$ and take unit parts to verify that the left side in each formula is the one-unit part of the corresponding right side. Combining formulas (10), (11) with Corollary 3.2.2, we are able to determine all algebraic relations among those arithmetic gamma values:

Corollary 3.3.3. *Fix a positive integer ℓ. Let L be the field over $\bar{k}$ generated by*

$$S_\ell := \left\{\left(\frac{1}{1-q^\ell}\right)!, \left(\frac{2}{1-q^\ell}\right)!, \ldots, \left(\frac{q^\ell - 2}{1-q^\ell}\right)!\right\}.$$

Then we have

$$\operatorname{tr.deg}_{\bar{k}} L = \ell.$$

Proof. Suppose $q > 2$ or $\ell > 1$. By (2) we see that

$$\operatorname{tr.deg}_{\bar{k}} L = \operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\left(\frac{1}{1-q^\ell}\right)!, \left(\frac{q}{1-q^\ell}\right)!, \ldots, \left(\frac{q^{\ell-1}}{1-q^\ell}\right)!\right) \leqslant \ell.$$

From (10) and (11) we observe that

$$\bar{k}\big(\Omega_\ell(\theta), \Omega_\ell^{(-1)}(\theta), \dots, \Omega_\ell^{(-(\ell-1))}(\theta)\big) \subseteq L$$

so that Corollary 3.2.2 gives $\operatorname{tr.deg}_{\bar{k}} L \geqslant \ell$. Thus, $\operatorname{tr.deg}_{\bar{k}} L = \ell$.

For $q = 2$ and $\ell = 1$, we interpret S_1 as $\{(-1)!\}$ and thus the theorem holds in that case too. $\square$

Remark 3.3.4. A uniform framework for arithmetic, geometric and classical gamma functions is described in [18, Sec. 7], [20, Sec. 4.12]. In particular, a 'bracket criterion' for the transcendence of gamma monomials at proper fractions is described. Our result implies that a set of arithmetic gamma monomials is $\bar{k}$-linearly dependent exactly when the ratio of some pair of them satisfies the bracket criterion. (The exact parallel statement is proved for geometric gamma monomials in [3], see [20, Thm. 10.5.3].) In fact, by the proof of [20, Thm. 4.6.4] a given arithmetic gamma monomial satisfies the bracket criterion precisely when, by integral translations of arguments, it is expressible as a monomial in $(q^j/(1-q^\ell))!$'s (with fixed ℓ and $0 \leqslant j < \ell$ and up to an element of k) and the latter monomial is trivial. Hence Corollary 3.3.3 implies that all algebraic relations over $\bar{k}$ among special arithmetic gamma values are generated by their bracket relations.

4. Algebraic independence of gamma values and zeta values

If $q = 2$, the zeta value $\zeta_C(n)$ for any positive integer n is a rational multiple of $\tilde{\pi}^n$, with $\tilde{\pi} = \tilde{\pi}_1$ (see Section 4.2). Thus one can easily determine all the algebraic relations among these special zeta values and the arithmetic gamma values put together via Section 3.3. The question which remains is the algebraic independence of $\zeta_C(n)$ along with arithmetic gamma values for $q > 2$. Thus we assume $q > 2$ throughout this section.

4.1. The main theorem

4.1.1. Carlitz motives and their tensor powers

For convenience, we let $C := M_1$ be the t-motive associated to the Carlitz $\mathbb{F}_q[t]$-module and let $\Omega := \Omega_1$ (cf. Section 3.2). For each $n \in \mathbb{N}$, we introduce the n-th tensor power $C^{\otimes n}$ of the Carlitz motive C. Its underlying space is $\bar{k}(t)$ with σ-action $\sigma f := (t-\theta)^n f^{(-1)}$ for $f \in C$. Thus Ω^n provides a rigid analytic trivialization for $C^{\otimes n}$. The Galois group of $C^{\otimes n}$ is isomorphic to $\mathbb{G}_m$ over $\mathbb{F}_q(t)$ because Ω is transcendental over $\bar{k}(t)$ (cf. Theorem 2.2.1).

4.1.2. Polylogarithms and $L_{n,\alpha}(t)$

Recall the Carlitz logarithm $\log_C(z)$ of the Carlitz $\mathbb{F}_q[t]$-module. As a power series, it is the inverse of $\exp_{C_1}(z)$ with respect to composition, and it can be written as

$$\log_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})}.$$

For each $n \in \mathbb{N}$, the n-th polylogarithm is defined by

$$\log_C^{[n]}(z) := z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)^n (\theta - \theta^{q^2})^n \cdots (\theta - \theta^{q^i})^n}.$$

It converges on the disc $\{|z|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}\}$.

For $n \in \mathbb{N}$ and $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, we consider the power series

$$L_{\alpha,n}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t - \theta^q)^n (t - \theta^{q^2})^n \cdots (t - \theta^{q^i})^n},$$

which converges on the disc $\{|t|_\infty < |\theta|_\infty^q\}$ and satisfies $L_{\alpha,n}(\theta) = \log_C^{[n]}(\alpha)$. Moreover, the action of σ on $L_{\alpha,n}(t)$ gives rise to

$$L_{\alpha,n}^{(-1)}(t) = \alpha^{(-1)} + \frac{L_{\alpha,n}(t)}{(t-\theta)^n},$$

and hence we have the functional equation

$$\left(\Omega^n L_{\alpha,n}\right)^{(-1)} = \alpha^{(-1)} (t-\theta)^n \Omega^n + \Omega^n L_{\alpha,n}. \tag{12}$$

4.1.3. Review of the Chang–Yu theorem

Fixing a positive integer s, we define

$$U(s) := \left\{n \in \mathbb{N};\ 1 \leqslant n \leqslant s,\ p \nmid n,\ (q-1) \nmid n\right\}.$$

For each $n \in U(s)$ we fix a finite set of $1 + m_n$ elements

$$\{\alpha_{n0}, \ldots, \alpha_{nm_n}\} \subseteq \bar{k}^\times$$

so that

$$|\alpha_{nj}|_\infty < |\theta|_\infty^{\frac{nq}{q-1}} \quad \text{for } j = 0, \ldots, m_n, \tag{13}$$

$$\tilde{\pi}^n, \mathcal{L}_{n0}(\theta), \ldots, \mathcal{L}_{nm_n}(\theta) \quad \text{are linearly independent over } k, \tag{14}$$

where

$$\mathcal{L}_{nj} := L_{\alpha_j,n} \quad \text{for } j = 0, \ldots, m_n.$$

For $n \in U(s)$, define $\Phi_n \in \mathrm{GL}_{m_n+2}(\bar{k}(t)) \cap \mathrm{Mat}_{m_n+2}(\bar{k}[t])$ and $\Psi_n \in \mathrm{GL}_{m_n+2}(\mathbb{T})$ by

$$\Phi_n = \Phi(\alpha_{n0}, \ldots, \alpha_{nm_n}) := \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ \alpha_{n0}^{(-1)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{nm_n}^{(-1)}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix},$$

$$\Psi_n = \Psi(\alpha_{n0}, \ldots, \alpha_{nm_n}) := \begin{bmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n \mathcal{L}_{n0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n \mathcal{L}_{nm_n} & 0 & \cdots & 1 \end{bmatrix}.$$

Then by (12) we have $\Psi_n^{(-1)} = \Phi_n \Psi_n$. Note that all the entries of Ψ_n are inside $\mathbb{E}$ and that Φ_n defines a t-motive which is an extension of the $(m_n + 1)$-dimensional trivial t-motive over $\bar{k}(t)$ by $C^{\otimes n}$ (cf. [10, Lemma A.1]).

Fixing a positive integer s, we define the block diagonal matrices,

$$\Phi_{(s)} := \bigoplus_{n \in U(s)} \Phi_n,$$

$$\Psi_{(s)} := \bigoplus_{n \in U(s)} \Psi_n.$$

Each $\Phi_{(s)}$ defines a t-motive $M_{(s)}$ which is the direct sum of the t-motives defined by Φ_n, $n \in U(s)$. Using (1), we see that any element in $\Gamma_{\Psi_{(s)}}$ is of the form

$$\bigoplus_{n \in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 1 & \cdots & 1 \end{bmatrix},$$

where the block matrix at the position corresponding to $n \in U(s)$ has size $m_n + 2$.

Since the Carlitz motive C is a sub-t-motive of $M_{(s)}$, by Tannakian category theory we have a natural surjective map

$$\pi : \Gamma_{\Psi_{(s)}} \twoheadrightarrow \mathbb{G}_m, \tag{15}$$

which coincides with the projection on the upper left corner of any element of $\Gamma_{\Psi_{(s)}}$ (cf. [10, Sec. 4.3]). Let $V_{(s)}$ be the kernel of π so that one has an exact sequence of linear algebraic groups, $1 \to V_{(s)} \to \Gamma_{\Psi_{(s)}} \twoheadrightarrow \mathbb{G}_m \to 1$. From the projection map π, we see that $V_{(s)}$ is contained in the $(\sum_{n \in U(s)} (m_n + 1))$-dimensional vector group $G_{(s)}$ in $\Gamma_{\Psi_{(s)}}$ which consists of all block diagonal matrices of the form

$$\bigoplus_{n \in U(s)} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix},$$

where the block matrix at the position corresponding to $n \in U(s)$ has size $m_n + 2$. Here we shall note that $G_{(s)}$ can be identified with the direct product $\prod_{n \in U(s)} \mathbb{G}_a^{m_n+1}$, with the block matrix corresponding to $n \in U(s)$ identified with points in $\mathbb{G}_a^{m_n+1}$ having coordinates $(x_{n0}, \ldots, x_{nm_n})$.

Theorem 4.1.4. *(See Chang and Yu [10].) Fix any $s \in \mathbb{N}$, let $\Phi_{(s)}$, $\Psi_{(s)}$, $V_{(s)}$, and $G_{(s)}$ be defined as above. Then we have $V_{(s)} = G_{(s)}$, and hence*

$$\dim \Gamma_{\Psi_{(s)}} = 1 + \sum_{n \in U(s)} (m_n + 1).$$

In particular, the union

$$\{\tilde{\pi}\} \cup_{n \in U(s)} \bigcup_{i=0}^{m_n} \{\mathcal{L}_{ni}(\theta)\}$$

is an algebraically independent set over $\bar{k}$.

4.1.5. The main theorem

Given positive integers ℓ and s, we consider the t-motive $M := M_{(s)} \oplus M_\ell$, where M_ℓ is the t-motive defined by Φ_ℓ with rigid analytic trivialization Ψ_ℓ (see Section 3.2). More precisely, M is defined by $\Phi := \Phi_{(s)} \oplus \Phi_\ell$ with a rigid analytic trivialization $\Psi := \Psi_{(s)} \oplus \Psi_\ell$. The main theorem of this subsection can be stated as follows.

Theorem 4.1.6. *Given any two positive integers s and ℓ, let (M, Φ, Ψ) be defined as above. Then the dimension of the Galois group Γ_Ψ of M is*

$$\ell + \sum_{n \in U(s)} \{m_n + 1\}.$$

In particular, the following set

$$\bigcup_{n \in U(s)} \bigcup_{j=0}^{m_n} \{\mathcal{L}_{nj}(\theta)\} \cup \left\{ \left(\frac{1}{1-q^\ell}\right)!, \left(\frac{q}{1-q^\ell}\right)!, \ldots, \left(\frac{q^{\ell-1}}{1-q^\ell}\right)! \right\}$$

is a transcendence basis of $\bar{k}(\Psi(\theta))$ over $\bar{k}$.

Proof. First we note that by (1) any element of $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ is of the form

$$\bigoplus_{n \in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix} \oplus B,$$

where

$$\bigoplus_{n \in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix} \in \Gamma_{\Psi_{(s)}}\big(\overline{\mathbb{F}_q(t)}\big), \quad B \in \Gamma_{\Psi_\ell}\big(\overline{\mathbb{F}_q(t)}\big).$$

Define $\Phi_D := \bigoplus_{n\in U(s)}[(t-\theta)^n] \oplus \Phi_\ell$, $\Psi_D := \bigoplus_{n\in U(s)}[\Omega^n] \oplus \Psi_\ell$. Then we have $\Psi_D^{(-1)} = \Phi_D\Psi_D$, and we note that such Φ_D defines a t-motive M_D, which is the direct sum of the t-motives $\bigoplus_{n\in U(s)} C^{\otimes n} \oplus M_\ell$. Moreover, the t-motive M_D is a sub-t-motive of M, and hence using the same argument as for the surjection of π (15) we have a surjective map

$$\pi_D : \Gamma_\Psi \twoheadrightarrow \Gamma_{\Psi_D},$$

which coincides with the projection map given by

$$\bigoplus_{n\in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix} \oplus B \mapsto \bigoplus_{n\in U(s)} [x^n] \oplus B.$$

Put $V := \operatorname{Ker}\pi_D$ and note that V is a vector group.

Since $\det\Psi_\ell(\theta) \sim \Omega(\theta)$, we have $\bar{k}(\Psi_D(\theta)) = \bar{k}(\Psi_\ell(\theta))$, and hence by Theorem 2.2.1 and Lemma 3.2.1 we see that $\dim\Gamma_{\Psi_D} = \dim\Gamma_{\Psi_\ell} = \ell$. Therefore, to prove this theorem it is equivalent to prove that $\dim V = \sum_{n\in U(s)}(m_n+1)$.

Consider the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & V \arrow[r] \arrow[d] & \Gamma_\Psi \arrow[r, two heads, "\pi_D"] \arrow[d, two heads, "\pi_s"] & \Gamma_{\Psi_D} \arrow[r] \arrow[d, two heads, "\chi_D"] & 1 \\ 1 \arrow[r] & V_{(s)} \arrow[r] & \Gamma_{\Psi_{(s)}} \arrow[r, two heads, "\pi"] & \mathbb{G}_m \arrow[r] & 1, \end{tikzcd} \tag{16}$$

where the right hand square is given by

$$\begin{tikzcd} \bigoplus_{n\in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix} \oplus B \arrow[r, mapsto, "\pi_D"] \arrow[d, mapsto, "\pi_s"] & \bigoplus_{n\in U(s)}[x^n] \oplus B \arrow[d, mapsto, "\chi_D"] \\ \bigoplus_{n\in U(s)} \begin{bmatrix} x^n & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{bmatrix} \arrow[r, mapsto, "\pi"'] & [x]. \end{tikzcd}$$

Here we note that the projection maps π_s and π_D are surjective since $M_{(s)}$ is a sub-t-motive of M and the Carlitz motive C is a sub-t-motive of M_D (cf. (15)).

By Theorem 4.1.4 we have $V_{(s)} = G_{(s)}$, which is identified with the product space $\prod_{n\in U(s)} \mathbb{G}_a^{m_n+1}$ canonically. For a double index nj with $n \in U(s)$, $0 \leqslant j \leqslant m_n$, we let the (nj)-coordinate space in $V_{(s)}$ be the one-dimensional vector subgroup consisting of points whose coordinates all vanish except the coordinate x_{nj}. Hence, we need only to show that $\pi_s|_V$ is surjective onto each (nj)-coordinate space in $V_{(s)}$ for $n \in U(s)$, $0 \leqslant j \leqslant m_n$.

Given a nonzero element v of the (nj)-coordinate space in $V_{(s)}(\overline{\mathbb{F}_q(t)})$, we can pick $\gamma \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ so that $\pi_s(\gamma) = v$ since π_s is surjective. Further, pick $a \in \overline{\mathbb{F}_q(t)}^\times \setminus \overline{\mathbb{F}_q}^\times$ and let $\delta \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ for which

$$\chi_D \circ \pi_D(\delta) = a.$$

Using the property that Γ_{Ψ_ℓ} is commutative (cf. Lemma 3.2.1), direct calculation shows that

$$\delta^{-1}\gamma\delta\gamma^{-1} \in V\big(\overline{\mathbb{F}_q(t)}\big)$$

and $\pi_s(\delta^{-1}\gamma\delta\gamma^{-1})$ belongs to the (nj)-coordinate space in $V_{(s)}(\overline{\mathbb{F}_q(t)})$. Moreover, for such choice of a, we see that $\pi_s(\delta^{-1}\gamma\delta\gamma^{-1})$ is nonzero and complete the proof. $\square$

4.2. *Application to zeta values*

The key for applying Section 4.1 to our problems on zeta values is the following fact concerning the special zeta value $\zeta_C(n)$ and n-th polylogarithms.

Theorem 4.2.1. *(See Anderson and Thakur [4].) Given a positive integer n, one can explicitly find a finite sequence $h_{n,0}, \ldots, h_{n,l_n} \in k$, $l_n < \frac{nq}{q-1}$, such that*

$$\zeta_C(n) = \sum_{i=0}^{l_n} h_{n,i} L_{\theta^i,n}(\theta). \tag{17}$$

Given any positive integer n not divisible by $q-1$, set

$$N_n := k\text{-span}\big\{\tilde{\pi}^n, L_{1,n}(\theta), L_{\theta,n}(\theta), \ldots, L_{\theta^{l_n},n}(\theta)\big\}.$$

By (17) we have $\zeta_C(n) \in N_n$ and $m_n + 2 := \dim_k N_n \geqslant 2$ since $\zeta_C(n)$ and $\tilde{\pi}^n$ are linearly independent over k. For each such n we fix once and for all a finite subset

$$\{\alpha_{n0}, \ldots, \alpha_{nm_n}\} \subseteq \big\{1, \theta, \ldots, \theta^{l_n}\big\}$$

such that both

$$\big\{\tilde{\pi}^n, \mathcal{L}_{n0}(\theta), \ldots, \mathcal{L}_{nm_n}(\theta)\big\}$$

and

$$\big\{\tilde{\pi}^n, \zeta_C(n), \mathcal{L}_{n1}(\theta), \ldots, \mathcal{L}_{nm_n}(\theta)\big\}$$

are bases of N_n over k, where $\mathcal{L}_{nj}(t) := L_{\alpha_{nj},n}(t)$ for $j = 0, \ldots, m_n$. This can be done because of Theorem 4.2.1.

Given a positive integer s, Theorem 4.1.4 then implies that all the zeta values $\zeta_C(n)$, $n \in U(s)$, and $\tilde{\pi}$, are algebraically independent over k. In other words, we have

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\big(\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(s)\big) = s - \lfloor s/p \rfloor - \big\lfloor s/(q-1)\big\rfloor + \big\lfloor s/p(q-1)\big\rfloor + 1. \tag{18}$$

This also implies that all $\bar{k}$-algebraic relations among Carlitz zeta values are those relating the $\zeta_C(m)$, with $m \notin U(s)$, and those $\zeta_C(n)$ with $n \in U(s)$ for given $s \in \mathbb{N}$.

These relations come from the Frobenius p-th power relations and the Euler–Carlitz relations, which we recall briefly. The Frobenius p-th power relations among zeta values (p is the characteristic) are

$$\zeta_C\left(p^m n\right) = \zeta_C(n)^{p^m} \quad \text{for } m, n \in \mathbb{N}.$$

Also the Euler–Carlitz relations among the $\zeta_C(n)$, for n divisible by $q-1$, and $\tilde{\pi}$ are

$$\zeta_C(n) = \frac{B_n}{\Gamma_{n+1}} \tilde{\pi}^n.$$

The Bernoulli–Carlitz 'numbers' B_n in k are given by the following expansion from the Carlitz exponential series

$$\frac{z}{\exp_{C_1}(z)} = \sum_{n=0}^{\infty} \frac{B_n}{\Gamma_{n+1}} z^n,$$

where Γ_{n+1} is the Carlitz factorial of n (cf. Section 1). Thus the formula in (18) is obtained by inclusion–exclusion.

Now, applying Theorem 4.1.6 to this setting we can determine the transcendence degree of the field generated by all arithmetic gamma values and zeta values put together.

Theorem 4.2.2. *Given any two positive integers s and ℓ, let E be the field over $\bar{k}$ generated by the set*

$$\left\{\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(s)\right\} \cup \left\{\left(\frac{c}{1-q^\ell}\right)!;\ 1 \leqslant c \leqslant q^\ell - 2\right\}.$$

Then the transcendence degree of E over $\bar{k}$ is

$$s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/p(q-1) \rfloor + \ell.$$

Remark 4.2.3. In the classical case, a conjecture about the Riemann zeta function at positive integers asserts that the Euler relations, i.e., $\zeta(2n)/(2\pi\sqrt{-1})^{2n} \in \mathbb{Q}$ for $n \in \mathbb{N}$, account for all the $\overline{\mathbb{Q}}$-algebraic relations among the special zeta values $\{\zeta(2), \zeta(3), \zeta(4), \ldots\}$. For special Γ-values, i.e., values of the Euler Γ-function at proper fractions, there are the 'natural' $\overline{\mathbb{Q}}$-algebraic relations among them coming from the translation, reflection and Gauss multiplication identities satisfied by the Γ-function, which are referred to as the standard relations. The Rohrlich–Lang conjecture predicts that these relations account for all $\overline{\mathbb{Q}}$-algebraic relations among special Γ-values. One is lead to the conjectures that the Euler relations among special zeta values and the standard algebraic relations among special Γ-values account for all the $\overline{\mathbb{Q}}$-algebraic relations among the special zeta values and special Γ-values put together.

Remark 4.2.4. Our Corollary 3.3.3 asserts that the standard relations among the arithmetic gamma values account for all $\bar{k}$-algebraic relations among the arithmetic gamma values. Theorem 4.2.2 asserts that the Euler–Carlitz relations, the Frobenius p-th power relations and the standard relations among the arithmetic gamma values account for all $\bar{k}$-algebraic relations among the arithmetic gamma values and Carlitz zeta values put together.

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