Note

A constructive characterization of total domination vertex critical graphs

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Abstract

Let $G$ be a graph of order $n$ and maximum degree $\Delta(G)$ and let $\gamma_t(G)$ denote the minimum cardinality of a total dominating set of a graph $G$. A graph $G$ with no isolated vertex is the total domination vertex critical if for any vertex $v$ of $G$ that is not adjacent to a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of $G$. We call these graphs $\gamma_t$-critical. For any $\gamma_t$-critical graph $G$, it can be shown that $n \leq \Delta(G)(\gamma_t(G) - 1) + 1$. In this paper, we prove that: Let $G$ be a connected $\gamma_t$-critical graph of order $n$ ($n \geq 3$), then $n = \Delta(G)(\gamma_t(G) - 1) + 1$ if and only if $G$ is regular and, for each $v \in V(G)$, there is an $A \subseteq V(G) - \{v\}$ such that $N(v) \cap A = \emptyset$, the subgraph induced by $A$ is 1-regular, and every vertex in $V(G) - A - \{v\}$ has exactly one neighbor in $A$.

Keywords: Total domination set; Total domination number; Vertex critical graphs; Cayley graphs; Corona

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). For notation and terminology not presented here, we in general follow [1]. In what follows, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

Let $v$ be a vertex of $G$, the open neighborhood of $v$ is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. For a subset $S$ of vertices, we define the open neighborhood $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood $N[S] = \bigcup_{v \in S} N[v]$. Let $N_S(v) = N(v) \cap S$, and $d_S(v) = |N_S(v)|$. The degree $d_G(v)$ of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$, and the vertex $v$ is called a vertex of degree $k$ if $d_G(v) = k$. An edge $e \in E(G)$ is called a pendant edge if $e$ is incident with a vertex of degree 1. If $A$ is another subset of $V(G)$ disjoint from $S$, we let $E(A, S) = \{uv \in E(G) \mid u \in A$ and $v \in S\}$, $e(A, S) = |E(A, S)|$, and let $(A)$ denote the subgraph of $G$ induced by $A$.

The set $S \subseteq V(G)$ is a dominating set of $G$ if $N[S] = V(G)$, and a total dominating set if $N[S] = V(G)$. The minimum cardinality among all total dominating sets of $G$ is the total domination number of $G$, denoted by $\gamma_t(G)$. A total dominating set with cardinality $\gamma_t(G)$ we call a $\gamma_t$-set.

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Let $S \subseteq V(G)$ and let $G - S$ denote the graph obtained from $G$ by deleting all the vertices of $S$ together with all the edges with at least one endvertex in $S$. When $S = \{x\}$, we simplify this notation to $G - x$. A vertex $v \in V(G)$ is called $\gamma_t$-critical if $\gamma_t(G - v) < \gamma_t(G)$. Let $D_1(G) = \{v \in V(G) \mid d_G(v) = 1\}$ and $W(G) = N(D_1(G))$. A graph $G$ is said to be total domination vertex critical, or $\gamma_t$-critical if each vertex of $V(G) - W(G)$ is critical. $G$ is called a $k$-$\gamma_t$-critical graph if $G$ is $\gamma_t$-critical, and $\gamma_t = k$. For example, the 5-cycle is 3-$\gamma_t$-critical. Throughout this paper, we let $n = |V(G)|$ denote the order of $G$.

We organize this paper as follows. In Section 2, we define a family of graphs, and study a few properties of the graphs defined. In Section 3, we give our main theorem. Finally, in Section 4 we propose two open problems.

2. The properties of the class of graphs

Note that a graph is $\gamma_t$-critical if and only if each component is $\gamma_t$-critical, so we only need to consider connected graphs. Furthermore, $K_2$ is 2-$\gamma_t$-critical, indicated by Goddard et al. [2]. We define that $K_1$ is 1-$\gamma_t$-critical.

**Definition 2.1.** We define a family of graphs $\Psi$ as follows:

1. $K_1, K_2 \in \Psi$.
2. Let $G$ be a connected graph with at least 3 vertices. $G \in \Psi$ if and only if both the following two conditions hold:
   1. $G$ is a regular graph;
   2. For any $v \in V(G)$, there exists an $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$, $(A)$ is 1-regular, $d_A(y) = 1$ for each $y \in V(G) - A - \{v\}$.

Here is an example showing that such a graph exists in $\Psi$. Let $G$ be a cycle of length 9. Suppose $G = C_9 = v_0v_1 \cdots v_8v_0$ and $v = v_8$. Then $G - v$ is a path and $A = \{v_1, v_2, v_5, v_6\}$, $V(G) - A - \{v\} = \{v_0, v_3, v_4, v_7\}$.

The following proposition is useful when we construct a graph in $\Psi$.

**Proposition 2.2.** Let $G \in \Psi$. If $G$ is a connected graph of order $n$ ($n \geq 3$). Then $n$ is odd and $G$ is $k$-regular, where $k$ is even.

**Proof.** Since $G \in \Psi$, by **Definition 2.1**, $G$ is $k$-regular and, for any $v \in V(G)$, there is an $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$ and $(A)$ is 1-regular, and $d_A(y) = 1$ for each $y \in V(G) - A - \{v\}$. Since $(A)$ is 1-regular, we have that $m$ is even. Since $G$ is $k$-regular, $(A)$ is 1-regular and $d_A(y) = 1$ for each $y \in V(G) - A$, we have that $|V(G) - A| = m(k - 1)$. Since $|V(G)| = |A| + |V(G) - A| + |\{v\}|$, $n = m + m(k - 1) + 1 = mk + 1$. Thus, $n$ is odd. Since $2|E(G)| = \sum_{v \in V(G)} d(v)$, which implies that $kn$ is even, we have that $k$ is even. 

The following proposition characterize the cycles in $\Psi$.

**Proposition 2.3.** A cycle $G = C_n$ is in $\Psi$ if and only if $n \equiv 1 \text{mod } 4$.

**Proof.** “⇒” Assume that $G = C_n$ is in $\Psi$. Let $v \in V(G)$. By **Definition 2.1**, there exists $A \subseteq V(G) - v$ such that $N(v) \cap A = \emptyset$, $(A)$ is 1-regular, $d_A(y) = 1$ for each $y \in V(G) - A$, and $(A)$ is 1-regular, we have that $|A|$ is even, suppose $|A| = 2m$. Since $G$ is $2$-regular, $(A)$ is 1-regular and $d_A(y) = 1$ for each $y \in V(G) - A$, we have that $|V(G) - A| = |A| = 2m$. Thus, $|V(G)| = 4m + 1$. Since $G$ is a cycle, $G$ is isomorphic to a cycle of length $4m + 1$. Thus, $n \equiv 1 \text{mod } 4$.

“⇐” Assume that $G = C_n$ and $n \equiv 1 \text{mod } 4$. By symmetry, choose $v = 0$ and $(A) = \{2, 3, 6, 7, \ldots, 4m - 2, 4m - 1\}$. It is easy to check that $G \in \Psi$.

Let $H$ be a group and $S$ be a generating subset of $H$ such that the identity element $1 \not\in S$ and $x^{-1} \in S$ for each $x \in S$. A Cayley graph $G(H; S)$ on group $H$ is defined to be the graph with the elements of $H$ as its vertices and edges joining $h$ and $hs$ for all $h \in H$ and $s \in S$. $S$ is called the connection set.

When $k \geq 4$, we construct a Cayley graph as follows. Let $H$ be a cyclic group $Z_{33}$ Then, we have that Cayley graph $G(H; S)$ with connection set $S = \{6, 27\}$ is in $\Psi$. By symmetry, we choose $v = 0$ and $(A) = \{(2, 8), (11, 12), (21, 22), (25, 31)\}$. It is easy to check that $G(H; S) \in \Psi$. 


3. Main theorem and its proof

To prove our main theorem, we need some necessary results also on $\gamma_1$-critical graphs.

**Observation 3.1** ([2]). If $G$ is a $\gamma_1$-critical graph, then $\gamma_1(G - v) = \gamma_1 - 1$ for any $v \in V(G) - W(G)$. Furthermore, each $\gamma_1(G - v)$-set contains no neighbor of $v$.

A corona of a graph $H$, denoted by $cor(H)$, is the graph obtained from $H$ by adding a pendant edge to each vertex of $H$. Goddard et al. [2] characterized the $\gamma_1$-critical graphs with at least one vertex of degree 1 as follows.

**Theorem 3.2** ([2]). Let $G$ be a connected graph with at least 3 vertices and at least one vertex of degree 1. Then, $G$ is $k$-$\gamma_1$-critical if and only if $G = cor(H)$ for some connected graph $H$ with $k$ vertices and $\delta(H) \geq 2$.

The next lemma plays a key role in the proof of our main theorem.

**Lemma 3.3.** Let $G$ be a connected $\gamma_1$-critical graph of order $n$ ($n \geq 3$). If $n = \Delta(G)(\gamma_1(G) - 1) + 1$, then the following results hold:

(i) $\delta(G) \geq 2$;

(ii) For any $v \in V$, there is an $A \subseteq V(G - v)$ such that $A$ is a $\gamma_1(G - v)$-set and $N(v) \cap A = \emptyset$. Moreover, $\langle A \rangle$ is $1$-regular, $d_G(x) = \Delta(G)$ for any $x \in A$ and $d_A(y) = 1$ for any $y \in V(G) - A - \{v\}$.

**Proof.** (i) Suppose, to the contrary, that $\delta(G) = 1$. Let $G$ be a $k$-$\gamma_1$-critical graph. By Theorem 3.2, $G = cor(H)$ for some connected graph $H$ with $k$ vertices and $\delta(H) \geq 2$, which implies that $k \geq 3$. Furthermore, we have that $n = 2k$ and $3 \leq \Delta(G) \leq k$, from which we obtain $1 + \frac{1}{\Delta(G)-2} \leq 2$ and $k > 1 + \frac{1}{\Delta(G)-2}$. Therefore we have that $2k < \Delta(G)(k - 1) + 1$ and $n < \Delta(G)(k - 1) + 1$, which contradicts $n = \Delta(G)(\gamma_1(G) - 1) + 1$. Thus, $\delta(G) \geq 2$.

(ii) Since $\delta(G) \geq 2$, we have that $W(G) = \emptyset$. Let $G$ be a $k$-$\gamma_1$-critical graph, by Observation 3.1, $\gamma_1(G - v) = k - 1$ for any $v \in V(G)$. Let $A$ be a $\gamma_1(G - v)$-set of $G - v$ and let $B = V(G - v) - A$, by Observation 3.1, we have that $N(v) \cap A = \emptyset$. So we have that $|A| = k - 1$ and $|B| = n - k$. Since $A$ is a $\gamma_1(G - v)$-set, we have that $d_A(x) \geq 1$ for any $x \in A$ and $d_A(y) \geq 1$ for any $y \in B$, which means that each vertex of $A$ has at most $\Delta(G) - 1$ neighbors in $B$ and each vertex of $B$ has at least one neighbor in $A$. Hence,

$$n - k \leq e(A, B) \leq (k - 1)(\Delta(G) - 1).$$

That is,

$$n \leq \Delta(G)(\gamma_1(G) - 1) + 1. \quad (2)$$

The equality $n = \Delta(G)(\gamma_1(G) - 1) + 1$ holds if and only if both the equalities of (1) hold. So we have that $n - k = e(A, B) = (k - 1)(\Delta(G) - 1)$, which means that each vertex of $A$ has exactly $\Delta(G) - 1$ neighbors in $B$, and each vertex of $B$ has exactly one neighbor in $A$. Therefore, there exists a partition of $V(G) - v$ for any $v \in V$, i.e., there exist $A \subseteq V(G) - v$ and $B = V(G - v) - A$, such that $A$ is a $\gamma_1(G - v)$-set and $N(v) \cap A = \emptyset$. Moreover, $\langle A \rangle$ is 1-regular, $d_G(x) = \Delta(G)$ for any $x \in A$ and $d_A(y) = 1$ for any $y \in B$. This completes the proof of Lemma 3.3. \hfill $\blacksquare$

Let $f$ be a map from set $A$ to set $B$. We say that $f$ is injective if $f(a) \neq f(a')$ for every pair $a \neq a' \in A$. We say that $f$ is surjective if there exists $x$ in $A$ such that $f(x) = y$ for every $y$ in $B$.

In the following, we give the main theorem of this paper.

**Theorem 3.4.** Let $G$ be a connected $\gamma_1$-critical graph of order $n$. Then $n = \Delta(G)(\gamma_1(G) - 1) + 1$ if and only if $G \in \Psi$.

**Proof.** It is easy to check that when $n = 1$ or 2, Theorem 3.4 is true. So we may assume that $G$ is a graph with at least three vertices.

“\(\Rightarrow\)” Assume that $n = \Delta(G)(\gamma_1(G) - 1) + 1$ and $\gamma_1(G) = k$. We will show that $G \in \Psi$.

By Lemma 3.3, for any $v \in V(G)$, there is an $A \subseteq V(G - v)$ such that $A$ is a $\gamma_1(G - v)$-set and $N(v) \cap A = \emptyset$. Let $B = V(G) - A - \{v\}$. Moreover, $\langle A \rangle$ is 1-regular, $d_G(x) = \Delta(G)$ for any $x \in A$ and $d_A(y) = 1$ for any $y \in B$.

Now we only need to prove that $\delta(G) = \Delta(G)$, which implies that $G$ is $\Delta(G)$-regular.
Suppose, to the contrary, that there exists a vertex \( v \in V(G) \) such that \( d_G(v) = \delta(G) < \Delta(G) \). Consider the subgraph \( G - v \). By Lemma 3.3, \( \delta(G) \geq 2 \), so \( W(G) = \emptyset \). By Lemma 3.3, there is an \( A \subseteq V(G - v) \) such that \( A \) is a \( \gamma_1(G - v) \)-set and \( N(v) \cap A \) is \( \emptyset \), \( \langle A \rangle \) is 1-regular. Suppose \( A = \{x_1, x_2, \ldots, x_{k-1}\} \), and \( E(A) = \{x_1x_2, x_3x_4, \ldots, x_{k-2}x_{k-1}\} \), and \( B_i = N_B(x_i) \) for each \( 1 \leq i \leq k - 1 \), then \( B \) is the disjoint union of \( B_i \), \( 1 \leq i \leq k - 1 \). (See Fig. 1).

We consider the subgraph \( G - x_1 \). Since \( G \) is a \( k - \gamma_1(G) \)-critical graph, by Observation 3.1, we have that \( \gamma_1(G - x_1) = k - 1 \). By Lemma 3.3, we obtain:

(\( \bullet \)) (There is an \( A' \subseteq V(G) - x_1 \) such that \( N(x_1) \cap A' = \emptyset \) and, \( A' \) is a \( \gamma_1(G - x_1) \)-set with \( |A'| = k - 1 \). \( A' \) is 1-regular, \( d_G(u) = \Delta(G) \) for any \( u \in A' \). Let \( B' = V(G - x_1) - A' \), then \( d_{A'}(y) = 1 \) for any \( y \in B' \).

Claim 1. \( A' \cap A \neq \emptyset \).

**Proof.** Suppose, to the contrary, \( A' \cap A = \emptyset \). Then the following two results hold:

(a) \( A' \subseteq \bigcup_{i=2}^{k-1} B_i \).

Proof. Since \( V(G) = A \cup B \cup \{v\} = A' \cup B' \cup \{x_1\} \) and \( A \cap B = \emptyset \) and \( A' \cap B' = \emptyset \), \( v \notin A \), \( v \notin B \) and, \( x_1 \notin A' \), \( x_1 \notin B' \), then \( A \subseteq B' \cup \{x_1\} \) and \( A' \subseteq B \cup \{v\} \), which give \( A - x_1 \subseteq B' \) and \( A' - v \subseteq B \). Since \( d_G(v) = \delta(G) < \Delta(G) \), then \( v \notin A' \), therefore \( A' \subseteq B \). Since \( N(x_1) \cap A' = \emptyset \), that is, \( (x_2 \cup B_1) \subseteq B' \). Therefore \( A' \subseteq \bigcup_{i=2}^{k-1} B_i \).

(b) \( |A' \cap B_i| \leq 1 (2 \leq i \leq k - 1) \).

Proof. Suppose, to the contrary, that there exist some \( j, 2 \leq j \leq k - 1 \) such that \( |B_j \cap A'| \geq 2 \). Since \( B_j = N_B(x_j) \), then \( |N_B(x_j) \cap A'| \geq 2 \). Since \( x_j \in A \) and \( A - x_1 \subseteq B' \), then \( x_j \in B' \). Thus \( d_{A'}(x_j) \geq 2 \). But by (\( \bullet \)), \( d_{A'}(x_j) = 1 (2 \leq j \leq k - 1) \). This is a contradiction.

By (a) and (b), \( |A'| \leq k - 2 \). This contradicts the fact that \( A' \) is a \( \gamma_1(G - x_1) \)-set with \( |A'| = k - 1 \). \( \square \)

If there exists \( x \in A' - A \), then \( x \notin A \). Therefore, \( x \in B \). By Lemma 3.3, there exists a unique vertex \( z \in A \) such that \( xz \in E(G) \). If \( z \in A \cap A' \), then there exists a unique vertex \( z_1 \in A - A' \) such that \( z_1z \in E(\langle A \rangle) \) since \( \langle A \rangle \) is 1-regular. Let \( f : A' - A \rightarrow A - A' \) be defined by (See Fig. 2)

\[
f(x) = \begin{cases} z & \text{if } x \in A' - A, N_A(x) = \{z\}, \text{ and } z \notin A' \cap A; \\
z_1 & \text{if } x \in A' - A, N_A(x) = \{z\}, z \in A' \cap A, \text{ and } N_A(z) = \{z_1\}. 
\end{cases}
\]

Claim 2. \( f \) is well defined.

**Proof.** Firstly, we prove that \( f(x) \in A - A' \). Consider the two Cases:

Case 1. \( x \in A' - A \), \( N_A(x) = \{z\} \), and \( z \notin A' \cap A \).

By the definition of \( f \), \( f(x) = z \). Since \( N_A(x) = \{z\} \), then \( z \in A \). Since \( z \notin A' \cap A \), then \( z \notin A' \). Therefore, \( z \in A - A' \), that is, \( f(x) \in A - A' \).

Case 2. \( x \in A' - A \), \( N_A(x) = \{z_1\} \), \( z_1 \in A' \cap A \), and \( N_A(z) = \{z_1\} \)

By the definition of \( f \), \( f(x) = z_1 \). Since \( N_A(z) = \{z_1\} \), then \( z_1 \in A \). If \( z_1 \in A' \), then \( z, z_1 \in A' \cap A \) since \( z \in A' \cap A \). Thus, \( x, z, z_1 \in A' \) and \( \{x, z_1\} \subseteq N_A(z_1) \), that is, \( d_{A'}(z) \geq 2 \), which contradicts the fact that \( A' \) is 1-regular. Therefore, \( z_1 \notin A' \), that is, \( z_1 \in A - A' \).
Lemma 3.3

Fig. 3

Claim 2

For two vertices $x, x' \in A' - A$, $x \neq x'$, and $f(x) = z$, $f(x') = z'$. We will show that $z \neq z'$. Suppose, to the contrary, $z = z'$. By the definition of $f$, $z \in A - A'$. Thus, $z \in B'$. We have the following cases:

Case (1). $x \in A' - A$, $N_A(x) = \{z\}$, and $z \notin A' \cap A$.

Subcase (1.1). $x' \in A' - A$, $N_A(x') = \{z'\}$, and $z' \notin A' \cap A$. (See Fig. 3(a)). Since $N_A(x) = \{z\}$, $N_A(x') = \{z'\}$, $xz \in E(G)$. Therefore $\{x, x'\} \subseteq N_A(z)$, that is, $d_{A'}(z) \geq 2$. This contradicts ($\clubsuit$).

Subcase (1.2). $x' \in A' - A$, $N_A(x') = \{z_1\}$, $z_1 \in A' \cap A$, and $N_A(z_1) = \{z_1\}$. (See Fig. 3(b)). Since $N_A(x) = \{z\}$, $xz \in E(G)$. Since $z_1 \in A \cap A'$ and $N_A(z_1) = \{z_1\}$, $z_1, x \subseteq N_A'$. Thus $d_{A'}(z) \geq 2$. This contradicts ($\clubsuit$).

Case (2), $x \in A' - A$, $N_A(x) = \{z_1\}$, $z_1 \in A' \cap A$, and $N_A(z_1) = \{z_1\}$.

Subcase (2.1). $x' \in A' - A$, $N_A(x') = \{z_1\}$, and $z' \notin A' \cap A$. (See Fig. 3(c)). Since $N_A(z_1) = \{z_1\}$ and $N_A(x') = \{z_1\}$, $z_1, x' \notin E(G)$. Since $z_1 \in A' \cap A$, $[z_1, x'] \subseteq N_A(z)$. Thus, $d_{A'}(z) \geq 2$. This contradicts ($\clubsuit$).

Subcase (2.2). $x' \in A' - A$, $N_A(x') = \{z_1\}$, $z_1 \in A' \cap A$, and $N_A(z_1) = \{z_1\}$. (See Fig. 3(d)). Then $z_1, x' \notin E(G)$ and $z_1, x' \notin E(G)$. If $z_1 \neq x'$, then $d_{A'}(z) \geq 2$ since $z_1 \notin A' \cap A$. This contradicts the fact that $\langle A' \rangle$ is 1-regular. If $z_1 = x'$, then $d_{A'}(z_1) \geq 2$ since $N_A(x) = \{z_1\}$ and $N_A(x') = \{z_1\}$. This contradicts the fact that $\langle A' \rangle$ is 1-regular.

This completes the proof of Claim 2. $\blacksquare$

Claim 3. $f$ is injective.

Proof. For two vertices $x, x' \in A' - A$, $x \neq x'$. Suppose $f(x) = z$, $f(x') = z'$. We will show that $z \neq z'$. Assume that $z = z'$. Then $d_{A'}(z) = 1$. Since $z \in A - A'$, this is impossible. This completes the proof of Claim 3. $\blacksquare$

Claim 4. $f$ is not surjective.

Proof. By ($\clubsuit$), $N(x_1) \cap A' = \emptyset$, then $x_1$ has no neighbor in $A'$. Furthermore, $x_1$ has no neighbor in $A' - A$. Since $x_1, x_2 \in E(\langle A' \rangle)$, $x_2 \in A \cap B'$. Thus, $x_2 \notin A' \cap A$, $x_2 \notin E(\langle A' \rangle)$ for each $x \in A' - A$. Note that $x_1 \in A - A'$. Thus, $f(x) \neq x_1$ for each $x \in A' - A$. Hence $f$ is not surjective. $\blacksquare$

From Claim 2 to Claim 4, it follows that $|A' - A| < |A - A'|$, which contradicts the fact that $|A| = |A'| = k - 1$. The contradiction shows that $G$ is $\Delta(G)$-regular. Thus we obtain $G \notin \Psi$.

"$\Leftarrow$" Assume that $G$ is a $k$-$\gamma_t$-critical graph with $n$ vertices and $G \in \Psi$. We will show that $n = \Delta(G)(\gamma_t(G) - 1) + 1$. 

Fig. 2.

A map $f : A' - A \rightarrow A - A'$

Fig. 3.

$x, x' \in A' - A$, $x \neq x'$, and $f(x) = z$, $f(x') = z'$, $z = z'$.
Since \( G \in \Psi \), then \( G \) is \( \Delta(G) \)-regular and \( \Delta(G) \geq 2 \). Moreover, for any \( v \in V(G) \), there exists an \( A \subseteq V(G - v) \) such that \( N(v) \cap A = \emptyset \) and \( \langle A \rangle \) is 1-regular, and let \( B = V(G - v) - A \), then \( d_A(y) = 1 \) for any \( y \in B \), which means that \( d_A(x) = 1 \) for any \( x \in V(G - v) \). So we have that
\[
|A| = e(A, B) = |A|(\Delta(G) - 1),
\]
that is,
\[
n - |A| - 1 = e(A, B) = |A|(\Delta(G) - 1),
\]
(3)
Since \( d_A(x) = 1 \) for any \( x \in V(G - v) \), \( A \) is a total dominating set of \( G - v \). By Observation 3.1, we have that
\[
|A| \geq \gamma_t(G - v) = \gamma_t(G) - 1.
\]
(4)
By (3) and (4), we obtain
\[
n \geq \Delta(G)(\gamma_t(G) - 1) + 1.
\]
(5)
On the other hand, since \( G \) is a \( k\)-\( \gamma_t \)-critical graph, by Observation 3.1, we have that \( \gamma_t(G - v) = k - 1 \) for any \( v \in V(G) \). Let \( A' \) be a \( \gamma_t(G - v) \)-set of \( G - v \) and \( B' = V(G - v) - A' \). So we have that \( |A'| = k - 1 \) and \( |B'| = n - k \). Since \( A' \) is a \( \gamma_t(G - v) \)-set, we have that \( d_{A'}(x) \geq 1 \) for any \( x \in A' \) and \( d_{A'}(y) \geq 1 \) for any \( y \in B' \), which means that each vertex of \( A' \) has at most \( \Delta(G) - 1 \) neighbors in \( B' \) and each vertex of \( B' \) has at least one neighbor in \( A' \). Hence,
\[
n - k \leq e(A', B') \leq (k - 1)(\Delta(G) - 1).
\]
(6)
that is,
\[
n \leq \Delta(G)(\gamma_t(G) - 1) + 1.
\]
(7)
By (5) and (7), we have that \( n = \Delta(G)(\gamma_t(G) - 1) + 1 \).
This completes the proof of Theorem 3.4. ■

4. Open problems

We close with two open problems.

1. Does there exists a graph in \( \Psi \) which is not a Cayley graph?
2. Motivated by Propositions 2.2 and 2.3, we propose the problem: Characterize \( 2k \)-regular graphs in \( \Psi \), for each \( k \geq 2 \)?

References