Discrete Mathematics 129 (1994) 19–27
North-Holland

Questions about linear spaces

Francis Buekenhout
C.P. 216 Université Libre de Bruxelles, Bd du Triomphe, B-1050 Bruxelles, Belgium

Received 1 March 1992

Abstract

We present three themes of interest for future research that require the cooperation of fairly large teams:
(1) linear spaces as building blocks;
(2) data for an Atlas of linear spaces;
(3) morphisms of linear spaces.

1. About the early years of linear spaces in Brussels

I would like to dedicate this paper to the memory of Paul Libois (1901–1990). Unaware of earlier work on linear spaces like the famous de Bruijn–Erdős theorem [15], Libois coined the name 'linear space' and started lecturing on the subject in 1961. He saw linear spaces as the simplest common generalization of projective and affine spaces [23, 24]. His students would soon take over. I realized that linear spaces were related to conceptual developments such as linear subspace, dimension, hyperplane and that these were perfect tools to build the common generalization of projective and affine geometry wanted by Libois.

A fairly big manuscript of mine, written in 1967, remained unpublished [4]. When Dembowski's [16] book appeared in 1968, I browsed through it in one night before giving a lecture in Frankfurt and realized that the problem of classifying all linear spaces whose planes are affine was apparently open. The solution was a nice reward for the theory of linear spaces [5] and that theme was later used by many other works. In 1968, Doyen [17] wrote the dissertation for his first degree, on the linear spaces of few points and on the behavior of their number for a given number of points. He went on with a thesis on Steiner systems [18] and became quickly a world expert on linear spaces and Steiner systems. Doyen and Hubaut [19] obtained a strong reduction toward the classification of the finite linear spaces that are locally projective.

Correspondence to: Francis Buekenhout, Mathematics Department, University of Brussels, CP 216, 1050 Brussels, Belgium.

SSDI 0012-365X(92)00502-W

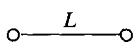
This should give an idea of the early consequences of Libois's move and of the rise of a school on linear spaces among his students. Later this would grow further, in particular with Teirlinck and Delandtsheer.

All of this brings me to the future of linear spaces. In 1968, I had become so enthusiastic about them that I saw linear spaces as the right setting in order to develop abstract incidence geometry. Doyen and I started writing a book [8] whose leading idea was: 'Do it with points and lines'. We wrote about 100 pages and did never get further. In the 1970s, Crapo and Rota [14] wrote along lines which were very close to the preceding idea. They felt that matroids were the right concept.

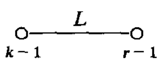
Why did I give up with the idea? Dembowski [16] showed already that there were other important structures. The main stroke came from elsewhere. I was aware of Tits' work. He was the prominent figure among Libois' students. He was launching the theory of buildings. I started work on projective quadrics. I still did it with points and lines but not with linear spaces anymore. Buildings were opening new roads. Linear spaces were no longer the all embracing frame for incidence geometry. Projective spaces became the most important particular case of a huge variety of new structures that were as powerful as the former. There was little room left for the linear spaces. Around 1975, inspiration by diagram geometries, mainly the sporadic groups and their geometry, gave still another perspective on incidence geometry [6]. Today, incidence geometries seem to provide the right setting for the foundations. Linear spaces appear as one of the best candidates in order to be admitted among future 'quasi-buildings' but the nature of these has to be clarified further by research. Linear spaces need to undergo the same fate as projective spaces. They need further internal exploration. However, instead of being the universal object from which there is no escape, they have to be studied as building blocks of more elaborate structures.

2. Linear spaces as building blocks

We shall use the terminology and conventions of diagram geometry, as in Buekenhout [7]. Let



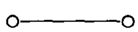
denote the class of linear spaces having at least two lines. Then



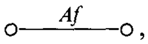
denotes the Steiner systems with k points per line and r lines per point.

Particular subclasses of interest are:

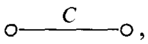
— the projective planes represented by

 ,

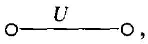
— the affine planes represented by



— the 'circles' (linear spaces whose lines have two points) represented by

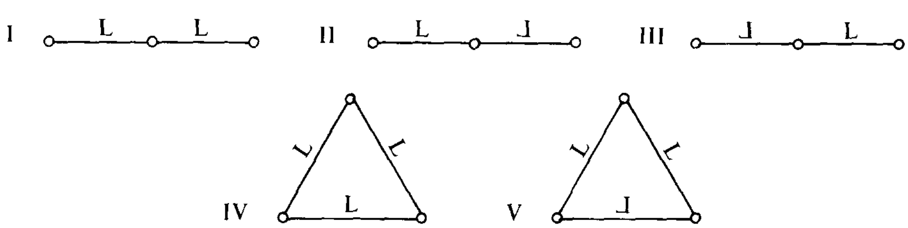


— the unitals



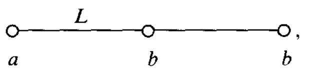
— etc.

What can be said of rank-3-geometries over diagrams using only L and the digons? These are:

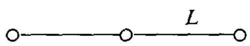


The incidence geometry Γ over such a diagram may be submitted to some global axioms such as (F) firm, (RC) residually connected, (IP) intersection property, etc.

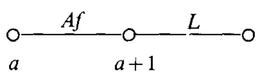
Class I: Assuming (F) and (RC), these are the planar spaces [6]. If we specialize to the finite



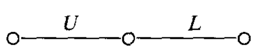
we are exactly dealing with the Doyen–Hubaut situation [19]. Similarly,



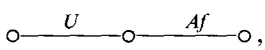
is giving the projective spaces of any dimension $d \geqslant 3$,

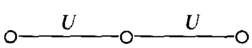


is giving the affine spaces of dimension $d \geqslant 3$, provided that $a \geqslant 3$ [6]. What can be said of diagrams like



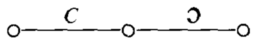
and specializations such as





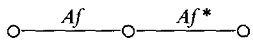
How to go further with Doyen–Hubaut's result?

Class II: A lot of work, assuming (F), (RC), (IP) has been done on the special case



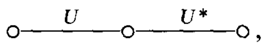
(semi-biplanes).

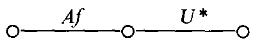
Neat results have been obtained with



[21,22].

May we hope something for

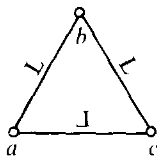


, 

?

Class III: Assuming (F), (RC), (IP), a striking classification was obtained by Sprague [30]. It would be most interesting to push this further.

Class IV: Assuming finiteness, (F), (RC) and orders, the De Bruijn–Erdős theorem gives us $a \leqslant b \leqslant c \leqslant a$ so $a = b = c$ and

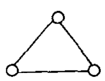


therefore all residues are projective planes. Hence, class V vanishes into class IV. Would this hold true without orders? Probably yes. And without the finiteness assumption? I doubt it.

Class V: What can be said?

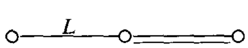
As a matter of fact, higher-rank diagrams are of interest too. These explorations require team work. At some time, they might unearth major new trends and lead to a less naive approach.

There are other valuable themes. For instance, if Γ is equipped with a chamber-transitive group of automorphisms, a lot is known about the diagram


.

In the finite case, the planes have order 2 and the group induced on them is the Frobenius group 7:3, see [32]. However, every projective plane can occur in such a building [17].

I must also mention that much is known about



(see [26]).

3. Data for an Atlas of linear spaces

3.1. We consider only finite linear spaces. The project can be worked out with or without computers.

3.2. My source of inspiration is group theory. The Atlas of finite groups [13], the databases included in [10] and many other data published or not, allow to derive many facts concerning geometries equipped with a fairly large automorphism group.

3.3. In geometry, such a trend has been developed locally for at least 35 years and there is now a sudden interest for systematization at various places. This occurs far behind the movement in algebra. To my knowledge, there are few databases so far. A good example is provided by the tables in [2]. The linear spaces of at most 9 points were listed by Doyen [17]. As I understand from Doyen, the case of 10 points was settled on a computer by Glynn (1987), who gets 5250 such linear spaces, together with their automorphism group and Betten (1989) has shown that there are at least 232 923 linear spaces on 11 points. All data known to us could usefully be collected, organized, completed and made available to a broad audience, especially young mathematicians who want to enter the field.

3.4. Here are some suggestions that may help. Enumeration of

(1) all linear spaces whose number of points is small,

(2) same with additional conditions such as

(3) a constant number of lines on each point (semi-Steiner systems),

(4) Steiner systems $S(2,k,v)$ with v small. There are at least 2 111 276 systems $S(2,3\ 19)$ [31]. A list of all known projective planes of order $n\leqslant 100$.

Lists of small planar spaces, small projective spaces, etc.

3.5. Given a linear space L, we want an easy control over all linear subspaces of L. Several equivalences are of interest on the set of linear subspaces of L: orbits under the automorphism group, isomorphism, weak isomorphisms (some specified invariants are equal).

3.6. Given L as above, we may want to know all of its restrictions to a subset of the point set.

This interferes with themes such as Baersubplanes, unitals, curves, etc.

4. Morphisms of linear spaces

4.1. In projective geometry, there is a rich tradition concerning various kinds of (homo)morphisms (e.g. [11, 12, 20, 25]). Rather deep questions have been solved. Some remain open.

4.2. As to morphisms in the context of linear spaces, I can only mention my unpublished work [4] and Rossi [29]. Delandtsheer suggests to look at the literature on matroids.

4.3. My feeling is that the problems in this area have not yet been presented in a unified context and that the subject requires a good deal of clarification. This is important for two reasons at least. In present day mathematics, a lot of attention is paid to categories and functors. The emphasis is laid on relations between classes of structures. This theme is not a standard one in incidence geometry and this may contribute to the relative isolation of the subject. The second reason is of another nature. Many mathematicians believe that projective geometry is of no use because it is just a translation of linear algebra. This is only true at the surface. If it were true in depth, the statement would obviously apply to the various morphisms of projective geometry and this is not the case at all.

4.4. In [4], I came up with a list of 14 pairwise nonequivalent concepts of morphism for a linear space and with the various implications among them.

I would like to come back on that matter and display now more than 10^5 definitions. How is this possible?

4.5. As usual in such situations, the 'secret' lays in various equivalent ways to define a linear space.

(1) Consider a linear space $L=(\mathscr{P},\mathscr{L})$. Its structure may be described in terms of points and lines. But there are other ways. In addition to $\mathscr{P}$, and instead of $\mathscr{L}$, we may give one of

(2) the ternary collinearity relation C;

(3) the set of all linear subspaces $\mathscr{S}$;

(4) the closure operator $C\mathscr{O}$: for $X\subseteq\mathscr{P}$, it gives $\langle X\rangle$ the linear subspace generated by X;

(5) the set $\mathscr{F}$ of all free sets;

(6) the set $C\mathscr{S}$ of all connected subspaces, i.e. subspaces that are not the union of two disjoint and nonempty linear subspaces

(7) the set of *connected subsets* $C\mathscr{S}\mathscr{S}$ namely those sets of points in which no pair of points constitutes a line.

(8)–(14) The complementary sets of $\mathscr{L},\dots,C\mathscr{S}\mathscr{S}$ in the set $2^{\mathscr{P}}$ of all subsets of $\mathscr{P}$.

There is a paper by Bartolozzi [1] using such an idea. The data provided by σ and one of $\mathscr{L},C,\mathscr{S},C\mathscr{O},\mathscr{F},C\mathscr{S},C\mathscr{S}\mathscr{S}$, or their complements are equivalent to $(\mathscr{P},\mathscr{L})$.

Here is a proof for (6). The minimal connected subspaces containing two points at least are the lines with more than two points. It is likely that more of such families can be found. Bruen suggested the set $J\mathscr{S}$ of all *intersection sets* namely those subsets of $\mathscr{P}$ that have a nonempty intersection with each line. I must confess that I do not see a proof, nor a counterexample. Cameron and Mazzocca [9] show that blocking sets do not work in general.

4.6. A side question is to axiomatize linear spaces in terms of $C,\mathscr{S},\dots,C\mathscr{S}\mathscr{S}$ or their complements. For C this was done in [8]. It is rather obvious for $\mathscr{S}$ and $C\mathscr{O}$. How about other cases?

4.7. Let L be a linear space, so $L=(\mathscr{P},\mathscr{L})$ or better $L=(\mathscr{P},\mathscr{L},C,\mathscr{S},C\mathscr{O},\mathscr{F},C\mathscr{S}\mathscr{S},\text{etc.})$ and let $L'=(\mathscr{P}',\mathscr{L}',C',\dots)$ be another linear space. Defining a *morphism* α from L to L' involves three components:

- the set theoretic nature of α,
- the sense of the conservation requirement: on α or on α^{-1},
- the conservation requirement itself.

4.7.1. The set theoretic nature of α may be

(1) a bijection from $\mathscr{P}$ onto $\mathscr{P}'$,

(2) an injective mapping from $\mathscr{P}$ into $\mathscr{P}'$,

(3) a mapping of $\mathscr{P}$ onto $\mathscr{P}'$,

(4) a function of $\mathscr{P}$ into $\mathscr{P}'$.

There are meaningful examples for each.

The empty set may be allowed to interfere in various ways. The list is not necessarily exhaustive. Ronse [28] working in the context of shape recognition, defines 'digital isomorphisms' and this time, α is a relation. How does this interfere with linear spaces? From now on, let me assume that (3) holds.

4.7.2. The sense of the conservation requirement provides two choices.

4.7.3. The conservation requirement can be expressed in 14 ways (at least) using one of $\mathscr{L}, C, \mathscr{S}$, etc. I mean, for instance, that $\alpha(\mathscr{L}) \subseteq \mathscr{L}'$, but we could introduce further variations like $x \in \mathscr{S}$ implies $\alpha(x) \subseteq x'$ where $x' \in \mathscr{S}'$ (or $\alpha(x) \supseteq x'$ with $x' \in \mathscr{S}'$, etc. Actually, our count is not correct since any set among 14 requirements, i.e. $(2^{14}-1)$ may be considered.

4.7.4. Thus, we get easily more than $2^2 \cdot 2 \cdot (2^{14}-1) = 131\,064$ kinds of morphisms for linear spaces that are distinct at least in their phrasing. It is rather likely that the actual number of such definitions may grow to infinity.

4.8. Of course, this is not very serious. Nobody can believe that such a simple-minded structure as linear spaces may require thousands of concepts of morphism.

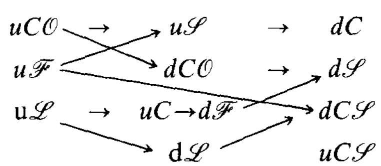
The point is to find an explanation and understanding for this situation.

4.9. It may be useful, to some extent, to determine the implications and equivalences among some of the morphisms displayed above. Why? Linear spaces are too wild a structure for classification purposes. Morphisms of some kind may help to determine interesting subclasses.

4.10. What happens if the preceding definitions are applied to the cases where L or L' (or both) are projective spaces?

If L, L' are projective spaces over a division ring, how do the preceding morphisms translate in terms of algebra? Much is known already on this matter (see, for instance, [11, 12, 20]). Interesting algebraic concepts such as field monomorphisms and valuations are involved in this.

4.11. Some results. Let L, L' be linear spaces and let α be a mapping of $\mathscr{P}$ onto $\mathscr{P}'$. Let us write, for instance, $d\mathscr{S}$ to mean that α preserves $\mathscr{S}$ in the direct way while $u\mathscr{S}$ means that it does so in the undirect way, i.e. for any subspace S' of L', $\alpha^{-1}(S') \in \mathscr{S}$. Then the following implications and no more, hold among 12 (out of the 14) 'elementary' types of morphisms defined earlier:



This calls for more attention to $uC\mathscr{S}$ and to the 'weakest' morphisms dC (a classical one), $d\mathscr{S}$ and $dC\mathscr{S}$.

Next, assume that α is a bijection from $\mathscr{P}$ onto $\mathscr{P}'$. Then the following implications and no other, hold among the indicated types of morphisms:

$$\begin{array}{lllllllll}
\text{isomorphism} & \longrightarrow & dC\mathscr{S} & \rightarrow & uC & \leftrightarrow & d\mathscr{S} & \leftrightarrow & d\mathscr{F} \\
\updownarrow & & & & & & & & \\
d\mathscr{L} \leftrightarrow u\mathscr{L} \leftrightarrow u\mathscr{C}\mathscr{O} & \text{dually} & & & & & & & \\
\updownarrow dC\mathscr{O} & \longrightarrow & uC\mathscr{S} & \rightarrow & dC & \leftrightarrow & u\mathscr{S} & \leftrightarrow & u\mathscr{F}
\end{array}$$

4.12. A suggestion made by Marchi. Forget $L=(\mathscr{P},\mathscr{L})$ and consider a 'functional approach', i.e. $L=(\mathscr{P},f)$ where f is a function defined on the set of pairs of $\mathscr{P}$ with values in the set of subsets of $\mathscr{P}$. Then other inspiration for morphisms may arise.

References

[1] F. Bartolozzi, Le immagini M-epimorfe di un M-piano, Annali di Mat. 21 (1974) 307–325.
[2] D. Beth, D. Jungnickel and H. Lenz, Design Theory, Mannheim (1985).
[3] F. Buekenhout, Espaces à fermeture, Bull. Soc. Math. Belg. 19 (1967) 147–178.
[4] F. Buekenhout, Espaces linéaires, Université de Bruxelles (1967) 83 pp.
[5] F. Buekenhout, Une caractérisation des espaces affins basée sur la notion de droite, Math. Z. 111 (1969) 367–371.
[6] F. Buekenhout, Diagrams for geometries and groups, J. Combin. Theory Ser. A 27 (1979) 121–151.
[7] F. Buekenhout, The basic diagram of a geometry, in: Geometries and Groups, Lecture Notes in Math. Vol. 893 (Springer, Berlin, 1981) 1–29.
[8] F. Buekenhout and J. Doyen, Do it with Points and Lines, Université de Bruxelles (1970), 96 pp.
[9] P. Cameron and F. Mazzocca, Bijections which preserve blocking sets, Geo. Ded. 21 (1986) 219–229.
[10] J. Cannon, A language for group theory, Univ. Sydney. 1982.
[11] D. Carter and A. Vogt, Collinearity-preserving functions between Desarguesian planes, Mem. Amer. Math. Soc. 235 (1980).
[12] P.V. Ceccherini, Collineazioni e semicollineazioni tra spazi affini o proiettivi, Rend. Math. e appl. 26 (1967) 309–348.
[13] J. Conway et al., Atlas of Finite Groups (Oxford Univ. Press, Oxford, 1985).
[14] H. Crapo and G.C. Rota, On the foundations of combinatorial theory II. Combinatorial geometries, Stud. Appl. Math. 49 (1970) 109–133.
[15] N.G. de Bruijn and P. Erdős, On a combinatorial problem, Kon. Ned. Akad. Weten. Proc. 51 (1948) 1277–1279.
[16] P. Dembowski, Finite Geometries (Springer, Berlin, 1968).
[17] J. Doyen, Sur les espaces linéaires finis, Mémoire de licence, Université Libre de Bruxelles, 1967.
[18] J. Doyen, Sur les systèmes de Steiner, Thèse, Université de Bruxelles, 1970.
[19] J. Doyen and X. Hubaut, Finite regular locally projective spaces, Math. Z. 119 (1971) 83–88.
[20] W. Klingenberg, Projektive Geometrien mit Homomorphismus, Math. Ann. 132 (1956) 180–200.
[21] C. Lefèvre-Percsy and L. Van Nypelseer, Finite rank 3 geometries with affine planes and dual affine point residues, Discrete Math. 84 (1990) 161–167.
[22] C. Lefèvre-Percsy, Infinite (Af, Af^*)-geometries, J. Combin. Theory Ser. A 55 (1990) 133–139.
[23] P. Libois, La géométrie de la relativité restreinte, Bull. Soc. Math. Belg. 14 (1962) 381–388.
[24] P. Libois, Quelques espaces linéaires, Bull. Soc. Math. Belg. 16 (1964) 13–22.
[25] P. Maroscia, Semicollineazioni e semicorrelazioni tra spazi lineari, Rend. Math. 3 (1970) 1–15.
[26] A. Pasini and S. Yoshiara, Flag-transitive Buekenhout geometries, in: A. Barlotti et al., eds., Combinatorics '90 (Elsevier, Amsterdam, 1992).
[27] M. Ronan, Triangle geometries, J. Combin. Theory Ser. A 37 (1984) 294–319.
[28] C. Ronse, An isomorphism for digital images, J. Combin. Theory Ser. A 39 (1985) 132–159.

[29] F. Rossi, Quasicollineazioni in geometrie sopra un insienne qualunque, Rend. Ist. Mat. Univ. Trieste 4 (1972) 1–21.

[30] A. Sprague, Rank 3 incidence structures admitting dual-linear, linear diagram, J. Combin. Theory. Ser. A 38 (1985) 254–259.

[31] D.R. Stinson and H. Ferch, 2000000 Steiner triple systems of order 19, Math. Comput. 44 (1985) 533–535.

[32] F. Timmesfeld, Tits geometries and parabolic systems in finitely generated groups I, II, Math. Z. 184 (1983) 377–396, 449–487.