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General neighbour-distinguishing index via chromatic number ☆

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ABSTRACT

An edge colouring of a graph G without isolated edges is neighbour-distinguishing if any two adjacent vertices have distinct sets consisting of colours of their incident edges. The general neighbour-distinguishing index of G is the minimum number $\mathrm{gndi}(G)$ of colours in a neighbour-distinguishing edge colouring of G. Győri et al. [E. Győri, M. Horňák, C. Palmer, M. Woźniak, General neighbour-distinguishing index of a graph, Discrete Math. 308 (2008) 827–831] proved that $\mathrm{gndi}(G) \in \{2, 3\}$ provided G is bipartite and gave a complete characterisation of bipartite graphs according to their general neighbour-distinguishing index. The aim of this paper is to prove that if $\chi(G) \geq 3$, then $\lceil \log_2 \chi(G) \rceil + 1 \leq \mathrm{gndi}(G) \leq \lfloor \log_2 \chi(G) \rfloor + 2$. Therefore, if $\log_2 \chi(G) \notin \mathbb{Z}$, then $\mathrm{gndi}(G) = \lceil \log_2 \chi(G) \rceil + 1$.

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All graphs we are dealing with in this paper are simple, finite and nonoriented. Let G be a graph and let k be a positive integer. A *k-edge-colouring* of G is a mapping $\varphi : E(G) \to \{1, \ldots, k\}$. The *colour set* of a vertex $x \in V(G)$ with respect to φ is the set

$$S_\varphi(x) := \{\varphi(xy) : xy \in E(G)\}.$$

The colouring φ is *neighbour-distinguishing* provided $S_\varphi(x) \neq S_\varphi(y)$ whenever $xy \in E(G)$. The notion of the colour set can be naturally extended to *partial* edge colourings of G in which some edges may be uncoloured.

Clearly, a neighbour-distinguishing colouring of G does exist if and only if G has no isolated edges. In such a case the *neighbour-distinguishing index* of G is the minimum k such that there is a proper k-edge-colouring of G that is neighbour-distinguishing; let $\mathrm{ndi}(G)$ denote the neighbour-distinguishing index of G. Evidently, $\chi'(G)$ is a trivial lower bound for $\mathrm{ndi}(G)$. The invariant has been introduced by Zhang et al. in the paper [6]. The authors conjecture that if G is a connected graph, $G \notin \{K_2, C_5\}$, then $\mathrm{ndi}(G) \leq \Delta(G) + 2$. The conjecture is known to be true for bipartite graphs and for graphs with maximum degree at most three; see Balister et al. [1]. In Edwards et al. [2] it was proved that if G is a plane graph with $\Delta(G) \geq 12$, then even $\mathrm{ndi}(G) \leq \Delta(G) + 1$. According to Hatami [5], $\mathrm{ndi}(G) \leq \Delta(G) + 300$ for any graph G satisfying $\Delta(G) > 10^{20}$.

Here we are interested in a generalised version of the problem in which we admit also edge colourings that are not proper. The corresponding invariant (first investigated by Győri et al. in [3]) is the *general neighbour-distinguishing index* of a graph G, in symbols $\mathrm{gndi}(G)$. Evidently, if G has connected components $G_1, \ldots, G_l$, then

$$\mathrm{gndi}(G) = \max(\mathrm{gndi}(G_i) : i = 1, \ldots, l).$$

Therefore, we shall restrict our attention to connected graphs distinct from K_2.

In [3] bipartite graphs are characterised from the point of view of their general neighbour-distinguishing index. Namely, if G is bipartite, then $2 \leq \mathrm{gndi}(G) \leq 3$; furthermore, $\mathrm{gndi}(G) = 2$ if and only if there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has at least one neighbour in both X_1 and X_2. If $\chi(G) \geq 3$, in [3] it is shown that $\mathrm{gndi}(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$.

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Győri and Palmer in [4] improved the upper bound for the general neighbour-distinguishing index of G to $3\lceil \frac{1}{2}\log_2 \chi(G)\rceil$. They have also proved that if $\chi(G) \geq 5$, then

$$\mathrm{gndi}(G) \leq \lceil \log_2(\chi(G)-3)\rceil + 5.$$

The aim of the present paper is to show that if $\chi(G) \geq 3$, then

$$\lceil \log_2 \chi(G)\rceil + 1 \leq \mathrm{gndi}(G) \leq \lfloor \log_2 \chi(G)\rfloor + 2.$$

Let p, q be integers. We shall denote as $[p, q]$ the *integer interval* bounded by p and q, i.e., the set $\{z \in \mathbb{Z} : p \leq z \leq q\}$. Analogously, we define $[p, \infty) := \{z \in \mathbb{Z} : z \geq p\}$. For $q \in [2, \infty)$ and $m \in \mathbb{Z}$ let $(m)_q$ be the (unique) integer in $[0, q-1]$ satisfying $(m)_q \equiv m (\mathrm{mod}\ q)$. If $k \in [1, \infty)$, a proper k-vertex-colouring of G can be viewed as a decomposition $\mathcal{V} = \{V_i : i \in [1, k]\}$ of the set $V(G)$ in which all sets $V_1, \ldots, V_k$ are independent. Without loss of generality we may assume that $|V_i| \leq |V_j|$ whenever $i, j \in [1, k]$, $i < j$. The nonincreasing sequence $(|V_k|, \ldots, |V_1|)$ is then called the *colour frequency sequence* of $\mathcal{V}$.

Theorem 1. *If $G \neq K_2$ is a connected graph, then* $\mathrm{gndi}(G) \geq \lceil \log_2 \chi(G)\rceil + 1$.

Proof. Put $k := \mathrm{gndi}(G)$ and consider a neighbour-distinguishing colouring $\varphi : E(G) \to [1, k]$. For $A \subseteq [1, k]$ let $\bar{A} := [1, k] - A$ and $V_A := \{x \in V(G) : S_\varphi(x) = A\}$. Clearly, V_A is an independent set of vertices of G. Moreover, if $x \in V_A$ and $y \in V_{\bar{A}}$, then $\emptyset = A \cap \bar{A} = S_\varphi(x) \cap S_\varphi(y)$, and so $xy \notin E(G)$ (notice that $xy \in E(G)$ implies $\varphi(xy) \in S_\varphi(x) \cap S_\varphi(y)$). Thus, $\{V_A \cup V_{\bar{A}} : A \subseteq [1, k]\}$ is a proper vertex colouring of G using at most $\frac{1}{2} \cdot 2^k = 2^{k-1}$ colours, which leads to $\chi(G) \leq 2^{k-1}$, $\mathrm{gndi}(G) = k \geq \log_2 \chi(G) + 1$, and the desired inequality follows. ∎

Theorem 2. *If G is a connected graph with $\chi(G) \geq 3$, then* $\mathrm{gndi}(G) \leq \lfloor \log_2 \chi(G)\rfloor + 2$.

Proof. Let $\chi := \chi(G)$ and let $\{V_i : i \in [1, \chi]\}$ be a proper vertex colouring for which the colour frequency sequence $(|V_\chi|, \ldots, |V_1|)$ of length χ (i.e., a shortest one) is lexicographically maximal. Then

$$\forall i \in [1, \chi-1]\ \forall j \in [i+1, \chi]\ \forall x_i \in V_i\ \exists x_j \in V_j\ x_i x_j \in E(G).$$

Indeed, provided a vertex x_i of V_i with $i \in [1, \chi-1]$ has no neighbour in some V_j with $j \in [i+1, \chi]$, the proper vertex colouring

$$\{V_l : l \in [1, \chi] - \{i, j\}\} \cup \{V_j \cup \{x_i\}, V_i - \{x_i\}\}$$

would have the colour frequency sequence that is lexicographically greater than $(|V_\chi|, \ldots, |V_1|)$, a contradiction.

If k is the integer determined by the inequalities $2^k \leq \chi < 2^{k+1}$, then $k \leq \log_2 \chi < k+1$ and $k = \lfloor \log_2 \chi\rfloor \geq 1$. Our theorem will be proved by finding a neighbour-distinguishing colouring $\varphi : E(G) \to [1, k+2]$. With

$$\mathcal{A} := \{A \subseteq [1, k+2] : k+2 \in A\} - \{\{1, k+2\}, \{1, 2, k+2\}\} \cup \{[1, k+1]\}$$

we have $|\mathcal{A}| = 2^{k+1} - 1 \geq \chi$, hence there is an injection $f : [1, \chi] \to \mathcal{A}$ satisfying

$$f(\chi - j) = \{j, k+2\}, \quad j \in [2, k+1],$$
$$f(\chi - 1) = [1, k+1],$$
$$f(\chi) = \{k+2\}.$$

Thus, $|f(i)| \geq 2$ for any $i \in [1, \chi-2]$. For $i \in [1, \chi]$ and $x \in V_i$ put

$$A_x := f(i).$$

We shall subsequently define partial colourings

$$\varphi_j : E(G) \to [1, k+2], \quad j = 1, 2, 3, 4, 5, 6,$$

in such a way that φ_j is a continuation of φ_{j-1}, $j = 2, 3, 4, 5, 6$; the colouring $\varphi := \varphi_6$ will have the required properties.

Any edge of G is incident with vertices $x \in V_i$ and $y \in V_j$ where $i \in [1, \chi-1]$ and $j \in [i+1, \chi]$. Thus, we can define

$$(i, j) \neq (\chi-1, \chi) \Rightarrow \varphi_1(xy) := \min(A_x \cap A_y).$$

If $j \neq \chi - 1$, then $k+2 \in A_x \cap A_y$. On the other hand, the assumption $j = \chi - 1$ yields $|A_x| \geq 2$, and so $A_x \cap A_y = A_x \cap [1, k+1] \neq \emptyset$. Therefore, the partial colouring φ_1 is correctly defined and

$$\forall z \in V(G) \qquad S_{\varphi_1}(z) \subseteq A_z.$$

Let us show that

$$\forall i \in [1, \chi-2]\ \forall x \in V_i \quad S_{\varphi_1}(x) = A_x.$$

First, $k+2 \in S_{\varphi_1}(x)$, since $\varphi_1(xy) = k+2$ for any edge xy with $y \in V_\chi$. Further, assume that $i \in [1, \chi-k-2]$. If $1 \in A_x$, then $1 \in S_{\varphi_1}(x)$ as $\varphi_1(xy) = 1$ for any edge xy with $y \in V_{\chi-1}$. If $j \in [2, k+1] \cap A_x$, then $j \in S_{\varphi_1}(x)$ because $\varphi_1(xy) = j$ provided $y \in V_{\chi-j}$. Finally, $i = \chi - j, j \in [2, k+1]$, leads to $j \in S_{\varphi_1}(x) \cap A_x$ for $\varphi_1(xy) = j$ whenever $y \in V_{\chi-1}$.

It remains to define colours of edges xy with $x \in V_{\chi-1}$ and $y \in V_\chi$. The colouring φ_2 is determined by

$$S_{\varphi_1}(y) = \{k+2\} \Rightarrow \varphi_2(xy) := 1.$$

Let us denote

$$V^0_{\chi-1} := \{x \in V_{\chi-1} : S_{\varphi_2}(x) = \emptyset\},$$
$$V^1_{\chi-1} := \{x \in V_{\chi-1} : S_{\varphi_2}(x) \neq \emptyset\}.$$

Further, for $t_0, t_1 \in \{0, 1\}$ let $V_\chi^{t_0 t_1}$ be the set of all those vertices $y \in V_\chi$ for which the statement "y has a neighbour in $V^i_{\chi-1}$" has the truth value t_i, $i = 0, 1$. The mapping φ_3 colours edges xy with $y \in V_\chi^{11}$:

$$\varphi_3(xy) := 1, \qquad S_{\varphi_2}(x) \neq \emptyset,$$
$$\varphi_3(xy) := k+2, \qquad S_{\varphi_2}(x) = \emptyset.$$

The mapping φ_4 is defined in several steps. First, for $y \in V_\chi^{01}$ let $N^-(y)$ be the set of all those neighbours x of y for which the edge xy is not coloured under φ_3. Put $\psi_0 := \varphi_3$ and, provided ψ_j is already determined, let ψ_{j+1} be a continuation of ψ_j colouring edges incident with a fixed vertex $y \in V_\chi^{01}$. The mapping ψ_{j+1} works according to the following rules:

If there is minimal $i \in \{1, 2\}$ such that $S_{\psi_j}(x) \neq \{i\}$ for every $x \in N^-(y)$, put

$$\psi_{j+1}(xy) := i, \quad x \in N^-(y).$$

If there are $x_1, x_2, x_3 \in N^-(y)$ satisfying $S_{\psi_j}(x_1) = \{1\}, S_{\psi_j}(x_2) = \{2\}$ and $S_{\psi_j}(x_3) = \{1, 2\}$, proceed as follows:

$$S_{\psi_j}(x) \neq \{1, 2\} \Rightarrow \psi_{j+1}(xy) := 1,$$
$$S_{\psi_j}(x) = \{1, 2\} \Rightarrow \psi_{j+1}(xy) := k+2.$$

If none of the above assumptions is fulfilled, define $\varphi_4 := \psi_j$.

If a vertex $y \in V_\chi^{01}$ is incident with an edge xy that is not coloured under φ_4, there are $x_1, x_2 \in N^-(y)$ with $S_{\varphi_4}(x_i) = \{i\}$, $i = 1, 2$, and $S_{\varphi_4}(x) \neq \{1, 2\}$ for every $x \in N^-(y)$. This allows us to define:

$$S_{\varphi_4}(x) = \{1\} \Rightarrow \varphi_5(xy) := 1,$$
$$S_{\varphi_4}(x) \neq \{1\} \Rightarrow \varphi_5(xy) := 2.$$

Now consider the set

$$W := V_\chi^{10} \cup V^0_{\chi-1}.$$

From the definition of the colouring φ_5 it is clear that any path joining a vertex $w \in W$ to a vertex $y \in W_0 := V_\chi^{11}$ has all its internal vertices in W. From the connectedness of G we see that $\{W_l : l \in [1, \infty)\}$ with

$$W_l := \{w \in W : \min(d_G(w, z) : z \in W_0) = l\},$$

where $d_G(w, z)$ is the distance between x and z in G, is a decomposition of the set W. Therefore, if $uv \in E(G)$ is an edge for which $\varphi_5(uv)$ is not determined, there is $m \in [1, \infty)$ such that $u \in W_m$ and $v \in W_{m+1}$. To define $\varphi_6(uv)$ it is useful to introduce an auxiliary name 0 for the colour $k+2$. Under that assumption we proceed in the following way:

$$(\exists m \in [1, \infty)\ (u \in W_m \wedge v \in W_{m+1})) \Rightarrow \varphi_6(uv) := (-m)_3;$$

notice that if $u \in W_0$ and $v \in W_1$, then $\varphi_6(uv) = \varphi_3(uv) = k+2 = (-0)_3$, and so the above definition is valid also for $m = 0$.

Let us now prove that $\varphi := \varphi_6$ is a neighbour-distinguishing colouring. For that purpose let $xy \in E(G)$ be an edge with $x \in V_i$ and $y \in V_j$ where $i < j$.

If $j \le \chi - 2$, then $S_\varphi(x) = S_{\varphi_1}(x) = f(i) \neq f(j) = S_{\varphi_1}(y) = S_\varphi(y)$ (recall that f is an injection).

If $j = \chi - 1$, then $i \le \chi - 2$ and $y \in V^1_{\chi-1} = V_{\chi-1} - W$. Therefore, the set $S_\varphi(x) = S_{\varphi_1}(x) \in \mathcal{A}$ contains $k+2$, but is distinct from $\{1, 2, k+2\}$. On the other hand, if the set $S_\varphi(y) = S_{\varphi_5}(y)$ contains $k+2$, then $S_\varphi(y) = \{1, 2, k+2\} \neq S_\varphi(x)$.

If $j = \chi$ and $i \le \chi - 2$, then $y \in V_\chi - W$ and $S_\varphi(y) = S_{\varphi_2}(y) \in \{\{k+2\}, \{1, k+2\}\}$, while $S_\varphi(x) = S_{\varphi_1}(x) \in \mathcal{A} - \{\{k+2\}\}$, and so $S_\varphi(x) \neq S_\varphi(y)$.

It remains to consider the case $i = \chi - 1$ and $j = \chi$.

First suppose that $x \in V_{\chi-1} - W$ which implies $S_\varphi(x) = S_{\varphi_5}(x) \neq \{1, k+2\}$. If $y \in V_\chi^{11}$, then $S_\varphi(y) = S_{\varphi_3}(y) = \{1, k+2\} \neq S_\varphi(x)$. Henceforth we may assume that $y \in V_\chi - (V_\chi^{11} \cup W)$ and $S_\varphi(y) = S_{\varphi_5}(y) \in \{\{1\}, \{2\}, \{1, 2\}, \{1, k+2\}\}$. If there is

$l \in \{1, 2\}$ such that $S_\varphi(y) = \{l\}$, then $S_\varphi(x) = S_{\varphi_5}(x)$, $|S_{\varphi_5}(x)| \geq 2$ and $S_\varphi(x) \neq S_\varphi(y)$. On the other hand, $S_\varphi(y) = \{1, 2\}$ implies $S_\varphi(x) \neq \{1, 2\}$ because of the definition of φ_5.

If $x \in W$, then $y \in W_0 \cup W$ and there are two possibilities. If $x \in W_{2l-1}$ and $y \in W_{2l}$ for some $l \in [1, \infty)$, then $S_\varphi(x) = S_{\varphi_6}(x) = \{(1-2l)_3, (2-2l)_3\}$ and $S_\varphi(y) = S_{\varphi_6}(y)$ is either $\{(-2l)_3, (1-2l)_3\}$ (if y has a neighbour in W_{2l+1}) or $\{(1-2l)_3\}$ (otherwise), in both cases $S_\varphi(x) \neq S_\varphi(y)$. Similarly, if there is $m \in [0, \infty)$ such that $x \in W_{2m+1}$ and $y \in W_{2m}$, then $S_\varphi(y) = \{(-2m)_3, (1-2m)_3\}$, $S_\varphi(x) \in \{\{(-2m)_3\}, \{(-2m)_3, (2-2m)_3\}\}$, and hence $S_\varphi(x) \neq S_\varphi(y)$. ■

Corollary 3. *If G is a connected graph with $\chi(G) \geq 3$ and $\chi(G)$ is not an integer power of two, then* $\text{gndi}(G) = \lceil \log_2 \chi(G) \rceil + 1$. ■

If $\chi(G) = 2^k$ with $k \in \mathbb{Z}$, Theorems 1 and 2 yield $\text{gndi}(G) \in [k+1, k+2]$. In the case $k = 1$ both possibilities $\text{gndi}(G) = 2$ and $\text{gndi}(G) = 3$ can apply. (In [3] there are classified bipartite graphs with respect to the general neighbour-distinguishing index.) Provided $k = 2$, the first upper bound for $\text{gndi}(G)$ given by [4] is $3\lceil \frac{1}{2} \log_2 4 \rceil = 3 = k + 1$ so that the upper bound of Theorem 2 is not attained. This leads to the following natural question:

Problem 1. Does there exist $k \in [3, \infty)$ and a connected graph G such that $\chi(G) = 2^k$ and $\text{gndi}(G) = k + 2$?

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