THE REGULAR TWO-GRAH ON 276 VERTICES

J.M. GOETHALS
M.B.I.E. Research Laboratory, Brussels, Belgium

and

J.J. SEIDEL
Technological University, Eindhoven, The Netherlands

Received 15 February 1974
Revised 25 September 1974

There is a unique regular two-graph on 276 vertices. This provides a characterization of Conway's group \( \cdot 3 \). The proof is based on \( 276 = 3 \times 11 + 3^5 \), and uses the ternary Golay code. The paper contains a list of the known strongly regular graphs with the eigenvalue \(-5\).

1. Introduction

The aim of the present paper is to prove the following theorem.

**Theorem A.** There is a unique non-trivial regular two-graph on 276 vertices (up to taking complements).

This can be reworded as follows.

**Theorem B.** Conway's sporadic simple group \( \cdot 3 \) is characterized by its representation as the automorphism group of a regular two-graph on 276 vertices.

Regular two-graphs have been introduced by Higman, and investigated by Taylor [17, 18], and Seidel [16]. The case 276 provides a combinatorial setting for the doubly transitive representation of the simple group \( \cdot 3 \) discovered by Conway [3]. This regular two-graph has been characterized in [17, 18] under the additional assumption that there
exists a set of 23 independent vertices (that is, 23 vertices none of whose triples belong to the triple set of the two-graph). The present theorem shows that this additional assumption is superfluous.*

Section 2 contains an introduction to regular two-graphs. In Section 3 we specialize to 276 vertices, eigenvalues 55 and -5, and prove that each 3-clique is in ten 6-cliques. This yields a dissection into strongly regular subgraphs on 33 and on 243 vertices, whose relation in Section 4 leads to the orthogonal complement of the ternary Golay code. Thus, in Section 5, Theorem A is proved. In addition, the 276-two-graph is shown to contain a strongly regular 276-graph of valency 140. Some special attention is paid to the 243-graph, which was first discovered by Delsarte [4]. Finally, we list the known strongly regular graphs with eigenvalue -5, wondering whether this list is complete.

2. Regular two-graphs

Let \( \Omega \) denote a finite set of \( n \) elements. Let \( \Omega^{(t)} \) denote the set of all \( t \)-subsets of \( \Omega \). An ordinary graph consists of a vertex set \( \Omega \) and an edge set \( E \subseteq \Omega^{(2)} \).

**Definition 2.1.** A regular two-graph \((\Omega, \Delta)\) is a pair of a vertex set \( \Omega \) and a triple set \( \Delta \subseteq \Omega^{(3)} \), such that

(i) each 4-subset of \( \Omega \) contains an even number of triples of \( \Delta \),

(ii) each 2-subset of \( \Omega \) is contained in a constant number, say \( a \), of triples of \( \Delta \).

Condition (i) implies that, for any \( \omega \in \Omega \), the triple set \( \Delta \) is determined by its triples containing \( \omega \). Indeed, \( \{\omega, \omega_1, \omega_2, \omega_3\} \in \Delta \) whenever an odd number of the other 3-subsets of \( \{\omega, \omega_1, \omega_2, \omega_3\} \) belongs to \( \Delta \). If only condition (i) is required, then \((\Omega, \Delta)\) is called a two-graph, and may be interpreted as a switching class of graphs on \( n \) vertices. Condition (ii) means that these graphs are strong graphs with 2 eigenvalues. These notions are briefly recalled hereafter. For more details, we refer to [16].

**Definition 2.2.** The switching class of graphs belonging to the two-graph

* Meanwhile, Rosemary Rowley-Bailey informed the authors to have obtained, by different methods, the same result in her Ph. D. Thesis, Finite Permutation Groups, Oxford, 1974.
Given any two-graph \((\Omega, \Delta)\), its switching class is obtained as follows: Select any \(\omega \in \Omega\), and partition \(\Omega \setminus \{\omega\}\) into any 2 disjoint sets \(\Omega_1\) and \(\Omega_2\). Let \(E\) consist of the following pairs:

\[
\begin{align*}
\{\omega, \omega_1\} & \quad \text{for all } \omega_1 \in \Omega_1; \\
\{\omega_1, \omega'_1\} & \quad \text{for all } \omega_1, \omega'_1 \in \Omega_1, \text{ with } \{\omega, \omega_1, \omega'_1\} \in \Delta; \\
\{\omega_2, \omega'_2\} & \quad \text{for all } \omega_2, \omega'_2 \in \Omega_2, \text{ with } \{\omega, \omega_2, \omega'_2\} \in \Delta; \\
\{\omega_1, \omega_2\} & \quad \text{for all } \omega_1 \in \Omega_1, \omega_2 \in \Omega_2, \text{ with } \{\omega, \omega_1, \omega_2\} \notin \Delta.
\end{align*}
\]

Then \((\Omega, E)\) belongs to the switching class of \((\Omega, \Delta)\). Conversely, every graph in the switching class of \((\Omega, \Delta)\) is obtained in this way. The graph obtained by taking \(\Omega_1 = \emptyset\) in the above construction is called the derived graph of \((\Omega, \Delta)\), with respect to \(\omega \in \Omega\).

With respect to any labeling of \(\Omega\), any graph \((\Omega, E)\) is described by its \((-1, 1)\)-adjacency matrix \(A\) as follows: The elements of \(A\) are \(a_{ii} = 0\) for all \(i \in \Omega\), \(a_{xy} = -1\) for adjacent \(x, y \in \Omega\), and \(a_{uv} = 1\) for non-adjacent \(u, v \in \Omega\). Thus, \(A\) is a symmetric matrix with zero diagonal of the order \(n\). If \((\Omega, E)\) has the adjacency matrix \(A\), then any graph \((\Omega, E')\) in its switching class has the adjacency matrix \(A' = DA\bar{D}\), for some diagonal matrix \(D\) of order \(n\) with diagonal elements \(\pm 1\). We shall say that \((\Omega, E')\) is obtained from \((\Omega, E)\) by switching with respect to the vertices of \(\Omega\) which correspond to the elements \(-1\) of \(D\).

**Definition 2.3.** A strong graph is a graph whose adjacency matrix \(A\) satisfies, for some real \(\rho_1, \rho_2\),

\[
(A - \rho_1 I)(A - \rho_2 I) = (n - 1 + \rho_1 \rho_2)J, \quad \rho_1 > \rho_2.
\]

For an equivalent set-theoretic definition, and for further properties, to be recalled hereafter, we refer to [14]. Strong graphs have 2 or 3 eigenvalues according as \(n - 1 + \rho_1 \rho_2 = 0\) or \(\neq 0\). A strong graph with \(n - 1 + \rho_1 \rho_2 \neq 0\) is regular, hence strongly regular. A strong graph with \(n - 1 + \rho_1 \rho_2 = 0\) may or may not be regular; each graph in its switching class has the eigenvalues \(\rho_1\) and \(\rho_2\), which are odd integers unless \(\rho_1 + \rho_2 = 0\).
Theorem 2.4. The switching class of a regular two-graph consists of strong graphs having 2 eigenvalues \( \rho_1 \) and \( \rho_2 \), which are odd integers unless their sum vanishes, and

\[
n = 1 - \rho_1 \rho_2, \quad a = -\frac{1}{3}(\rho_1 + 1)(\rho_2 + 1).
\]

This theorem, for whose proof we refer to [16], enables us to investigate regular two-graphs \((\Omega, \Delta)\) on the basis of adjacency matrices of the order \( n \) satisfying

\[
(A - \rho_1 I)(A - \rho_2 I) = 0.
\]

From \( \text{tr} \ A = 0 \) it follows that the eigenvalues \( \rho_1 \) and \( \rho_2 \) have the multiplicities

\[
\mu_1 := \frac{-\rho_2 (1 - \rho_1 \rho_2)}{\rho_1 - \rho_2}, \quad \mu_2 := \frac{\rho_1 (1 - \rho_1 \rho_2)}{\rho_1 - \rho_2},
\]

respectively. The complete two-graph, with \( \Delta = \Omega^{(3)} \), has \( \rho_1 = 1, \rho_2 = 1 - n, \mu_1 = n - 1, \mu_2 = 1 \). The void two-graph, with \( \Delta = \emptyset \), has \( \rho_1 = n - 1, \rho_2 = -1, \mu_1 = 1, \mu_2 = n - 1 \). These are called trivial regular two-graphs. Non-trivial regular two-graphs have \( \rho_1 > 1, \rho_2 < -1, 1 < \mu_1, \mu_2 < n - 1 \).

Definition 2.5. A clique of a regular two-graph is a complete sub-two-graph; a coclique is a void sub-two-graph.

Theorem 2.6. In a regular two-graph each 3-clique is contained in a constant number of 4-cliques. This number equals

\[
b := 1 - \frac{1}{3}(\rho_1 + 3)(\rho_2 + 3),
\]

where \( \rho_1, \rho_2 \) are the eigenvalues of the regular two-graph \((\Omega, \Delta)\).

Proof. Let \( \{\alpha, \beta, \gamma\} \in \Delta \), and le: the number of 4-cliques \( \{\alpha, \beta, \gamma, \xi\} \) be \( b \). Since the number of 3-cliques containing any pair of vertices is a constant, \( a \) say, and since any 4-set contains an even number of 3-cliques, we have

\[
n = 3 + b + 3(a - 1 - b) = 3a - 2b,
\]

and the result follows from Theorem 2.4.
Theorem 2.7. The cardinality of a clique $\Gamma$ of a non-trivial regular two-graph with the eigenvalues $\rho_1, \rho_2$ and the multiplicities $\mu_1, \mu_2$ satisfies

$$|\Gamma| < \min \{1 - \rho_2, \mu_1, \mu_2\}.$$ 

Proof. Assuming the existence of a clique of size $c$, we represent the regular two-graph by the adjacency matrix

$$A = \begin{bmatrix} I - J & N \\ N^T & C \end{bmatrix},$$

with $I - J$ of size $c$. If $\mu_1 < c$, then $C$ has size $n - c < \mu_2$, hence $I - J$ has the eigenvalue $\rho_2 \neq 1$ with the multiplicity

$$1 = \mu > \mu_2 = (n - c) = c - \mu_1.$$

Hence $\rho_2 = 1 - c = -\mu_1$, which implies

$$1 - \rho_1 \rho_2 = \rho_1 - \rho_2, \quad (1 - \rho_1)(1 + \rho_2) = 0.$$

contrary to the assumption that the two-graph is non-trivial. Analogously, $\mu_2 < c$ is impossible. From $(A - \rho_1 I)(A - \rho_2 I) = 0$ it follows that

$$NN^T = -(1 - \rho_1)(1 - \rho_2)I + (2 - \rho_1 - \rho_2 - c)J.$$

The eigenvalues of this matrix are non-negative, whence

$$-(1 - \rho_1)(1 - \rho_2) + (2 - \rho_1 - \rho_2 - c)c \geq 0,$$

which implies $c \leq 1 - \rho_2$.

Remark 2.8. The present theorem also yields bounds for the cardinality of cocliques. Indeed, consider the complement of the two-graph $(\Omega, \Delta)$, which is defined to be the two-graph $(\Omega, \Omega^{(3)} \setminus \Delta)$. Obviously, a regular two-graph with the eigenvalues $\rho_1, \rho_2$ has a regular complement with the eigenvalues $-\rho_2 - \rho_1$. 


3. The case \( n = 276 \)

Let \((\Omega, \Delta)\) denote a non-trivial regular two-graph on 276 vertices. From

\[
276 = 1 - \rho_1 \rho_2, \quad \mu_1 = \frac{-276\rho_2}{\rho_1 - \rho_2}, \quad \mu_2 = \frac{276\rho_1}{\rho_1 - \rho_2}
\]

it follows that \(\rho_1 + \rho_2 \neq 0\), and that \(\rho_1 = 11, \rho_2 = -25\) and \(\rho_1 = 25, \rho_2 = -11\) are impossible. Hence in view of Theorem 2.4, without loss of generality, we may put \(n = 276, \rho_1 = 55, \rho_2 = -5, \mu_1 = 23, \mu_2 = 253, a = 112\).

By Theorem 2.6, each 3-clique is contained in 30 cliques of size 4. The adjacency matrix of these 3 + 30 vertices has the smallest eigenvalue \(-5\), with the multiplicity \(\geq 253 - (276 - 33) = 10\). Hence this matrix, augmented by \(5I\), to be denoted by

\[
\begin{bmatrix}
6I_3 - J_3 & -J_3 x_{30} \\
-J_3 x_{30} & 5I_{30} + C_{30}
\end{bmatrix},
\]

is positive semi-definite of rank \(\leq 23\). Therefore, it may be interpreted as the matrix of the inner products (Gram matrix) of 33 vectors, \(u, v, w, x_1, x_2, ..., x_{30}\) say, in \(\mathbb{R}^{23}\). It follows that the 30 vectors

\[
y_i = \frac{1}{2} x_i + \frac{1}{6} (u + v + w), \quad i = 1, 2, ..., 30,
\]

are perpendicular to \(u, v, w\), hence are in \(\mathbb{R}^{20}\). Since

\[
(y_i, y_j) = \frac{1}{4} (x_i, x_j) - \frac{1}{4},
\]

the vectors \(y_1, ..., y_{30}\) are unit vectors having mutual inner products 0 and \(-\frac{1}{2}\). We apply the following lemma, for the proof of which we refer to [9, Theorems 4.2 and 5.1].

**Lemma 3.1.** The maximum number of unit vectors in \(\mathbb{R}^d\) having mutual inner products 0 and \(-\frac{1}{2}\), equals \(d + \lfloor \frac{1}{2} d \rfloor \). For even \(d\) the maximum is achieved iff the vectors constitute \(\frac{1}{2} d\) mutually perpendicular stars of 3 planar vectors at the angle \(\frac{2}{3} \pi\).

Therefore, \(y_1, ..., y_{30}\) may be arranged in 10 perpendicular triples
with inner products $-\frac{1}{3}$. In the graph, the vectors $u, v, w, x_1, \ldots, x_{30}$ correspond to 33 vertices arranged in 11 mutually non-adjacent triangles, and we have:

**Lemma 3.2.** In $(\Omega, \Delta)$ each 3-clique is contained in 10 cliques of size 6.

By switching, any 6-clique in $(\Omega, \Delta)$ can be dissected into 2 disjoint 3-cliques in 10 ways. Following Lemma 3.2 each such dissection determines 27 further vertices, which are arranged in 9 cliques of size 3. The 10 sets of 27 vertices are mutually disjoint, and exhaust the 270 vertices outside of the 6-clique. They are called the 10 *pillars* with respect to the 6-clique. Hence we have:

**Lemma 3.3.** With respect to any 6-clique the remaining 270 vertices of $(\Omega, \Delta)$ are dissected into 10 pillars, each characterized by a dissection of the 6-clique, and each containing 9 disjoint 3-cliques.

We now prove:

**Lemma 3.4.** $(\Omega, \Delta)$ contains in its switching class a graph, which consists of 11 mutually non-adjacent triangles, and 243 further vertices each of which is adjacent to exactly one vertex of each triangle.

**Proof.** Start from a 3-clique $\{u, v, w\}$. By Lemma 3.2, there exists a graph having 11 mutually non-adjacent triangles $\{u, v, w\}, \{x_1, x_2, x_3\}, \ldots, \{x_{28}, x_{29}, x_{30}\}$, say. Switch with respect to $\{u, v, w\}$, and consider any of the 10 available 6-cliques $\{u, v, w, x_1, x_2, x_3\}$, say. By switching, each of the remaining 243 vertices is made adjacent to 2, and non-adjacent to 1 vertex of $\{u, v, w\}$. By Lemma 3.3, this vertex is adjacent to 1, and non-adjacent to 2 vertices of $\{x_1, x_2, x_3\}$, since it corresponds to a dissection of $\{u, v, w, x_1, x_2, x_3\}$. Now switch back with respect to $\{u, v, w\}$, then the lemma is proved.

**Lemma 3.5.** The 243 vertices of Lemma 2.4 carry a subgraph which is strongly regular.

**Proof.** Represent $(\Omega, \Delta)$ by a graph with the adjacency matrix:

$$A = \begin{bmatrix} B & N \\ N^T & C \end{bmatrix}.$$
where $B$ is the adjacency matrix of a subgraph consisting of 11 mutually non-adjacent triangles. Then

$$(B - I)(B + 5I) = 27J, \quad Rj = 28j$$

From $(A - 55I)(A + 5I) = 0$ it follows that

$$NN^T = 27(2B + 10I - J), \quad (C - 25I)^2 = 900I - N^TN.$$

Since $NN^T$ has the eigenvalues $(891)^1, (324)^{22}, (0)^{10}$, it follows that $(C - 25I)^2$ has the eigenvalues $(9)^1, (576)^{22}, (900)^{22}$. Hence $C$ has the eigenvalues $(22 \text{ or } 28)^1, (-1)^\alpha, (49)^{22-\alpha}, (55)^\beta, (-5)^{220-\beta}$, say. But $\text{tr } C = 0$, rank $(C + 5I) \leq 23$ imply $\alpha = \beta = 0$, hence

$$(C + 5I)(C - 49I) = -3I, \quad Cj = 22j;$$

$$N^TN = 3(2C + 10I + J), \quad Nj = 81j, \quad N^Tj = 11j.$$

4. Ternary codes

In Section 3 we have dissected the 276 vertices of $(\Omega, \Delta)$ into 2 parts of sizes 33 and 243, each carrying a strongly regular subgraph. Their mutual adjacencies are determined by $N$.

**Lemma 4.1.** The $33 \times 243$ matrix $N$ determines a ternary code $\Gamma$ of 243 code words of length 11, with the distances 6 and 9. Any code word is at distance 6 from 132, and 9 from 110 other code words.

**Proof.** We let correspond a coordinate position to each of the 11 triangles of the 33-graph, and a ternary value $\in \text{GF}(3)$ to each of the 3 vertices of any triangle. Lemma 3.4 implies that each of the vertices of the 243-graph corresponds to a vector of the vector space $V(11, 3)$ of dimension 11 over $\text{GF}(3)$. We take the correspondence in such a way that one of the vertices corresponds to the zero vector of $V(11, 3)$. Notice that this still leaves the freedom of multiplying any coordinate by $-1$, and of permuting the coordinates.

Following the proof of Lemma 3.5, the off-diagonal elements of $N^TN$ are 9 and $-3$. Hence any 2 distinct vectors differ in 6 or 9 coordinate positions. In terms of coding theory (cf. [10]) the above says
that the 243-graph corresponds to a ternary code $\Gamma$ of length 11, any pair of whose code words have the Hamming distance 6 or 9. From $C_j = 22j$ it follows that any code word has the Hamming distance 6 to 132, and 9 to 110 other code words.

**Lemma 4.2.** The code $\Gamma$ is a linear code.

**Proof.** Since $0 \in \Gamma$, Lemma 4.1 implies that over GF(3) we have

$$\sum_{i=1}^{11} x_i^2 = 0, \quad \sum_{i=1}^{11} y_i^2 = 0, \quad \sum_{i=1}^{11} (x_i - y_i)^2 = 0$$

for any $x = (x_1, x_2, ..., x_{11})$ and $y = (y_1, y_2, ..., y_{11})$ in $\Gamma$. Hence the inner product of $x$ and $y$ vanishes over GF(3):

$$(x, y) := \sum_{i=1}^{11} x_i y_i = 0.$$ 

It follows that the linear span of $\Gamma$ is contained in its orthogonal complement with respect to the inner product, hence has dimension $\leq 5$. Since $\Gamma$ contains precisely 243 vectors, it must be linear of dimension 5.

**Lemma 4.3.** The generator matrix of $\Gamma$ may be taken to be

$$G = [I_5 \quad -I_5 \quad S_5], \quad S_5 = \text{circ}(0, -1, 1, 1, -1).$$

**Proof.** By suitably combining linearly we start with a basis for $\Gamma$ whose vectors are the rows of the matrix

$$[I_5 \quad P_{5 \times 6}].$$

Since the distances to $0 \in \Gamma$ of these vectors are 6 or 9, hence 6, each row of $P$ contains one entry 0. These entries 0 occur at different coordinate positions since otherwise the rows of the matrix cannot be orthogonal. We permute the last 6 coordinate positions so as to obtain a 6th column without zero entries, and a matrix with zero diagonal at the coordinate positions 7 through 11. Any entry 1 of the 6th column is changed into $-1$ by taking the negative of the corresponding row vector, and 1 and $-1$ are interchanged at the corresponding of the first 5 coordinate positions. The matrix thus obtained is denoted by
From $GG^T = 0$ we obtain $SS^T = -I - J$ over $GF(3)$, that is,

$$SS^T = 5I - J$$

over $R$. Multiplication by $-1$ of any of the last 5 coordinate positions is still allowed, as well as simultaneous interchange of pairs of rows and of the corresponding columns of $S$. This readily leads to the circulant form of $S$ announced in the lemma.

**Remark 4.4.** Lemma 4.3 implies that the orthogonal complement of $\Gamma$ is the perfect ternary Golay code of length 11 and dimension 6. Addition of a parity check coordinate yields the extended Golay code of length 12 and dimension 6, having the generator matrix

$$[C_6 \quad I_6], \quad C_6 = \begin{bmatrix} 0 & I_5^T \\ I_5 & -S_5 \end{bmatrix}.$$

Lemma 4.3 essentially is Pless' result [13] that the ternary Golay code is characterized as a linear ternary code of length 11, dimension 6, having minimum distance 5. For characterizations without assuming linearity, see [5].

**Remark 4.5.** The columns of the matrix $G$ of Lemma 4.3, to be denoted by $x_1, x_2, \ldots, x_{11}$, give rise to another strongly regular graph on 243 vertices, with $\rho_0 = 198, \rho_1 = 9, \rho_2 = -9$. Indeed, the vertices are the 22 vectors $\pm x_i$ and the vectors $\pm x_i \pm x_j$, two vertices being adjacent whenever the difference of the corresponding vectors is one of $\pm x_1, \pm x_2, \ldots, \pm x_{11}$ (cf. [1]). This graph and the 243-graph of Lemma 3.5 are called dual by Delsarte [4].

5. The 276-two-graph

**Theorem 5.1.** There is a unique non-trivial regular two-graph on 276 vertices (up to taking complements).

**Proof.** Any non-trivial regular two-graph on $2^m$ vertices (or its comple-
ment) has the eigenvalues $-5$ and $55$. Any such two-graph may be reconstructed from the orthogonal complement of the ternary Golay code. This follows from Lemmas 4.3; 4.1, 3.4, 3.5. Indeed, the unique code $\Gamma$ determines the $33 \times 243$ matrix $N$, whence $B, C$ and $A$, and the regular two-graph is fixed.

From now on we shall refer to the unique non-trivial regular two-graph on 276 vertices with the eigenvalues $-5$ and $55$ as to the 276-two-graph.

In view of Theorem 5.1, the following theorem is a consequence of results by Conway [3] and Taylor [17, 18].

**Theorem 5.2.** The automorphism group of the 276-two-graph is Conway's group $\cdot 3$, which acts 2-transitively on the vertices, and transitively on the 23-cliques.

**Theorem 5.3.** The 276-two-graph contains in its switching class a strongly regular graph with $\mu_0 = -5$.

**Proof.** By use of Lemma 3.3 the 276-two-graph is represented by the matrix

$$
\begin{bmatrix}
I - J & D_0 & \ldots & D_9 \\
D_0^T & E_{00} & \ldots & E_{09} \\
\vdots & \vdots & \ddots & \vdots \\
D_9^T & E_{90} & \ldots & E_{99}
\end{bmatrix}
$$

This means the following: $I - J$ denotes any 6-clique. For $i = 0, 1, \ldots, 9$, the bordering block $D_i$, of size $6 \times 27$, consists of 27 repetitions of the column $d_i$, which is one of the 10 permutations of the column $(-1, -1, -1, 1, 1, 1)^T$, opposite columns being identified by switching. The diagonal blocks $E_{ii}$, of size $27 \times 27$, are the adjacency matrices of the 10 pillars, each consisting of 9 mutually non-adjacent triangles. Any off-diagonal block $E_{ij}$, $i \neq j$, of size $27 \times 27$, is dissected into $9 \times 9$ small matrices of size $3 \times 3$, according to the triangles in the pillars $i$ and $j$. By application of Lemma 3.4, the pillar $j$ may be switched such that each of the $9 \times 9$ small matrices of any block $E_{ij}$ becomes a permutation of the matrix $J_3 - 2J_3$. If so, then $D_i^T D_j = 2J$. However, fur-
ther switching with respect to one of the pillars $i$ and $j$ yields $D_j^T D_j = -2J$ and transforms all small matrices of $E_{ij}$ into permutations of $2I_3 - J_3$. Therefore, the pillars themselves constitute a graph, the pillars being taken adjacent whenever $E_{ij}$ has the $2I_3 - J_3$ form, that is, whenever $D_j^T D_j = -2J$, that is, $d_i^T d_j = -2$. If by switching all columns $d_i$ are taken with first coordinate $-1$, that is

$$D := \begin{bmatrix} d_0 & d_1 & \ldots & d_9 \end{bmatrix} = \begin{bmatrix} -1 & -1 & \ldots & -1 \\ e_0 & e_1 & \ldots & e_9 \end{bmatrix},$$

then the matrix obtained from $D$ by deleting the first row is the $(-1,1)$-incidence matrix of the block design $(v, k, b, r, \lambda) = (5, 2, 10, 4, 1)$, and the pillar graph is the Petersen graph, of valency 3. Now the pillars may be switched in such a way that the matrix $D$ becomes the $(-1,1)$-incidence matrix of the block design $(v, k, b, r, \lambda) = (6, 3, 10, 5, 2)$; then the pillar graph is the complement of the Petersen graph, of valency 6. The adjacency matrix of the 276-graph, thus obtained, has constant row sums. Indeed, the row sum corresponding to any vertex of the 6-clique equals $-5 + 27 \times 0 = -5$. The row corresponding to any vertex in any pillar has 6 entries corresponding to some $d_i$, 27 entries $0^1(-1)^21^4$ in its diagonal block $E_{ii}$, 9 times $(1, -1, -1)$ in each of 6 off-diagonal blocks $E_{ij}$, and 9 times $(-1, 1, 1)$ in each of 3 off-diagonal blocks $E_{ij}$. Hence this row has the sum

$$0 + (-2 + 24) + 6 \times 9 \times (-1) + 3 \times 9 \times 1 = -5,$$

which proves the theorem.

5.4. Two-graphs with $\rho_2 = -5$. The following regular two-graphs with a 2-transitive automorphism group are sub-two-graphs of the 276-two-graph (cf. [16–18]):

<table>
<thead>
<tr>
<th>$n$</th>
<th>276</th>
<th>176</th>
<th>126</th>
<th>36</th>
<th>26</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>55</td>
<td>35</td>
<td>25</td>
<td>7</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>Aut = Con. 3</td>
<td>$HiS$</td>
<td>$PGU(3,5^2)$</td>
<td>Sp(6,2)</td>
<td>$PSL(2,5^2)$</td>
<td>$V(4,2) \cdot Sp(4,2)$</td>
<td></td>
</tr>
</tbody>
</table>

These parameters cover the possibilities for non-trivial regular two-graphs with $\rho_2 = -5$, possibly apart from the cases $n = 96, \rho_1 = 19$, and $n = 76, \rho_1 = 15$, which are unknown to exist. For $n = 36$ and $n = 26$, there are at least 90, and 3, other regular two-graphs with $\rho_2 = -5$. 
5.5. Strongly regular graphs with $\rho_2 = -5$. There are many strongly regular graphs with $\rho_2 = -5$. Infinite series (and many graphs for each parameter set) are constructed from the Steiner triple systems of order $m$, and from the Latin squares of order $m$:

$$ \text{St}(m) : n = \frac{1}{6}m(m-1)(m-2), \quad \rho_0 = -\frac{1}{6}(m-3)(m-16), \quad \rho_1 = m - 8, \quad \rho_2 = -5; $$
$$ \text{L}_3(m) : n = m^2, \quad \rho_0 = -(m-1)(m-5), \quad \rho_1 = 2m-5, \quad \rho_2 = -5. $$

Apart from these, the following strongly regular graphs are known:

<table>
<thead>
<tr>
<th>$n$</th>
<th>276</th>
<th>275</th>
<th>253</th>
<th>243</th>
<th>176</th>
<th>175</th>
<th>162</th>
<th>126</th>
<th>125</th>
<th>120</th>
<th>112</th>
<th>105</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0$</td>
<td>-5</td>
<td>50</td>
<td>28</td>
<td>22</td>
<td>35</td>
<td>30</td>
<td>49</td>
<td>-5</td>
<td>20</td>
<td>35</td>
<td>51</td>
<td>40</td>
<td>55</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>55</td>
<td>55</td>
<td>51</td>
<td>49</td>
<td>35</td>
<td>35</td>
<td>31</td>
<td>25</td>
<td>25</td>
<td>23</td>
<td>19</td>
<td>19</td>
<td>15</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>-5</td>
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</tr>
</tbody>
</table>

For $n = 253, 176, 120, 105, 100, 77, 56$, we refer to [6]. From the 126-two-graph the graphs with $n = 125, 126$ may be obtained by switching. The graph with $n = 50$ is the Moore graph (cf. [1]). For $n = 36, 35$, we refer to [2], for $n = 24, 25$ to [12], and for $n = 16, 15$ to [14].

From the 243-graph of Lemma 3.3, close on 81, 112, 162 vertices are obtained as follows. Arrange the 243 vertices according to the values $0, 1, -1$ taken by any of the 11 coordinates of the vectors of the code $\Gamma$ of Lemma 4.1. To be precise, arrange the linear code $\Gamma$ following

$$ \Gamma = \Gamma' \cup (\Gamma' + a) \cup (\Gamma' - a), $$

where

$$ \Gamma' := \{ e \in \Gamma : c_1 = 0 \}, \quad a \in \Gamma, \quad a \notin \Gamma'. $$

*Meanwhile, the existence of the following graphs came to our attention: $(126, -75, 15, -5)$ and $(330, -203, 37, -5)$ by R. Mathon, private communication, and $(117, -44, 19, -5)$ in W.D. Wallis, Bull. Austral. Math. Soc. 4 (1971) 41-49; 5 (1971) 43.
The adjacency matrix of the 243-graph reads

\[ C = \begin{bmatrix}
A & B & B^T \\
B^T & A & B \\
B & B^T & A
\end{bmatrix}. \]

Put

\[ D = \begin{bmatrix}
I & -B \\
-B^T & A
\end{bmatrix}. \]

\( \Gamma' \) is a linear code of dimension 4 in \( V(10, 3) \) with Hamming distances 6 and 9. From calculations involving eigenvalues it follows that \( A \) is the adjacency matrix of the 81-graph, and \( D \) that of the 162-graph. Also the 112-graph and McLaughlin's 275-graph \([11]\) may be obtained from the 243-graph, on the basis of \( 112 = 1 + 30 + 81 \) and \( 275 = 1 + 112 + 162 \).

For the two non-isomorphic 40-graphs, for the 45-graph, and for the 64-graph, we refer to \([8, p. 285]\). It is curious to observe that the 45-graph cannot be a subgraph of some 276-graph, because of its eigenvalue \(-20\). The 49-graph is defined on the elements of \( GF(49) \), adjacency iff the difference is a cube.

Finally, we mention that the graphs with \( n = 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, \ldots, 10^2 \) are negative Latin square graphs \( NL_2(m) \), with the parameters \( (4 \leq m \leq 10) \)

\[ n = m^2, \quad \rho_0 = (m+1)(m-5), \quad \rho_1 = 2m-5, \quad \rho_2 = -5. \]

Other values of \( m \) are impossible. Indeed, take \( r = 2 \) in the following theorem.

**Theorem 5.6.** Negative Latin square graphs \( NL_r(m) \) can only exist for \( m \leq r(r+3) \).

**Proof.** Negative Latin square graphs are defined as strongly regular graphs on \( m^2 \) vertices whose adjacency matrix \( A \) has the eigenvalues

\[ \rho_0 = (m+1)(m-1-2r), \quad \rho_1 = 2(m-r) - 1, \quad \rho_2 = -1 - 2r. \]
Their $(1,0)$-adjacency matrix $D := \frac{1}{2}(J - I - A)$ satisfies

$$(D - rJ)(D + (m - r)I) = r(r + 1)J,$$

$$D^2 = r(m - r)I + r(r + 1)J - (m - 2r)D.$$ 

whence

$$r(r + 1) - (m - 2r) > 0.$$ 

5.7. Questions

(i) Is our list of strongly regular graphs with $\rho_2 = -5$ complete?

(ii) Which of the strongly regular graphs with $\rho_2 = -5$ are subgraphs of some graph in the switching class of the 276-two-graph?

These questions are related to Higman's question VII in [7]:

(iii) What are the primitive rank 3 graphs with eigenvalue $-5$?

References


