On the Boundary Spike Layer Solutions to a Singularly Perturbed Neumann Problem

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1. INTRODUCTION

In this paper, we are concerned with stationary solutions to the following reaction-diffusion system, which was proposed by Gierer and Meinhardt [5] to model biological pattern formation:

\begin{align*}
\frac{\partial a}{\partial t} &= D_a \Delta a - \mu_a a + \rho_a \left( \frac{a^p}{h^r} + \rho_0 \right), \\
\frac{\partial h}{\partial t} &= D_h \Delta h - \mu_h h + \rho_h \frac{a'}{r}.
\end{align*}

Here, the unknowns $a = a(x, t)$ and $h = h(x, t)$ represent the respective concentrations at point $x \in \mathbb{R}^N$ and at time $t$ of the biochemical called an activator and an inhibitor; $D_a$, $D_h$, $\mu_a$, $\mu_h$, $c_a$, $c_h$, $\rho_a$, $\rho_h$ are all positive constants, while $\rho_0$ is a nonnegative constant; $\Delta = \sum_{j=1}^N \partial^2 / \partial x_j^2$ is the Laplace operator in $\mathbb{R}^N$. The exponents $p$, $q$, $r$, $s$ are assumed to satisfy the condition

\begin{equation}
(A) \quad p > 1, q > 0, r > 0, s \geq 0, \quad \text{and} \quad 0 < \frac{p - 1}{q} < \frac{r}{s + 1}.
\end{equation}

We assume that the activator and the inhibitor occupy a bounded domain $\Omega$ in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and that there is no flux through the boundary; i.e., we impose the homogeneous Neumann boundary condition

\begin{equation}
\text{BC} \quad \frac{\partial a}{\partial v} = \frac{\partial h}{\partial v} = 0 \quad \text{on } \partial \Omega,
\end{equation}

in which $v$ denotes the unit outer normal to $\partial \Omega$.

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By numerical simulation of the activator-inhibitor system (GM), it is observed that, when the ratio $D_a/D_h$ is small, (GM) seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\bar{\Omega}$. Moreover, as $D_a \to 0$ the pattern exhibits a “spike layer phenomenon,” by which we mean that the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain number of points as $D_a \to 0$, whereas the maximum value of the activator concentration diverges to $+\infty$.

In this paper we would like to construct stationary solutions with a single boundary spike and classify the locations of single concentrations on the boundary for the stationary solutions when $\rho_0 = 0$. For the case $\rho_0 > 0$, we have partial progress in [20] (it is known (see [19]) that for $\rho_0 > 0$ we have a priori estimates for the stationary solutions).

Therefore, we consider the stationary problem

$$
\varepsilon^2 \Delta A - A + \frac{A^p}{H^q} = 0 \quad \text{in } \Omega,
$$

$$
D_h \Delta H - \mu H + \frac{A^p}{H^q} = 0 \quad \text{in } \Omega \quad (1.1)
$$

$$
\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
$$

where we have normalized the unknowns $a(x)$, $h(x)$ and the equations by putting

$$
a(x) = \left\{ \left( \frac{\mu_a}{\epsilon_h \mu_a} \right)^q \left( \frac{\epsilon_p \mu_a}{\mu_a} \right)^{s+1} \right\}^{\frac{1}{q}} A(x)
$$

$$
h(x) = \left\{ \left( \frac{\mu_a}{\epsilon_h \mu_a} \right)^{p-1} \left( \frac{\epsilon_p \mu_a}{\mu_a} \right)^{s+1} \right\}^{\frac{1}{q}} H(x)
$$

and

$$
\lambda = \frac{1}{qr - (p-1)(s+1)},
$$

$$
\varepsilon = \sqrt{D_a \mu_a}, \quad D = \frac{D_h}{\mu_a}, \quad \mu = \frac{\mu_h}{\mu_a}
$$

Note that $\lambda > 0$ by (A).
If we let $D_h \to \infty$ and suppose that the quantity $-\mu H + A'/H'$ remains bounded, then for $AH \to 0$, $\partial H/\partial v = 0$ on $\partial \Omega$, we find that $H(x) \to \zeta$, a constant. To derive the equation for $\zeta$, we integrate both sides of (1.1) over $\Omega$ and observe that $\int_\Omega AH \, dx = 0$ due to the boundary condition. Hence in the limit of $D_h \to \infty$, we obtain two independent equations,

$$\begin{cases}
\varepsilon^2 A u - u + u'' = 0 \\
u > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega.
\end{cases}$$

(1.2)

and

$$-|\Omega| \mu \zeta + \varepsilon^{q/p-1-s} \int_\Omega u' \, dx = 0,$$

(1.3)

where we put

$$A(x) = \varepsilon^{q/p-1} u(x).$$

Thus we are reduced to studying the single equation (1.2). The purpose of this paper is to study the role of the mean curvature of the boundary in the solutions of (1.2). Throughout this paper, we always assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\varepsilon > 0$, $1 < p < (N+2)/(N-2)$ when $N \geq 3$ and $1 < p < \infty$ when $N = 1, 2$.

Equation (1.2) is also known as the stationary equation of the Keller–Segal system in chemotaxis; see, e.g., [8].

Lin, Ni, and Takagi first established the existence of least-energy solutions in [8] and Ni and Takagi in [10, 11] showed that for $\varepsilon$ sufficiently small the least-energy solution has only one local maximum point $P \in \partial \Omega$ (therefore the least-energy solutions have a boundary spike layer). Moreover, $H(P) \to \max_{P \in \partial \Omega} H(P)$ as $\varepsilon \to 0$, where $H(P)$ is the mean curvature of $P$ at $\partial \Omega$. In [12] Ni and Takagi constructed multiple spike solutions with spike on the boundary in an axially symmetric domains.

In this paper, we shall study the general solutions in general domains. In particular, we investigate the role of the mean curvature of the boundary in the general solutions of problem (1.2). The simplest general solutions are the so-called single-boundary-peaked solutions (see definition below). We shall characterize all local maximum points of single-boundary-peaked solutions. It seems this paper is the first one in the literature dealing with the effect of mean curvature on the general solutions of (1.2) in general domains.

To state our results, we need to introduce some notation.
It is known that the solution of the problem

\[
\begin{align*}
Aw - w + w^p &= 0, & \text{in } R^N \\
w > 0, \quad w(z) \to 0 & \quad \text{as } |z| \to \infty \\
w(0) &= \max_{x \in R^N} w(z)
\end{align*}
\]  

(1.4)
is radial [6] and unique [7]. We denote this solution as \( w \).

Let \( u \in H^1(\Omega) \) and

\[
I_v(u) = \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} u^2 - \frac{1}{p+1} \int_{\Omega} u^{p+1}.
\]

Let

\[
I(w) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1}.
\]

Family of solutions \( u_v \) of (1.2) are called single-boundary-peaked if \( \lim_{\varepsilon \to 0} \varepsilon^{-N} I_v(u_v) = \frac{1}{2} I(w) \). Certainly least energy solutions are single-boundary-peaked.

Our first result concerns the locations of the maximum points of single-boundary-peaked solutions.

**Theorem 1.1.** If \( u_v \) is a solution of (1.2) and \( \lim_{\varepsilon \to 0} \varepsilon^{-N} I_v(u_v) = \frac{1}{2} I(w) \), then for \( \varepsilon \) sufficiently small \( u_v \) has only one local (hence global) maximum point, \( P_v \), and \( P \in \partial \Omega \). Moreover, \( \nabla_{\varepsilon \tau} H(P_v) \to 0 \) as \( \varepsilon \to 0 \) where \( \nabla_{\varepsilon \tau} \) is the tangential derivative at \( P_v \).

Our second result is a converse of Theorem 1.1.

**Theorem 1.2.** Let \( P_0 \in \partial \Omega \). Suppose that \( P_0 \) is a nondegenerate critical point of the mean curvature function \( H(P) \). Then for \( \varepsilon \) sufficiently small there exists a solution \( u_v \) to (1.2) such that \( \varepsilon^{-N} I_v(u_v) \to \frac{1}{2} I(w) \), \( u_v \) has only one local maximum point \( P_v \), and \( P \in \partial \Omega \). Moreover, \( P_v \to P_0 \).

We note that when \( p = (N+2)/(N-2) \), similar results for the boundary spike layer solutions have been obtained by [3, 9, etc.] Other related concentration phenomena are found in [13–18, 23, etc.]. Our results here are the first ones in constructing general boundary peak solutions for Eq. (1.2) in general domains. Going back to the system (GM), it would be an interesting and important question to study how the geometry of the boundary affects the stability of the solutions constructed in Theorem 1.2.

Our strategy in proving Theorem 1.1 and 1.2 is the following.
First step: Fixing a point $P \in \partial \Omega$, we introduce a good approximate function $P_\varepsilon w([x-P]/\varepsilon)$ which satisfies the Neumann boundary condition and concentrates at $P$.

Second step: Using the functions $P_\varepsilon w([x-P]/\varepsilon)$ for $P \in \partial \Omega$, we establish a coordinate system $(\rho, P, v)$ for each function $u$ with energy near $\varepsilon^N \frac{1}{2} H(w)$, where $\rho$ is the scale of $u$, $P$ is the center of $u$, and $v \in H^1(\Omega)$ is the error term, which is in the orthogonal space of the kernel.

Third step: We solve $v$ and expand $v$ in terms of $\varepsilon$. The problem is then reduced to an $(N+1)$-dimensional problem.

Final step: We solve the finite-dimensional problem.

The organization of our paper is as follows. In Section 2, we analyze the projection of $w$ in $H^1(\Omega)$. Then we set up the technical framework for the problem and reduce the problem to a finite-dimensional problem in Section 3. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2. Finally all the technical quantities are proved in Appendices A, B, and C.

In this paper we denote various generic constants by $C$. We use $O(A)$, $o(A)$ to mean $|O(A)| \leq C|A|$, $o(A)/|A| \to 0$ as $|A| \to 0$, respectively. Whenever we have a repeated index, we mean summation over that index from 1 to $N-1$ unless otherwise specified.

2. PRELIMINARIES AND ANALYSIS OF PROJECTIONS

In this section, we introduce a nice approximate function (see (2.4)) and derive the asymptotic expansion of the function as well as its tangential derivatives.

We first transform the boundary.

Let $P \in \partial \Omega$. Since $\partial \Omega$ is smooth, we can find $R_0 > 0$, $\rho : B'(R_0) \to R$ a smooth function such that $\rho(0) = 0$, $\nabla \rho(0) = 0$, and

$$\Omega_1 = \Omega \cap B(R_0) = \{ (x', x_N) \in B(R_0) \mid x_N - P_N > \rho(x' - P') \}$$

$$\omega_1 = \partial \Omega \cap B(R_0) = \{ (x', x_N) \in B(R_0) \mid x_N - P_N = \rho(x' - P') \}$$

where $x' = (x_1, \ldots, x_{N-1})$ and

$$B(R_0) = \{ x \in R^N \mid |x| < R_0 \}, \ B'(R_0) = \{ x' \in R^{N-1} \mid |x'| < R_0 \}.$$

Note that $H(P_0) = A \rho(0)$. By Taylor’s expansion, we can assume that

$$\rho(a) = \frac{1}{2} \rho_A(0) a, a + \frac{1}{6} \rho_{\text{sk}}(0) a, a, a + O(|a|^3)$$
for $a \in R^{N-1}$ small, where

$$\rho_{i} = \frac{\partial \rho}{\partial x_{i}}, \quad \rho_{ij} = \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{j}}, \quad \text{etc.}$$

For $z \in \partial \Omega$, let $\nu(z)$ denote the unit outward normal at $z$, let $\partial / \partial \nu$ denote the normal derivative, let $(\tau_{1}(z), ..., \tau_{N-1}(z))$ denote the $(N-1)$ linearly independent tangent vectors, and let $(\partial / \partial \tau_{1}, ..., \partial / \partial \tau_{N-1})$ denote the corresponding $(N-1)$ tangential derivatives at $z$.

For $x \in \omega_{1}$, we have

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla_{x} \rho|^{2}}} \left( \nabla_{x} \rho(x'-P), -1 \right)$$

$$\frac{\partial}{\partial \nu} = \frac{1}{\sqrt{1 + |\nabla_{x} \rho|^{2}}} \left( \rho \frac{\partial}{\partial x_{i}} - \frac{\partial}{\partial x_{N}} \right)_{x_N - P_N - \rho(x'-P)}$$

$$\tau_{i}(x) = \left(0, ..., 1, ..., 0, \frac{\partial \rho}{\partial x_{i}}(x'-P)\right), \quad i = 1, ..., N-1$$

$$\frac{\partial}{\partial \tau_{i}} = \frac{\partial}{\partial x_{i}} + \rho_{i}(x'-P) \frac{\partial}{\partial x_{N}} \bigg|_{x_N - P_N - \rho(x'-P)}, \quad i = 1, ..., N-1.$$

Let $w$ be the unique solution of (1.4). We set $P_{\Omega} w(\lfloor x - P \rfloor / \varepsilon)$ to be the unique solution of

$$\begin{cases}
\varepsilon^{2} \Delta u - u + w \left( \frac{x - P}{\varepsilon} \right) = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(2.4)

By the Maximum Principle, $P_{\Omega} w(\lfloor x - P \rfloor / \varepsilon) > 0$. In the following, we will write $P_{\Omega} w(\lfloor x - P \rfloor / \varepsilon)$ as $P_{\Omega} w$ or $P_{w}$ when there is no confusion.

For each $u, v \in H^{1}(\Omega)$, we define

$$\langle u, v \rangle_{x} = \varepsilon^{-N} \int_{\partial \Omega} (\varepsilon^{2} \nabla u \cdot \nabla v + uv).$$

We denote $\langle u, u \rangle_{x}$ as $\|u\|_{x}^{2}$.

We now analyze $P_{\Omega} w(\lfloor x - P \rfloor / \varepsilon)$. To this end, we set

$$h_{\varepsilon, P}(x) = w \left( \frac{x - P}{\varepsilon} \right) - P_{\Omega} w \left( \frac{x - P}{\varepsilon} \right).$$

We introduce the following functions.
Let $v_1$ be the unique solution of
\[
\begin{aligned}
\begin{cases}
(Dv - v = 0 \text{ in } R_+^N, \\ v \in H^1(R_+^N) \\
\frac{\partial v}{\partial y_N} = \frac{1}{2} \frac{w'}{|y|} \rho_{y_j}(0) y_i y_j \text{ on } \partial R_+^N.
\end{cases}
\end{aligned}
\]
(2.5)

Let $v_2$ be the unique solution of
\[
\begin{aligned}
\begin{cases}
(Dv - v - 2 \sum_{i,j=1}^{N-1} \rho_{y_j}(0) y_i \frac{\partial^2 v_1}{\partial y_j y_N} = 0 \text{ in } R_+^N, \\ v \in H^1(R_+^N) \\
\frac{\partial v}{\partial y_N} = \rho_{y_j}(0) y_i \frac{\partial v_1}{\partial y_j} \text{ on } \partial R_+^N.
\end{cases}
\end{aligned}
\]
(2.6)

and let $v_3$ be the unique solution of
\[
\begin{aligned}
\begin{cases}
(Dv - v = 0 \text{ in } R_+^N, \\ v \in H^1(R_+^N) \\
\frac{\partial v}{\partial y_N} = \frac{1}{3} \frac{w'}{|y|} \rho_{y_k}(0) y_i y_j y_k \text{ on } \partial R_+^N.
\end{cases}
\end{aligned}
\]
(2.7)

Note that $v_1, v_2$ are even functions in $y = (y_1, \ldots, y_{N-1})$ (i.e., $v_i(y', y_N) = v_i(-y', y_N)$, $v_2(y', y_N) = v_2(-y', y_N)$). Similarly $v_3$ is an odd function in $y'$. Moreover, it is easy to see that $|v_i| \leq Ce^{-h|y|}$ for $i = 1, 2, 3$ and some $0 < h < 1$.

Let $\gamma(a)$ be a function such that $\gamma(a) = 1$ for $a \in B(0, R_0)$, $\gamma(a) = 0$ for $a \in B(0, R_0)\backslash B(0, R_0)$.

Set
\[
\begin{aligned}
ev' = x' - P', \quad ey_N = x_N - P_N - \rho(x' - P'),
\end{aligned}
\]
\[
\begin{aligned}
h_{x, \rho}(x) = ev_j(y) \gamma(x - P) + e^2(v_{y_j}(x - P) + v_j(y) \gamma(x - P)) + e^3 v_j'(x).
\end{aligned}
\]
Then we have

**PROPOSITION 2.1.** $\|e v_j\|_* \leq C$.

To prove Proposition 2.1, we expand $h_{x, \rho}(x)$. We first note that the Laplacian operator and the boundary derivative operator become
\[
\begin{aligned}
\varepsilon^2 A_x = A_x + |\nabla_x \rho|^2 \frac{\partial^2}{\partial y_N^2} - 2 \sum_{i=1}^{N-1} \rho \frac{\partial^2}{\partial y_i \partial y_N} - \varepsilon A_{x, \rho} \rho \frac{\partial}{\partial y_N},
\end{aligned}
\]
(2.8)

\[
\begin{aligned}
\sqrt{1 + |\nabla_x \rho|^2} \frac{\partial}{\partial y_N} = \frac{1}{\varepsilon} \left( \nabla_{y_N} \rho \frac{\partial}{\partial y_N} - (1 + |\nabla_x \rho|^2) \frac{\partial}{\partial y_N} \right).
\end{aligned}
\]
(2.9)
We are in need of the following

**Lemma 2.2.** Let \( u \) be a solution of

\[
\varepsilon^2 \Delta u - u + f = 0 \quad \text{in } \Omega
\]

\[
\frac{\partial u}{\partial v} = g \quad \text{on } \partial \Omega
\]

and \( \int_{\Omega} |f|^2 \leq C \varepsilon^N \), \( \int_{\Omega} |g|^2 \leq C \varepsilon^{N-1} \). Then

\[ |u|_\infty \leq C. \]

**Proof.** Multiplying the equation by \( u \), we have

\[
\varepsilon^2 \int_{\Omega} |\nabla u|^2 + \int_{\Omega} u^2 = \int_{\Omega} fu + \varepsilon^2 \int_{\partial \Omega} gu.
\]

By the following interpolation inequality (see for example the proof of Theorem 3.1 in [1]),

\[ \varepsilon^{-(N-1)} \int_{\partial \Omega} u^2 \, dx \leq C \| u \|_\infty^2. \]

Lemma 2.2 follows easily.

We now prove Proposition 2.1.

**Proof of Proposition 2.1.** We first compute

\[-\varepsilon^2 \Delta v'_{\varepsilon}(x) + v''_{\varepsilon}(x)\]

\[= \frac{1}{\varepsilon^2} \left[ \varepsilon^2 \left( \varepsilon A_1 \partial_1^2 + \varepsilon^2 (v_2 \lambda + v_3 \lambda) \right) - \varepsilon^2 \lambda - \varepsilon^2 \lambda - \varepsilon^2 \lambda \right] \]

\[= \frac{1}{\varepsilon^2} \left[ \lambda A_1 \left( \frac{\partial^2 v_1}{\partial y_0} - 2p_1 \frac{\partial^2 v_1}{\partial y_0} \right) - e A_1 \frac{\partial v_1}{\partial y_0} - v_1 \right] \lambda \]

\[+ \varepsilon \left( A_2 \varepsilon + |\nabla_\varepsilon| \right) \left( \frac{\partial^2 v_2}{\partial y_0} - 2p_1 \frac{\partial^2 v_2}{\partial y_0} - e A_2 \frac{\partial v_2}{\partial y_0} - v_2 \right) \lambda \]

\[+ \varepsilon \left( A_3 + |\nabla_\varepsilon| \right) \left( \frac{\partial^2 v_3}{\partial y_0} - 2p_1 \frac{\partial^2 v_3}{\partial y_0} - e A_3 \frac{\partial v_3}{\partial y_0} - v_3 \right) \lambda + E(\lambda) \]
where $E_\epsilon(\rho)$ denotes all the terms involving derivatives of $\rho$. Since $|v_1|$, $|v_2|$, $|v_3| \leq \exp(-\mu |y|)$ for $\mu < 1$ we have

$$|f_r| \leq C e^{\mu' |y|} \quad \text{for some} \quad 0 < \mu' < \mu$$

for some $0 < \mu' < \mu < 1$. Hence

$$\int_\Omega f_r' \leq C e^\mu$$

for $r > 1$.

On the boundary $\partial \Omega$

$$\frac{\partial v_r(x)}{\partial v} = \frac{1}{\epsilon^2} \left[ \frac{\partial w}{\partial v} - \epsilon \frac{\partial (v_1 x)}{\partial v} - \epsilon^2 \left( \frac{\partial (v_2 x)}{\partial v} + \frac{\partial (v_3 x)}{\partial v} \right) \right].$$

Note that

$$\frac{\partial w}{\partial v} \sqrt{1 + |\nabla x|^2}$$

$$= w(\frac{x - P}{\epsilon} - v_{1} \epsilon) - \frac{v_{1} \epsilon}{\epsilon |x - P|}$$

$$= \frac{w([x - P]/\epsilon)}{\epsilon |x - P|} \left[ \frac{1}{2} \rho_y(0)(x_i - P_j)(x_j - P_j) \right.$$

$$+ \frac{1}{3} \rho_{y^2}(0)(x_i - P_j)(x_j - P_i)(x_k - P_k) + \epsilon \exp(\mu |y|) \left] \right.$$
\begin{align*}
\sqrt{1 + |\nabla \varphi|^2} \frac{\partial v_1}{\partial \varphi} &= \frac{1}{\varepsilon} \left\{ \rho_k \frac{\partial v_1}{\partial y_k} - (1 + |\nabla \rho|^2) \frac{\partial v_1}{\partial y_N} \right\}, \\
\varepsilon \frac{\partial v_i^e}{\partial \varphi} (x) &= \frac{1}{\sqrt{1 + |\nabla \rho|^2}} \left[ w' \left( \frac{1}{2} \rho_y \varphi_{yy} + \frac{\varepsilon}{3} \rho_{y0}(0) \varphi_y \right) + e^2(\exp(-\mu |y|)) \\
&- \rho_k \frac{\partial v_1}{\partial y_k} + |\nabla \rho|^2 \frac{\partial v_1}{\partial y_N} \\
&- \varepsilon \rho_k \frac{\partial v_2}{\partial y_k} + \frac{\varepsilon}{3} |\nabla \rho|^2 \frac{\partial v_2}{\partial y_N} \\
&- \varepsilon \rho_k \frac{\partial v_3}{\partial y_k} + \frac{\varepsilon}{3} |\nabla \rho|^2 \frac{\partial v_3}{\partial y_N} + E(\varphi) \right] \\
&= g_i(y).
\end{align*}

This implies that
\[ g_i \leq \exp(-\mu' |x - P|/\varepsilon) \quad \text{for some} \quad 0 < \mu' < \mu. \]

Therefore,
\[ \int_\Omega g_i^2(x) \leq C e^{N-1}. \]

By Lemma 2.2
\[ \|v_i^*\| \leq C. \]

Proposition 2.1 is thus proved.

We next analyze \( \partial / \partial \tau_{P_j} P_j \) where \( \partial / \partial \tau_{P_j} \) is given in (2.3). By the coordinate system we choose, we can assume that
\[ \partial / \partial \tau_{P_j} = \partial / \partial \tau_{P_j}. \]

Then \( \partial / \partial \tau_{P_j} h_{\varepsilon, \mu}(x) \) satisfies
\[ \varepsilon^2 \Delta v - v = 0 \quad \text{in} \ \Omega, \]
\[ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} \varepsilon \frac{\partial}{\partial P_j} \left( \frac{x - P_j}{\varepsilon} \right) \quad \text{on} \ \partial \Omega. \]
We compute
\begin{align*}
(1 + |\nabla r|^2) & \frac{\partial}{\partial y} \frac{\partial}{\partial P_j} \left( \frac{x - P}{\varepsilon} \right) \\
& = \frac{\partial}{\partial x_i} \frac{\partial}{\partial P_j} \left( \frac{x - P}{\varepsilon} \right) \rho_i - \frac{\partial}{\partial x_N} \frac{\partial}{\partial P_j} \left( \frac{x - P}{\varepsilon} \right) \\
& = - \left[ \frac{\partial^2 w}{\partial x_i \partial x_N} \left( \frac{x - P}{\varepsilon} \right) \rho_i - \frac{\partial^2 w}{\partial x_N \partial x_j} \left( \frac{x - P}{\varepsilon} \right) \right].
\end{align*}
Now we have for \( x = P + \varepsilon z, \ y_j = x_j - P_j, \ y_N = x_N - P_N - \rho(x' - P'), \)
\begin{align*}
\frac{\partial w}{\partial z_j} (z) & = w' z_j, \\
\frac{\partial^2 w}{\partial z_j \partial z_j} & = w' z_j^2 + w' \left( \frac{\partial}{\partial z_j} z_j \right), \\
\frac{\partial^2 w((x - P)/\varepsilon)}{\partial x_N \partial x_j} & = \frac{1}{\varepsilon} \left( \frac{w' z_j}{|z|^3} \right), \\
\frac{\partial^2 w((x - P)/\varepsilon)}{\partial x_i \partial x_j} & = \frac{1}{\varepsilon} \left( \frac{w' \delta_{ij}}{|z|^3} \right) \rho_i.
\end{align*}
\begin{align*}
(1 + |\nabla r|^2) & \frac{\partial}{\partial y} \frac{\partial}{\partial P_j} \left( \frac{x - P}{\varepsilon} \right) = - \left[ \frac{1}{\varepsilon} \left( \frac{w' z_j}{|z|^3} \right) + w' \left( \frac{\partial}{\partial z_j} z_j \right) \right] \varepsilon \rho_i y_k \\
& - \frac{1}{\varepsilon} \left( \frac{w' y_j}{|y|^3} - w' \frac{y_j}{|y|^3} \right) \frac{\varepsilon^2}{2} \rho_{kl} y_k y_l \\
& = \frac{1}{\varepsilon} \left[ \frac{1}{2} \rho_{kl} \left( \frac{w'}{|y|^3} - \frac{w'}{|y|^3} \right) y_j y_k + \frac{w'}{|y|^3} \rho_{kl} y_j y_k \right] \\
& + \text{smaller term.}
\end{align*}
Let
\begin{align*}
\frac{\partial w}{\partial t_{P_i}} - \frac{\partial}{\partial t_{P_i}} P_\omega w \left( \frac{x - P}{\varepsilon} \right) = w_1(y) \gamma(x - P) + \varepsilon w'(x).
\end{align*}
Here \( w_1 \) is the unique solution of
\begin{align*}
Aw - v & = 0 & \text{on } R^N_+, \ v \in H^1(R^N_+) \\
\frac{\partial v}{\partial y_N} & = \frac{1}{2} \left( \frac{w'}{|y|^3} - \frac{w'}{|y|^3} \right) \rho_{\partial}(0) y_j y_k + \frac{w'}{|y|^3} \rho_{\partial}(0) y_j y_k & \text{on } \partial R^N_+.
\end{align*}
Note that $|w_1| \leq C \exp(-\mu |y|)$ for some $\mu < 1$ and $w_1$ is an odd function in $y'$. Then $w_2^\varepsilon$ satisfies
\begin{align*}
\left\{ \begin{array}{l}
\varepsilon^2 Aw_2^\varepsilon - w_2^\varepsilon + \frac{1}{\varepsilon} \left[ \varepsilon^2 Axw_2^\varepsilon - w_2^\varepsilon \right] = 0 \quad \text{in } \Omega, \\
\frac{\partial w_2^\varepsilon}{\partial \nu} = \frac{1}{\varepsilon} \left[ \frac{\partial w}{\partial \nu} \varepsilon^{-1} + \frac{\partial}{\partial \nu} w_1(y) \chi(x - P) \right] \quad \text{on } \partial \Omega.
\end{array} \right.
\end{align*}
(2.11)

By Lemma 2.2, similar to the proof of Proposition 2.1, we obtain

**Proposition 2.3.**
\[
\frac{\partial}{\partial \tau_P} P_{\partial \Omega} w \left( \frac{x - P}{\varepsilon} \right) = w_1(y) \chi(x - P) + \varepsilon w_2^\varepsilon(x),
\]
where $w_1$ is as defined above and
\[
\|w_2^\varepsilon\| \leq C.
\]

3. TECHNICAL FRAMEWORK

In this section, we set up a technical framework. Our idea is to decompose $u$ into two parts. One is the major part, which involves the peak point; the other is the error part, which will be small.

For each $a > 0$, we define
\[
F = \left\{ P_{\partial \Omega} w \left( \frac{x - P}{\varepsilon} \right) \mid P \in \partial \Omega \right\},
\]
\[
V_a = \left\{ (x, P) \in \mathbb{R} \times \partial \Omega \mid |x - 1| < a, P \in \partial \Omega \right\}.
\]

Let $d(u, F) = \inf_{v \in F} \|u - v\|_s$.

We first have the following lemma.

**Lemma 3.1.** If $u_c$ is a solution of (1.2) such that $\varepsilon^{-N}I_c(u_c) \to \frac{1}{2}I(w)$, then
\[
\lim_{\varepsilon \to 0} d(u_c, F) = 0.
\]

**Proof.** Let $u_c$ be a solution of (1.2). Arguments similar to those in the proof of Theorem 2.2 in [11] show that $u_c$ has only one local maximum point, $P_c \in \partial \Omega$. Moreover
\[
\|u_c - w((x - P_c)/\varepsilon)\|_s \to 0.
\]
Note that
\[\|P_\omega w((x - P_\epsilon)/\epsilon) - w((x - P_\epsilon)/\epsilon)\|_\infty \to 0.\]
Hence \(d(u_\epsilon, F) \to 0\) as \(\epsilon \to 0\).

Next we state a decomposition lemma, the proof of it is delayed until Appendix A.

**Lemma 3.2.** There exists \(\eta_0 > 0, a_0 > 0\) such that if \(u \in H^1(\Omega)\) satisfies \(d(u, F) < \eta_0\) then the problem

\[
\minimize_{\mathcal{M}} \left| u - \alpha P_\Omega \left( \frac{x - P}{\epsilon} \right) \right|^2
\]

with respect to \((x, P) \in V_a\) has a unique solution in the open set \(V_a\) for \(a < a_0\).

Therefore, there exists a diffeomorphism between a neighborhood of the possible single-boundary-peaked solutions of (1.2) we are interested in and the open set

\[
\mathcal{M} = \left\{ m = (x, P, v) \mid (x, P, v) \in \mathcal{M}' \right\}
\]

with \(\eta > 0\) some suitable constant and

\[
E_{a_\eta} = \left\{ v \in H^1(\Omega) \mid \langle v, P_\omega w \rangle_{x_\epsilon} = 0, i = 1, \ldots, N - 1 \right\}
\]

(Recall that \(P_\omega w = P_\omega w([x - P]/\epsilon)\).)

Let us now define the functional

\[
K_\varepsilon : M_\eta \to R
\]

\[
m = (x, P, v) \to e^{-N_\varepsilon} \left( \alpha P_\Omega \left( \frac{x - P}{\epsilon} \right) + v \right).
\]

It follows then that

**Proposition 3.3.** \(m = (x, P, v) \in M_\eta\) is a critical point of \(K_\varepsilon\) if and only if \(u = \alpha P_\Omega w + v\) is a critical point of \(K_\varepsilon\), i.e., if and only if there exists \((A, B) \in R \times R^{N-1}\) such that
The results of Theorem 1.1 and Theorem 1.2 will be obtained by a careful analysis of (E) on $M'$. We first deal with the $v$-part of $u$.

Analysis of (E$_v$):

**Proposition 3.4.** There exists an $\eta_0 > 0, \varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0, \eta < \eta_0$ then there exists a smooth map which, to any $(\varepsilon, \pi, P)$ such that $(\pi, P, 0) \in M_{\pi}$ associates $\bar{v} \in E_{\varepsilon, \pi}$, $||\bar{v}|| < \eta$ such that (E) is satisfied for some $(A, B) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Such a $\bar{v}$ is unique and minimizes $K(\pi, P, v)$ with respect to $v$ in \{\(v \in E_{\varepsilon, \pi} | ||v|| < \eta\}, and we have

$$\frac{\partial \bar{v}}{\partial \varepsilon} = 0 \text{ on } \partial \Omega, \quad ||\bar{v}|| < O(\varepsilon).$$

For the proof, see Appendix B.

Once $\bar{v} = \bar{v}(\pi, P)$ is obtained, we can estimate $A$ and $B$. In fact, we have

$$\left< \partial_{\varepsilon} K_{\varepsilon}, P_{\varepsilon} \right> = A \left< P_{\varepsilon}, P_{\varepsilon} \right> \varepsilon + \sum_{i=1}^{N-1} B_i \left< \partial_{\varepsilon} P_{\varepsilon}, P_{\varepsilon} \right> \varepsilon$$

and by (C.1), (C.2), and (C.3) in Appendix C, we obtain

$$\left< P_{\varepsilon}, P_{\varepsilon} \right> = \int_{\mathbb{R}^n} w^{p+1} + O(\varepsilon)$$

$$\left< \partial_{\varepsilon} P_{\varepsilon}, P_{\varepsilon} \right> = O(\varepsilon)$$

$$\left< \partial_{\varepsilon} P_{\varepsilon}, \partial_{\varepsilon} P_{\varepsilon} \right> = \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^n} p w^{p-1}(w')^2 \delta_{e} + O(\varepsilon) \right).$$
On the other hand

\[ \frac{\partial K_v}{\partial v} P_{\omega w} = \frac{\partial K_v}{\partial \alpha} \]

\[ \frac{\partial K_v}{\partial v} \frac{\partial P_{\omega w}}{\partial \tau_p} = \frac{1}{\alpha} \frac{\partial K_v}{\partial \tau_p} \]

Let us now estimate \( \partial K_v/\partial \alpha \) and \( \partial K_v/\partial \tau_p \). To this end, we need the following two important lemmas, the proofs of which are postponed to the end of this section.

The first is about the expansion of \( v \).

**Lemma 3.5.** If \( \alpha \) satisfies \((E)_\alpha\), then \( \alpha = 1 + \varepsilon \alpha_0 + O(\varepsilon^{1+\sigma}) \), where

\[ \alpha_0 = \frac{\int_{R^N} p w^p v_1}{(p-1) \int_{R^N} w^p}, \quad \sigma = \min(p - 1, 1) \]

(3.7)

and \( v_1 \) is defined by (2.5).

We next expand \( \tilde{v}_s \). Then we have

**Lemma 3.6.** If \( \alpha \) satisfies

\[ \alpha = 1 + \varepsilon \alpha_0 + O(\varepsilon^{1+\sigma}) \]

(3.8)

and \( \tilde{v}_s \) satisfies \((E)_{\tilde{v}_s}\), then we have

\[ \tilde{v}_s = \varepsilon \Phi_0(y) \chi(x - P) + \Psi_{\tilde{v}_s, P} \]

(3.9)

where \( \Phi_0 \) is the unique solution of

\[ \Delta \Phi_0 - \Phi_0 + p w^p - 1 \Phi_0 - (p - 1) \alpha_0 w^p - p w^p v_1 = 0, \quad \text{in } R^N_+ \]

\[ \frac{\partial \Phi_0}{\partial y_i} = 0 \quad \text{on } \partial R^N_+ \]

(3.10)

\[ \Phi_0 \perp H^1(R^N_+) \text{ span } \left\{ \frac{\partial w}{\partial y_j}, j = 1, \ldots, N - 1 \right\} \]

and \( \Psi_{\tilde{v}_s, P} \) satisfies

\[ \| \Psi_{\tilde{v}_s, P} \| \leq C \varepsilon^{1+\sigma} \]
Assuming that $\alpha = 1 + \varepsilon \alpha_0 + O(\varepsilon^{1+\varphi})$ and (E.) is satisfied, we then have, by (C.4) and (C.5) in Appendix C and (3.8),
\[
\frac{\partial K}{\partial \alpha} = \alpha \int_{\Omega_{\alpha \tau}} w^p Pw - \int_{\Omega_{\alpha \tau}} (\alpha Pw + \varepsilon)^p Pw
\]
\[
= \alpha \int_{\Omega_{\alpha \tau}} w^p Pw - \int_{\Omega_{\alpha \tau}} (\alpha Pw)^p Pw - p \int_{\Omega_{\alpha \tau}} (\alpha Pw)^{p-1} \varepsilon Pw
\]
\[
+ \int_{\Omega_{\alpha \tau}} ((\alpha Pw)^p + p(\alpha Pw)^{p-1} \varepsilon - (\alpha Pw + \varepsilon)^p) Pw
\]
\[
= (\alpha - \alpha^p) \int_{\Omega_{\alpha \tau}} w^p Pw + \alpha^p \int_{\Omega_{\alpha \tau}} (w^p Pw - (Pw)^{p+1}) + O(\varepsilon^{1+\varphi})
\]
\[
= O(\varepsilon^{1+\varphi}).
\]

By (C.6) and (C.7) in Appendix C, we have
\[
\frac{\partial K}{\partial \tau_{P_i}} = \alpha^2 \int_{\Omega_{\alpha \tau}} w^p \frac{\partial Pw}{\partial \tau_{P_i}} - \alpha \int_{\Omega_{\alpha \tau}} (\alpha Pw + \varepsilon)^p \frac{\partial Pw}{\partial \tau_{P_i}}
\]
\[
= \alpha \int_{\Omega_{\alpha \tau}} w^p \frac{\partial Pw}{\partial \tau_{P_i}} Pw - \int_{\Omega_{\alpha \tau}} (\alpha Pw)^p \frac{\partial Pw}{\partial \tau_{P_i}} - p \int_{\Omega_{\alpha \tau}} (\alpha Pw)^{p-1} \varepsilon \frac{\partial Pw}{\partial \tau_{P_i}}
\]
\[
+ \int_{\Omega_{\alpha \tau}} ((\alpha Pw)^p + p(\alpha Pw)^{p-1} \varepsilon - (\alpha Pw + \varepsilon)^p) \frac{\partial Pw}{\partial \tau_{P_i}}
\]
\[
= O(\varepsilon^{1+\varphi}).
\]

Combining all these, by Eqs. (3.6) and (3.7), we have
\[
A = O\left(\left|\frac{\partial K}{\partial \alpha}\right| + \varepsilon^2\right) = O(\varepsilon^{1+\varphi})
\]
\[
B = \varepsilon^2 O\left(\sum_{i=1}^{N} \left|\frac{\partial K}{\partial \tau_{P_i}}\right| + \varepsilon^2\right) = O(\varepsilon^{3+\varphi}).
\]

We can now estimate equation (E.):
\[
\frac{\partial K}{\partial \tau_{P_i}} = \sum_{i=1}^{N-1} B_i \left\langle \frac{\partial^2 P_{\Omega \tau}}{\partial \tau_{P_i} \partial \tau_{P_j}}, v \right\rangle_{\alpha, \tau} \leq \sum_{i=1}^{N-1} |B_i| \left\| \frac{\partial^2 P_{\Omega \tau}}{\partial \tau_{P_i} \partial \tau_{P_j}} \right\|_\infty \left\| v \right\|_\infty
\]
\[
= O(\varepsilon^{(3/2)+\varphi}).
\]

Finally, in this section, we prove Lemmas 3.5 and 3.6.
Proof of Lemma 3.5. From Eq. \((E)\) and Appendix C, we have
\[
\begin{align*}
\alpha & \int_{\Omega} w^p P_w - \int_{\Omega} (x P_w + \bar{v})^p P_w = 0 \\
\alpha & \int_{\Omega} w^p P_w - \int_{\Omega} (x P_w)^p P_w - \int_{\Omega} (x P_w)^{p-1} \bar{v} P_w \\
& - \int_{\Omega} \left\{(x P_w + \bar{v})^p - (x P_w)^p - p x P_w (x P_w)^{p-1} \bar{v} \right\} P_w = 0 \\
\alpha & \int_{\Omega} w^p P_w - \int_{\Omega} (x P_w)^p P_w = O(\varepsilon^{1+\sigma}).
\end{align*}
\]
Let \(\alpha = 1 + \varepsilon \beta\). Then it is easy to see that
\[
\beta = \frac{\int_{\Omega} w^p v_1}{(p-1)} - p \frac{\varepsilon^{1+\sigma}}{\varepsilon}.\]

Lemma 3.5 is proved.

Proof of Lemma 3.6. Let \(K_{\varepsilon, p} = \text{span}\{w^p, (\partial^2 w^p / \partial \tau_p), j = 1, ..., N - 1\}\). By Appendix B, if \(v \in E_{\varepsilon, p}\), then
\[
\varepsilon^{-N} \left( \varepsilon^2 \int_{\Omega} |v|^2 + v^2 - P(P_w)^{p-1} v^2 \right) \geq \rho \|v\|_{\varepsilon}^2
\]
for some \(\rho > 0\).

By Eq. (3.10), we have
\[
|\Phi_0| \leq C \exp(-\mu |y|) \quad \text{for} \quad \mu < 1
\]
\[
\Phi_0 \perp \text{span}\left\{w, \frac{\partial w}{\partial y_j}, j = 1, ..., N - 1\right\}. \tag{3.13}
\]

Let \(\text{Proj}(\Phi_0 Z)\) be the projection of \(\Phi_0 Z\) onto \(E_{\varepsilon, p}\). Then because of (3.13),
\[
\|\Phi_0 Z - \text{Proj}(\Phi_0 Z)\|_{\varepsilon} \leq C\varepsilon.
\]

Moreover \(N_{\varepsilon} = \varepsilon^2 A(\Phi_0 Z - \text{Proj}(\Phi_0 Z)) - (\Phi_0 Z - \text{Proj}(\Phi_0 Z)) \in K_{\varepsilon, p}\).

Let \(N \text{Proj}(\Phi_0 Z)\) be the unique solution of
\[
\varepsilon^2 A v - v = \varepsilon^2 A \text{Proj}(\Phi_0 Z) - \text{Proj}(\Phi_0 Z) \quad \text{in} \quad \Omega,
\]
\[
\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega. \tag{3.14}
\]
Since \(\partial \Phi_0 / \partial y_N = 0\) on \(\partial R_N\), we have
\[
\|N \text{Proj}(\Phi_0 Z) - \Phi_0 Z\|_{\varepsilon} \leq C\varepsilon.
\]
We write $v = c_N \text{Proj}(\Phi_0 \chi) = \Psi^1_{\epsilon, p}$. Then
\[ \| \Psi^1_{\epsilon, p} - \Psi^1_{\epsilon, p} \|_\epsilon \leq C \epsilon \]
and the equation for $\Psi^1_{\epsilon, p}$ is
\[
\varepsilon^2 \Delta \Psi^1_{\epsilon, p} - \Psi^1_{\epsilon, p} = - (x P_\Omega w + \varepsilon_1) + \varepsilon w^\beta \chi - \varepsilon ((\epsilon^2 \Delta \Phi_0 \chi - \Phi_0 \chi) + N_0) + \varepsilon P_\Omega w + \varepsilon v_i x + \varepsilon (p - 1) \sigma_0 w \chi - p(\lambda P_\Omega w)^{p-1} \Psi^1_{\epsilon, p} \nonumber \\
= (x P_\Omega w)^p + p(x P_\Omega w)^{p-1} v_i + \varepsilon (p - 1) \sigma_0 w \chi - p(\lambda P_\Omega w)^{p-1} \Psi^1_{\epsilon, p} \nonumber \\
- \varepsilon \sigma_0 \Phi_0 \chi + N_0, \nonumber
\]
where $N_0 \in K_{\epsilon, p}$. By (3.7) and Eq. (3.10)
\[
xw^\beta - (x P_\Omega w)^p - \varepsilon w^p x_i x - \varepsilon (p - 1) \sigma_0 w \chi = O(\epsilon^1 + \sigma). \nonumber
\]

On the other hand,
\[
(\lambda P_\Omega w)^p + p(\lambda P_\Omega w)^{p-1} v_i = (x P_\Omega w)^p + p(x P_\Omega w)^{p-1} v_i = O(\epsilon^2) \nonumber \\
- \varepsilon \sigma_0 \Phi_0 \chi = O(\epsilon^1 + \sigma). \nonumber
\]

Now $\Psi^1_{\epsilon, p} \in E_{\epsilon, p}$ and $\partial \Psi^1_{\epsilon, p}/\partial v = 0$ on $\partial \omega$. Multiplying the equation for $\Psi^1_{\epsilon, p}$ by $\Psi^1_{\epsilon, p}$, using (3.12), we have $\| \Psi^1_{\epsilon, p} \|_\epsilon \leq C \epsilon^{1 + \sigma}$. Lemma 3.6 is proved.

4. PROOF OF THEOREMS 1.1 AND 1.2

We can now finish the proofs of Theorems 1.1 and 1.2.

Let $u_\epsilon$ be a sequence of single-boundary-peaked solutions. By the same argument as in the proof of Theorem 2.1 of Ni and Takagi [10], $u_\epsilon$ has only one local maximum points, $P_\epsilon$, and $P_\epsilon \in \partial \omega$. Then by Lemma 3.1 and Proposition 3.3
\[
u_\epsilon = \sigma_0 P_\Omega w \left( \frac{x - P_\epsilon}{\epsilon} \right) + v', \tag{4.1}
\]
where $(\sigma_0, P_\epsilon, v')$ satisfies Eq. (E). It follows that $v' = \varepsilon_0 (\sigma_0, P_\epsilon)$. Then (E) yields, by Lemma 3.5,
\[
\sigma_0 = 1 + \varepsilon \sigma_0 + O(\epsilon^{1 + \sigma}).
\]
By Lemma 3.6, (E) together with (E) shows that
\[
v' = \varepsilon \Phi_0 (\gamma) \chi (x - P) + \Psi^1_{\epsilon, p}.
\]
Substituting into Eq. (E₁), we have

**Lemma 4.1.** Suppose that $\alpha$ satisfies

$$\alpha = 1 + \varepsilon \alpha_0 + O(\varepsilon^{1+\sigma}),$$

(4.2)

where $\alpha_0$ is defined by (3.7) and (E₁) is satisfied; then Eq. (E₁) are equivalent to

$$\frac{\partial K_i}{\partial \tau_{P_0}} = c \gamma \nabla_{\tau_{P_0}} H(P) + O(\varepsilon^{1+\sigma}), \quad i = 1, \ldots, N - 1,$$

(4.3)

where

$$\gamma = -\frac{1}{3} \int_{\mathbb{R}^{n-1}} (w')^2 \, y_i^2 \, dy \neq 0.$$

Hence if $P_0 \to P_0'$ we have

$$\nabla_{\tau_{P_0}} H(P_0) = 0,$$

(4.4)

where $\nabla_{\tau_{P_0}}$ are the tangential derivatives at $P_0'$. Moreover if $H(P)$ has a nondegenerate critical point at $P_0$, then $P_0 - P_0' = O(\varepsilon^\gamma)$, which proves Theorem 1.1.

We now begin to prove Theorem 1.2.

Let $P_0 \in \partial \Omega$ be such that $\nabla_{P_0} H(P_0) = 0$ and the matrix $(\nabla_{P_0}^2 H(P_0))$ is nondegenerate. We set $P = P_0 + \varepsilon \xi$, $\alpha = 1 + \varepsilon \alpha_0 + \varepsilon^{\gamma/2} \beta$, $M(P_0) = (\nabla_{P_0}^2 H(P_0))$.

Then according to previous computations, the system (E) is equivalent to

$$\beta = V_x(\varepsilon, \beta, \xi)$$

(4.5)

$$M(P_0) \xi = V_P(\varepsilon, \beta, \xi),$$

(4.6)

where $V_x, V_P$ are smooth functions and satisfy

$$V_x = O(\varepsilon^{\gamma/2} + \beta^2)$$

(4.7)

$$V_P = O(\varepsilon^{\gamma/2} + \xi^2 + |\beta|).$$

(4.8)

Since the matrix $M(P_0)$ is nondegenerate and so it is invertible, we have by Brouwer's fixed point theorem that the system (E) has a solution $(\beta^*, P^*)$ for $\varepsilon$ small enough. Moreover

$$\beta^* = O(\varepsilon^{\gamma/3})$$

(4.9)

$$\xi = O(\varepsilon^{\gamma/3}).$$

(4.10)
By construction, the corresponding $u_\varepsilon \in H^1(\Omega)$ is a critical point of $J_\varepsilon$; i.e., $u_\varepsilon$ satisfies on $\Omega$

$$\varepsilon^2 A u_\varepsilon - u_\varepsilon + |u_\varepsilon|^{p-1} u_\varepsilon = 0, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \tag{4.11}$$

Multiplying both sides by $u_\varepsilon^- = \max(0, -u_\varepsilon)$ and integrating by parts on $\Omega$, we obtain

$$\left( \int_{\Omega_{\varepsilon, r}} |u_\varepsilon^-|^{p+1} \right)^{2/(p+1)} \leq C \int_{\Omega_{\varepsilon, r}} |u_\varepsilon^-|^{p+1}. \tag{4.12}$$

Hence either $u_\varepsilon^- \equiv 0$ or $\int_{\Omega_{\varepsilon, r}} |u_\varepsilon^-|^{p+1} \rightarrow C > 0$. By our construction $\int_{\Omega_{\varepsilon, r}} |u_\varepsilon^-|^{p+1} \rightarrow 0$. Hence $u_\varepsilon^- \equiv 0$. By the Maximum Principle, $u_\varepsilon > 0$. Moreover $\varepsilon^{-N} I_i(u_\varepsilon) \rightarrow \frac{1}{2} H(w)$. So $u_\varepsilon$ is a single-boundary-peaked solution which concentrates at $P_0$.

Finally in this section, we prove Lemma 4.1.

Proof of Lemma 4.1. To simplify our notation, we denote $P_{\varepsilon} w([x - P]/\varepsilon)$ as $P_{\varepsilon}$. From Eq. (E-P), we have

$$\frac{\partial K_{\varepsilon}}{\partial \nu} = \frac{\partial K_{\varepsilon}}{\partial P_j}$$

$$= x \int_{\Omega_{\varepsilon, r}} \frac{\partial P_{\varepsilon} w}{\partial P_j} - x \int_{\Omega_{\varepsilon, r}} (x P_{W} + v')^p \frac{\partial P_{W}}{\partial P_j}$$

$$= (x^2 - x^{p+1}) \int_{\Omega_{\varepsilon, r}} w^p \frac{\partial P_{W}}{\partial P_j} + x^{p+1} \int_{\Omega_{\varepsilon, r}} \left( w^p \frac{\partial P_{W}}{\partial P_j} + (x P_{W} + v')^p \frac{\partial P_{W}}{\partial P_j} \right)$$

$$- x^{p+1} \int_{\Omega_{\varepsilon, r}} (x P_{W} + v')^p \frac{\partial P_{W}}{\partial P_j}$$

$$+ x \int_{\Omega_{\varepsilon, r}} ((x P_{W})^p + p(x P_{W})^{p-1} v' - (x P_{W} + v')^p) \frac{\partial P_{W}}{\partial P_j}$$

$$= I_1^* + I_2^* + I_3^* + I_4^*,$$

where $I_j^*$, $i = 1, 2, 3, 4$, are defined at the last equality.

We first compute $I_1^*$:

$$I_1^* = (x^2 - x^{p+1}) \int_{\Omega_{\varepsilon, r}} w^p \left( \frac{\partial w}{\partial P_j} + w_1 + cw_2 \right) = O(\varepsilon^0).$$
Then for $I_2$, 

$$
\int_{\Omega} (w^\rho - (Pw)^\rho) \frac{\partial Pw}{\partial P_j}
= \int_{\Omega} (w^\rho - (Pw)^\rho - pw^{\rho-1}(w - Pw)) \frac{\partial Pw}{\partial P_j} + \int_{\Omega} pw^{\rho-1}(w - Pw) \frac{\partial Pw}{\partial P_j}
= I_{2.1}^e + I_{2.2}^e,
$$

where

$$
I_{2.2}^e = p \int_{\Omega} w^{\rho-1} v_3^2 \left( \frac{\partial w}{\partial P_j} + O(\varepsilon^2) \right)
= -p \int_{\Omega} w^{\rho-1} v_3 \frac{\partial w}{\partial P_j} + O(\varepsilon^2)
= \gamma \varepsilon \sqrt{\gamma} H(P) + O(\varepsilon^2),
$$

where

$$
\gamma = \sum_{l, m=1}^{N-1} - \frac{1}{3} \int_{\mathbb{R}^2} \frac{(w')^2}{|y|} y_j^2 y_j dy \neq 0
$$

and

$$
I_{2.1}^e = \int_{\Omega} (w^\rho - (Pw)^\rho - pw^{\rho-1}(w - Pw) - p(p - 1) w^{\rho-2}(w - Pw)^2) \frac{\partial w}{\partial P_j}
+ p(p - 1) \int_{\Omega} w^{\rho-2}(w - Pw)^2 \frac{\partial w}{\partial P_j}
+ \int_{\Omega} (w^\rho - (Pw)^\rho - pw^{\rho-1}(w - Pw))(w_1 + \varepsilon w_2)
= I_{2.1.1}^e + I_{2.1.2}^e + I_{2.1.3}^e,
$$

where $I_{2.1.1}^e, i = 1, 2, 3,$ are defined at the last equality.
We have

$$|I_{2.1,1}^\sigma| \leq C \int_I |w|^{\sigma - 2 + \tau} |Pw - w|^{2 + \tau} \left| \frac{\partial w}{\partial P_j} \right|$$

(if $p < 3$)

$$\leq C \int_I (|w|^{\sigma - 3 + \tau} + |Pw|^{\sigma - 3 + \tau}) |Pw - w|^{3} \left| \frac{\partial w}{\partial P_j} \right|$$

(if $p \geq 3$)

$$= O(\varepsilon^{1 + \sigma})$$

for $\sigma = \min(1, p - 1)$.

$$I_{2.1,3}^\varepsilon = p(p - 1) \varepsilon^2 \int \omega \sum_{i=1}^{\varepsilon^{2}} \left( \frac{\partial w}{\partial P_j} \right) + O(\varepsilon^3) = O(\varepsilon^3)$$

since $r_i$ is even.

Similarly

$$I_{2.1,3}^\varepsilon = O(\varepsilon^{1 + \sigma}).$$

Summing up we have

$$I_2^\varepsilon = \gamma \varepsilon V / H(P) + O(\varepsilon^{1 + \sigma}).$$

For the other two terms, see (C.6) and (C.7) in Appendix C:

$$I_5^\varepsilon = O(\varepsilon^{1 + \sigma}), \quad I_\varepsilon = O(\varepsilon^{1 + \sigma}).$$

Hence we have

$$\frac{\partial K_\varepsilon}{\partial \varepsilon} = \gamma \varepsilon V / H(P) + O(\varepsilon^{1 + \sigma}).$$

APPENDIX A. A DECOMPOSITION LEMMA

In this Appendix, we shall prove the decomposition Lemma 3.2 in Section 3. Since the proof is very similar to the proof of Lemma 3.1 in [23], we just sketch the idea. We start with some lemmas.

**LEMMA 5.1.** Let $(\varepsilon_k)$ be a sequence with $\varepsilon_k > 0$, $\lim_{k \to \infty} \varepsilon_k = 0$, and $\lim_{k \to \infty} \eta_k = 0$. Let $(\alpha_k, P_k) \in \mathcal{A}_m$ be such that

$$\lim_{k \to \infty} \left\| \alpha_k P_{\Omega} \left( \frac{x - P_k}{\varepsilon_k} \right) - \tilde{\alpha}_k P_{\Omega} \left( \frac{x - \tilde{P}_k}{\varepsilon_k} \right) \right\|_{r_q} = 0. \quad (5.1)$$


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Then we have

\[
\lim_{k \to \infty} |x_k - \tilde{x}_k| = 0, \quad (5.2)
\]

\[
\lim_{k \to \infty} \frac{|P_k - \bar{P}_k|}{\varepsilon_k} = 0. \quad (5.3)
\]

**Proof.** Let us define \( P_w = P_{\Omega w}([x - P_k] / \varepsilon_k) \) and \( P_{\tilde{w}} = P_{\Omega w}([x - \bar{P}_k] / \varepsilon_k) \). Then we have

\[
\lim_{k \to \infty} \frac{\|P_k\|^2}{\varepsilon_k} = \lim_{k \to \infty} \frac{\|P_{\tilde{w}}\|^2}{\varepsilon_k} = A \quad (5.4)
\]

\[
\lim_{k \to \infty} \langle P_{\tilde{w}}, P_w \rangle_{\alpha, \Omega} = \begin{cases} 0 & \text{as } |P_k - \bar{P}_k| / \varepsilon_k \to \infty \\ a \in \mathbb{R}^p \wedge (\cdot - a) & (\langle A if \ a \neq 0; \ = A if \ a = 0) \text{ as } (P_k - \bar{P}_k) / \varepsilon_k \to a. \end{cases} \quad (5.5)
\]

We have

\[
\|x_k P_w - \tilde{x}_k P_{\tilde{w}}\|_{\varepsilon_k}^2 = (x_k)^2 (A + o(1)) + (\tilde{x}_k)^2 (A + o(1)) - 2x_k \tilde{x}_k \langle P_w, P_{\tilde{w}} \rangle_{\alpha, \Omega}. \quad (5.6)
\]

On the other hand, it is easy to see that

\[
(x_k)^2 - (\tilde{x}_k)^2 = o(1). \quad (5.7)
\]

Combining (5.5) and (5.6), we obtain the Lemma. \( \square \)

We now prove Lemma 3.2. We will follow closely the proof in [23]. We argue by contradiction. Suppose there exists \( \varepsilon_k \to 0, \eta_k \to 0 \) such that

\[
\inf_{v \in A_{\eta_k}} \|u - v\|_{\alpha, \eta_k} < \eta_k,
\]

and \((x_k, P_k), (\tilde{x}_k, \bar{P}_k) \in A_{\eta_k}\), such that if \( v^k = u_k - x_k P_w, \ \bar{v}^k = u_k - \tilde{x}_k P_{\tilde{w}}, \)

\[
\langle v^k, P_w \rangle_{\alpha, \Omega} = 0, \quad (5.7)
\]

\[
\langle v^k, \bar{\partial}_i P_w \rangle_{\alpha, \Omega} = 0, \quad (5.8)
\]

\[
\langle \bar{v}^k, P_{\tilde{w}} \rangle_{\alpha, \Omega} = 0, \quad (5.9)
\]

\[
\langle \bar{v}^k, \bar{\partial}_i P_{\tilde{w}} \rangle_{\alpha, \Omega} = 0, \quad (5.10)
\]

where \( \bar{\partial}_i = \partial / \partial x_i, \ i = 1, ..., N - 1. \)
Let \( a_k = P_k - \tilde{P}_k / \varepsilon_k \), \( \mu_k = \alpha_k - \beta_k \). Then by Lemma 5.1, \( |a_k| = o(1) \), \( |\mu_k| = o(1) \).

We denote \( C \) as various constants which do not depend on \( k \). We first observe that
\[
|w^\rho(y) - w^\rho(y - a_k)| \leq C |a_k| w^\rho(y).
\] (5.11)

By the Maximum Principle
\[
|Pw - \tilde{P}w| \leq C |a_k| Pw(y).
\] (5.12)

From (5.7) and (5.9), we obtain
\[
\langle \tilde{v}_k, Pw - \tilde{P}w \rangle_{\alpha_k} = \langle v_k - \tilde{v}_k, Pw \rangle_{\alpha_k}
= \langle \pi_k Pw - \beta_k P\tilde{w}, Pw \rangle_{\alpha_k}
= \langle \pi_k Pw - \beta_k P\tilde{w}, Pw \rangle_{\alpha_k}.
\]

We now compute the right hand side as
\[
\text{right hand side} = \mu_k \langle Pw, Pw \rangle_{\alpha_k} + \beta_k \langle Pw - \tilde{P}w, Pw \rangle_{\alpha_k}
= \mu_k (A + o(1)) + \beta_k O(|a_k|^2)
\]
since
\[
\langle Pw - \tilde{P}w, Pw \rangle_{\alpha_k} = \int_{\Omega_k} (w^\rho - \tilde{w}^\rho) Pw
= O(|a_k|^2 + o(1)|a_k|).
\]

On the other hand, we have
\[
\langle \tilde{v}_k, Pw - \tilde{P}w \rangle_{\alpha_k} \leq \| \tilde{v}_k \|_{\alpha_k} \| Pw - \tilde{P}w \|_{\alpha_k}
= o(1) \| Pw - \tilde{P}w \|_{\alpha_k},
\]
\[
\| Pw - \tilde{P}w \|_{\alpha_k}^2 = \int_{\Omega_k} [w^\rho - \tilde{w}^\rho(\cdot - a_k)](Pw - \tilde{P}w)
= o(1) |a_k|.
\]

Combining all these together, we have
\[
\mu_k = o(1)(|\mu_k| + |a_k|).
\] (5.13)
As in [23], we use (5.8) and (5.10) to obtain
\[ a_k = o(1) \left( |p_k| + |a_k| \right). \] (5.14)
Therefore, we conclude that \( \sigma_k = \tilde{\sigma}_k, \ P_k = \tilde{P}_k. \)

APPENDIX B. ANALYSIS OF \((E_v)\)

In this Appendix, we prove Proposition 3.1.

Proof: We first expand \( K(x, P, v) \) at \((x, P, 0)\); we have
\[ K(x, P, v) = K(x, P, 0) + f_{x, P, v}(v) + Q_{x, P, v}(v) + R_{x, P, v}(v), \] (6.1)
where
\[ K(x, P, 0) = e^{-N} I_x(xP_0 w) \]
\[ f_{x, P, v}(v) = e^{-N} \left[ \int_\Omega (xPw)^p v \right] \]
\[ Q_{x, P, v}(v) = e^{-N} \left[ \int_\Omega |v|^2 + v^2 - p \int_\Omega (xPw)^{p-1}v^2 \right] \]
and \( R_{x, P, v} \) satisfies
\[ R_{x, P, v}(v) = O(\|v\|_{E_0}^{\min(3, p + 1)}), \] (6.2)
\[ R_{x, P, v}(v) = O(\|v\|_{E_0}^{\min(2, p)}), \] (6.3)
\[ R_{x, P, v}(v) = O(\|v\|_{E_0}^{\min(1, p - 1)}). \] (6.4)

Since \( f_{x, P, v}(v) \) is continuous from \( E_0, P \) equipped with \( \langle \cdot, \cdot \rangle_\varepsilon \) scalar product we may write
\[ f_{x, P, v}(v) = \langle F_{x, P, v}, v \rangle_\varepsilon \quad \text{for some} \quad F_{x, P, v} \in E_0, P. \] (6.5)

Since \( Q_{x, P, v} \) is a continuous quadratic form on \( E_0, P \), there exists a continuous and symmetric operator \( L_{x, P, v} \in L(E_0, P) \), a linear functional on \( E_0, P \), such that
\[ Q_{x, P, v}(v) = \langle L_{x, P, v}, v \rangle_\varepsilon. \] (6.6)

Moreover, we have
Lemma 6.1. There exists $\rho > 0$ such that for $\varepsilon$ small enough and $\eta$ small enough, we have

$$Q_{\varepsilon, \eta}(v) \geq \rho \|v\|_1, \quad \text{for all } v \in E_{\varepsilon, P}. \quad (6.7)$$

The proof of this lemma will be delayed until the end of this section.

Therefore $L_{\varepsilon, \eta}$ is coercive operator whose modulus of coercivity is bounded from below independently on $\varepsilon, P$.

The derivative of $K_{\varepsilon}$ with respect to $v$ on $E_{\varepsilon, P}$ may be written

$$F_{\varepsilon, \eta} + 2L_{\varepsilon, \eta}v + O(\|v\|_1^2).$$

Using the implicit function theorem, we derive the existence of a $C^2$-map $T$ which to each $(\varepsilon, \eta, P)$ associates $v = (\eta, \eta, P) \in E_{\varepsilon, P}$ such that

$$K_{\varepsilon}(v) = 0 \quad (\varepsilon, \eta, P) \in E_{\varepsilon, P} \subset 0$$

and

$$\|E_{\varepsilon}\|_\infty = O(\|F_{\varepsilon, \eta}\|_\infty). \quad (6.8)$$

We now claim that

$$\|f_{\varepsilon, \eta}(v)\|_\infty \leq O(\varepsilon) \|v\|_1, \quad (6.9)$$

which by (6.8) and (6.9) proves Proposition 3.1.

Recall that

$$f_{\varepsilon, \eta}(v) = -\varepsilon^{-N} \left( \int_{\Omega} (Pv)^\rho \right)$$

since

$$\int_{\Omega} (Pv)^\rho = \int_{\Omega}((Pv)^\rho - w^\rho) \quad (\varepsilon \in E_{\varepsilon, P}).$$

We now calculate, by Proposition 2.1, that

$$\varepsilon^{-N} \int_{\Omega}((Pv)^\rho - w^\rho)^2 \leq O(\varepsilon^2).$$

Thus (6.9) is established.
Since \( \tilde{v}_e \) minimizes \( K_e \) over \( E \), we have that \( \tilde{v}_e \) satisfies
\[
\int_\Omega e^2 \nabla \tilde{v}_e \cdot \nabla \varphi + \tilde{v}_e \varphi - \int_\Omega (\pi P_{\omega} w + \tilde{v}_e)^p \varphi = 0 \quad \text{for all } \varphi \in E_{\omega, P} \tag{6.10}
\]
and \( \partial \tilde{v}_e / \partial v = 0 \) on \( \partial \Omega \).

Finally, we prove Lemma 6.1. The proof of Lemma 6.1 follows from the following observation:

**LEMMA 6.2.** The eigenvalue problem
\[
\begin{cases}
A u - u + \mu w^{p-1} u = 0 & \text{in } R^N_+ \\
u \in H^1(R^N_+), \frac{\partial u}{\partial y_N} = 0 & \text{on } \partial R^N_+ 
\end{cases}
\tag{6.11}
\]
has eigenvalues \( \mu_1 = 1 < \mu_2 = \cdots = \mu_N = p < \mu_{N+1} < \cdots \). Moreover, the eigenspace
\[
V_1 = \{ w \}, \quad V_\mu = \text{span} \left\{ \frac{\partial w}{\partial y_j}, j = 1, \ldots, N-1 \right\}.
\tag{6.12}
\]

Hence there is a \( \rho > 0 \) such that for any \( v \perp \{ w, \partial w / \partial y_j, j = 1, \ldots, N-1 \} \), we have
\[
\int_{R^N_+} (|v|^2 + v^2) \geq (p + \rho) \int_{R^N_+} w^{p-1} v^2.
\tag{6.13}
\]

**Proof.** Let \( u \) be a solution of (6.11). Let \( v(y) = u(y) \) for \( y_N \geq 0 \) and \( v(y) = u(y', -y_N) \) if \( y_N \leq 0 \). Then \( v \in H^1(R^N) \) and is a solution of
\[
Av - v + pw^{p-1} v = 0 \quad \text{in } R^N.
\]
By Lemma 4.1 in \([22]\), \( v \in \text{span} \{ w, \partial w / \partial y_j, j = 1, \ldots, N \} \) and \( \mu_1 = 1 < \mu_2 = \cdots = \mu_N = p < \mu_{N+1} \). Since \( \partial v / \partial y_N = 0 \), (6.12) follows easily.

Next consider the following minimizing problem:
\[
\mu = \min \left\{ \int_{R^N_+} (|\nabla u|^2 + u^2), \int_{R^N_+} w^{p-1} u^2 = 1, u \perp \{ w, \frac{\partial w}{\partial y_j}, j = 1, \ldots, N-1 \} \right\}.
\]
By the above results \( \mu > p \), hence (6.13) holds.

**APPENDIX C. VARIOUS ESTIMATES**

In this Appendix, we provide all the estimates we need. Recall that \( \sigma = \min(1, p-1) \).
By using Proposition 2.1 and Proposition 2.2, we have

\[ (C.1) \quad \langle P_w, w \rangle_z = \int_{\Omega_r} w^p P_w + O(\varepsilon), = \int_{K^2} w^{p+1} + O(\varepsilon) \]

\[ (C.2) \quad \left( \frac{\partial P_w}{\partial \tau_j} \right)_z = \int_{\Omega_r} w^p \frac{\partial P_w}{\partial P_j} = O(\varepsilon) \]

\[ (C.3) \quad \left( \frac{\partial P_w}{\partial \tau_j} \right) = \frac{1}{\varepsilon^2} \left( \int_{K^2} p w^{p-1}(w')^2 \frac{\partial}{\partial y} + O(\varepsilon) \right) \]

By Proposition 2.1 and Lemma 3.6, we have

\[ (C.4) \int_{\Omega_r} (P_w)^p \tilde{v}_x = \int_{\Omega_r} \left( (P_w)^p - w^p \right) \tilde{v}_x = p \int_{\Omega_r} w^{p-1} e \tilde{v}_x \Phi_0 + O(\varepsilon^2 + \varepsilon) \]

\[ = p \int_{K^2} w^{p-1} e \tilde{v}_x \Phi_0 + O(\varepsilon) \]

\[ (C.5) \quad \varepsilon^{-N} \int_{\Omega} (wP_w + \tilde{v}_x)^p - (wP_w)^p - p(wP_w)^{p-1} \tilde{v}_x \tilde{v}_x = \varepsilon^{-N} \int_{\Omega} \left( (wP_w + \tilde{v}_x)^p - (wP_w)^p \right. \]

\[ - p(wP_w)^{p-1} \tilde{v}_x \tilde{v}_x - p(p-1)(wP_w)^{p-2} \tilde{v}_x^2 \frac{\partial P_w}{\partial P_j} \]

\[ + p(p-1) \int_{\Omega} (wP_w)^{p-2} \tilde{v}_x \frac{\partial P_w}{\partial P_j} \]

\[ = J_1 + J_2. \]

We estimate \( J_1 \) and \( J_2 \) as follows

\[ J_1 \leq C \int_{\Omega_r} |P_w|^{p-\sigma} e^\sigma \tilde{v}_x^2 + |\tilde{v}_x|^p \frac{\partial P_w}{\partial P_j} \] if \( p < 3 \)

\[ \leq C \int_{\Omega_r} (|P_w|^{p-3} + |\tilde{v}_x|^{p-3}) \tilde{v}_x^2 \frac{\partial P_w}{\partial P_j} \] if \( p \geq 3 \)

\[ = O(\varepsilon^{2\sigma}). \]
For $J_2$, since $\Phi_0$ is even, we have

$$J_2 = \varepsilon^2 \int_{\Omega_2} \left( (P_w)^{p-2} \frac{\partial P_w}{\partial P_j} - (w)^{p-2} \frac{\partial w}{\partial P_j} \right) \Phi_0^2 + O(\varepsilon^{2+\sigma})$$

$$= O(\varepsilon^{2+\sigma}).$$

Therefore

$$(C.5) \quad \varepsilon^{-N} \int_{\Omega} \left[ (\varepsilon P_w + \varepsilon_j)^p - (\varepsilon P_w)^p - p(\varepsilon P_w)^{p-1} \varepsilon_j \right] P_w$$

$$= O(\varepsilon^{2+\sigma}).$$

Similarly we have

$$(C.6) \quad \int_{\Omega_2} (P_w)^{p-1} \frac{\partial P_w}{\partial P_n} \varepsilon_j = O(\varepsilon^{1+\sigma}),$$

$$(C.7) \quad \varepsilon^{-N} \int_{\Omega} \left[ (\varepsilon P_w + \varepsilon_j)^p - (\varepsilon P_w)^p - p(\varepsilon P_w)^{p-1} \varepsilon_j \right] \frac{\partial P_w}{\partial P_j}$$

$$= \int_{\Omega} \left[ (\varepsilon P_w + \varepsilon_j)^p - (\varepsilon P_w)^p - p(\varepsilon P_w)^{p-1} \varepsilon_j - p(p-1) \right]$$

$$\times (\varepsilon P_w)^{p-2} \varepsilon_j^2 \frac{\partial P_w}{\partial P_j}$$

$$+ p(p-1) \int_{\Omega} (\varepsilon P_w)^{p-2} \varepsilon_j^2 \frac{\partial P_w}{\partial P_j}$$

$$= O(\varepsilon^{1+\sigma}).$$

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