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Note

A constructive characterization of total domination vertex critical graphs[☆]

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Abstract

Let G be a graph of order n and maximum degree $\Delta(G)$ and let $\gamma_t(G)$ denote the minimum cardinality of a total dominating set of a graph G. A graph G with no isolated vertex is the total domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of G. We call these graphs γ_t-critical. For any γ_t-critical graph G, it can be shown that $n \leq \Delta(G)(\gamma_t(G) - 1) + 1$. In this paper, we prove that: Let G be a connected γ_t-critical graph of order n ($n \geq 3$), then $n = \Delta(G)(\gamma_t(G) - 1) + 1$ if and only if G is regular and, for each $v \in V(G)$, there is an $A \subseteq V(G) - \{v\}$ such that $N(v) \cap A = \emptyset$, the subgraph induced by A is 1-regular, and every vertex in $V(G) - A - \{v\}$ has exactly one neighbor in A.

Keywords: Total domination set; Total domination number; Vertex critical graphs; Cayley graphs; Corona

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). For notation and terminology not presented here, we in general follow [1]. In what follows, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

Let v be a vertex of G, the *open neighborhood* of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a subset S of vertices, we define the open neighborhood $N(S) = \bigcup_{v\in S} N(v)$, and the closed neighborhood $N[S] = \bigcup_{v\in S} N[v]$. Let $N_S(v) = N(v) \cap S$, and $d_S(v) = |N_S(v)|$. The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, and the vertex v is called a vertex of degree k if $d_G(v) = k$. An edge $e \in E(G)$ is called a *pendant* edge if e is incident with a vertex of degree 1. If A is another subset of $V(G)$ disjoint from S, we let $E(A, S) = \{uv \in E(G) \mid u \in A \text{ and } v \in S\}$, $e(A, S) = |E(A, S)|$, and let $\langle A\rangle$ denote the subgraph of G induced by A.

The set $S \subseteq V(G)$ is a *dominating* set of G if $N[S] = V(G)$, and a *total dominating* set if $N(S) = V(G)$. The minimum cardinality among all total dominating sets of G is the *total domination number* of G, denoted by $\gamma_t(G)$. A total dominating set with cardinality $\gamma_t(G)$ we call a γ_t-*set*.

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Let $S \subseteq V(G)$ and let $G - S$ denote the graph obtained from G by deleting all the vertices of S together with all the edges with at least one endvertex in S. When $S = \{x\}$, we simplify this notation to $G - x$. A vertex $v \in V(G)$ is called *γ_t-critical* if $\gamma_t(G - v) < \gamma_t(G)$. Let $D_1(G) = \{v \in V(G) \mid d_G(v) = 1\}$ and $W(G) = N(D_1(G))$. A graph G is said to be *total domination vertex critical*, or *γ_t-critical* if each vertex of $V(G) - W(G)$ is critical. G is called a *k-γ_t-critical* graph if G is γ_t-critical, and $\gamma_t = k$. For example, the 5-cycle is 3-γ_t-critical. Throughout this paper, we let $n = |V(G)|$ denote the order of G.

We organize this paper as follows. In Section 2, we define a family of graphs, and study a few properties of the graphs defined. In Section 3, we give our main theorem. Finally, in Section 4 we propose two open problems.

2. The properties of the class of graphs

Note that a graph is γ_t-critical if and only if each component is γ_t-critical, so we only need to consider connected graphs. Furthermore, K_2 is 2-γ_t-critical, indicated by Goddard et al. [2]. We define that K_1 is 1-γ_t-critical.

Definition 2.1. We define a family of graphs Ψ as follows:

(1) $K_1, K_2 \in \Psi$.

(2) Let G be a connected graph with at least 3 vertices. $G \in \Psi$ if and only if both the following two conditions hold:

(i) G is a regular graph;

(ii) For any $v \in V(G)$, there exists an $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$, $\langle A \rangle$ is 1-regular, $d_A(y) = 1$ for each $y \in V(G) - A - \{v\}$.

Here is an example showing that such a graph exists in Ψ. Let G be a cycle of length 9. Suppose $G = C_9 = v_0 v_1 \cdots v_8 v_0$ and $v = v_8$. Then $G - v$ is a path and $A = \{v_1, v_2, v_5, v_6\}$, $V(G) - A - \{v\} = \{v_0, v_3, v_4, v_7\}$.

The following proposition is useful when we construct a graph in Ψ.

Proposition 2.2. *Let $G \in \Psi$. If G is a connected graph of order n ($n \geq 3$). Then n is odd and G is k-regular, where k is even.*

Proof. Since $G \in \Psi$, by Definition 2.1, G is k-regular and, for any $v \in V(G)$, there is an $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$ and $\langle A \rangle$ is 1-regular, and $d_A(y) = 1$ for any $y \in V(G - v) - A$. Let $|A| = m$. Since $\langle A \rangle$ is 1-regular, we have that m is even. Since G is k-regular, $\langle A \rangle$ is 1-regular and $d_A(y) = 1$ for any $y \in V(G - v) - A$, we have that $|V(G - v) - A| = m(k - 1)$. Since $|V(G)| = |A| + |V(G - v) - A| + |\{v\}|$, $n = m + m(k - 1) + 1 = mk + 1$. Thus, n is odd. Since $2|E(G)| = \sum_{v \in V(G)} d(v)$, which implies that kn is even, we have that k is even. ■

The following proposition characterize the cycles in Ψ.

Proposition 2.3. *A cycle $G = C_n$ is in Ψ if and only if $n \equiv 1 (\mathrm{mod}\ 4)$.*

Proof. "⇒" Assume that $G = C_n$ is in Ψ. Let $v \in V(G)$. By Definition 2.1, there exists $A \subseteq V(G - v)$ such that $N(v) \cap A = \emptyset$ and $\langle A \rangle$ is 1-regular, $d_A(y) = 1$ for any $y \in V(G - v) - A$. Since $\langle A \rangle$ is 1-regular, we have that $|A|$ is even, suppose $|A| = 2m$. Since G is 2-regular, $\langle A \rangle$ is 1-regular and $d_A(y) = 1$ for any $y \in V(G - v) - A$, we have that $|V(G - v) - A| = |A| = 2m$. Thus, $|V(G)| = 4m + 1$. Since G is a cycle, G is isomorphic to a cycle of length $4m + 1$. Thus, $n \equiv 1 (\mathrm{mod}\ 4)$.

"⇐" Assume that $G = C_n$ and $n \equiv 1 (\mathrm{mod}\ 4)$. By symmetry, choose $v = 0$ and $\langle A \rangle = \{(2, 3), (6, 7), \ldots, (4m - 2, 4m - 1)\}$. It is easy to check that $G \in \Psi$. ■

Let H be a group and S be a generating subset of H such that the identity element $1 \notin S$ and $x^{-1} \in S$ for each $x \in S$. A *Cayley graph* $G(H; S)$ on group H is defined to be the graph with the elements of H as its vertices and edges joining h and hs for all $h \in H$ and $s \in S$. S is called the connection set.

When $k \geq 4$, we construct a Cayley graph as follows. Let H be a cyclic group Z_{33}. Then, we have that Cayley graph $G(H; S)$ with connection set $S = \{6, 27\}$ is in Ψ. By symmetry, we choose $v = 0$ and $\langle A \rangle = \{(2, 8), (11, 12), (21, 22), (25, 31)\}$. It is easy to check that $G(H; S) \in \Psi$.

3. Main theorem and its proof

To prove our main theorem, we need some necessary results also on γ_t-critical graphs.

Observation 3.1 (*[2]*). *If G is a γ_t-critical graph, then $\gamma_t(G-v)=\gamma_t-1$ for any $v\in V(G)-W(G)$. Furthermore, each $\gamma_t(G-v)$-set contains no neighbor of v.*

A *corona* of a graph H, denoted by $cor(H)$, is the graph obtained from H by adding a pendant edge to each vertex of H. Goddard et al. [2] characterized the γ_t-critical graphs with at least one vertex of degree 1 as follows.

Theorem 3.2 (*[2]*). *Let G be a connected graph with at least 3 vertices and at least one vertex of degree 1. Then, G is k-γ_t-critical if and only if $G=cor(H)$ for some connected graph H with k vertices and $\delta(H)\geq 2$.*

The next lemma plays a key role in the proof of our main theorem.

Lemma 3.3. *Let G be a connected γ_t-critical graph of order n ($n\geq 3$). If $n=\Delta(G)(\gamma_t(G)-1)+1$, then the following results hold:*
(i) $\delta(G)\geq 2$;
(ii) *For any $v\in V$, there is an $A\subseteq V(G-v)$ such that A is a $\gamma_t(G-v)$-set and $N(v)\cap A=\emptyset$. Moreover, $\langle A\rangle$ is 1-regular, $d_G(x)=\Delta(G)$ for any $x\in A$ and $d_A(y)=1$ for any $y\in V(G)-A-\{v\}$.*

Proof. (i) Suppose, to the contrary, that $\delta(G)=1$. Let G be a k-γ_t-critical graph. By Theorem 3.2, $G=cor(H)$ for some connected graph H with k vertices and $\delta(H)\geq 2$, which implies that $k\geq 3$. Furthermore, we have that $n=2k$ and $3\leq\Delta(G)\leq k$, from which we obtain $1+\frac{1}{\Delta(G)-2}\leq 2$ and $k>1+\frac{1}{\Delta(G)-2}$. Therefore we have that $2k<\Delta(G)(k-1)+1$ and $n<\Delta(G)(k-1)+1$, which contradicts $n=\Delta(G)(\gamma_t(G)-1)+1$. Thus, $\delta(G)\geq 2$.

(ii) Since $\delta(G)\geq 2$, we have that $W(G)=\emptyset$. Let G be a k-γ_t-critical graph, by Observation 3.1, $\gamma_t(G-v)=k-1$ for any $v\in V(G)$. Let A be a $\gamma_t(G-v)$-set of $G-v$ and let $B=V(G-v)-A$, by Observation 3.1, we have that $N(v)\cap A=\emptyset$. So we have that $|A|=k-1$ and $|B|=n-k$. Since A is a $\gamma_t(G-v)$-set, we have that $d_A(x)\geq 1$ for any $x\in A$ and $d_A(y)\geq 1$ for any $y\in B$, which means that each vertex of A has at most $\Delta(G)-1$ neighbors in B and each vertex of B has at least one neighbor in A. Hence,

$$n-k\leq e(A,B)\leq (k-1)(\Delta(G)-1). \tag{1}$$

That is,

$$n\leq \Delta(G)(\gamma_t(G)-1)+1. \tag{2}$$

The equality $n=\Delta(G)(\gamma_t(G)-1)+1$ holds if and only if both the equalities of (1) hold. So we have that $n-k=e(A,B)=(k-1)(\Delta(G)-1)$, which means that each vertex of A has exactly $\Delta(G)-1$ neighbors in B, and each vertex of B has exactly one neighbor in A. Therefore, there exists a partition of $V(G)-v$ for any $v\in V$, i.e., there exist $A\subseteq V(G)-v$ and $B=V(G-v)-A$, such that A is a $\gamma_t(G-v)$-set and $N(v)\cap A=\emptyset$. Moreover, $\langle A\rangle$ is 1-regular, $d_G(x)=\Delta(G)$ for any $x\in A$ and $d_A(y)=1$ for any $y\in B$. This completes the proof of Lemma 3.3. ■

Let f be a map from set A to set B. We say that f is *injective* if $f(a)\neq f(a')$ for every pair $a\neq a'\in A$. We say that f is *surjective* if there exists x in A such that $f(x)=y$ for every y in B.

In the following, we give the main theorem of this paper.

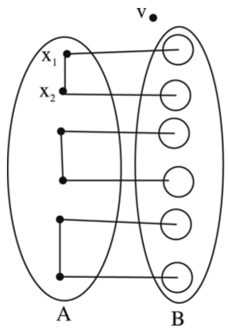
Theorem 3.4. *Let G be a connected γ_t-critical graph of order n. Then $n=\Delta(G)(\gamma_t(G)-1)+1$ if and only if $G\in\Psi$.*

Proof. It is easy to check that when $n=1$ or 2, Theorem 3.4 is true. So we may assume that G is a graph with at least three vertices.

"$\Rightarrow$" Assume that $n=\Delta(G)(\gamma_t(G)-1)+1$ and $\gamma_t(G)=k$. We will show that $G\in\Psi$.

By Lemma 3.3, for any $v\in V(G)$, there is an $A\subseteq V(G-v)$ such that A is a $\gamma_t(G-v)$-set and $N(v)\cap A=\emptyset$. Let $B=V(G)-A-\{v\}$. Moreover, $\langle A\rangle$ is 1-regular, $d_G(x)=\Delta(G)$ for any $x\in A$ and $d_A(y)=1$ for any $y\in B$.

Now we only need to prove that $\delta(G)=\Delta(G)$, which implies that G is $\Delta(G)$-regular.



Partition of $V(G)-v$

Fig. 1.

Suppose, to the contrary, that there exists a vertex $v \in V(G)$ such that $d_G(v) = \delta(G) < \Delta(G)$. Consider the subgraph $G - v$. By Lemma 3.3, $\delta(G) \geq 2$, so $W(G) = \emptyset$. By Lemma 3.3, there is an $A \subseteq V(G-v)$ such that A is a $\gamma_t(G-v)$-set and $N(v) \cap A = \emptyset$, $\langle A\rangle$ is 1-regular. Suppose $A = \{x_1, x_2, \ldots, x_{k-1}\}$, and $E(\langle A\rangle) = \{x_1x_2, x_3x_4, \ldots, x_{k-2}x_{k-1}\}$, and $B_i = N_B(x_i)$ for each $1 \leq i \leq k-1$, then B is the disjoint union of B_i, $1 \leq i \leq k-1$. (See Fig. 1).

We consider the subgraph $G - x_1$. Since G is a k-$\gamma_t(G)$-critical graph, by Observation 3.1, we have that $\gamma_t(G-x_1) = k-1$. By Lemma 3.3, we obtain:

(♣) {*There is an* $A' \subseteq V(G) - x_1$ *such that* $N(x_1) \cap A' = \emptyset$ *and,* A' *is a* $\gamma_t(G-x_1)$*-set with* $|A'| = k-1$. $\langle A'\rangle$ *is 1-regular,* $d_G(u) = \Delta(G)$ *for any* $u \in A'$.*Let* $B' = V(G-x_1) - A'$, *then* $d_{A'}(y) = 1$ *for any* $y \in B'$.}

Claim 1. $A' \cap A \neq \emptyset$.

Proof. Suppose, to the contrary, $A' \cap A = \emptyset$. Then the following two results hold:

(a) $A' \subseteq \cup_{i=2}^{k-1} B_i$.

Proof. Since $V(G) = A \cup B \cup \{v\} = A' \cup B' \cup \{x_1\}$ and $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$, $v \notin A$, $v \notin B$ and, $x_1 \notin A'$, $x_1 \notin B'$, then $A \subseteq B' \cup \{x_1\}$ and $A' \subseteq B \cup \{v\}$, which give $A - x_1 \subseteq B'$ and $A' - v \subseteq B$. Since $d_G(v) = \delta(G) < \Delta(G)$, then $v \notin A'$, therefore $A' \subseteq B$. Since $N(x_1) \cap A' = \emptyset$, that is, $(x_2 \cup B_1) \subseteq B'$. Therefore $A' \subseteq \cup_{i=2}^{k-1} B_i$.

(b) $|A' \cap B_i| \leq 1 (2 \leq i \leq k-1)$.

Proof. Suppose, to the contrary, that there exist some j, $2 \leq j \leq k-1$ such that $|B_j \cap A'| \geq 2$. Since $B_j = N_B(x_j)$, then $|N_B(x_j) \cap A'| \geq 2$. Since $x_j \in A$ and $A - x_1 \subseteq B'$, then $x_j \in B'$. Thus $d_{A'}(x_j) \geq 2$. But by (♣), $d_{A'}(x_j) = 1$ $(2 \leq j \leq k-1)$. This is a contradiction.

By (a) and (b), $|A'| \leq k-2$. This contradicts the fact that A' is a $\gamma_t(G-x_1)$-set with $|A'| = k-1$. ■

If there exists $x \in A' - A$, then $x \notin A$. Therefore, $x \in B$. By Lemma 3.3, there exists a unique vertex $z \in A$ such that $xz \in E(G)$. If $z \in A \cap A'$, then there exists a unique vertex $z_1 \in A - A'$ such that $z_1 z \in E(\langle A\rangle)$ since $\langle A\rangle$ is 1-regular. Let $f : A' - A \to A - A'$ be defined by (See Fig. 2)

$$f(x) = \begin{cases} z & \text{if } x \in A' - A,\ N_A(x) = \{z\}, \text{ and } z \notin A' \cap A; \\ z_1 & \text{if } x \in A' - A,\ N_A(x) = \{z\}, z \in A' \cap A, \text{ and } N_A(z) = \{z_1\}. \end{cases}$$

Claim 2. *f is well defined.*

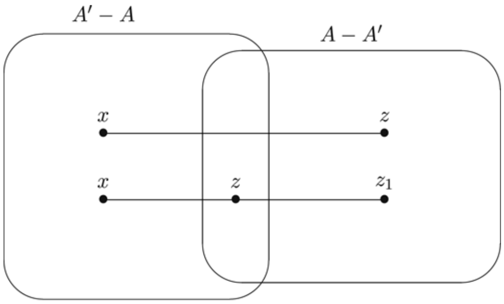
Proof. Firstly, we prove that $f(x) \in A - A'$. Consider the two Cases:

Case (1). $x \in A' - A$, $N_A(x) = \{z\}$, and $z \notin A' \cap A$.

By the definition of f, $f(x) = z$. Since $N_A(x) = \{z\}$, then $z \in A$. Since $z \notin A' \cap A$, then $z \notin A'$. Therefore, $z \in A - A'$, that is, $f(x) \in A - A'$.

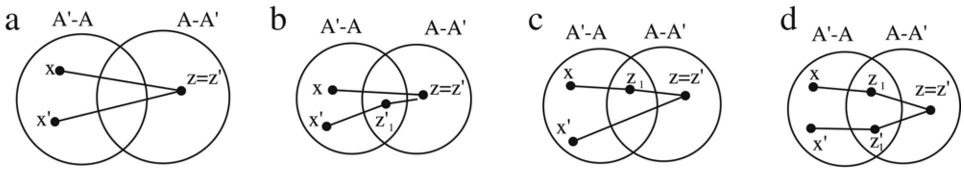
Case (2). $x \in A' - A$, $N_A(x) = \{z\}$, $z \in A' \cap A$, and $N_A(z) = \{z_1\}$

By the definition of f, $f(x) = z_1$. Since $N_A(z) = \{z_1\}$, then $z_1 \in A$. If $z_1 \in A'$, then $z, z_1 \in A' \cap A$ since $z \in A' \cap A$. Thus, $x, z, z_1 \in A'$ and $\{x, z_1\} \subseteq N_{A'}(z)$, that is, $d_{A'}(z) \geq 2$, which contradicts the fact that $\langle A'\rangle$ is 1-regular. Therefore, $z_1 \notin A'$, that is, $z_1 \in A - A'$.



A map $f : A' - A \to A - A'$

Fig. 2.



$x, x' \in A' - A$, $x \neq x'$, and $f(x) = z$, $f(x') = z'$, $z = z'$.

Fig. 3.

By Case (1) and Case (2), $f(x) \in A - A'$.

Secondly, by Lemma 3.3, for each $x \in B$, there exists a unique vertex $z \in A$ such that $xz \in E(G)$. Note that $\langle A \rangle$ is 1-regular. Thus, $f(x)$ is unique for each vertex $x \in A' - A$.

This completes the proof of Claim 2. ■

Claim 3. *f is injective.*

Proof. For two vertices $x, x' \in A' - A$, $x \neq x'$. Suppose $f(x) = z$, $f(x') = z'$. We will show that $z \neq z'$. Suppose, to the contrary, $z = z'$. By the definition of f, $z \in A - A'$. Thus, $z \in B'$. We have the following cases:

Case (1). $x \in A' - A$, $N_A(x) = \{z\}$, and $z \notin A' \cap A$.

Subcase (1.1). $x' \in A' - A$, $N_A(x') = \{z'\}$, and $z' \notin A' \cap A$. (See Fig. 3(a)). Since $N_A(x) = \{z\}$, $N_A(x') = \{z'\}$, $xz, x'z' \in E(G)$. Therefore $\{x, x'\} \subseteq N_{A'}(z)$, that is, $d_{A'}(z) \geq 2$. This contradicts (♣).

Subcase (1.2). $x' \in A' - A$, $N_A(x') = \{z_1'\}$, $z_1' \in A' \cap A$, and $N_A(z_1') = \{z'\}$. (See Fig. 3(b)). Since $N_A(x) = \{z\}$, $xz \in E(G)$. Since $z_1' \in A \cap A'$ and $N_A(z_1') = \{z'\}$, $\{z_1', x\} \subseteq N_{A'}(z)$. Thus $d_{A'}(z) \geq 2$. This contradicts (♣).

Case(2). $x \in A' - A$, $N_A(x) = \{z_1\}$, $z_1 \in A' \cap A$, and $N_A(z_1) = \{z\}$.

Subcase (2.1). $x' \in A' - A$, $N_A(x') = \{z'\}$, and $z' \notin A' \cap A$. (See Fig. 3(c)). Since $N_A(z_1) = \{z\}$ and $N_A(x') = \{z'\}$, $z_1 z, x'z' \in E(G)$. Since $z_1 \in A' \cap A$, $\{z_1, x'\} \subseteq N_{A'}(z)$. Thus, $d_{A'}(z) \geq 2$. This contradicts (♣).

Subcase (2.2). $x' \in A' - A$, $N_A(x') = \{z_1'\}$, $z_1' \in A' \cap A$, and $N_A(z_1') = \{z'\}$. (See Fig. 3(d).) Then $z_1 z \in E(G)$ and $z_1' z' \in E(G)$. If $z_1 \neq z_1'$, then $d_A(z) \geq 2$ since $z_1, z_1' \in A' \cap A$. This contradicts the fact that $\langle A \rangle$ is 1-regular. If $z_1 = z_1'$, then $d_{A'}(z_1) \geq 2$ since $N_A(x) = \{z_1\}$ and $N_A(x') = \{z_1'\}$. This contradicts the fact that $\langle A' \rangle$ is 1-regular.

This completes the proof of Claim 3. ■

Claim 4. *f is not surjective.*

Proof. By (♣), $N(x_1) \cap A' = \emptyset$, then x_1 has no neighbor in A'. Furthermore, x_1 has no neighbor in $A' - A$. Since $x_1 x_2 \in E(\langle A \rangle)$, $x_2 \in A \cap B'$. Thus, $x_2 \notin A' \cap A$, $xx_2 \notin E(\langle A' \rangle)$ for each $x \in A' - A$. Note that $x_1 \in A - A'$. Thus, $f(x) \neq x_1$ for each $x \in A' - A$. Hence f is not surjective. ■

From Claim 2 to Claim 4, it follows that $|A' - A| < |A - A'|$, which contradicts the fact that $|A| = |A'| = k - 1$. The contradiction shows that G is $\Delta(G)$-regular. Thus we obtain $G \in \Psi$.

"$\Leftarrow$" Assume that G is a k-γ_t-critical graph with n vertices and $G \in \Psi$. We will show that $n = \Delta(G)(\gamma_t(G) - 1) + 1$.

Since $G \in \Psi$, then G is $\Delta(G)$-regular and $\Delta(G) \geq 2$. Moreover, for any $v \in V(G)$, there exists an $A \subseteq V(G-v)$ such that $N(v) \cap A = \emptyset$ and $\langle A \rangle$ is 1-regular, and let $B = V(G-v) - A$, then $d_A(y) = 1$ for any $y \in B$, which means that $d_A(x) = 1$ for any $x \in V(G-v)$. So we have that

$$n - |A| - 1 = e(A, B) = |A|(\Delta(G) - 1),$$

that is,

$$n = \Delta(G)|A| + 1. \tag{3}$$

Since $d_A(x) = 1$ for any $x \in V(G-v)$, A is a total dominating set of $G-v$. By Observation 3.1, we have that

$$|A| \geq \gamma_t(G-v) = \gamma_t(G) - 1. \tag{4}$$

By (3) and (4), we obtain

$$n \geq \Delta(G)(\gamma_t(G) - 1) + 1. \tag{5}$$

On the other hand, since G is a k-γ_t-critical graph, by Observation 3.1, we have that $\gamma_t(G-v) = k-1$ for any $v \in V(G)$. Let A' be a $\gamma_t(G-v)$-set of $G-v$ and $B' = V(G-v) - A'$. So we have that $|A'| = k-1$ and $|B'| = n-k$. Since A' is a $\gamma_t(G-v)$-set, we have that $d_{A'}(x) \geq 1$ for any $x \in A'$ and $d_{A'}(y) \geq 1$ for any $y \in B'$, which means that each vertex of A' has at most $\Delta(G) - 1$ neighbors in B' and each vertex of B' has at least one neighbor in A'. Hence,

$$n - k \leq e(A', B') \leq (k-1)(\Delta(G) - 1). \tag{6}$$

that is,

$$n \leq \Delta(G)(\gamma_t(G) - 1) + 1. \tag{7}$$

By (5) and (7), we have that $n = \Delta(G)(\gamma_t(G) - 1) + 1$.

This completes the proof of Theorem 3.4. ■

4. Open problems

We close with two open problems.

1. Does there exists a graph in Ψ which is not a Cayley graph?
2. Motivated by Propositions 2.2 and 2.3, we propose the problem: Characterize $2k$-regular graphs in Ψ, for each $k \geq 2$?

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