Existence and Regularity of a Class of Weak Solutions to the Navier–Stokes Equations

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We construct a class of weak solutions to the Navier–Stokes equations, which have second order spatial derivatives and one order time derivatives, of $p$ power summability for $1 < p \leq 5/4$. Meanwhile, we show that $u \in L^1(0,T;W^{2,r}(\Omega))$ with $1/s + 3/2r = 2$ for $1 < r \leq 5/4$. $r$ can be relaxed not to exceed $3/2$ if we consider only in the interior of $\Omega$. In the end, we extend the classical regularity theorem. Our results show that $u$ is a regular solution if $\nabla u \in L^1(0,T;L^r(\Omega))$ with $1/s + 3/2r = 1$ for $\Omega$ satisfying (1.3), with $1/s + 1/r = 5/6$ for arbitrary domain in $\mathbb{R}^3$ and $1 < s \leq 2$. For $\Omega = \mathbb{R}^n$ with $n \geq 3$, this result was previously obtained by H. Beirão da Veiga (Chinese Ann. Math. Ser. B 16, 1995, 407–412).

1. INTRODUCTION

Since Leray [13] and Hopf [8] proved the existence of global-in-time weak solutions to the Navier–Stokes equations for the initial value problem or for the initial boundary value problem, various global weak solutions have been constructed by different methods. Because of the absence of uniqueness for weak solutions to the Navier–Stokes equations, it is not clear whether the properties of various weak solutions obtained by different methods coincide. Thus, many efforts have been made to construct global solutions which are more regular than the one known.

In this paper, we first consider a class of weak solutions to the initial boundary value problem for the Navier–Stokes equations, which are more regular than the Leray–Hopf weak solutions. Namely, we prove global-in-time existence of solutions to the Navier–Stokes equations corresponding to initial data $a \in \mathcal{F}^{2-2/p,1}(\Omega) \cap H$, which have time generalized deriv-
tives and second order spatial generalized derivatives, of $p$ power summability for $1 < p \leq 5/4$. If $p = 5/4$, this result was first obtained by Ladyzhenskaya [11] for the Cauchy problem. Later, the same result was obtained for $p = 5/4$ and $\Omega$ bounded in [2] or for $p = 5/4$ and exterior domains in [6]. The result, together with the local energy inequality, was proved for any $p$ such that $10/9 < p < 5/4$ and $\Omega$ bounded in [3]. When $10/9 < p \leq 5/4$, the weak solutions verify the local energy inequality in the sense specified in [2]. Thus, the one dimensional Hausdorff measure of the set of the interior singularities of the suitable weak solutions is zero, in view of the result of [2]. Meanwhile, we show that the weak solutions belong to $L^\infty(0,T; W^{s,2}(\Omega))$ with $1/s + 3/2r = 2$ for $1 < r \leq 5/4$. If we consider only in the interior bounded subdomain $\Omega'$, $r$ can be relaxed not to exceed $3/2$. In particular, we show that $u \in L^\infty(0,T; L^1(\Omega'))$. However, the properties of the solutions are not sufficient to ensure that it is regular.

The second purpose of this paper is to extend the classical regularity result. Many people showed that uniqueness and regularity for solutions to the Navier–Stokes equations hold if it belongs to $L^s(0,T; L^r(\Omega))$ with

$$\frac{1}{s} + \frac{3}{2r} = \frac{1}{2}, \quad s \geq 2, r_1 \geq 3.$$ (1.1)

For details, see [5, 7, 11, 14, 15, 19] and their literature. In this paper, we show that $u$ is regular if

$$\nabla u \in L^s(0,T; L^r(\Omega)), \quad \frac{1}{s} + \frac{3}{2r} = 1, 1 < s \leq 2,$$ (1.2)

and domain $\Omega$ such that

$$\left\| e^{-tA} f \right\|_q \leq C t^{-(3/2)(1/p - 1/q)} \left\| f \right\|_p, \quad 1 < p \leq q < +\infty$$ (1.3)

for every $f \in L^p(\Omega)$, where $A$ is the Stokes operator. When $\Omega = \mathbb{R}^n$, the same result was obtained by [4]. If $\Omega$ is an arbitrary three dimensional domain and $\nabla u \in L^s(0,T; L^r(\Omega))$ with

$$\frac{1}{s} + \frac{1}{r} = \frac{5}{6}, \quad 1 < s \leq 2,$$ (1.4)

then $u$ is a regular solution. If $r = 3$, condition (1.4) coincides with (1.2). While $r > 3$, (1.4) implies

$$\frac{1}{s} + \frac{3}{2r} < 1.$$
We think that condition (1.2) is also sufficient to ensure the regularity of weak solutions to the Navier–Stokes equations in arbitrary domain in $R^3$. Unfortunately, we don’t show this. If $s > 2$, by the Sobolev inequality, our conditions (1.2), (1.4) yield exactly $u \in L^s(0, T; L^r(\Omega))$ with $(s, r_1)$ satisfying (1.1). This argument shows that our result is just the natural extension of the classical regularity criteria. When $1 < s \leq 2, r > 3$, by the Sobolev imbedding theorem, $u$ is a Hölder continuous function on a spatial variable. As pointed out by Beirão da Veiga in [4], this shows that the loss of regularity in the time variable can be balanced by further regularity in the spatial variable. It is worth pointing out that $r_1$ reaches its maximum $\frac{1}{2}$ for $s = 2$ in the classical regularity criteria, while our result shows that $W^{1,3}(\Omega)$ is a regularity class for arbitrary domain in $R^3$. This couldn’t follow from the classical criteria, because $W^{3,2}(\Omega)$ cannot be imbedded into $L^3$.

2. PRELIMINARIES AND NOTATIONS

Let $\Omega$ be a arbitrary three dimensional domain. Let $L^p(\Omega), 1 \leq p \leq +\infty$, represent the usual Lebesgue space of scalar functions as well as that of vector functions with norm denoted by $\| \cdot \|_p$. Let $C_0^\infty(\Omega)$ denote the set of all $C^\infty$ real vector functions $\phi = (\phi_1, \phi_2, \phi_3)$ with compact support in $\Omega$, such that $\text{div} \phi = 0$. $W^{m,p}(\Omega)$ is the usual Sobolev space of order $(m, p)$ of functions on $\Omega$. By $W^{s,p}(\Omega)$ with $s \geq 0$ and $p \geq 1$, we denote the Sobolev space of functions on $\Omega$ with derivatives of fractional order $(s, p)$ endowed with the intrinsic norm $\| \cdot \|_{s,p}$ (cf. [1]). Let $J^p(\Omega)$ = completion of $C_0^\infty(\Omega)$ in $L^p(\Omega)$, while $J^{s,p}(\Omega)$ = completion of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$. Especially, let $J^2(\Omega) = H, J^{1,2}(\Omega) = V$. Let $L^p_w, 1 \leq p < \infty$, denote the Marcinkiewicz space defined by

$$L^p_w = \{ u \ | \ |u|_{L^p_w(\Omega)} = \inf \{ A \ | \ \text{meas}\{ x \in \Omega \ | \ |u(x)| > \lambda \}^{1/p} \leq A, \forall \lambda > 0 \} < \infty \}.$$

Finally, given a Banach space $X$ with norm $\| \cdot \|_X$, we denote by $L^p(0, T; X), 1 \leq p \leq +\infty$, the set of functions $f(t)$ defined on $(0, T)$ with values in $X$ such that $\int_0^T \| f(t) \|_X^p \, dt < +\infty$. $P$ is the Helmholtz projection from $L^p(\Omega)$ to $J^p(\Omega)$. Then the Stokes operator $A$ is defined by $A = -P\Delta$ with $D(A) = \{ u \in W^2, p(\Omega), u|_{\partial \Omega} = 0 \} \cap J^p(\Omega)$. For details, see Teman [19, Chap I]. In the end, by symbol $C$, we represent a generic constant whose value is unessential to our aims, and it may change from line to line.

We shall consider the following initial boundary value problem for Navier–Stokes equations concerning the unknown velocity field $u =$
u(x, t) = (u_1, u_2, u_3) and the scalar pressure p = p(x, t),

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u &= -\nabla p + f, & \text{in } \Omega \times (0, T), \\
\text{div } u &= 0, & \text{in } \Omega \times (0, T), \\
u u(x, t) &= 0, & \text{for } x \in \partial \Omega, t > 0, \\
u u(x, 0) &= a(x), & \text{in } \Omega, \\
u u \to 0, & \text{as } |x| \to +\infty,
\end{aligned}
\]

where \( \nu > 0 \) is the viscosity, \( a = a(x) \) is a given initial velocity vector field, and \( f = f(x, t) = (f_1, f_2, f_3) \) is the external force vector. If \( \Omega = \mathbb{R}^3 \), the boundary conditions in (2.1) must be dropped, while if \( \Omega \) is a bounded domain, the condition on the velocity at infinity must be omitted.

First, we cite some results concerning the linearized equations of (2.1), which we write as

\[
\begin{aligned}
\frac{\partial w}{\partial t} - \nu \Delta w &= -\nabla p + f, & \text{in } \Omega \times (0, T), \\
\text{div } w &= 0, & \text{in } \Omega \times (0, T), \\
w(x, t) &= 0, & \text{for } x \in \partial \Omega, t > 0, \\
w(x, 0) &= w_0, & \text{in } \Omega, \\
w \to 0, & \text{as } |x| \to +\infty.
\end{aligned}
\]

Similarly, if \( \Omega = \mathbb{R}^3 \), the boundary conditions in (2.2) must be dropped, while if \( \Omega \) is a bounded domain, the condition on the velocity at infinity must be omitted. We need the following results.

**Lemma 2.1.** Let \( f \in L^p(0, T; L^p(\Omega)) \) and \( w_0 \in L^{2+2/p} \cap \cap (\Omega) \) for \( p > 1 \). There exists a unique solution \((w, p)\), which satisfies

\[
\begin{aligned}
\| \frac{\partial w}{\partial t} \|_{L^p(0, T; L^p(\Omega))} + \| w \|_{L^p(0, T; W^{2, p}(\Omega))} + \| \nabla p \|_{L^p(0, T; L^p(\Omega))} \\
&\leq C(T) (\| f \|_{L^p(0, T; L^p(\Omega))} + \| w_0 \|_{2+2/p, p}).
\end{aligned}
\]

This lemma is a consequence of Theorem 4.2 and Remark 2 in [17, p. 495].

**Lemma 2.2.** Suppose \( f \in C^\infty_0 (\Omega \times (0, T)) \) and \( w_0 \in C^\infty_0 (\Omega) \). Then there exists a unique solution \((w, p)\) satisfying (2.3) for all \( p > 1 \).

Lemma 2.2 can be found in [6].
Lemma 2.3. Let $1 < p \leq 2$, then $C^\infty_{0,\sigma}(\Omega)$ is dense in $\tilde{H}^{2-2/p,p}(\Omega) \cap H$.

Proof. When $p = 2$, it is obvious. So we only consider $1 < p < 2$. For arbitrary $w_0 \in H \cap \tilde{H}^{2-2/p,p}(\Omega)$, let $w$ be the solution of (2.2) with $f = 0$. By Lemma 2.1, it is obvious that

$$w \in L^p(0, T; W^{2,p}(\Omega)) \cap L^2(0, T; V), \quad \frac{\partial w}{\partial t} \in L^p(0, T; L^p(\Omega)).$$

(2.4)

By the intermediate derivative [22, Theorem 2.3, Chap. I],

$$D_t^{1/p}w \in L^p(0, T; W^{2-2/p,p}(\Omega)), \quad D_t = \frac{\partial}{\partial t}$$

with the continuous dependence on initial data (cf. [16, 17]), we deduce that

$$w\left(\cdot, \frac{1}{n}\right) \to w_0 \quad \text{in } \tilde{H}^{2-2/p,p}(\Omega) \cap H \quad \text{as } n \to \infty. \quad (2.5)$$

Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $\tilde{H}^{2,p}(\Omega)$ and $W(\cdot, 1/n) \in \tilde{H}^{2,p}(\Omega)$, there exist $(\phi_{n,k})_{k \in N} \subset C^\infty_{0,\sigma}(\Omega)$ such that

$$\phi_{n,k} \to w\left(\cdot, \frac{1}{n}\right) \quad \text{in } \tilde{H}^{2,p}(\Omega) \quad \text{as } k \to \infty. \quad (2.6)$$

When $1 < p \leq 3/2$, by the Sobolev imbedding theorem (cf. [1]), $W^{2,p}(\Omega)$ is continuously imbedded into $L^2(\Omega)$. So

$$\phi_{n,k} \to w\left(\cdot, \frac{1}{n}\right) \quad \text{in } \tilde{H}^{2-2/p,p}(\Omega) \cap H \quad \text{as } k \to \infty,$$

which with (2.5) shows that

$$\phi_{n,k} \to w_0 \quad \text{in } \tilde{H}^{2-2/p,p}(\Omega) \cap H \quad \text{as } k, n \to \infty.$$

This implies our result when $1 < p \leq 3/2$.

When $3/2 < p < 2$, the Sobolev imbedding theorem, $W^{2,p}(\Omega)$ may be continuously imbedded into $W^{7/2-3/p,2}(\Omega)$. We replace $\tilde{H}^{2,p}(\Omega)$ by $\tilde{H}^{7/2-3/p,2}(\Omega)$, and similar to the agreement above, we have our conclusion. \[\blacksquare\]
3. existence theorem

In this section, we shall now prove a result of the existence of global weak solutions to problem (2.1). Namely

**Theorem 3.1.** Let \( a \in \mathcal{J}^{2-2/p,p}(\Omega) \cap H \) and \( f \in L^1_{loc}(0, \infty; L^2(\Omega)) \cap L^p_{loc}(0, \infty; L^p(\Omega)) \) with \( 1 < p \leq 5/4 \). Then there exists a weak solution \((u(x,t), p(x,t))\) to (2.1) such that

\[
\begin{align*}
\frac{\partial u}{\partial t}, \quad \nabla p & \in L^p_{loc}(0, \infty; L^p(\Omega)).
\end{align*}
\]

Moreover, \((u, p)\) verifies the energy inequality

\[
\|u(t)\|^2 + 2\nu \int_0^t \|\nabla u(s)\|^2 \, ds \leq \|a\|^2 + 2 \int_0^t f \cdot u \, ds
\]

and estimations for \( T > 0 \)

\[
\|u(t)\|^2 \leq \|a\|^2 + \int_0^T \|f(s)\|^2 \, ds, \quad \text{for } t \leq T,
\]

\[
\nu \int_0^T \|\nabla u(s)\|^2 \, ds \leq \|a\|^2 + 2 \left( \int_0^T \|f(s)\|^2 \, ds \right)^2,
\]

\[
\|\frac{\partial u}{\partial t}\|_{L^p(0,T;L^p(\Omega))} + \|u\|_{L^p(0,T;W^{1,p}(\Omega))} + \|\nabla p\|_{L^p(0,T;L^p(\Omega))} \leq C \left( \|a\|_{2-2/p,p}^2 + \|a\|^2 + \|f\|_{L^p(0,T;L^p(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))} \right). \tag{3.6}
\]

If \( 10/9 < p \leq 5/4 \), the local energy estimate

\[
\int_\Omega |u(t)|^2 \phi(t) + 2\nu \int_0^t \int_\Omega |\nabla u|^2 \phi \\
\leq \int_\Omega |a|^2 \phi(0) + \int_0^t \int_\Omega \left( \frac{\partial \phi}{\partial t} + \nu \Delta \phi \right) |u|^2 \\
+ \int_0^t \int_\Omega (|u|^2 + 2p) u \cdot \nabla \phi + 2 \int_0^t \int_\Omega f \cdot u \phi
\]

holds for every \( t > 0 \) and for every \( \phi \in C^2_0(\Omega \times (0,t)), \phi \geq 0 \) on \( \Omega \times (0,t) \).

**Remark.** The solutions obtained above exist globally without restrictions on the size of the data and have time derivatives and second order spatial derivatives. These solutions are more regular than the Leray–Hopf weak solutions.
(2) If \( p = 5/4 \), \( \Omega = \mathbb{R}^3 \) or \( \Omega \) bounded, the solutions with properties (3.1), (3.2) were obtained by Ladyzhenskaya [11]. If \( \Omega \) is a exterior domain in \( \mathbb{R}^3 \), the solutions were found by Galdi and Maremonti [6]. If \( \Omega \) is an open bounded subset of \( \mathbb{R}^3 \), Theorem 3.1 has already been proved by Beirão da Veiga [3].

(3) If \( 10/9 < p \leq 5/4 \), the solutions verify the local energy inequality (3.7). Therefore the one dimensional Hausdorff measure of the set of the interior singularities of the suitable weak solutions is zero, in light of the result of Caffarelli, Kohn, and Nirenberg [2].

**Proof of Theorem 3.1.** We first construct the sequences of approximate solutions. For this purpose, we select \( a^k \in C_0^\infty(\Omega) \) such that

\[
a^k \to a \quad \text{in} \quad W^{2-2/p,p}(\Omega) \cap H \quad \text{strongly},
\]

and

\[
\|a^k\|_2 \leq C\|a\|_2, \quad \|a^k\|_{w^{2-2/p,p}(\Omega)} \leq C\|a\|_{w^{2-2/p,p}(\Omega)}.
\]

We now consider the linearized Navier–Stokes equations in \( \Omega \times (0,T) \) for arbitrary \( T > 0 \)

\[
\begin{align*}
\frac{\partial u^0}{\partial t} - \nu \Delta u^0 &= -\nabla p^0 + f^0, \quad \text{in} \quad \Omega \times (0,T) \\
div u^0 &= 0, \quad \text{in} \quad \Omega \times (0,T), \\
u \partial u^0(x,t) &= 0, \quad \text{in} \quad \partial \Omega \times (0,T), \\
u u^0(x,t) &= a^0(x), \quad \text{as} \quad |x| \to +\infty, \\
u u^0(x,0) &= a^0(x), \quad \text{in} \quad \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial u^k}{\partial t} - \nu \Delta u^k + (u^{k-1} \cdot \nabla) u^k &= -\nabla p^k + f^k, \quad \text{in} \quad \Omega \times (0,T), \\
div u^k &= 0, \quad \text{in} \quad \Omega \times (0,T), \\
u \partial u^k(x,t) &= 0, \quad \text{on} \quad \partial \Omega \times (0,T) \\
u u^k(x,t) &= 0, \quad \text{as} \quad |x| \to +\infty, \\
u u^k(x,0) &= a^k(x), \quad \text{in} \quad \Omega,
\end{align*}
\]

for \( k \geq 1 \), where \( f^k \in L^2(0,T;L^2(\Omega)) \cap L^p(0,T;L^p(\Omega)) \) such that

\[
f^k \to f \quad \text{in} \quad L^1(0,T;L^2(\Omega)) \cap L^p(0,T;L^p(\Omega)) \quad \text{strongly}.
\]
It is well known (cf. [11]) that there exists a unique solution \((u^0, p^0)\) to (3.8) satisfying

\[
\frac{\partial u^0}{\partial x_i}, \frac{\partial u^0}{\partial t}, \frac{\partial^2 u^0}{\partial x_j \partial x_i}, \frac{\partial p^0}{\partial x_i} \in L^2(\Omega \times (0, T))
\]

for \(i, j = 1, 2, 3\). Utilizing the Sobolev embedding theorem, it is easy to show

\[
u \leq L^p(0, T; L^p(\Omega)) \quad \text{for} \quad \frac{1}{q} + \frac{3}{2p} \leq 1.
\]

Then applying Theorem 1' of [11, Chap. 3], we obtain a unique solution \((u^k, p^k)\) to (3.9) which satisfies

\[
\frac{\partial u^k}{\partial x_i}, \frac{\partial u^k}{\partial t}, \frac{\partial^2 u^k}{\partial x_j \partial x_i}, \frac{\partial p^k}{\partial x_i} \in L^2(\Omega \times (0, T))
\]

and

\[
u \leq L^p(0, T; L^p(\Omega)),
\]

for \(i, j = 1, 2, 3, k \geq 1\).

In the following, we need to establish the uniform estimates of solutions \((u^k, p^k)\) for \(k \geq 1\). So let us consider the linearized problem (2.2) with \(f \) replaced by \(g = f^k - (u^{k-1} \cdot \nabla)u^k\) and \(w(0) = a^k(x)\), along the lines of Ladyzhenskaya [11]. It is easy to prove

\[
\|g\|_{L^p(0, T; L^p(\Omega))} \leq \|f\|_{L^p(0, T; L^p(\Omega))} + C(T) \|u^{k-1}\|_2^{3/p-2} \|\nabla u^{k-1}\|_{L^1(0, T; L^2(\Omega))}^{3-2/p} \|\nabla u^k\|_2.
\]

(3.10)

From (3.8), (3.9), it is obvious that

\[
u \int_0^T \|\nabla u^k\|_2^2 \, ds \leq (\|a\|_2 + \|f\|_{L^1(0, T; L^2(\Omega))})^2 = A^2_1
\]

(3.12)

hold uniformly for \(k \geq 1\). By taking into account the estimates (3.10), it follows that

\[
u \leq L^p(0, T; L^p(\Omega)) \leq \|f\|_{L^p(0, T; L^p(\Omega))} + C(T) A^2_1.
\]

(3.13)
So problem (2.2) admits a unique solution \( w \) enjoying the estimate (2.3) with \( 1 < p \leq 5/4 \). We need to prove \( w = u^k \). Therefore, we set \( v = w - u^k \). One easily deduces that \( v \) satisfies the identity

\[
\int_0^T \int_\Omega v \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right) \, dx \, dt = 0
\]

for all solenoidal \( \phi \) such that \( \phi, \partial \phi/\partial t, \partial^2 \phi/\partial x_i \partial x_j, \in L^2(0, T; L^2(\Omega)) \) for \( i, j = 1, 2, 3 \). \( \phi \) vanishes on \( \partial \Omega \) and \( \phi(x, T) = 0 \). Following the argument of [11], let \( \phi(x, t) \) be the solution of the adjoint problem of (2.2) corresponding to an external force \( F(x, t) \in C_0^\infty(\Omega \times (0, T)) \) and zero data at time \( T \). Thus, it follows that

\[
\int_0^T \int_\Omega v(x, t) \cdot F(x, t) \, dx \, dt = 0.
\]

By the arbitrariness of \( F \), the last fact implies \( v = 0 \). Therefore \( (u^k, p^k) \) verifies the estimates (2.3) for all \( k \geq 1 \), i.e.,

\[
\left\| \frac{\partial u^k}{\partial t} \right\|_{L^p(0, T; L^q(\Omega))} + \left\| u^k \right\|_{L^p(0, T; W^{2,q}(\Omega))} + \left\| \nabla p^k \right\|_{L^p(0, T; L^q(\Omega))} \\
\leq C(T) \left( \left\| f \right\|_{L^p(0, T; L^q(\Omega))} + \| a \|_{2 - 1/p, p} + A_2^2 \right). \tag{3.14}
\]

Employing estimates (3.11), (3.12), (3.14), it is now routine to show the existence of solutions \( (u, p) \) satisfying (3.1)–(3.6). For details, see Hopf [8], Ladyzhenskaya [11], Galdi and Maremonti [6]. The proof of (3.7) is exactly the same as that in Beirão da Veiga [3].

**Theorem 3.2.** Suppose the boundary \( \partial \Omega \in C^2 \) and conditions of Theorem 3.1 hold.

(i) If \( a \in L^p(\Omega) \) for \( 1 < p \leq 3/2 \), then the solution \( (u, p) \) satisfies

\[
\left\| \int_0^t u(s) \, ds \right\|_{W^{2,q}(\Omega)} + \left\| \int_0^t \nabla p(s) \, ds \right\|_p \leq C \tag{3.15}
\]

for \( t > 0 \) and \( 1 < p \leq 3/2 \). Moreover

\[
u \in W^{-1,q}(0, T; W^{2,p}(\Omega)), \quad \nabla p \in W^{-1,q}(0, T; L^p(\Omega)). \tag{3.16}\]

(ii) Let \( 1 < r \leq 5/4 \), then the solution \( (u, p) \) satisfies

\[
u \in L^r(0, T; W^{2,r}(\Omega)), \quad \nabla p \in L^r(0, T; L^r(\Omega)). \tag{3.17}\]
with $1/s + 3/2r = 2$, which imply that
\begin{equation}
\frac{1}{s_1} + \frac{3}{2r_1} = 1, \quad 3 < r_1 \leq \frac{15}{2}.
\end{equation}

**Proof.** We return to the approximate solutions used in the proof of Theorem 3.1. We only need to show that this approximate solution verifies the uniform estimates of the norm on the above space for $k \geq 1$.

Let
\begin{align*}
U(t) &= \int_0^t u^k(s) \, ds, \\
\beta(t) &= \int_0^t (u^{k-1}(s) \cdot \nabla) u^k(s) \, ds, \\
F(t) &= \int_0^t f^k(s) \, ds, \\
P(t) &= \int_0^t p^k(s) \, ds.
\end{align*}

Then from (3.9), $(U(t), P(t))$ satisfies
\begin{equation}
\begin{cases}
-\nu \Delta U(t) + \nabla P(t) = G(t), \\
\text{div} \, U(t) = 0, \\
U(t)|_{\partial \Omega} = 0,
\end{cases}
\end{equation}
where $G(t) = F(t) - \beta(t) - u^k(t) + a^k$.

Since $a \in L^p(\Omega)$ for $1 < p \leq 3/2$, modifying in a suitable way Lemma 3.2 of [6], we may be obtain that
\begin{equation}
\|u^k(t)\|_p \leq C
\end{equation}
holds uniformly for $k \geq 0$. By the Minkowski inequality
\begin{equation}
\|\beta(t)\|_p \leq \int_0^t \| (u^{k-1}(s) \cdot \nabla) u^k(s) \|_p \, ds
\end{equation}
and by (3.10), (3.13), it follows that
\begin{equation}
\|\beta(t)\|_p \leq CA^2, \quad \text{for } 1 < p \leq \frac{3}{2}.
\end{equation}

So
\begin{equation}
\|G(t)\|_p \leq C, \quad \text{for } 0 < t \leq T, \ 1 < p \leq \frac{3}{2}.
\end{equation}
Applying Proposition 2.2 of [19, p. 33], we deduce that

\[ \|U(t)\|_{w^{2,1}(\Omega)} + \|\nabla P(t)\|_p \leq C \|G(t)\|_p \leq C \]  

(3.21)

holds uniformly for \( k \geq 0 \). Taking the limit as \( k \to \infty \), we show (3.15). Therefore (3.16) is obvious.

Let \( 1 < r \leq 5/4 \),

\[ U(t) = \int_t^{t+h} u^k(s) \, ds, \]

\[ \beta(t) = \int_t^{t+h} (u^{k-1}(s) \cdot \nabla)u^k(s) \, ds, \]

\[ F(t) = \int_t^{t+h} f^k(s) \, ds, \]

\[ P(t) = \int_t^{t+h} p^k(s) \, ds, \]

\[ G(t) = \int_t^{t+h} g(s) \, ds, \]

\[ g(t) = f^k(t) - (u^{k-1}(t) \cdot \nabla)u^k(t) - \frac{\partial u^k}{\partial t}. \]

Then \((U(t), P(t))\) satisfies system (3.19). Since

\[ \|G(t)\|_p \leq \int_t^{t+h} \|g(s)\|_p \, ds \]

and

\[ \|g\|_r \leq \|f^k\|_r + A_1^{2-3/r} \|\nabla u^k\|_2 \|\nabla u^{k-1}\|_2^{3-3/r} + \left\| \frac{\partial u^k}{\partial t} \right\|_r, \]

then similar to the deducement of (3.21), we have

\[ \left\| \int_t^{t+h} u^k(s) \, ds \right\|_{w^{2,1}(\Omega)} + \left\| \int_t^{t+h} \nabla p(s) \, ds \right\|_p \leq C \|G(t)\|_p \]

\[ \leq C \int_t^{t+h} \|g(s)\|_p \, ds. \]
Because of

$$u^k(t) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} u^k(s) \, ds$$

and

$$\left| u^k(t) \right| \leq \lim_{h \to 0} \frac{1}{h'} |U(t)|,$$

by Fatou's Lemma (cf. Yosida [21, p. 17]), it follows that

$$\int_{\Omega} \left| u^k(t) \right|' \, dx = \int_{\Omega} \lim_{h \to 0} \inf \frac{1}{h'} |U|^r \, dx \leq \lim_{h \to 0} \inf \frac{1}{h'} \int_{\Omega} |U|^r \, dx
\leq C \lim_{h \to 0} \inf \frac{1}{h'} \|G\|_r \leq C \lim_{h \to 0} \inf \frac{1}{h'} \left( \int_t^{t+h} \|g(s)\|_r \, ds \right)'
\leq C(T) \|g(t)\|_r. \quad (3.22)$$

Similarly

$$\int_{\Omega} \left| \nabla u^k \right|^r \, dx + \int_{\Omega} \left| \frac{\partial^2 u^k}{\partial x_i \partial x_j} \right|^r \, dx + \int_{\Omega} \left| \nabla p^k \right|^r \, dx \leq C \|g(t)\|_r'. \quad (3.23)$$

Since \( g \in L^r(0, T; L^r(\Omega)) \) for \( 1/s + 3/2r = 2 \), then (3.22), (3.23), imply (3.17). By the Sobolev imbedding theorem, (3.18) follows from (3.17).  \( \blacksquare \)

4. INTERIOR PROPERTY OF WEAK SOLUTIONS

In this section, we apply the fundamental solution of the heat equation to obtain the more regular property of weak solutions in the interior domain than that in the whole domain. However, this property is not sufficient to inform us about the interior regularity of weak solutions. For simplicity, let \( \text{div} f = 0 \), and \( \int_{\Omega} p^k = 0 \).

**Lemma 4.1.** Let \( \Omega' \) be a bounded subdomain such that \( \Omega' \subseteq \Omega \) and the conditions of Theorem 3.1 be satisfied. Then

$$\int_0^T \|\nabla p^k\|_r \leq C, \quad \text{for } \frac{1}{s} + \frac{3}{2r} \geq 2, 1 < r \leq \frac{3}{2} \quad (4.1)$$

holds uniformly for \( k \geq 1 \). Therefore,

$$\nabla p \in L^r(0, T; L^r(\Omega)) \quad \text{for } \frac{1}{s} + \frac{3}{2r} \geq 2, 1 < r \leq \frac{3}{2}. \quad (4.2)$$
Proof. For the pressure, we observe that (3.9) implies that
\[
\Delta p^k = -\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (u_i^{k-1}u_j^k) \quad \text{in } \Omega \times (0, T). \quad (4.3)
\]
Since \( \Omega' \Subset \Omega \), let \( \phi \in C_0^\infty(\Omega) \) with \( \phi(x) = 1 \) in a neighbourhood of \( \Gamma' \). Then at any time
\[
\phi(x)p^k(x, t) = -\frac{3}{4\pi} \int_{R^3} \frac{1}{|x-y|} \Delta_y (\phi p^k) \, dy
\]
\[
= -\frac{3}{4\pi} \int_{R^3} \frac{1}{|x-y|} \left\{ p^k \Delta \phi + 2(\nabla \phi \cdot \nabla p^k) + \phi \Delta p^k \right\} \, dy.
\]
(4.4)

Substituting (4.3) for \( \Delta p^k \) into (4.4) and integrating by parts, we obtain the expression for \( \phi p^k \) as
\[
\phi p^k = p^k_1 + p^k_2 + p^k_3,
\]
\[
p^k_1 = \frac{3}{4\pi} \int_{R^3} \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{|x-y|} \right) \phi u_i^{k-1}u_j^k \, dy,
\]
\[
p^k_2 = \frac{3}{4\pi} \int_{R^3} \sum_{i,j} \frac{x_i - y_i}{|x-y|^3} \frac{\partial \phi}{\partial y_j} u_i^{k-1}u_j^k \, dy
\]
\[
+ \frac{3}{4\pi} \int_{R^3} \frac{1}{|x-y|} \sum_{i,j} \frac{\partial^2 \phi}{\partial y_i \partial y_j} \phi u_i^{k-1}u_j^k \, dy,
\]
\[
p^k_3 = \frac{3}{4\pi} \int_{R^3} \frac{1}{|x-y|} p(y) \Delta_y \phi \, dy
\]
\[
+ \frac{3}{4\pi} \int_{R^3} \sum_{i} \frac{x_i - y_i}{|x-y|^3} p(y) \frac{\partial \phi}{\partial y_i} \, dy.
\]

The Calderón–Zygmund theory (cf. Stein [18]) on a singular integral yields the estimates
\[
\|\nabla p^k_1\|_p \leq C\|u^{k-1}\|_p \|\nabla u^k\|_p \leq C A_1^{3/3-p} \|\nabla u^{k-1}\|_p^{3-3/p} \|\nabla u^k\|_p.
\]
Since \( \phi = 1 \) for a neighborhood of \( \Gamma' \), then for \( x \in \Omega' \)
\[
|\nabla p^k_2| + |\nabla p^k_3| \leq C \int_{\Omega} (|p^k| + |u^{k-1}| |u^k|) \, dx \leq C \int_{\Omega'} |p^k| + CA_1^2.
\]
The last fact and (3.12) and (3.17) give (4.1), (4.2).
Theorem 4.2. Let $\Omega'$ be a bounded domain such that $\overline{\Omega'} \subset \Omega$ and the conditions of Theorem 3.1 be satisfied. If $a \in W^{1,p}(\Omega)$ with $1 < p \leq 3/2$, then

$$u \in L^s(0,T;W^{2,p}(\Omega')) \quad \text{for} \quad \frac{1}{s} + \frac{3}{2p} \geq 2, \ 1 < p < \frac{3}{2}, \quad (4.5)$$

$$u \in L^1_n(0,T;W^{2,3/2}(\Omega')). \quad (4.6)$$

Proof. Similar to the proof of Theorem 3.1, we only need to establish the uniform estimate of the approximate solutions on the norm of the above space. Let $\phi \in C^0_0(\Omega)$, such that $\phi = 1$ for $x \in \Omega'$. Multiplying both sides of (3.9) by $\phi$, we get

$$\frac{\partial}{\partial t} (\phi u^k) - \nu \Delta(\phi u^k) + (u^{k-1} \cdot \nabla)(\phi u^k) = -\phi \nabla p^k + h, \quad (4.7)$$

where $h = \phi f + (u^{k-1} \cdot \nabla \phi)u^k - \nu u^k \Delta \phi - 2\nu (\nabla \phi \cdot \nabla)u^k$. Applying the fundamental solution of the heat equation, $\phi u^k$ can be expressed as

$$\phi u^k(x,t) = (4\pi)^{-3/2} \int_{R^3} (t - s)^{-3/2} e^{-|x - y|^2 / 4t} dy$$

$$- \int_0^t \int_{R^3} (t - s)^{-3/2} (u^{k-1} \cdot \nabla)(\phi u^k)(y) e^{-|x - y|^2 / 4(t - s)} dy ds$$

$$- \int_0^t \int_{R^3} (t - s)^{-3/2} \nabla p^k(y) \phi(y) e^{-|x - y|^2 / 4(t - s)} dy ds$$

$$+ \int_0^t \int_{R^3} (t - s)^{-3/2} h(y) e^{-|x - y|^2 / 4(t - s)} dy ds \bigg). \quad (4.8)$$

Utilizing $\tau^a e^{-\tau t} \leq C$, we deduce that

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} (\phi u^k) \right| \leq C \left( t^{-2} \int_{R^3} |\nabla u^k| e^{-|x - y|^2 / 4t} dy + \int_0^t \int_{R^3} (t - s)^{-5/2} |u^{k-1}| |\nabla u^k|(y) e^{-|x - y|^2 / 4(t - s)} dy ds + \int_0^t \int_{R^3} (t - s)^{-5/2} |\nabla p^k|(y) e^{-|x - y|^2 / 4(t - s)} dy ds + \int_0^t \int_{R^3} (t - s)^{-5/2} |h(y)| e^{-|x - y|^2 / 4(t - s)} dy ds \right).$$
Since $\phi = 1$ for $x \in \Omega'$, by the Hausdorff–Young inequality, we deduce that
\[
\left\| \frac{\partial^2 u^k}{\partial x_i \partial x_j} \right\|_p \leq C \left( t^{-1/2} \| a \|_{W^{1,\infty}(\Omega)} + \int_0^t (t-s)^{-1} \left\| u^{k-1} \right\| \| \nabla u^k \|_p \, ds + \int_0^t (t-s)^{-1} \| h(s) \|_p \, ds \right).
\]

Since
\[
\| h \|_p \leq \| f \|_p + C \| u^{k-1} \|_{L^p/(2-p)} \| \nabla u^k \|_2 + C \| u^k \|_p + C \| \nabla u^k \|_p,
\]
then
\[
h \in L^s(0, T; L^p(\Omega)) \quad \text{for} \quad \frac{1}{s} + \frac{3}{2p} \geq 2, 1 < p \leq \frac{3}{2}.
\]
The $L^p$-boundedness, $1 < p < \infty$, of the Hilbert transform implies
\[
\left( \int_0^T \left\| \frac{\partial^2 u^k}{\partial x_i \partial x_j} \right\|_p^s \, d\tau \right)^{1/s} \leq CT^{1/s-1/2} + CA_2^2 + \| \nabla p^k \|_{L^p(0, T; L^p(\Omega))}
\]
\[+ \| h \|_{L^p(0, T; L^p(\Omega))},
\]
which gives us (4.5). Weak $L^1$-boundedness of the Hilbert transform shows (4.6).

By virtue of the Sobolev inequality, (4.5) shows that $u \in L^1(0, T; L^q(\Omega'))$ for $q < \infty$, but $q \neq +\infty$. For $q = \infty$, we have
\[
\text{Theorem 4.3. Assume the conditions of Theorem 3.1 hold. Then}
\]
\[
\text{u} \in L^1_w(0, T; L^\infty(\Omega')) \quad \text{(4.9)}
\]

Proof: Utilizing the Hausdorff–Young inequality, expression (4.8) shows us that
\[
\| u^k \|_{L^1(\Omega')} \leq C \left( t^{-3/4} \| a \|_2 + \int_0^t (t-s)^{-1} \left\| (u^{k-1} \cdot \nabla) u^k \right\|_{3/2} \, ds + \int_0^t (t-s)^{-1} \| \nabla p^k \|_{3/2} \, ds + \int_0^t (t-s)^{-3/4} \| h \|_2 \, ds \right).
\]

Expression (4.1) and weak $L^1$-boundedness of Hilbert transform allow us to get
\[
\| u^k \|_{L^1_w(0, T; L^\infty(\Omega'))} \leq C.
\]
Taking the limit as $k \to +\infty$, the last fact implies (4.9).
Remark. From the procedure of the proof of Theorems 4.2, 4.3, we only consider in the interior of the domain owing to the estimate of (4.1) on pressure. If \( \Omega = R^3 \), Theorems 4.2, 4.3 are valid for \( \Omega' = R^3 \).

5. AN EXTENSION OF THE CLASSICAL REGULARITY RESULT

The aim of this section is to extend the classical regularity result. Namely, we show that \( u \) is a regular solution whenever \( \nabla u \in L^r(0, T; L'(\Omega)) \) for \( 1/s + 3/2r = 1 \) if \( \Omega \) satisfies assumption (1.3), while \( 1/s + 1/r = 5/6 \) if \( \Omega \) is an arbitrary domain in \( R^3 \) and \( 1 < s \leq 2 \). This result is the natural extension to \( s \in (1, 2] \) of the classical regularity class \( u \in L^r(0, T; L'(\Omega)) \) with \( 1/s + 3/2r = 1/2 \) for \( s \geq 2 \). Meanwhile, our result shows that \( L^2(0, T; W^{1,3}(\Omega)) \) is a regularity class. This result cannot be obtained from the classical result that \( L^2(0, T; L^r(\Omega)) \) is a regularity class, because \( W^{1,3}(\Omega) \) cannot be imbedded in \( L^r(\Omega) \). For simplicity, let \( f = 0 \).

**Theorem 5.1.** Let \( \Omega \) be a three-dimensional domain such that the semigroup generated by the Stokes operator \( A \) satisfies

\[
\| e^{-tA}g \|_q \leq C t^{-3/2(1/p - 1/q)} \| g \|_p
\]

for \( g \in L^p(\Omega), 1 < p \leq q < +\infty \). Then if \( a \in \dot{J}'(\Omega), \) and

\[
\nabla u \in L^r(0, T; L'(\Omega)) \quad \text{with} \quad \frac{1}{s} + \frac{3}{2r} = 1 \quad \text{and} \quad 1 < s \leq 2,
\]

we hold

\[
u \in L^r(0, T; L'(\Omega)),
\]

which implies that \( u \) is a regular solution.

Remark. (1) If \( \Omega \) is \( R^3, R^3_+ \), the bounded domain, or the exterior domain, the semigroup generated by the Stokes operator \( A \) verifies \( L^p \rightarrow L^q \) estimation (5.1), cf. [9, 10, 20].

(2) In the case \( \Omega = R^n \ (n \geq 3) \), a similar result has already been proved by Beirão da Veiga [4].

**Proof of Theorem 5.1.** As is standard practice, the solution of (2.1) can be expressed as follows, via the corresponding integral equation

\[
u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}P(u \cdot \nabla)u \, ds.
\]
Let $r \geq 3$. By the $L^p - L^q$ estimate (5.1), we deduce that
\[ \|u(t)\| \leq C \|a\| + C \int_0^t (t - s)^{-3/2r} \|u\| \|\nabla u\| \, ds. \]

Employing the Gronwall inequality, the last fact gives us (5.3), which implies that $u$ is a regular solution by the classical regularity result.

**Theorem 5.2.** Let $\Omega$ be an arbitrary domain and $a \in V$. Then if
\[ \nabla u \in L^s(0, T; L^r(\Omega)) \quad \text{for} \quad \frac{1}{s} + \frac{1}{r} = \frac{5}{6}, 1 < s \leq 2, \]
we have
\[ \|\nabla u\|_2^2 \leq \|\nabla a\|_2^2 \exp\left( C \int_0^t \|\nabla u\|_2^2 \, d\tau \right) \]
\[ \nabla u \in L^r(0, T; L^2(\Omega)), \quad Au \in L^2(0, T; L^2(\Omega)), \]
which implies that $u$ is a regularity solution.

**Remark.** If $1 < s \leq 2$ and $1/s + 1/r = 5/6$, then $1/s + 3/2r < 1$. We think that Theorem 5.2 is valid whenever (5.2) holds. Unfortunately, we don't show this.

**Proof of Theorem 5.2.** Applying the Helmholtz projection $P$ to (2.1), we reformulate the equations as
\[ \frac{\partial u}{\partial t} - \nu Au + P(u \cdot \nabla)u = 0. \]

Multiplying both sides of (5.7) by $Au$ and integrating over $\Omega$, we get
\[ \frac{d}{dt} \|\nabla u\|_2^2 + 2\nu \|Au\|_2^2 \leq 2 \|u\|_\infty \|\nabla u\|_2 \|Au\|_2, \quad \text{if} \quad r > 3, \]
\[ \frac{d}{dt} \|\nabla u\|_2^2 + 2\nu \|Au\|_2^2 \leq 2 \|u\|_6 \|\nabla u\|_3 \|Au\|_2, \quad \text{for} \quad r = 3. \]

by the Young inequality, it follows that
\[ \frac{d}{dt} \|\nabla u\|_2^2 + \nu \|Au\|_2^2 \leq \nu^{-1} \|u\|_\infty^2 \|\nabla u\|_2^2 \quad \text{for} \quad r > 3. \]

by Theorem 7.1 of [12, p. 14],
\[ \|u\|_\infty \leq C \|u\|_2^{(2r+6)/(5r-6)} \|\nabla u\|_r^{3r/(5r-6)} \quad \text{for} \quad r > 3. \]
So
\[
\frac{d}{dt} \| \nabla u \|^2_2 + \nu \| Au \|^2_2 \leq C \| \nabla u \|^6_6 \| \nabla u \|^2_2,
\]
i.e.,
\[
\frac{d}{dt} \left( \exp \left( -C \int_0^t \| \nabla u \|^6_6 \, d\tau \right) \right) + \nu \exp \left( -C \int_0^t \| \nabla u \|^6_6 \, d\tau \right) \| Au \|^2_2 \leq 0
\]
which implies (5.5), (5.6), for \( r > 3 \). If \( r = 3 \), the Sobolev inequality and (5.9) give us
\[
\frac{d}{dt} \| \nabla u \|^2_2 + \nu \| Au \|^2_2 \leq \nu^{-1} \| \nabla u \|^2_2 \| \nabla u \|^2_2.
\]
(5.10)

By an analogous deducment, (5.5), (5.6) follow from (5.10).

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