Discrete Mathematics 66 (1987) 175–190
North-Holland

ON A CLASS OF TRANSLATION PLANES OF SQUARE ORDER

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Received 29 May 1985
Revised 14 August 1986

A class of translation planes of order q^2, where $q = p^r$, p is a prime, $p \geqslant 7$, $p \not\equiv \pm 1 \pmod{10}$ and r is an odd natural number is constructed and the translation complements of these planes are determined. A property shared by all these planes is that the translation complement fixes a distinguished point and divides the remaining distinguished points into two orbits of lengths q and $q^2 - q$. The order of the translation complement is $rq(q-1)^2$ except for $q = 7$ and $q = 13$. The translation complements of these exceptional cases are also briefly studied. The class of planes considered in this paper are distinct from the classes of translation planes of S.D. Cohen and M.J. Ganley [Quart. J. Math. Oxford, 35 (1984) 101–113].

1. Introduction

The study and construction of translation planes of square order has received remarkable attention in recent years and several papers have been devoted to the study of translation planes which possess a particular collineation group.

In [9], Jha raised the following problem:

> Let Γ be a spread whose components are (n-dimensional) subspaces of $V(2n, q)$. Suppose $G \leqslant \operatorname{Aut} \Gamma$ leaves a set Δ of $q+1$ components invariant while acting transitively on $\Gamma \setminus \Delta$.

Find the possibilities for Γ or, more generally, the possibilities for (G, Γ, n, q).

Special cases of the above problem were studied by several authors (see [2, 3, 6, 7 and 8]). Jha [6] and Cohen et al. [3] considered (Δ-transitive) planes of order p^2 which admit a linear autotopism group (a subgroup of the linear translation complement fixing at least two points on the line at infinity) having an orbit of length $p^2 - p$ on the line at infinity. In this paper we have given a possibility for Γ of Jha's problem (above) when $n = 2$ by exhibiting a class of 1-spread sets over GF(q), where q is an odd power of a prime, in which 5 is a nonsquare and studied the translation planes associated with Γ and their translation complements. A salient property shared by all these planes is that the translation complement fixes a distinguished point and divides the remaining distinguished points into two orbits of lengths q and $q^2 - q$. Further the order of the translation complement of each plane is $rq(q-1)^2$ except for $q = 7$ and $q = 13$. The translation complements of these exceptional cases are also studied. It is noticed that the group of all collineations that fix at least two distinguished points is not transitive on $q^2 - q$ distinguished points. Thus the class of planes considered in this paper are not isomorphic to the planes discussed by Jha [6] and Cohen et al. [3].

The matter is presented as follows. In Section 2, the description of the translation planes associated with a class of 1-spread sets is given. Section 3 deals with some collineations. Section 4 is devoted to determine the translation complements. The translation complements of the exceptional cases are briefly given in Section 5. Finally, Section 6 is devoted to show that the class of translation planes discussed in this paper are not isomorphic to the translation planes considered by Jha [6] and Cohen et al. [3].

2. A class of 1-spread sets and the translation planes associated with them

We begin with a criterion for 5 to be a square in GF(p), where p is a prime.

Lemma 2.1. *In* GF(p), *5 is a square if and only if* $p \equiv \pm 1 \pmod{10}$.

Proof. We quote the following number theoretic result [12, Problem 22.6, p. 151].

Let b and p be distinct odd primes with $b \equiv 1 \pmod 4$. Then $(b/p) = +1$ if and only if p has the form

$$p \equiv b + a(b+1) \pmod{2b},$$

where $(a/b) = +1$.

Taking $a = \pm 1$, $b = 5$ in the above result we get that $x^2 \equiv 5 \pmod p$ if and only if $p \equiv 5 \pm 6 \pmod{10}$. Hence the lemma. □

Lemma 2.2. *Let p be a prime, $p \not\equiv \pm 1 \pmod{10}$ and r be an odd natural number. Then 5 is a nonsquare in* GF(q), *where* $q = p^r$.

Proof. Follows from Lemma 2.1 and the fact that r is an odd natural number. □

Throughout this paper q denotes p^r, where p is a prime, $p \geqslant 7$, $p \not\equiv \pm 1 \pmod{10}$ and r is an odd natural number.

Let s be a nonzero element of GF(q) and $M(a, b; s)$ be a 2×2 matrix over GF(q) defined by

$$M(a, b; s) = \begin{bmatrix} a & b \\ -\frac{1}{5}s^2 b^5 & a + sb^3 \end{bmatrix}, \quad a, b \in \mathrm{GF}(q).$$

We observe that $M(a, b; s) \in \mathrm{GL}(2, q)$ if $(a, b) \neq (0, 0)$ and

$$M(a, b; s) = aI + M(0, b; s)$$

for all $a, b \in \mathrm{GF}(q)$, where I is the 2×2 identity matrix.

Let

$$\Gamma(s) = \{M(a, b; s) \mid a, b \in \mathrm{GF}(q)\}.$$

Lemma 2.3. *The set $\Gamma(s)$ is a 1-spread set over* GF(q).

Proof. It is necessary to prove that the difference of any two distinct matrices of $\Gamma(s)$ is nonsingular since the set $\Gamma(s)$ has q^2 matrices including the zero and identity matrices. Let D be the discriminant of the characteristic polynomial of the difference of the matrices $M(0, b; s)$ and $M(0, d; s)$ of $\Gamma(s)$. It is straight forward to see that

$$D = \tfrac{1}{5}s^2(b-d)^2(b^2 + 3bd + d^2)^2$$

is a nonsquare in GF(q) if $b \neq d$. Suppose that $L, N \in \Gamma(s)$, $L \neq N$ and $L = M(a, b; s)$, $N = M(c, d; s)$. Then

$$L - N = (a-c)I + [M(0, b; s) - M(0, d; s)],$$

if $b \neq d$, then $L - N$ is nonsingular, since the characteristic polynomial of $M(0, b; s) - M(0, d; s)$ is irreducible. If $b = d$ then $a \neq c$ and $L - N$ is obviously nonsingular. From this it follows that the difference of any two distinct matrices of $\Gamma(s)$ is nonsingular. Thus the set $\Gamma(s)$ is a 1-spread set over GF(q).

Lemma 2.4. *If s is a square in* GF(q), *then*

$$\{\sqrt{s}\, M \mid M \in \Gamma(s)\} = \Gamma(1).$$

If s is a nonsquare in GF(q), *then*

$$\{\sqrt{\tfrac{1}{5}s}\, M \mid M \in \Gamma(s)\} = \Gamma(5).$$

Further,

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} M \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, M \in \Gamma(1)\right\} = \Gamma(5).$$

Proof. Let s be a square in GF(q) and let

$$\frac{1}{\sqrt{s}}\Gamma(1) = \left\{\frac{1}{\sqrt{s}} M \mid M \in \Gamma(1)\right\}.$$

Taking $M = M(a, b; 1)$ in the above and putting $x = a/\sqrt{s}$ and $y = b/\sqrt{s}$, we find that

$$\frac{1}{\sqrt{s}}\Gamma(1) = \Gamma(s).$$

Let s be a nonsquare in GF(q) and let $\sqrt{\tfrac{1}{5}s}\,\Gamma(s)=\{\sqrt{\tfrac{1}{5}s}\,M(a,b;s)\}$. Taking $x=\sqrt{\tfrac{1}{5}s}\,a$ and $y=\sqrt{\tfrac{1}{5}s}\,b$, we get that

$$\sqrt{\tfrac{1}{5}s}\,\Gamma(s)=\Gamma(5).$$

Finally

$$\begin{aligned}\begin{bmatrix}1&0\\0&5\end{bmatrix}\Gamma(1)\begin{bmatrix}5&0\\0&1\end{bmatrix}&=\left\{\begin{bmatrix}1&0\\0&5\end{bmatrix}M\begin{bmatrix}5&0\\0&1\end{bmatrix}\;\middle|\;M\in\Gamma(1)\right\}\\&=\left\{\begin{bmatrix}5a&b\\-5b^5&5a+5b^3\end{bmatrix}\;\middle|\;a,b\in\mathrm{GF}(q)\right\}\\&=\{M(x,y;5)\mid x=5a,\ y=b;\ a,b\in\mathrm{GF}(q)\}\\&=\Gamma(5).\end{aligned}$$

Hence the lemma. □

Define for every matrix $M\in\Gamma(s)$,

$$V(M)=\{(w,x,y,z)\mid w,x\in\mathrm{GF}(q),\ (y,z)=(w,x)M\},$$

and let

$$V(\infty)=\{0,0,y,z)\mid y,z\in\mathrm{GF}(q)\}.$$

Each $V(M)$, $M\in\Gamma(s)$ and $V(\infty)$ are 2-dimensional subspaces of $V(4,q)$, the 4-dimensional vector space over GF(q). Thus

$$\mathcal{S}=\{V(M)\mid M\in\Gamma(s)\}\cup\{V(\infty)\}$$

is a collection of 2-dimensional subspaces of $V(4,q)$ having the property that each point of $V(4,q)$ is contained in one and only one member of $\mathcal{S}$. The incidence structure with $\mathcal{S}$ and their cosets in the additive group of $V(4,q)$ as lines, the vectors of $V(4,q)$ as points and inclusion as incidence relation is the translation (affine) plane $\pi(s)$ of order q^2 associated with the 1-spread set $\Gamma(s)$ over GF(q). Since the 1-spread set $\Gamma(s)$ is not a ring, the plane $\pi(s)$ is non-Desargusian [2, Section 11], [4, p. 220]. In view of [8, Proposition 3, p. 487] and Lemma 2.4 the plane $\pi(s)$ is isomorphic to $\pi(1)$ for all $s\in\mathrm{GF}(q)$, $s\neq 0$. It is therefore sufficient to consider the class of planes associated with $\Gamma(1)$. In what follows, $\Gamma(1)=\Gamma$, $\pi(1)=\pi$, $M(a,b;1)=M(a,b)$ for all $a,b\in\mathrm{GF}(q)$ and the distinguished point $V(M)$ denotes the distinguished point associated with $V(M)$. For convenience we denote $V(M(0,0))$ and $V(M(1,0))$ by $V(0)$ and $V(1)$ respectively.

It is well known that any nonsingular linear transformation on $V(4,q)$ induces a collineation of π if and only if it permutes the subspaces of $\mathcal{S}$ among themselves. The group of all collineations of π leaving the point corresponding to the zero vector of π invariant is called the translation complement of π.

3. Some collineations of the class of planes

In this section we give some collineations of the class of planes π and indicate their actions on the set of distinguished points of π.

Let $\alpha(x)$ be the mapping on Γ defined by

$$\alpha(x): M(a, b) \to M(a, b) + xI, \quad x \in \mathrm{GF}(q).$$

It is already noticed that $M(a, b) + xI \in \Gamma$, for all $M(a, b) \in \Gamma$ and $x \in \mathrm{GF}(q)$. Thus $\alpha(x)$ induces a collineation of π and its action on the set of distinguished points of π is given by:

$$\alpha(x): V(\infty) \to V(\infty), \quad V(M(a, b)) \to V(M(a+x, b)).$$

The collineation group

$$H = \langle \alpha(x) \mid x \in \mathrm{GF}(q) \rangle$$

fixes the distinguished point $V(\infty)$ and is transitive on the set of distinguished points

$$\{V(M(a, b)) \mid a \in \mathrm{GF}(q)\},$$

for a fixed $b \in \mathrm{GF}(q)$ and it is of order q.

Let $\varphi(c)$ be the mapping on Γ defined by

$$\varphi(c): M \to \begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad c \in \mathrm{GF}(q), \quad c \neq 0, \quad M \in \Gamma.$$

The relation

$$\begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M(a, b) \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix} = M(ac^3, bc) \in \Gamma$$

implies that $\varphi(c)$ induces a collineation of π which fixes $V(\infty)$ and $V(0)$ and maps $V(M(a, b))$ onto $V(M(ac^3, bc))$. Let the group generated by

$$\{\varphi(c) \mid c \in \mathrm{GF}(q), c \neq 0\}$$

be denoted by W. By letting c run through the nonzero elements of $\mathrm{GF}(q)$, we find that W is transitive on the set of distinguished points

$$\{V(M(0, b)) \mid b \in \mathrm{GF}(q), b \neq 0\}.$$

It now follows that the group of collineations $G = \langle W, H \rangle$ fixes the distinguished point $V(\infty)$ and is transitive on each of the following sets of distinguished points

$$Q = \{V(M(a, 0)) \mid a \in \mathrm{GF}(q)\},$$

and

$$R = \{V(M(a, b)) \mid a, b \in \mathrm{GF}(q), b \neq 0\}.$$

Let τ_x be the mapping from Γ defined by

$$\tau_x : M \to (xI)^{-1}M(xI), \quad x \in \mathrm{GF}(q), \quad x \neq 0; \quad M \in \Gamma.$$

This mapping τ_x induces a collineation fixing all the distinguished points of π. However τ_x moves the affine points other than the zero vector. Let S be the subgroup of collineations generated by $\{\tau_x : x \in \mathrm{GF}(q), x \neq 0\}$. The group S is called the group of scalar collineations and it is of order $(q-1)$.

Let γ be the mapping on $V(4, q)$ defined by

$$\gamma : (w, x, y, z) \to (w^p, x^p, y^p, x^p).$$

Obviously the mapping γ induces a collineation of π which fixes $V(\infty)$, $V(M(a, 0))$, $a \in \mathrm{GF}(p)$ and maps $V(M(a, b))$ onto $V(M(a^p, b^p))$. Let the group generated by γ be denoted by K. Obviously K is of order r. Throughout this paper by a collineation, we mean a collineation from the translation complement of π other than a collineation induced by an automorphism of $\mathrm{GF}(q)$.

It may be noted that this paper deals with the class of translation planes considered in [11] with a greater generality by considering a wider class of 1-spread sets $\Gamma(s)$ and determining the translation complements of the class of planes completely. This paper is in a way an elaboration of [11] with a different spirit than that of [11]. The reader is referred to [11] for the following results (see [11, Lemma 3.1 and Theorem 3.5]).

Lemma 3.1. *No collineation of π fixes $V(\infty)$ and moves $V(0)$ onto $V(M(a, b))$, $b \neq 0$.*

Theorem 3.2. *Every collineation of π fixes $V(\infty)$.*

In view of Lemma 3.1, Theorem 3.2 and the collineation group G, we get that the translation complement $\bar{G}$ of π fixes a distinguished point and divides the remaining distinguished points into two orbits Q and R of lengths q and $q^2 - q$ respectively.

4. Translation complement of π

In this section we determine the translation complements of the class of translation planes when $q \neq 7$ and $q \neq 13$. We now find the group of all collineations of π that fix the distinguished points $V(\infty)$, $V(0)$ and $V(1)$, that is to find all collineations of π induced by conjugations on the 1-spread set Γ. To do this we have to partition Γ into classes consisting of similar matrices. We quote the following result which is used in the sequel.

Result 4.1. The equation $z^3 = 1$ has exactly one solution in $\mathrm{GF}(q)$ if $p \not\equiv 1$

(mod 3), where as it has exactly three distinct solutions 1, w, w^2 in GF(q) if $p \equiv 1$ (mod 3).

In view of Result 4.1 we divide the discussion into two cases:

Case 1. $p \not\equiv 1 \pmod 3$.

Lemma 4.2. *Every nonzero matrix $M(y, x)$, $x \neq 0$ in Γ is conjugate to exactly two matrices in Γ.*

Proof. It is evident that $M(0, x)$, $x \neq 0$ is conjugate to itself. From the relation

$$E^{-1}M(0, x)E = M(x^3, -x), \quad \text{where } E = \begin{bmatrix} 1 & 0 \\ x^2 & -1 \end{bmatrix}, \tag{4.1}$$

we get that $M(0, x)$ is conjugate to $M(x^3, -x)$. Suppose that $M(0, x)$ is conjugate to $M(a, b)$. Equating their traces and determinants we get

$$2a + b^3 = x^3, \tag{4.2}$$

$$a^2 + ab^3 + \tfrac{1}{5}b^6 = \tfrac{1}{5}x^6, \tag{4.3}$$

solving the Eqs. (4.2) and (4.3) for a and b^3 we get $a = 0$ and $b^3 = x^3$ or $a = x^3$ and $b^3 = -x^3$. Using Result 4.1, we get $b = x$ if $a = 0$. Then $M(a, b) = M(0, x)$. Taking $a = x^3$ and making use of Result 4.1, we get $M(a, b) = M(x^3, -x)$. Hence the lemma. □

Lemma 4.3. *The general forms of the matrices $A_i(x) \in \mathrm{GL}(2, q)$, $i = 1, 2$ over* GF(q) *satisfying the relations*

$$[A_i(x)]^{-1}M(0, x)A_i(x) = \begin{cases} M(0, x), & \textit{if } i = 1, \quad (4.4) \\ M(x^3, -x), & \textit{if } i = 2, \quad (4.5) \end{cases}$$

are

$$A_1(x) = \begin{bmatrix} a & b \\ -\tfrac{1}{5}x^4b & a + x^2b \end{bmatrix}$$

and

$$A_2(x) = \begin{bmatrix} a & b \\ x^2a + \tfrac{1}{5}x^4b & -a \end{bmatrix},$$

where $a, b \in$ GF(q), $(a, b) \neq (0, 0)$.

Proof. The matrix $M(0, x)$ is in the field F of matrices given by,

$$F = \left\{ \begin{bmatrix} a & b \\ -\tfrac{1}{5}x^4b & a + x^2b \end{bmatrix} \middle|\, a, b \in \mathrm{GF}(q) \right\},$$

since the characteristic polynomial of $M(0, x)$ is irreducible over GF(q). Then by

Schur's Lemma [5, Chapter 4, p. 206] the centralizer of $M(0, x)$ is the multiplicative group of the field containing $M(0, x)$ and contained in $\mathrm{GL}(2, q)$, which is F, defined above. This proves the first part of the Lemma.

In view of relation (4.1), the set of matrices $A_2(x) \in \mathrm{GL}(2, q)$ satisfying the relation (4.5) is given by

$$\{A_1(x)E \mid x \in \mathrm{GF}(q), x \neq 0\}.$$

The general form of $A_2(x)$ satisfying (4.5) is therefore

$$\left\{\begin{bmatrix} r+sx^2 & -s \\ \frac{1}{5}x^4s + rx^2 + x^4s & -r-sx^2 \end{bmatrix} \middle| \; r, s \in \mathrm{GF}(q), (r, s) \neq (0, 0)\right\}.$$

Taking $r + sx^2 = a$, $-s = b$ and simplifying, we get the general form of $A_2(x)$ and it is given by

$$A_2(x) = \begin{bmatrix} a & b \\ x^2a + \frac{1}{5}x^4b & -a \end{bmatrix}, \quad a, b \in \mathrm{GF}(q), (a, b) \neq (0, 0).$$

Hence the lemma. □

It may be noted that

$$[A_2(x)]^{-1}M(x^3, -x)A_2(x) = M(0, x)$$

and the matrices $A_1(x)$ and $A_2(x)$ work without any restriction on p.

Lemma 4.4. *Suppose that x_0 is a fixed nonzero element of* $\mathrm{GF}(q)$. *Then*

(i) *there are exactly two elements x in* $\mathrm{GF}(q)$ *such that $A_1(x) = A_1(x_0)$, if $b \neq 0$ in the general form of $A_1(x)$;*

(ii) *there are exactly two elements x in* $\mathrm{GF}(q)$ *such that $A_2(x) = A_2(x_0)$, if $b = 0$ in the general form of $A_2(x)$, and*

(iii) *there are at most four elements x in* $\mathrm{GF}(q)$ *such that $A_2(x) = A_2(x_0)$, if $b \neq 0$ in the general form of $A_2(x)$.*

The elements x may be the same or different for different values of i.

Proof. Taking $b \neq 0$ in the general form of $A_1(x)$ and equating the expressions for the corresponding elements of $A_1(x)$ and $A_1(x_0)$ and simplifying we get $x^2 - x_0^2 = 0$, which has exactly two solutions for x in $\mathrm{GF}(q)$.

Equating the expressions for the corresponding elements of $A_2(x)$ and $A_2(x_0)$ we get

$$\tfrac{1}{5}bx^4 + ax^2 - \tfrac{1}{5}x_0^4b - x_0^2a = 0,$$

which has exactly two solutions for x if $b = 0$, and at most four solutions for x in $\mathrm{GF}(q)$ if $b \neq 0$. Hence the lemma. □

Lemma 4.5. *If the mapping*

$$M \to L^{-1}ML, \quad M \in \Gamma \tag{4.6}$$

for some $L \in \mathrm{GL}(2, q)$, induces a collineation of π, then L assumes one or both the forms stated in Lemma 4.3.

Proof. Follows from Lemma 4.4. □

Theorem 4.6. *The scalar collineations are the only collineations induced by conjugation mappings on Γ.*

Proof. Suppose that the mapping (4.6) induces a collineation of π then L assumes one or both the forms.

Suppose $b = 0$ in the general form of A_i. If $L = A_1(x_0)$ for some $x_0 \neq 0$, then the mapping (4.6) is τ_a and it induces a collineation of π. If $L = A_2(x_0)$ for some $x_0 \neq 0$, then by the second part of Lemma 4.4, there are exactly two nonzero elements $x \in \mathrm{GF}(q)$ such that L is of the form $A_2(x)$. But $\mathrm{GF}(q)$ contains more than two elements. Hence the mapping (4.6) does not induce a collineation if L assumes the second form and $b = 0$.

Let $b \neq 0$ in the general form of A_i. Suppose that there exist nonzero elements $x_1, x_2 \in \mathrm{GF}(q)$ such that $L = A_1(x_1) = A_2(x_2)$. For all other nonzero x in $\mathrm{GF}(q)$, L must be $A_i(x_i)$ for $i = 1$ or 2. But by Lemma 4.4 there are exactly two elements x in $\mathrm{GF}(q)$ such that $A_1(x) = A_1(x_1)$ and there are at most four elements x in $\mathrm{GF}(q)$ such that $A_2(x) = A_2(x_2)$. Thus there are at most six nonzero elements x in $\mathrm{GF}(q)$ such that L is of the form $A_i(x)$. Since $\mathrm{GF}(q)$ has more than six nonzero elements, the mapping (4.6) does not induce a collineation in this case. The theorem now follows. □

Lemma 4.7. *The group J_1 of all collineations incuded by the mappings $M \to A^{-1}MB$ on Γ is generated by W and S and it is of order $(q-1)^2$.*

Proof. Obviously $\langle W, S\rangle \subset J_1$. It is already noted that $\langle W\rangle$ fixes $V(\infty)$, $V(0)$ and it is transitive on

$$\{V(M(a, 0)) \mid a \in \mathrm{GF}(q), a \neq 0\}.$$

By Theorem 4.6 we have that S is the group of all collineations that fix $V(\infty)$, $V(0)$ and $V(1)$. A coset decomposition of J_1 by S is now given by

$$J_1 = \bigcup_{i=1}^{q-1} Sy_i,$$

where y_i is a collineation from J_1 sending $V(1)$ onto $V(M(i, 0))$. The collineation y_i may be taken from $\langle W\rangle$. Thus $J_1 = \langle W, S\rangle$ and the order of J_1 is the product of $(q-1)$ and the order of S.

Lemma 4.8. *The group J of all collineations which fix $V(\infty)$ and $V(0)$ is generated by J_1 and K and it is of order $r(q-1)^2$.*

Proof. Follows from Lemma 4.7. □

Theorem 4.9. *The translation complement $\bar{G}$ is generated by J and H and it is of order $rq(q-1)^2$.*

Proof. It is observed that the translation complement $\bar{G}$ fixes $V(\infty)$ and is transitive on Q. The subgroup J consists of all collineations from $\bar{G}$ which fix one distinguished point $V(0) \in Q$ and therefore a coset decomposition of $\bar{G}$ by J is thus given by

$$\bar{G} = \bigcup_{i=1}^{q} Jx_i,$$

where x_i is an element of $\bar{G}$ such that x_i fixes $V(\infty)$ and maps $V(0)$ onto $V(M(i, 0))$. The elements x_i may be taken from H and thus $\bar{G} = \langle J, H\rangle$. The order of $\bar{G}$ is given by

$$|\bar{G}| = q \times (\text{order of } J) = rq(q-1)^2.$$

Hence the theorem. □

Case 2. $p \equiv 1 \pmod 3)$

It is easy to see that the mapping

$$\delta_w : M \to C^{-1}MC, \quad M \in \Gamma, \quad \text{where } C = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix},$$

is a collineation of π and its action on the set of distinguished points of π is given by

$$\delta_w : \begin{cases} V(\infty) \to V(\infty), \quad V(M(a, 0)) \to V(M(a, 0)), \\ V(M(a, b)) \to V(M(a, bw)), \end{cases}$$

we denote the group generated by τ_a and δ_w by S_1.

Lemma 4.10. *Every nonzero matrix $M(y, x)$, $x \neq 0$ in Γ is conjugate to exactly six matrices in Γ.*

Proof. From the relations,

$$C^{-k}M(0, x)C^k = M(0, xw^k), \quad 0 \leq k \leq 2$$

and

$$(EC^k)^{-1}M(0, x)EC^k = M(x^3, -xw^k), \quad 0 \leq k \leq 2,$$

we get that $M(0, x)$ is conjugate to the following six matrices:

$$M(0, xw^k), M(x^3, -xw^k), \quad 0 \leq k \leq 2.$$

Suppose that $M(0, x)$ is conjugate to $M(a, b)$. Equating their traces and determinants as in the proof of Lemma 4.2 we get the Eqs. (4.2) and (4.3) and solving them for a and b^3 we get $a = 0$ and $b^3 = x^3$ or $a = x^3$ and $b^3 = -x^3$. By Result 4.1 the Eqs. $b^3 = x^3$ and $b^3 = -x^3$ have solutions $b = x$, xw, xw^2 and $b = -x$, $-xw^2$, $-xw^2$ respectively. Therefore $M(a, b)$ is one of the following six matrices

$$M(0, xw^k),\ M(x^3, -xw^k), \quad 0 \leqslant k \leqslant 2.$$

Hence the lemma. □

Lemma 4.11. *The general forms of the matrices* $B_i(x) \in \mathrm{GL}(2, q)$, $1 \leqslant i \leqslant 6$ *such that*

$$B_i^{-1}(x)M(0, x)B_i(x) = M(0, xw^{i-1}), \quad 1 \leqslant i \leqslant 3,$$
$$B_{3+i}^{-1}(x)M(0, x)B_{3+i}(x) = M(x^3, -xw^{i-1}), \quad 1 \leqslant i \leqslant 3$$

are

$$B_i(x) = \begin{bmatrix} a & b \\ -\frac{1}{5}x^4bw^{1-i} & w^{i-1}a + x^2b \end{bmatrix}, \quad 1 \leqslant i \leqslant 3,$$

$$B_{3+i}(x) = \begin{bmatrix} a & b \\ x^2a + \frac{1}{5}x^4bw^{1-i} & -w^{i-1}a \end{bmatrix}, \quad 1 \leqslant i \leqslant 3.$$

Proof. The general forms of B_1 and B_4 are same as A_1 and A_2 (Case 1) respectively and we can deduce the other forms of B_i by using Lemma 4.3 and the matrices C^k, $k = 1, 2$ in the place of E in the proof of Lemma 4.3. □

Lemma 4.12. *Suppose that* x_0 *is a fixed nonzero element of* GF(q). *Then*

(i) *there are exactly two elements* x *in* GF(q) *such that* $B_i(x) = B_i(x_0)$, $1 \leqslant i \leqslant 3$, *if* $b \neq 0$ *in the general form of* $B_i(x)$;
(ii) *there are exactly two elements* x *in* GF(q) *such that* $B_i(x) = B_i(x_0)$, $4 \leqslant i \leqslant 6$, *if* $b = 0$ *in the general form of* $B_i(x)$, *and*
(iii) *there are at most four elements* x *in* GF(q) *such that* $B_i(x) = B_i(x_0)$, $4 \leqslant i \leqslant 6$, *if* $b \neq 0$ *in the general form of* $B_i(x)$.

Proof. Equating the expressions for the corresponding elements of $B_i(x)$ and $B_i(x_0)$ and simplifying, we get the following:

(i) $x^2 = x_0^2$ when $b \neq 0$ in the general form of $B_i(x)$, $1 \leqslant i \leqslant 3$;
(ii) $x^2 = x_0^2$ when $b = 0$ in the general form of $B_i(x)$, $4 \leqslant i \leqslant 6$, and
(iii) $\frac{1}{5}x^4bw^k + x^2a - \frac{1}{5}x_0^4bw^k - x_0^2a = 0$ when $b \neq 0$ in the general form of $B_i(x)$, $4 \leqslant i \leqslant 6$.

The lemma now follows. □

Lemma 4.13. *If the mapping*

$$M \to L^{-1}ML, \quad M \in \Gamma \tag{4.6}$$

for some $L \in \mathrm{GL}(2, q)$ *induces a collineation of* π, *then* L *assumes one or more forms stated in Lemma* 4.11.

Proof. Follows from Lemma 4.12. □

Theorem 4.14. *The group of all collineations induced by conjugation mappings on* Γ *is* S_1 *and it is of order* $3(q-1)$.

Proof. Suppose that in the general forms B_i, $1 \leqslant i \leqslant 6$, $a \neq 0$ and $b = 0$. Then

$$B_i(x) = B_i(1), \quad 1 \leqslant i \leqslant 3,$$

for all x in $\mathrm{GF}(q)$, $x \neq 0$. Taking $L = B_i(1)$ the mapping (4.6) is $\tau_a \delta_w$ and hence induces a collineation of π. If $L = B_i(x_0)$ for some $x_0 \neq 0$, $4 \leqslant i \leqslant 6$ and in view of the second part of Lemma 4.12, there are exactly six nonzero elements $x \in \mathrm{GF}(q)$ such that L is of the form $B_i(x)$, $4 \leqslant i \leqslant 6$. The mapping (4.6) does not induce a collineation of π if L is of the form $B_i(x)$, $4 \leqslant i \leqslant 6$, since $\mathrm{GF}(q)$ contains more than 6 nonzero elements.

Suppose that $b \neq 0$ in the general form of $B_i(x)$, $1 \leqslant i \leqslant 6$ and suppose that there exist nonzero elements x_i, $1 \leqslant i \leqslant 6$ in $\mathrm{GF}(q)$ such that $L = B_i(x_i)$ for $1 \leqslant i \leqslant 6$. For the remaining nonzero x in $\mathrm{GF}(q)$, L must be $B_i(x_i)$ for some or other $i = 1, 2, \ldots, 6$. By using the first and third part of Lemma 4.12, we deduce that there are at most 18 nonzero elements x in $\mathrm{GF}(q)$ such that L is of the form $B_i(x)$. The mapping (4.6) does not induce a collineation of π if L is of the form $B_i(x)$, $1 \leqslant i \leqslant 6$ since the smallest prime power q in this case is 23.

In the above proof, it may very well happen that L may not assume some of the forms. The estimates given in the above proof are under the assumption that L assumes all forms. If L assumes fewer forms then the estimates may still come down. It may be noted that $q = 7$ and $q = 13$ have been excluded from the discussion. These two cases are discussed in Section 5. Thus we get that the group of all collineations that fix $V(0)$, $V(\infty)$ and $V(1)$ is generated by τ_a and δ_w which is S_1 and it is of order $3(q-1)$ and $S \subset S_1$.

Lemma 4.15. *The group* J_1 *of all collineations induced by the mappings*

$$M \rightarrow A^{-1}MB, \quad M \in \Gamma,$$

for some $A, B \in \mathrm{GL}(2, q)$ *fixes the distinguished points* $V(\infty)$, $V(0)$ *and partitions*

$$\{V(M(a, 0)) \mid a \in \mathrm{GF}(q), a \neq 0\}$$

into three orbits each of length $\frac{1}{3}(q-1)$.

Proof. The translation complement divides the set of distinguished points into three orbits, the first consisting of $V(\infty)$, the second consisting of $V(M(a, 0))$, $a \in \mathrm{GF}(q)$ and the third consisting of $V(M(a, b))$, $a, b \in \mathrm{GF}(q)$, $b \neq 0$. However the subgroup J_1 fixes $V(\infty)$ and $V(0)$ and maps the distinguished points $V(M(a, 0))$, $a \in \mathrm{GF}(q)$, $a \neq 0$ among themselves. Thus for a given $a \in \mathrm{GF}(q)$, $a \neq 0$,

$$A^{-1}M(a, 0)B = M(b, 0),$$

for some $b \in \mathrm{GF}(q)$, $b \neq 0$. This forces $B = nA$ for some $n \in \mathrm{GF}(q)$, $n \neq 0$. We now look for the possible values of n. Suppose the mapping

$$\eta : M \to nA^{-1}MA, \quad M \in \Gamma \tag{4.7}$$

induces a collineation of π. Choosing $M = M(-\frac{1}{2}x^3, x)$, $x \neq 0$ and using the fact that $nA^{-1}MA \in \Gamma$, we get that the trace and determinant of $nA^{-1}MA$ should be 0 and $-n^2x^6/20$ respectively. Any nonzero matrix in Γ with trace 0 is of the form $M(-\frac{1}{2}y^3, y)$ whose determinant is $-\frac{1}{20}y^6$. From this we obtain that $-n^2x^6/20 = -y^6/20$, for some $y \neq 0$. This forces n to be a cube in $\mathrm{GF}(q)$.

If the mapping (4.7) induces a collineation of π then η maps $V(M(a, 0))$ onto $V(M(an, 0))$ where n is a cube in $\mathrm{GF}(q)$. If n is a cube of a generator of the multiplicative group of $\mathrm{GF}(q)$, then n divides the set of distinguished points $\{V(M(a, 0)) \mid a \in \mathrm{GF}(q), a \neq 0\}$ into three orbits each of length $\frac{1}{3}(q-1)$ irrespective of the choice of n. Any other choice of n permutes the elements of any orbit among themselves. Choosing

$$A = \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } n = c^3, \quad \text{for any } c \in \mathrm{GF}(q),\ c \neq 0,$$

we get the already known collineations $\varphi(c)$, constituting W. In particular W fixes $V(0)$ and $V(\infty)$ and is transitive on the set of distinguished points

$$T = \{V(M(g^{3k}, 0)) \mid 0 \leq k < \tfrac{1}{3}(q-1)\},$$

where g is a generator of $\mathrm{GF}(q)$. Hence the lemma. □

Lemma 4.16. *The subgroup J_1 of all collineations induced by the mapping $M \to A^{-1}MB$ on Γ is generated by S_1 and W and it is of order $(q-1)^2$, $q \neq 7$, $q \neq 13$.*

Proof. Obviously $\langle W, S_1 \rangle \subset J_1$ and J_1 fixes $V(0)$ and $V(\infty)$ and it is transitive on the set T of the distinguished points. The set T contains $V(1)$ and the group of all collineations fixing $V(\infty)$, $V(0)$ and $V(1)$ is S_1 when $q \neq 7$, $q \neq 13$. A coset decomposition of J_1 by S_1 is therefore

$$J_1 = \bigcup_{i=0}^{q-2} S_1\varphi(g^i),$$

which in turn implies that $J_1 = \langle W, S_1 \rangle$ and $J_1 = |T|\,|S_1| = (q-1)^2$. Hence the lemma. □

We have the following lemma and theorem as in Case 1.

Lemma 4.17. *The subgroup J of all collineations which fix $V(\infty)$ and $V(0)$ is generated by J, H and it is of order $r(q-1)^2$, $q \neq 7$, 13.*

Theorem 4.18. *The translation complement $\bar{G}$ is generated by J, H and it is of order $rq(q-1)^2$, $q \neq 7$, 13.*

5. The exceptional cases

In this section we outline the collineations of the translation plane π when $q=7$ and 13. We observe that all the collineations exhibited in Case 2 of Section 4 also work in these two cases. However we find that there are some extra collineations. We list below all the collineations of the plane π for the cases $q=7$ and 13 respectively. The proofs and techniques are similar to those employed in Case 2 of Section 4. The details are omitted for want of space.

(a) The translation plane π when $q=7$

The translation complement $\bar{G}$ of π is generated by the following mappings:

(i) $\tau_x : M \to (xI)^{-1}M(xI), \quad x \in \mathrm{GF}(7), x \neq 0, M \in \Gamma,$

(ii) $\varphi(c) : M \to \begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad c \in \mathrm{GF}(7), c \neq 0, M \in \Gamma,$

(iii) $\alpha(x) : M \to M + xI, \quad x \in \mathrm{GF}(7), M \in \Gamma,$

(iv) $\delta_2 : M \to C^{-1}MC, C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M \in \Gamma,$

(v) $O : M \to A^{-1}MA, A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad M \in \Gamma.$

The extra collineations of π mentioned earlier are obtained as the combination of O with the other collineations. Finally $\bar{G} = \langle \tau_x, \varphi(c), \alpha(x), O, \delta_2 \rangle$ is of order 2016 and $\bar{G}$ divides the set of distinguished points into three orbits of lengths 1, 7 and 42.

(b) The translation plane π when $q=13$

The translation complement $\bar{G}$ of π is generated by the following mappings.

(i) $\tau_x : M \to (xI)^{-1}M(xI), \quad x \in \mathrm{GF}(13), x \neq 0, M \in \Gamma,$

(ii) $\varphi(c) : M \to \begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad c \in \mathrm{GF}(13), c \neq 0, M \in \Gamma,$

(iii) $\alpha(x) : M \to M + xI, \quad x \in \mathrm{GF}(13), M \in \Gamma,$

(iv) $\delta_3 : M \to C^{-1}MC, C = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad M \in \Gamma,$

(v) $\xi : M \to B^{-1}MB, B = \begin{bmatrix} 0 & 1 \\ \frac{1}{5} & 0 \end{bmatrix}, \quad M \in \Gamma.$

The extra collineations mentioned earlier are obtained as the combinations of ξ with the other collineations. The translation complement $\bar{G}$ is of order 3744 and $\bar{G}$ divides the set of distinguished points into three orbits of lengths 1, 13 and 156.

6. Conclusion

This section is devoted to show that the class of planes considered in this paper are distinct from the translation planes constructed by Jha [6] and Cohen et al. [3].

Theorem 6.1. *The group of all collineations that fix at least two distinguished points is not transitive on $q^2 - q$ distinguished points, while fixing the remaining $(q+1)$ distinguished points setwise.*

Proof. As we have already seen that the translation complement is invariant on $\Delta = Q \cup \{V(\infty)\}$ and is transitive on $S(=\Gamma \setminus \Delta)$, we have to consider the group of all collineations that fix at least two distinguished points of Δ. Since the translation complement fixes $V(\infty)$ and is transitive on $\Delta' = \Delta \setminus \{V(\infty)\}$, we have to consider collineations which fix $V(\infty)$ and some distinguished points of Δ'. We can choose $V(\infty)$ as the first distinguished point and $V(0)$ as the second and possibly some other distinguished points of Δ' as the fixed points. We now show that the theorem is true in the case of collineations that fix $V(\infty)$ and $V(0)$. These collineations are in $J = \langle W, S, K \rangle$ or $\langle W, S_1, K \rangle$, according as $p \not\equiv 1 \pmod 3$ or $p \equiv 1 \pmod 3$ respectively except $q = 7$ and $q = 13$. Consider the set of distinguished points

$$\{V(M(0, b)) \mid b \in \mathrm{GF}(q), b \neq 0\}.$$

This set of distinguished points is invariant under J. Thus J cannot be transitive on $\Gamma \setminus \Delta$. Now the group of all collineations that fix the distinguished points $V(\infty)$, $V(0)$ and some other points of Δ' is a subgroup of J and therefore cannot act transitively on $q^2 - q$ distinguished points. The proof of the theorem may be completed, using the transitivity of the translation complement on Δ.

In the case of $q = 7$ and $q = 13$ the set of distinguished points

$$\{V(M(0, bw^k)), V(M(b^3 - bw^k)) \mid b \in \mathrm{GF}(q), b \neq 0\}$$

is invariant under J and therefore G cannot act transitively on $q^2 - q$ distinguised points while fixing the remaining $q + 1$ distinguished points setwise. □

Theorem 6.2. *The class of translation planes considered in this paper are not isomorphic to the translation planes constructed by Jha* [6] *and Cohen et al.* [3].

Proof. The translation planes of Jha and Cohen et al. have the property that the subgroup of all collineations of the translation complement which fix at least two distinguished points is transitive on $q^2 - q$ distinguished points while fixing the remaining distinguished points setwise. Now the theorem follows from the Theorem 6.1. □

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