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Castelnuovo–Mumford regularity for complexes and weakly Koszul modules

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Abstract

Let A be a noetherian AS-regular Koszul quiver algebra (if A is commutative, it is essentially a polynomial ring), and $\operatorname{gr} A$ the category of finitely generated graded left A-modules. Following Jørgensen, we define the Castelnuovo–Mumford regularity $\operatorname{reg}(M^\bullet)$ of a complex $M^\bullet \in D^b(\operatorname{gr} A)$ in terms of the local cohomologies or the minimal projective resolution of $M^\bullet$. Let $A^!$ be the quadratic dual ring of A. For the Koszul duality functor $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$, we have $\operatorname{reg}(M^\bullet) = \max\{i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\}$. Using these concepts, we interpret results of Martinez-Villa and Zacharia concerning *weakly Koszul modules* (also called *componentwise linear modules*) over $A^!$. As an application, refining a result of Herzog and Römer, we show that if J is a monomial ideal of an exterior algebra $E = \bigwedge\langle y_1, \dots, y_d\rangle$, $d \geq 3$, then the $(d-2)$nd syzygy of E/J is weakly Koszul.

MSC: primary 16E05; secondary 13F55, 16S37, 16W50

1. Introduction

Let $S = K[x_1, \dots, x_d]$ be a polynomial ring over a field K. We regard S as a graded ring with $\deg x_i = 1$ for all i. The following is a well-known result.

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Theorem 1.1 (*cf.* [4]). *Let M be a finitely generated graded S-module. For an integer r, the following conditions are equivalent.*

(1) $H^i_{\mathfrak{m}}(M)_j = 0$ *for all* $i, j \in \mathbb{Z}$ *with* $i + j > r$.
(2) *The truncated module* $M_{\geq r} := \bigoplus_{i \geq r} M_i$ *has an* r*-linear free resolution.*

Here $\mathfrak{m} := (x_1, \ldots, x_d)$ *is the irrelevant ideal of* S, *and* $H^i_{\mathfrak{m}}(M)$ *is the* i*th local cohomology module.*

If the conditions of Theorem 1.1 are satisfied, we say M is r-*regular*. For a sufficiently large r, M is r-regular. We call $\operatorname{reg}(M) = \min\{r \mid M \text{ is } r\text{-regular}\}$ the *Castelnuovo–Mumford regularity* of M. This is a very important invariant in commutative algebra.

Let A be a noetherian AS-regular Koszul quiver algebra with the graded Jacobson radical $\mathfrak{m} := \bigoplus_{i \geq 1} A_i$. If A is commutative, A is essentially a polynomial ring. When A is connected (i.e., $A_0 = K$), it is the coordinate ring of a "noncommutative projective space" in noncommutative algebraic geometry. Let $\operatorname{gr} A$ be the category of finitely generated graded left A-modules and their degree preserving maps. (For a graded ring B, $\operatorname{gr} B$ means the similar category for B.) The local cohomology module $H^i_{\mathfrak{m}}(M)$ of $M \in \operatorname{gr} A$ behaves pretty much like in the commutative case. For example, we have the "Serre duality theorem" for the derived category $D^b(\operatorname{gr} A)$. See [11,23] and Theorem 2.7 below. By virtue of this duality, we can show that Theorem 1.1 also holds for *bounded complexes* in $\operatorname{gr} A$.

Theorem 1.2. *For a complex* $M^\bullet \in D^b(\operatorname{gr} A)$ *and an integer* r, *the following conditions are equivalent.*

(1) $H^i_{\mathfrak{m}}(M^\bullet)_j = 0$ *for all* $i, j \in \mathbb{Z}$ *with* $i + j > r$.
(2) *The truncated complex* $(M^\bullet)_{\geq r}$ *has an* r*-linear projective resolution.*

Here $(M^\bullet)_{\geq r}$ *is the subcomplex of* $M^\bullet$ *whose* i*th term is* $(M^i)_{\geq (r-i)}$.

For a sufficiently large r, the conditions of the above theorem are satisfied. The regularity $\operatorname{reg}(M^\bullet)$ of $M^\bullet$ is defined in the natural way. When A is connected, Jørgensen [10] has studied the regularity of complexes, and essentially proved the above result. See also [9,15]. (Even in the case when A is a polynomial ring, it seems that nobody had considered Theorem 1.2 before [10].) But his motivation and treatment are slightly different from ours.

For $M^\bullet \in D^b(\operatorname{gr} A)$, set $\mathcal{H}(M^\bullet)$ to be a complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and the differential maps are zero. Then we have $\operatorname{reg}(\mathcal{H}(M^\bullet)) \geq \operatorname{reg}(M^\bullet)$. The difference $\operatorname{reg}(\mathcal{H}(M^\bullet)) - \operatorname{reg}(M^\bullet)$ is a theme of the last section of this paper.

Let $A^!$ be the quadratic dual ring of A. For example, if $S = K[x_1, \ldots, x_d]$ is a polynomial ring, then $S^!$ is an exterior algebra $E = \bigwedge\langle y_1, \ldots, y_d\rangle$. It is known that $A^!$ is always Koszul, finite dimensional, and selfinjective. The Koszul duality functors $\mathcal{F} : D^b(\operatorname{gr} A^!) \to D^b(\operatorname{gr} A)$ and $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$ give a category equivalence $D^b(\operatorname{gr} A^!) \cong D^b(\operatorname{gr} A)$ (see [2]). It is easy to check that

$$\operatorname{reg}(M^\bullet) = \max\{i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\}$$

for $M^\bullet \in D^b(\operatorname{gr} A)$.

Let $\operatorname{gr} A^{\mathsf{op}}$ be the category of finitely generated graded *right* A-modules. The above results on $\operatorname{gr} A$ also hold for $\operatorname{gr} A^{\mathsf{op}}$. Moreover, we have

$$\operatorname{reg}(\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, \mathcal{D}^\bullet)) = -\min\{i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\}$$

for $M^\bullet \in D^b(\operatorname{gr} A)$. Here $\mathcal{D}^\bullet$ is a balanced dualizing complex of A, which gives duality functors between $D^b(\operatorname{gr} A)$ and $D^b(\operatorname{gr} A^{\mathsf{op}})$.

Let B be a noetherian Koszul algebra. For $M \in \operatorname{gr} B$ and $i \in \mathbb{Z}$, $M_{\langle i \rangle}$ denotes the submodule of M generated by the degree i component M_i of M. We say M is *weakly Koszul* if $M_{\langle i \rangle}$ has a linear projective resolution for all i. This definition is different from the original one given in [13], but they are equivalent. (Weakly Koszul modules are also called "componentwise linear modules" by some commutative algebraists.) Martinez-Villa and Zacharia proved that if $N \in \operatorname{gr} A^!$ then the ith syzygy $\Omega_i(N)$ of N is weakly Koszul for $i \gg 0$. For $N \in \operatorname{gr} A^!$, set

$$\operatorname{lpd}(N) := \min\{i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul}\}.$$

Let $N \in \operatorname{gr} A^!$ and $N' := \underline{\operatorname{Hom}}_{A^!}(N, A^!) \in \operatorname{gr} (A^!)^{\mathsf{op}}$ its dual. In Theorem 4.4, we show that N is weakly Koszul if and only if $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathsf{op}}(N')) = 0$, where $\mathcal{F}^{\mathsf{op}} : D^b(\operatorname{gr} (A^!)^{\mathsf{op}}) \to D^b(\operatorname{gr} A^{\mathsf{op}})$ is the Koszul duality functor. (Since $\operatorname{reg}(\mathcal{F}^{\mathsf{op}}(N')) = 0$, we have $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathsf{op}}(N')) \geq 0$ in general.) Moreover, we have

$$\operatorname{lpd}(N) = \operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathsf{op}}(N'))$$

(Theorem 4.7). As an application of this formula, we refine a result of Herzog and Römer on monomial ideals of an exterior algebra. Among other things, in Proposition 4.15, we show that if J is a monomial ideal of an exterior algebra $E = \bigwedge\langle y_1, \ldots, y_d\rangle$, $d \geq 3$, then $\operatorname{lpd}(E/J) \leq d - 2$.

Finally, we remark that Herzog and Iyengar [8] studied the invariant lpd and related concepts over noetherian commutative (graded) local rings. Among other things, they proved that $\operatorname{lpd}(N)$ is always finite over some "nice" local rings (e.g., complete intersections whose associated graded rings are Koszul).

2. Preliminaries

Let K be a field. The ring A treated in this paper is a (not necessarily commutative) K-algebra with some nice properties. More precisely, A is a noetherian AS-regular Koszul quiver algebra. If A is commutative, it is essentially a polynomial ring. But even in this case, most results in Section 4 and a few results in Section 3 are new. (In the polynomial ring case, many results in Section 3 were obtained in [3].) So one can read this paper assuming that A is a polynomial ring.

We sketch the definition and basic properties of graded quiver algebras here. See [5] for further information.

Let Q be a finite quiver. That is, $Q = (Q_0, Q_1)$ is an oriented graph, where Q_0 is the set of vertices and Q_1 is the set of arrows. Here Q_0 and Q_1 are finite sets. The path algebra KQ is a positively graded algebra with grading given by the lengths of paths. We denote the graded Jacobson radical of KQ by J. That is, J is the ideal generated by all arrows.

If $I \subset J^2$ is a graded ideal, we say $A = KQ/I$ is a *graded quiver algebra*. Of course, $A = \bigoplus_{i \geq 0} A_i$ is a graded ring such that the degree i component A_i is a finite-dimensional K-vector space for all i. The subalgebra A_0 is a product of copies of the field K, one copy for each element of Q_0. If $A_0 = K$ (i.e., Q has only one vertex), we say A is *connected*. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded algebra with $R_0 = K$ and $\dim_K R_1 =: n < \infty$. If R is generated by R_1 as a K-algebra, then it can be regarded as a graded quiver algebra over a quiver with one vertex and n loops. Let $\mathfrak{m} := \bigoplus_{i \geq 1} A_i$ be the graded Jacobson radical of A. Unless otherwise specified, we assume that A is left and right noetherian throughout this paper.

Let $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) be the category of graded left (resp. right) A-modules and their degree-preserving A-homomorphisms. Note that the degree i component M_i of $M \in \operatorname{Gr} A$ (or $M \in \operatorname{Gr} A^{\mathsf{op}}$) is an A_0-module for each i. Let $\operatorname{gr} A$ (resp. $\operatorname{gr} A^{\mathsf{op}}$) be the full subcategory of $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) consisting of finitely generated modules. Since we assume that A is noetherian, $\operatorname{gr} A$ and $\operatorname{gr} A^{\mathsf{op}}$ are abelian categories. In what follows, we will define several concepts for $\operatorname{Gr} A$ and $\operatorname{gr} A$. But the corresponding concepts for $\operatorname{Gr} A^{\mathsf{op}}$ and $\operatorname{gr} A^{\mathsf{op}}$ can be defined in the same way.

For $n \in \mathbb{Z}$ and $M \in \operatorname{Gr} A$, set $M_{\geq n} := \bigoplus_{i \geq n} M_i$ to be a submodule of M, and $M_{\leq n} := \bigoplus_{i \leq n} M_i$ to be a graded K-vector space. The nth shift $M(n)$ of M is defined by $M(n)_i = M_{n+i}$. Set $\sigma(M) := \sup\{ i \mid M_i \neq 0\}$ and $\iota(M) := \inf\{i \mid M_i \neq 0\}$. If $M = 0$, we set $\sigma(M) = -\infty$ and $\iota(M) = +\infty$. Note that if $M \in \operatorname{gr} A$ then $\iota(M) > -\infty$. For a complex $M^\bullet$ in $\operatorname{Gr} A$, set

$$\sigma(M^\bullet) := \sup\{\sigma(H^i(M^\bullet)) + i \mid i \in \mathbb{Z}\} \text{ and } \iota(M^\bullet) := \inf\{\iota(H^i(M^\bullet)) + i \mid i \in \mathbb{Z}\}.$$

For $v \in Q_0$, we have the idempotent e_v associated with v. Note that $1 = \sum_{v \in Q_0} e_v$. Set $P_v := Ae_v$ and ${}_vP := e_v A$. Then we have ${}_AA = \bigoplus_{v \in Q_0} P_v$ and $A_A = \bigoplus_{v \in Q_0} ({}_vP)$. Each P_v and ${}_vP$ are indecomposable projectives. Conversely, any indecomposable projective in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) is isomorphic to P_v (resp. ${}_vP$) for some $v \in Q_0$ up to degree shifting. Set $K_v := P_v/(\mathfrak{m} P_v)$ and ${}_vK := {}_vP/({}_vP\,\mathfrak{m})$. Each K_v and ${}_vK$ are simple. Conversely, any simple object in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) is isomorphic to K_v (resp. ${}_vK$) for some $v \in Q_0$ up to degree shifting.

We say a graded left (or right) A-module M is *locally finite* if $\dim_K M_i < \infty$ for all i. If $M \in \operatorname{gr} A$, then it is locally finite. Let $\operatorname{lf} A$ (resp. $\operatorname{lf} A^{\mathsf{op}}$) be the full subcategory of $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) consisting of locally finite modules.

Let $C^b(\operatorname{Gr} A)$ be the category of bounded cochain complexes in $\operatorname{Gr} A$, and $D^b(\operatorname{Gr} A)$ its derived category. We have similar categories for $\operatorname{Gr} A^{\mathsf{op}}$, $\operatorname{lf} A$, $\operatorname{lf} A^{\mathsf{op}}$, $\operatorname{gr} A$ and $\operatorname{gr} A^{\mathsf{op}}$. For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the pth translation of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$. Since $D^b(\operatorname{gr} A) \cong D^b_{\operatorname{gr} A}(\operatorname{Gr} A) \cong D^b_{\operatorname{gr} A}(\operatorname{lf} A)$, we freely identify these categories. A module M can be regarded as a complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the 0th term. We can regard $\operatorname{Gr} A$ as a full subcategory of $C^b(\operatorname{Gr} A)$ and $D^b(\operatorname{Gr} A)$ in this way.

For $M, N \in \operatorname{Gr} A$, set $\underline{\operatorname{Hom}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$ to be a graded K-vector space with $\underline{\operatorname{Hom}}_A(M, N)_i = \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$. Similarly, we can also define $\underline{\operatorname{Hom}}^\bullet_A(M^\bullet, N^\bullet)$, $\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, N^\bullet)$, and $\underline{\operatorname{Ext}}^i_A(M^\bullet, N^\bullet)$ for $M^\bullet, N^\bullet \in D^b(\operatorname{Gr} A)$.

If V is a K-vector space, V^* denotes the dual vector space $\mathrm{Hom}_K(V, K)$. For $M \in \mathrm{Gr}\, A$ (resp. $M \in \mathrm{Gr}\, A^{\mathsf{op}}$), $M^\vee := \bigoplus_{i \in \mathbb{Z}} (M_i)^*$ has a graded *right* (resp. *left*) A-module structure given by $(fa)(x) = f(ax)$ (resp. $(af)(x) = f(xa)$) and $(M^\vee)_i = (M_{-i})^*$. If $M \in \mathrm{lf}\, A$, then $M^\vee \in \mathrm{lf}\, A^{\mathsf{op}}$ and $M^{\vee\vee} \cong M$. In other words, $(-)^\vee$ gives exact duality functors between $\mathrm{lf}\, A$ and $\mathrm{lf}\, A^{\mathsf{op}}$, which can be extended to duality functors between $C^b(\mathrm{lf}\, A)$ and $C^b(\mathrm{lf}\, A^{\mathsf{op}})$, or between $D^b(\mathrm{lf}\, A)$ and $D^b(\mathrm{lf}\, A^{\mathsf{op}})$. In this paper, when we say W is an A–A bimodule, we always assume that $(aw)a' = a(wa')$ for all $w \in W$ and $a, a' \in A$. If W is a graded A–A bimodule, then so is $W^\vee$.

It is easy to see that $I_v := ({}_vP)^\vee$ (resp. ${}_vI := (P_v)^\vee$) is injective in $\mathrm{Gr}\, A$ (resp. $\mathrm{Gr}\, A^{\mathsf{op}}$). Moreover, I_v and ${}_vI$ are graded injective hulls of K_v and ${}_vK$ respectively. In particular, the A–A bimodule $A^\vee$ is injective both in $\mathrm{Gr}\, A$ and in $\mathrm{Gr}\, A^{\mathsf{op}}$.

Let W be a graded A–A-bimodule. For $M \in \mathrm{Gr}\, A$, we can regard $\underline{\mathrm{Hom}}_A(M, W)$ as a graded *right* A-module by $(fa)(x) = f(x)a$. We can also define $\mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet, W) \in D^b(\mathrm{Gr}\, A^{\mathsf{op}})$ and $\underline{\mathrm{Ext}}^i_A(M^\bullet, W) \in \mathrm{Gr}\, A^{\mathsf{op}}$ for $M^\bullet \in D^b(\mathrm{Gr}\, A)$ in this way. Similarly, for $M^\bullet \in D^b(\mathrm{Gr}\, A^{\mathsf{op}})$, we can make $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathsf{op}}}(M^\bullet, W)$ and $\underline{\mathrm{Ext}}^i_{A^{\mathsf{op}}}(M^\bullet, W)$ (bounded complex of) graded left A-modules. For $M \in \mathrm{Gr}\, A$, we can regard $\underline{\mathrm{Hom}}_A(W, M)$ as a graded *left* A-module by $(af)(x) = f(xa)$.

For the functor $\underline{\mathrm{Hom}}_A(-, W)$, we mainly consider the case when $W = A$ or $W = A^\vee$. But, we have $\underline{\mathrm{Hom}}_A(-, A^\vee) \cong (-)^\vee$. To see this, note that

$$
\begin{aligned}
(M^\vee)_i = \mathrm{Hom}_K(M_{-i}, K) &= \bigoplus_{v \in Q_0} \mathrm{Hom}_K(e_v M_{-i}, K) \\
&\cong \bigoplus_{v \in Q_0} \mathrm{Hom}_K(e_v M_{-i}, K_v) \\
&\cong \mathrm{Hom}_{A_0}(M_{-i}, A_0).
\end{aligned}
$$

Via the identification $(A^\vee)_0 \cong (A_0)^* \cong A_0$, $f \in (M^\vee)_i \cong \mathrm{Hom}_{A_0}(M_{-i}, A_0)$ gives a morphism $f' : M_{\geq -i} \to A^\vee(i)$ in $\mathrm{Gr}\, A$. Since $\mathrm{Hom}_{\mathrm{Gr}\, A}(M/M_{\geq -i}, A^\vee(i)) = 0$ and $A^\vee$ is injective, the short exact sequence $0 \to M_{\geq -i} \to M \to M/M_{\geq -i} \to 0$ induces a unique extension $f'' : M \to A^\vee(i)$ of f'. From this correspondence, we have $\underline{\mathrm{Hom}}_A(M, A^\vee) \cong M^\vee$.

Let $P^\bullet$ be a right bounded complex in $\mathrm{gr}\, A$ such that each P^i is projective. We say $P^\bullet$ is *minimal* if $d(P^i) \subset \mathfrak{m} P^{i+1}$ for all i. Here d is the differential map. Any complex $M^\bullet \in C^b(\mathrm{gr}\, A)$ has a minimal projective resolution, that is, we have a minimal complex $P^\bullet$ of projective objects and a graded quasi-isomorphism $P^\bullet \to M^\bullet$. A minimal projective resolution of $M^\bullet$ is unique up to isomorphism. We denote a graded module $A/\mathfrak{m}$ by A_0. Set $\beta^{i,j}(M^\bullet) := \dim_K \underline{\mathrm{Ext}}^{-i}_A(M^\bullet, A_0)_{-j}$. Let $P^\bullet$ be a minimal projective resolution of $M^\bullet$, and $P^i := \bigoplus_{l=1}^m T^{i,l}$ an indecomposable decomposition. Then we have

$$
\beta^{i,j}(M^\bullet) = \#\{l \mid T^{i,l}(j) \cong P_v \text{ for some } v\}.
$$

We can also define $\beta^{i,j}(M^\bullet)$ as the dimension of $\mathrm{Tor}^A_{-i}(A_0, M^\bullet)_j$. This definition must be much more familiar to commutative algebraists. Note that $\beta^{i,j}(-)$ is an invariant of isomorphism classes of the derived category $D^b(\mathrm{gr}\, A)$. Note that these facts on minimal projective resolutions also hold over any noetherian graded algebra.

Definition 2.1. Let A be a (not necessarily noetherian) graded quiver algebra. We say A is *Artin–Schelter regular* (AS-regular, for short), if

- A has finite global dimension d.
- $\underline{\mathrm{Ext}}^i_A(K_v, A) = \underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}({}_vK, A) = 0$ for all $i \neq d$ and all $v \in Q_0$.
- There are a permutation δ on Q_0 and an integer n_v for each $v \in Q_0$ such that $\underline{\mathrm{Ext}}^d_A(K_v, A) \cong {}_{\delta(v)}K(n_v)$ (equivalently, $\underline{\mathrm{Ext}}^d_{A^{\mathrm{op}}}({}_vK, A) \cong K_{\delta^{-1}(v)}(n_v)$) for all v.

Remark 2.2. The AS regularity is a very important concept in non-commutative algebraic geometry. In the original definition, it is assumed that an AS-regular algebra A is connected and there is a positive real number γ such that $\dim_K A_n < n^\gamma$ for $n \gg 0$, while some authors do not require the latter condition. We also remark that Martinez-Villa and coworkers called rings satisfying the conditions of Definition 2.1 *generalized Auslander regular algebras* in [6,11].

Definition 2.3. For an integer $l \in \mathbb{Z}$, we say $M^\bullet \in \operatorname{gr} A$ has an *l-linear (projective) resolution*, if

$$\beta^{i,j}(M^\bullet) \neq 0 \Rightarrow i + j = l.$$

If $M^\bullet$ has an l-linear resolution for some l, we say $M^\bullet$ has a *linear resolution*.

Definition 2.4. We say A is *Koszul*, if the graded left A-module A_0 has a linear resolution.

In the definition of the Koszul property, we can regard A_0 as a right A-module. (We get the equivalent definition.) That is, A is Koszul if and only if any simple graded left (or, right) A-module has a linear resolution.

Lemma 2.5. *If A is noetherian, AS-regular, Koszul, and has global dimension d, then $\underline{\mathrm{Ext}}^d_A(K_v, A) \cong {}_{\delta(v)}K(d)$ and $\underline{\mathrm{Ext}}^d_{A^{\mathrm{op}}}({}_vK, A) \cong K_{\delta^{-1}(v)}(d)$ for all v. Here δ is the permutation of Q_0 given in Definition 2.1.*

Proof. Since A is Koszul, P^{-d} of a minimal projective resolution $P^\bullet : 0 \to P^{-d} \to \cdots \to P^0 \to 0$ of K_v is generated by its degree d-part $(P^{-d})_d$ (more precisely, $P^{-d} = P_{\delta(v)}(-d)$). □

In the rest of this paper, A is always a noetherian AS-regular Koszul quiver algebra of global dimension d.

Example 2.6. (1) A polynomial ring $K[x, \ldots, x_d]$ is clearly a noetherian AS-regular Koszul (quiver) algebra of global dimension d. Conversely, if a regular noetherian graded algebra is connected and commutative, it is a polynomial ring.

(2) Let $K\langle x_1, \ldots, x_d\rangle$ be the free associative algebra, and $(q_{i,j})$ a $d \times d$ matrix with entries in K satisfying $q_{i,j}q_{j,i} = q_{i,i} = 1$ for all i, j. Then the quotient ring $A = K\langle x_1, \ldots, x_n\rangle / \langle x_jx_i - q_{i,j}x_ix_j \mid 1 \leq i, j \leq d\rangle$ is a noetherian AS-regular Koszul algebra with global dimension d. This fact must be well-known to specialists, but we will sketch a proof here for the reader's convenience. Since $x_1, \ldots, x_d \in A_1$ form a regular normalizing sequence with the quotient ring $K = A/(x_1, \ldots, x_d)$, A is a noetherian ring with a balanced dualizing complex by [15, Lemma 7.3]. It is not difficult to construct a

minimal free resolution of the module $K = A/\mathfrak{m}$, which is a "q-analog" of the Koszul complex of a polynomial ring $K[x_1, \ldots, x_d]$. So A is Koszul and has global dimension d. Since A has finite global dimension and admits a balanced dualizing complex, it is AS-regular (cf. [15, Remark 3.6 (3)]).

Artin et al. [1] classified connected AS-regular algebras of global dimension 3. (Their definition of AS regularity is stronger than ours. See Remark 2.2.) All of the algebras they listed are noetherian [1, Theroem 8.1]. But some are Koszul and some are not.

(3) A *preprojective algebra* is an important example of non-connected AS-regular algebras. See [6] and the references cited there for the definition of this algebra and further information. The preprojective algebra A of a finite quiver Q is a graded quiver algebra over the *inverse completion* $\overline{Q}$ of Q. If the quiver Q is connected (of course, it does not mean A is connected), then A is (almost) always an AS-regular algebra of global dimension 2, but it is not Koszul in some cases, and not noetherian in many cases. Let G be the bipartite graph of Q in the sense of [6, Section 3]. If G is Euclidean, then A is a noetherian AS-regular Koszul algebra of global dimension 2.

For $M \in \operatorname{Gr} A$, set

$$\Gamma_{\mathfrak{m}}(M) = \varinjlim \underline{\operatorname{Hom}}_A(A/\mathfrak{m}^n, M) = \{x \in M \mid A_n x = 0 \text{ for } n \gg 0\} \in \operatorname{Gr} A.$$

Then $\Gamma_{\mathfrak{m}}(-)$ gives a left exact functor from $\operatorname{Gr} A$ to itself. So we have a right derived functor $\mathbf{R}\Gamma_{\mathfrak{m}} : D^b(\operatorname{Gr} A) \to D^b(\operatorname{Gr} A)$. For $M^\bullet \in D^b(\operatorname{Gr} A)$, $H^i_{\mathfrak{m}}(M^\bullet)$ denotes the ith cohomology of $\mathbf{R}\Gamma_{\mathfrak{m}}(M^\bullet)$, and we call it the ith *local cohomology* of $M^\bullet$. It is easy to see that $H^i_{\mathfrak{m}}(M^\bullet) = \lim_{\to} \underline{\operatorname{Ext}}^i_A(A/\mathfrak{m}^n, M^\bullet)$. Similarly, we can define $\mathbf{R}\Gamma_{\mathfrak{m}^{\mathrm{op}}} : D^b(\operatorname{Gr} A^{\mathrm{op}}) \to D^b(\operatorname{Gr} A^{\mathrm{op}})$ and $H^i_{\mathfrak{m}^{\mathrm{op}}} : D^b(\operatorname{Gr} A^{\mathrm{op}}) \to \operatorname{Gr} A^{\mathrm{op}}$ in the same way. If M is an A–A bimodule, $H^i_{\mathfrak{m}}(M)$ and $H^i_{\mathfrak{m}^{\mathrm{op}}}(M)$ are also.

Let $I \in \operatorname{Gr} A$ be an indecomposable injective. Then $\Gamma_{\mathfrak{m}}(I) \neq 0$, if and only if $I \cong I_v(n)$ for some $v \in Q_0$ and $n \in \mathbb{Z}$, if and only if $\Gamma_{\mathfrak{m}}(I) = I$. Similarly, $\underline{\operatorname{Hom}}_A(A_0, I) \neq 0$ if and only if $I \cong I_v(n)$ for some $v \in Q_0$ and $n \in \mathbb{Z}$. In this case, $\underline{\operatorname{Hom}}_A(A_0, I) = K_v(n)$. The same is true for an indecomposable injective $I \in \operatorname{Gr} A^{\mathrm{op}}$.

Let $I^\bullet$ be a minimal injective resolution of A in $\operatorname{gr} A$. Since A is AS-regular, $I^i = 0$ for all $i > d$, $\Gamma_{\mathfrak{m}}(I^i) = 0$ for all $i < d$, and $\Gamma_{\mathfrak{m}}(I^d) = A^\vee(d)$. Hence $\mathbf{R}\Gamma_{\mathfrak{m}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A)$. By the same argument as [23, Proposition 4.4], we also have $\mathbf{R}\Gamma_{\mathfrak{m}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A^{\mathrm{op}})$. It does not mean that $H^d_{\mathfrak{m}}(A) \cong A^\vee(d)$ as A–A bimodules. But there is an A–A bimodule L such that $L \otimes_A H^d_{\mathfrak{m}}(A) \cong A^\vee(d)$ as A–A bimodules. Here the underlying additive group of L is A, but the bimodule structure is give by $A \times L \times A \ni (a, l, b) \mapsto \phi(a)lb \in A = L$ for a (fixed) K-algebra automorphism ϕ of A. In particular, $L \cong A$ as left A-modules and as right A-modules (separately). Note that $\phi(e_v) = e_{\delta(v)}$ for all $v \in Q_0$, where δ is the permutation on Q_0 appeared in Definition 2.1. If A is commutative, then ϕ is the identity map.

We give a new A–A bimodule structure L' to the additive group A by $A \times L' \times A \ni (a, l, b) \mapsto al\phi(b) \in A = L'$. Then $L' \cong \underline{\operatorname{Hom}}_A(L, A)$. Set $\mathcal{D}^\bullet := L'(-d)[d]$. Note that $\mathcal{D}^\bullet$ belongs both $D^b(\operatorname{gr} A)$ and $D^b(\operatorname{gr} A^{\mathrm{op}})$. We have $H^i_{\mathfrak{m}}(\mathcal{D}^\bullet) = H^i_{\mathfrak{m}^{\mathrm{op}}}(\mathcal{D}^\bullet) = 0$ for all $i \neq 0$ and $H^0_{\mathfrak{m}}(\mathcal{D}^\bullet) \cong H^0_{m^{\mathrm{op}}}(\mathcal{D}^\bullet) \cong A^\vee$ as A–A bimodules by the same argument as [23, Section 4]. Thus (an injective resolution of) $\mathcal{D}^\bullet$ is a *balanced dualizing complex* of

A in the sense of [23] (the paper only concerns connect rings, but the definition can be generalized in the obvious way).

Easy computation shows that $\underline{\mathrm{Hom}}_A(P_v, L') \cong {}_{\delta^{-1}(v)}P$ and $\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}({}_vP, L') \cong P_{\delta(v)}$ for all $v \in Q_0$. Since $\mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet, \mathcal{D}^\bullet)$ (resp. $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(M^\bullet, \mathcal{D}^\bullet)$) for $M^\bullet \in \mathrm{gr}\, A$ (resp. $M^\bullet \in \mathrm{gr}\, A^{\mathrm{op}}$) can be computed by a projective resolution of $M^\bullet$, $\mathbf{R}\underline{\mathrm{Hom}}_A(-, \mathcal{D}^\bullet)$ and $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(-, \mathcal{D}^\bullet)$ give duality functors between $D^b(\mathrm{gr}\, A)$ and $D^b(\mathrm{gr}\, A^{\mathrm{op}})$. (Of course, we can also prove this by the same argument as [23, Proposition 3.4].)

Theorem 2.7 (*Yekutieli [23, Theorem 4.18], Martinez-Villa [11, Proposition 4.6]*). *For $M^\bullet \in D^b(\mathrm{gr}\, A)$, we have*

$$\mathbf{R}\Gamma_{\mathfrak{m}}(M^\bullet)^\vee \cong \mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet, \mathcal{D}^\bullet).$$

In particular,

$$(H^i_{\mathfrak{m}}(M^\bullet)_j)^* \cong \underline{\mathrm{Ext}}_A^{-i}(M^\bullet, \mathcal{D}^\bullet)_{-j}.$$

Proof. The above result was proved by Yekutieli in the connected case. (In some sense, Martinez-Villa proved a more general result than ours, but he did not concern complexes.) But, the proof of [23, Theorem 4.18] only uses formal properties such as A is noetherian, $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(-, \mathcal{D}^\bullet), \mathcal{D}^\bullet) \cong \mathrm{Id}$, and $\mathbf{R}\Gamma_{\mathfrak{m}}\mathcal{D}^\bullet \cong A^\vee$. So the proof also works in our case. □

Definition 2.8 (*Jørgensen, [10]*). For $M^\bullet \in D^b(\mathrm{gr}\, A)$, we say

$$\mathrm{reg}(M^\bullet) := \sigma(\mathbf{R}\Gamma_{\mathfrak{m}}(M^\bullet)) = \sup\{i + j \mid H^i_{\mathfrak{m}}(M^\bullet)_j \neq 0\}$$

is the *Castelnuovo–Mumford regularity* of $M^\bullet$.

By Theorem 2.7 and the fact that $\mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet, \mathcal{D}^\bullet) \in D^b(\mathrm{gr}\, A^{\mathrm{op}})$, we have $\mathrm{reg}(M^\bullet) < \infty$ for all $M^\bullet \in D^b(\mathrm{gr}\, A)$.

Theorem 2.9 (*Jørgensen, [10]*). *If $M^\bullet \in C^b(\mathrm{gr}\, A)$, then*

$$\mathrm{reg}(M^\bullet) = \max\{i + j \mid \beta^{i,j}(M^\bullet) \neq 0\}. \tag{2.1}$$

When A is a polynomial ring and $M^\bullet$ is a module, the above theorem is a fundamental result obtained by Eisenbud and Goto [4]. In the non-commutative case, under the assumption that A is connected but not necessarily regular, this has been proved by Jørgensen [10, Corollary 2.8]. (If A is not regular, we have $\mathrm{reg}(A) > 0$ in many cases. So one has to assume that $\mathrm{reg} A = 0$ there.) In our case (i.e., A is AS-regular), we have a much simpler proof. So we will give it here. This proof is also different from one given in [4].

Proof. Set $Q^\bullet := \underline{\mathrm{Hom}}^\bullet_A(P^\bullet, L'(-d)[d])$. Here $P^\bullet$ is a minimal projective resolution of $M^\bullet$, and L' is the A–A bimodule given in the construction of the dualizing complex $\mathcal{D}^\bullet$. Recall that $\underline{\mathrm{Hom}}_A(P_v, L') \cong {}_{\delta^{-1}(v)}P$ for all $v \in Q_0$. Let s be the right-hand side of (2.1), and l the minimal integer with the property that $\beta^{l,s-l}(M^\bullet) \neq 0$. Then

$\iota(Q^{-d-l}) = l - s + d$, and $(Q^{-d-l+1})_{\leq(l-s+d-1)} = 0$ (Note that $\beta^{l-1,m}(M^\bullet) = 0$ for all $m \geq s - l + 1$.) Since $Q^\bullet$ is a minimal complex, we have

$$0 \neq H^{-d-l}(Q^\bullet)_{l-s+d} = \underline{\mathrm{Ext}}_A^{-d-l}(M^\bullet, \mathcal{D}^\bullet)_{l-s+d} = (H_{\mathfrak{m}}^{d+l}(M^\bullet)_{-l+s-d})^*.$$

Thus $\mathrm{reg}(M^\bullet) \geq \max\{i + j \mid \beta^{i,j}(M^\bullet) \neq 0\}$.

On the other hand, if $H_{\mathfrak{m}}^{d+l}(M^\bullet)_{-l+r-d} \neq 0$, we have that $\beta^{l,t-l}(M^\bullet) \neq 0$ for some $t \geq r$ by an argument similar to the above. Hence $\mathrm{reg}(M^\bullet) \leq \max\{i + j \mid \beta^{i,j}(M^\bullet) \neq 0\}$, and we are done. □

For $M^\bullet \in D^b(\mathrm{gr}\, A)$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and all differential maps are zero.

Lemma 2.10. *We have $\beta^{i,j}(\mathcal{H}(M^\bullet)) \geq \beta^{i,j}(M^\bullet)$ for all $M^\bullet \in D^b(\mathrm{gr}\, A)$ and all $i, j \in \mathbb{Z}$. In particular,* $\mathrm{reg}(\mathcal{H}(M^\bullet)) \geq \mathrm{reg}(M^\bullet)$.

The difference between $\mathrm{reg}(M^\bullet)$ and $\mathrm{reg}(\mathcal{H}(M^\bullet))$ can be arbitrary large. In the last section, we will study the relation between this difference and a work of Martinez-Villa and Zacharia [13].

Proof. The assertion easily follows from the spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}_A^p(H^{-q}(N^\bullet), A_0) \longrightarrow \underline{\mathrm{Ext}}_A^{p+q}(N^\bullet, A_0). \quad \square$$

For a complex $M^\bullet \in C^b(\mathrm{gr}\, A)$ and an integer r, $(M^\bullet)_{\geq r}$ denotes the subcomplex of $M^\bullet$ whose ith term is $(M^i)_{\geq(r-i)}$. Even if $M^\bullet \cong N^\bullet$ in $D^b(\mathrm{gr}\, A)$, we have $(M^\bullet)_{\geq r} \ncong (N^\bullet)_{\geq r}$ in general.

In the module case, the following is a well-known property of Castelnuovo–Mumford regularity.

Proposition 2.11. *Let $M^\bullet \in C^b(\mathrm{gr}\, A)$. Then $(M^\bullet)_{\geq r}$ has an r-linear resolution if and only if $r \geq \mathrm{reg}(M^\bullet)$.*

To prove the proposition, we need the following lemma.

Lemma 2.12. *For a module $M \in \mathrm{gr}\, A$ with $\dim_K M < \infty$, we have $H_{\mathfrak{m}}^0(M) = M$ and $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq 0$. In particular,* $\mathrm{reg}(M) = \sigma(M)$ *in this case.*

Proof. If $P^\bullet$ is a minimal projective resolution of $M^\vee \in \mathrm{gr}\, A^{\mathrm{op}}$, then $I^\bullet := (P^\bullet)^\vee$ is a minimal injective resolution of M. Since each indecomposable summand of I^i is isomorphic to $I_v(n)$ for some $v \in Q_0$ and $n \in \mathbb{Z}$, we have $\Gamma_{\mathfrak{m}}(I^\bullet) = I^\bullet$. □

Proof of Proposition 2.11. For a complex $T^\bullet \in D^b(\mathrm{gr}\, A)$, it is easy to see that $\iota(T^\bullet) = \min\{i + j \mid \beta^{i,j}(T^\bullet) \neq 0\}$. In particular, $\iota(T^\bullet) \leq \mathrm{reg}(T^\bullet)$. Hence $T^\bullet$ has an l-linear projective resolution if and only if $\iota(T^\bullet) = \mathrm{reg}(T^\bullet) = l$.

Consider the short exact sequence of complexes

$$0 \to (M^\bullet)_{\geq r} \to M^\bullet \to M^\bullet/(M^\bullet)_{\geq r} \to 0, \tag{2.2}$$

and set $N^\bullet := M^\bullet/(M^\bullet)_{\geq r}$. Note that $\dim_K H^i(N) < \infty$ for all i. By Lemmas 2.10 and 2.12, we have

$$r > \sigma(N^\bullet) = \max\{\operatorname{reg}(H^i(N^\bullet)) + i \mid i \in \mathbb{Z}\} = \operatorname{reg}(\mathcal{H}(N^\bullet)) \geq \operatorname{reg}(N^\bullet).$$

By the long exact sequence of $\underline{\operatorname{Ext}}_A^\bullet(-, A_0)$ induced by (2.2), we have

$$\begin{aligned} r \leq \iota((M^\bullet)_{\geq r}) \leq \operatorname{reg}((M^\bullet)_{\geq r}) &\leq \max\{\operatorname{reg}(N^\bullet) + 1, \operatorname{reg}(M^\bullet)\} \\ &\leq \max\{r, \operatorname{reg}(M^\bullet)\}. \end{aligned}$$

Moreover, if $r < \operatorname{reg}(M^\bullet)$ then we have $\operatorname{reg}(N^\bullet) + 1 < \operatorname{reg}(M^\bullet)$ and $\operatorname{reg}((M^\bullet)_{\geq r}) = \operatorname{reg}(M^\bullet) > r$. Hence $(M^\bullet)_{\geq r}$ has an r-linear resolution if and only if $r \geq \operatorname{reg}(M^\bullet)$. □

The following is one of the most basic results on Castelnuovo–Mumford regularity (see [4]). Jørgensen [9] proved the same result for $M \in \operatorname{gr} A$.

Let $S = K[x_1, \ldots, x_d]$ be a polynomial ring. If $M \in \operatorname{gr} S$ satisfies $H^0_{\mathfrak{m}}(M)_{\geq r+1} = 0$ and $H^i_{\mathfrak{m}}(M)_{r+1-i} = 0$ for all $i \geq 1$, then $r \geq \operatorname{reg}(M)$ (i.e., $H^i_{\mathfrak{m}}(M)_{\geq r+1-i} = 0$ for all $i \geq 1$).

The similar result also holds for $M^\bullet \in D^b(\operatorname{gr} A)$. Since a minor adaptation of the proof of [9, Theorem 2.4] also works for complexes, we leave the proof to the reader.

Proposition 2.13. *If $M^\bullet \in D^b(\operatorname{gr} A)$ with $t := \max\{i \mid H^i(M^\bullet) \neq 0\}$ satisfies*

- $H^i_{\mathfrak{m}}(M^\bullet)_{\geq r+1-i} = 0$ *for all* $i \leq t$
- $H^i_{\mathfrak{m}}(M^\bullet)_{r+1-i} = 0$ *for all* $i > t$,

then $r \geq \operatorname{reg}(M^\bullet)$ (i.e., $H^i_{\mathfrak{m}}(M^\bullet)_{\geq r+1-i} = 0$ for all $i > t$).

3. Koszul duality

In this section, we study the relation between the Castelnuovo–Mumford regularity of complexes and the Koszul duality. For precise information of this duality, see [2, Section 2]. There, the symbol A (resp. $A^!$) basically means a finite-dimensional (resp. noetherian) Koszul algebra. This convention is opposite to ours. So the reader should be careful.

Recall that $A = KQ/I$ is a graded quiver algebra over a finite quiver Q. Let Q^{op} be the *opposite quiver* of Q. That is, $Q_0^{\mathsf{op}} = Q_0$ and there is a bijection from Q_1 to Q_1^{op} which sends an arrow $\alpha : v \to u$ in Q_1 to the arrow $\alpha^{\mathsf{op}} : u \to v$ in Q_1^{op}. Consider the bilinear form $\langle -, - \rangle : (KQ)_2 \times (KQ^{\mathsf{op}})_2 \to A_0$ defined by

$$\langle \alpha\beta, \gamma^{\mathsf{op}}\delta^{\mathsf{op}} \rangle = \begin{cases} e_u & \text{if } \alpha = \delta \text{ and } \beta = \gamma, \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta, \gamma, \delta \in Q_1$. Here $u \in Q_0$ is the vertex with $\beta \in Ae_u$. Let $I^\perp \subset KQ^{\mathsf{op}}$ be the ideal generated by

$$\{y \in (KQ^{\mathsf{op}})_2 \mid \langle x, y \rangle = 0 \text{ for all } x \in I_2\}.$$

We say $KQ^{\mathsf{op}}/I^\perp$ is the *quadratic dual ring* of A, and denote it by $A^!$. Clearly, $(A^!)_0 = A_0$. Since A is Koszul, so is $A^!$. Since A is AS-regular, $A^!$ is a finite-dimensional selfinjective algebra with $A = \bigoplus_{i=0}^{d} A_i$ by [12, Theorem 5.1]. If A is a polynomial ring, then $A^!$ is the exterior algebra $\bigwedge(A_1)^*$.

Since $A^!$ is selfinjective, $\mathbf{D}_{A^!} := \underline{\mathrm{Hom}}_{A^!}(-, A^!)$ and $\mathbf{D}_{(A^!)^{\mathrm{op}}} := \underline{\mathrm{Hom}}_{(A^!)^{\mathrm{op}}}(-, A^!)$ give exact duality functors between $\mathrm{gr}\, A^!$ and $\mathrm{gr}\,(A^!)^{\mathrm{op}}$. They induce duality functors between $D^b(\mathrm{gr}\, A^!)$ and $D^b(\mathrm{gr}\,(A^!)^{\mathrm{op}})$, which are also denoted by $\mathbf{D}_{A^!}$and $\mathbf{D}_{(A^!)^{\mathrm{op}}}$. It is easy to see that $\mathbf{D}_{A^!}(N) \cong \mathrm{Hom}_K(N, K)(-d)$.

We say a complex $F^\bullet \in C(\mathrm{gr}\, A^!)$ is a projective (resp. injective) resolution of a complex $N^\bullet \in C^b(\mathrm{gr}\, A^!)$, if each term F^i is projective (= injective), $F^\bullet$ is right (resp. left) bounded, and there is a graded quasi-isomorphism $F^\bullet \to N^\bullet$ (resp. $N^\bullet \to F^\bullet$). We say a projective (or, injective) resolution $F^\bullet \in C^b(\mathrm{gr}\, A^!)$ is *minimal* if $d^i(F^i) \subset \mathfrak{n}F^{i+1}$ for all i, where $\mathfrak{n}$ is the graded Jacobson radical of $A^!$. (The usual definition of a minimal injective resolution is different from the above one. But they coincide in our case.) A bounded complex $N^\bullet \in C^b(\mathrm{gr}\, A^!)$ has a minimal projective resolution and a minimal injective resolution, and they are unique up to isomorphism. If $F^\bullet$ is a minimal projective (resp. injective) resolution of $N^\bullet$ then $\mathbf{D}_{A^!}(F^\bullet)$ is a minimal injective (resp. projective) resolution of $\mathbf{D}_{A^!}(N^\bullet)$.

For $N^\bullet \in D^b(\mathrm{gr}\, A^!)$, set

$$\mu^{i,j}(N^\bullet) := \dim_K \underline{\mathrm{Ext}}^i_{A^!}(A_0, N^\bullet)_j.$$

Then $\mu^{i,j}(N^\bullet)$ measures the size of a minimal injective resolution of $N^\bullet$. More precisely, if $F^\bullet$ is a minimal injective resolution of $N^\bullet$, and $F^i := \bigoplus_{l=1}^{m} T^{i,l}$ is an indecomposable decomposition, then we have

$$\begin{aligned}\mu^{i,j}(N^\bullet) &= \#\{l \mid \mathrm{soc}(T^{i,l}) = (T^{i,l})_j\} \\ &= \#\{l \mid T^{i,l}(j) \text{ is isomorphic to a direct summand of } A^!(d)\}.\end{aligned}$$

Let V be a finitely generated left A_0-module. Then $\mathrm{Hom}_{A_0}(A^!, V)$ is a graded *left* $A^!$-module with $(af)(a') = f(a'a)$ and $\mathrm{Hom}_{A_0}(A^!, V)_i = \mathrm{Hom}_{A_0}((A^!)_{-i}, V)$. Since $A^!$ is selfinjective, we have $\mathrm{Hom}_{A_0}(A^!, A_0) \cong A^!(d)$. Hence $\mathrm{Hom}_{A_0}(A^!, V)$ is a projective (and injective) left $A^!$-module for all V. If V has degree i (e.g., $V = M_i$ for some $M \in \mathrm{gr}\, A$), then we set $\mathrm{Hom}_{A_0}(A^!, V)_j = \mathrm{Hom}_{A_0}(A^!_{-j-i}, V)$.

For $M^\bullet \in C^b(\mathrm{gr}\, A)$, let $\mathcal{G}(M^\bullet) := \mathrm{Hom}_{A_0}(A^!, M^\bullet) \in C^b(\mathrm{gr}\, A^!)$ be the total complex of the double complex with $\mathcal{G}(M^\bullet)^{i,j} = \mathrm{Hom}_{A_0}(A^!, M^i_j)$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(f)(x) = \sum_{\alpha \in Q_1} \alpha f(\alpha^{\mathrm{op}} x), \qquad d''(f)(x) = \partial_{M^\bullet}(f(x))$$

for $f \in \mathrm{Hom}_{A_0}(A^!, M^i_j)$ and $x \in A^!$. The gradings of $\mathcal{G}(M^\bullet)$ is given by

$$\mathcal{G}(M^\bullet)^p_q := \bigoplus_{p=i+j,\, q=-l-j} \mathrm{Hom}_{A_0}((A^!)_l, M^i_j).$$

Each term of $\mathcal{G}(M^\bullet)$ is injective. For a module $M \in \mathrm{gr}\, A$, $\mathcal{G}(M)$ is a minimal complex. Thus we have

$$\mu^{i,j}(\mathcal{G}(M)) = \begin{cases} \dim_K M_i & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Similarly, for a complex $N^\bullet \in C^b(\mathrm{gr}\, A^!)$, we can define a new complex $\mathcal{F}(N^\bullet) := A \otimes_{A_0} N^\bullet \in C^b(\mathrm{gr}\, A)$ as the total complex of the double complex with $\mathcal{F}(N^\bullet)^{i,j} = A \otimes_{A_0} N^i_j$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(a \otimes x) = \sum_{\alpha \in Q_1} a\alpha \otimes \alpha^{\mathsf{op}} x, \qquad d''(a \otimes x) = a \otimes \partial_{N^\bullet}(x)$$

for $a \otimes x \in A \otimes_{A_0} N^i$. The gradings of $\mathcal{F}(N^\bullet)$ is given by

$$\mathcal{F}(N^\bullet)^p_q := \bigoplus_{p=i+j,q=l-j} A_l \otimes_{A_0} N^i_j.$$

Clearly, each term of $\mathcal{F}(N^\bullet)$ is a projective A-module. For a module $N \in \mathrm{gr}\, A^!$, $\mathcal{F}(N)$ is a minimal complex. Hence we have

$$\beta^{i,j}(\mathcal{F}(N)) = \begin{cases} \dim_K N_i & \text{if } i+j=0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is well known that the operations $\mathcal{F}$ and $\mathcal{G}$ define functors $\mathcal{F} : D^b(\mathrm{gr}\, A^!) \to D^b(\mathrm{gr}\, A)$ and $\mathcal{G} : D^b(\mathrm{gr}\, A) \to D^b(\mathrm{gr}\, A^!)$, and they give an equivalence $D^b(\mathrm{gr}\, A) \cong D^b(\mathrm{gr}\, A^!)$ of triangulated categories. This equivalence is called the *Koszul duality*. When A is a polynomial ring, this equivalence is called *Bernstein-Gel'fand-Gel'fand correspondence*. See, for example, [3].

We have the functors $\mathcal{F}^{\mathsf{op}} : D^b(\mathrm{gr}\,(A^!)^{\mathsf{op}}) \to D^b(\mathrm{gr}\, A^{\mathsf{op}})$ and $\mathcal{G}^{\mathsf{op}} : D^b(\mathrm{gr}\, A^{\mathsf{op}}) \to D^b(\mathrm{gr}\,(A^!)^{\mathsf{op}})$ giving $D^b(\mathrm{gr}\, A^{\mathsf{op}}) \cong D^b(\mathrm{gr}\,(A^!)^{\mathsf{op}})$.

Proposition 3.1 (*cf.* [3, *Proposition 2.3*]). *In the above situation, we have*

$$\beta^{i,j}(M^\bullet) = \dim_K H^{i+j}(\mathcal{G}(M^\bullet))_{-j} \quad \text{and} \quad \mu^{i,j}(N^\bullet) = \dim_K H^{i+j}(\mathcal{F}(N^\bullet))_{-j}.$$

Proof. While the assertion follows from Proposition 3.4 below, we give a direct proof here. We have

$$\begin{aligned} \underline{\mathrm{Ext}}^i_{A^!}(A_0, N^\bullet)_j &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A^!)}(A_0, N^\bullet[i](j)) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A)}(\mathcal{F}(A_0), \mathcal{F}(N^\bullet[i](j))) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A)}(A, \mathcal{F}(N^\bullet)[i+j](-j)) \\ &\cong H^{i+j}(\mathcal{F}(N^\bullet))_{-j}. \end{aligned}$$

Since $\mu^{i,j}(N^\bullet) = \dim_K \underline{\mathrm{Ext}}^i_{A^!}(A_0, N^\bullet)_j$, the second equation of the proposition follows. We can prove the first equation by a similar argument. But this time we use the contravariant functor $\mathbf{D}_{A^!} \circ \mathcal{G} : D^b(\mathrm{gr}\, A) \to D^b(\mathrm{gr}\,(A^!)^{\mathsf{op}})$ and the fact that $\mathbf{D}_{A^!} \circ \mathcal{G}(A_0) \cong \mathbf{D}_{A^!}(A^!(d)) \cong A^!(-d)$. □

Corollary 3.2. $\mathrm{reg}(M^\bullet) = \max\{i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\}$.

Proof. Follows Theorem 2.9 and Proposition 3.1. □

Recall that $\mathbf{D}_A := \mathbf{R}\underline{\mathrm{Hom}}_A(-, \mathcal{D}^\bullet)$ is a duality functor from $D^b(\mathrm{gr}\, A)$ to $D^b(\mathrm{gr}\, A^{\mathsf{op}})$.

Proposition 3.3. $\mathrm{reg}(\mathbf{D}_A(M^\bullet)) = -\min\{i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\}$.

Proof. Let L' be the A–A bimodule given in the construction of the dualizing complex $\mathcal{D}^\bullet$. Note that $\mathbf{D}_A(M^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_A(P^\bullet, L'(-d)[d]) =: Q^\bullet$ for a projective resolution $P^\bullet$ of $M^\bullet$. Since $\mathbf{D}_A(P_v) = {}_{\delta^{-1}(v)}P(-d)[d]$, $Q^\bullet$ is a complex of projectives. And $Q^\bullet$ is a minimal complex if and only if $P^\bullet$ is. Hence $\beta^{-i-d,-j+d}(\mathbf{D}_A(M^\bullet)) = \beta^{i,j}(M^\bullet)$. Therefore, the assertion follows from Proposition 3.1. □

We can refine Proposition 3.1 using the notion of *linear strands* of projective (or injective) resolutions, which was introduced by Eisenbud et al. (See [3, Section 3].) First, we will generalize this notion to our rings. Let B be a noetherian Koszul algebra (e.g., $B = A$ or $A^!$) with the graded Jacobson radical $\mathfrak{m}$, and $P^\bullet$ a *minimal* projective resolution of a bounded complex $M^\bullet \in D^b(\mathrm{gr}\, B)$. Consider the decomposition $P^i := \bigoplus_{j\in\mathbb{Z}} P^{i,j}$ such that any indecomposable summand of $P^{i,j}$ is isomorphic to a summand of $B(-j)$. For an integer l, we define the *l-linear strand* $\mathrm{proj.lin}_l(M^\bullet)$ of a projective resolution of $M^\bullet$ as follows. The term $\mathrm{proj.lin}_l(M^\bullet)^i$ of cohomological degree i is $P^{i,l-i}$ and the differential $P^{i,l-i} \to P^{i+1,l-i-1}$ is the corresponding component of the differential $P^i \to P^{i+1}$ of $P^\bullet$. So the differential of $\mathrm{proj.lin}_l(M^\bullet)$ is represented by a matrix whose entries are elements in B_1. Set $\mathrm{proj.lin}(M^\bullet) := \bigoplus_{l\in\mathbb{Z}} \mathrm{proj.lin}_l(M^\bullet)$. It is obvious that $\beta^{i,j}(M^\bullet) = \beta^{i,j}(\mathrm{proj.lin}(M^\bullet))$ for all i, j.

Using a spectral sequence argument, we can construct $\mathrm{proj.lin}(M^\bullet)$ from a (not necessarily minimal) projective resolution $Q^\bullet$ of $M^\bullet$. Consider the $\mathfrak{m}$-adic filtration $Q^\bullet = F_0 Q^\bullet \supset F_1 Q^\bullet \supset \cdots$ of $Q^\bullet$ with $F_p Q^i = \mathfrak{m}^p Q^i$ and the associated spectral sequence $\{E_r^{*,*}, d_r\}$. The associated graded object $\mathrm{gr}_{\mathfrak{m}} M := \bigoplus_{p\geq 0} \mathfrak{m}^p M/\mathfrak{m}^{p+1} M$ of $M \in \mathrm{gr}\, B$ is a module over $\mathrm{gr}_{\mathfrak{m}} B = \bigoplus_{p\geq 0} \mathfrak{m}^p/\mathfrak{m}^{p+1} \cong B$. Since $\mathfrak{m}^p M$ is a graded submodule of M, we can make $\mathrm{gr}_{\mathfrak{m}} M$ a graded module using the original grading of M (so $(\mathrm{gr}_{\mathfrak{m}} M)_i$ is *not* $\mathfrak{m}^i M/\mathfrak{m}^{i+1} M$ here). Under the identification $\mathrm{gr}_{\mathfrak{m}} B$ with B, we have $\mathrm{gr}_{\mathfrak{m}} M \ncong M$ in general. But if each indecomposable summand N of M is generated by $N_{\iota(N)}$ then $\mathrm{gr}_{\mathfrak{m}} M \cong M$. Since Q^t is a projective B-module, $Q_0^t := \bigoplus_{p+q=t} E_0^{p,q} = \bigoplus_{p\geq 0} \mathfrak{m}^p Q^t/\mathfrak{m}^{p+1} Q^t = \mathrm{gr}_{\mathfrak{m}} Q^t$ is isomorphic to Q^t. The maps $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ make $Q_0^\bullet$ a cochain complex of projective $\mathrm{gr}_{\mathfrak{m}} B$-modules. Consider the decomposition $Q^\bullet = P^\bullet \oplus C^\bullet$, where $P^\bullet$ is minimal and $C^\bullet$ is exact. (We always have such a decomposition.) If we identify Q_0^t with $Q^t = P^t \oplus C^t$, the differential d_0 of $Q_0^\bullet$ is given by $(0, d_{C^\bullet})$. Hence we have $Q_1^t = \bigoplus_{p+q=t} E_1^{p,q} \cong P^t$. The maps $d_1^{p,q} : E_1^{p,q} = \mathfrak{m}^p P^t/\mathfrak{m}^{p+1} P^t \to E_1^{p+1,q} = \mathfrak{m}^{p+1} P^{t+1}/\mathfrak{m}^{p+2} P^{t+1}$ make $Q_1^\bullet$ a cochain complex of projective $\mathrm{gr}_{\mathfrak{m}} B(\cong B)$-modules whose differential is the "linear component" of the differential $d_{P^\bullet}$ of $P^\bullet$. Thus the complex $(Q_1^\bullet, d_1)$ is isomorphic to $\mathrm{proj.lin}(M^\bullet)$.

Since $A^!$ is selfinjective, we can consider the linear strands of an injective resolution. More precisely, starting from a minimal injective resolution of $N^\bullet \in D^b(\mathrm{gr}\, A^!)$, we can construct its l-linear strand $\mathrm{inj.lin}_l(N^\bullet)$ in a similar way. Here, if I^i is the cohomological degree ith term of $\mathrm{inj.lin}_l(N^\bullet)$, then the socle of I^i coincides with $(I^i)_{l-i}$. In other words, any indecomposable summand of I^i is isomorphic to a summand of $A^!(i - l + d)$. Set $\mathrm{inj.lin}(N^\bullet) = \bigoplus_{l\in\mathbb{Z}} \mathrm{inj.lin}_l(N^\bullet)$. This complex can also be constructed using spectral sequences.

We have that $\mathbf{D}_{A^!}(\text{inj.lin}(N^\bullet)) \cong \text{proj.lin}(\mathbf{D}_{A^!}(N^\bullet))$ and $\mathbf{D}_{A^!}(\text{proj.lin}(N^\bullet)) \cong \text{inj.lin}(\mathbf{D}_{A^!}(N^\bullet))$.

Proposition 3.4 (*cf.* [3, *Corollary 3.6*]). *For $M^\bullet \in D^b(\text{gr}\, A)$ and $N^\bullet \in D^b(\text{gr}\, A^!)$, we have*

$$\text{proj.lin}(\mathcal{F}(N^\bullet)) = \mathcal{F}(\mathcal{H}(N^\bullet)) \qquad \textit{and} \qquad \text{inj.lin}(\mathcal{G}(M^\bullet)) = \mathcal{G}(\mathcal{H}(M^\bullet)).$$

More precisely,

$$\text{proj.lin}_l(\mathcal{F}(N^\bullet)) = \mathcal{F}(H^l(N^\bullet))[-l] \quad \textit{and} \quad \text{inj.lin}_l(\mathcal{G}(M^\bullet)) = \mathcal{G}(H^l(M^\bullet))[-l].$$

Proof. Set $Q^\bullet = \mathcal{F}(N^\bullet)$. Note that $Q^\bullet$ is a (non-minimal) complex of projective modules. We use the above spectral sequence argument (and the notation there). Then the differential $d_0^t : Q_0^t \cong \mathcal{F}^t(N^\bullet) \to Q_0^{t+1} \cong \mathcal{F}^{t+1}(N^\bullet)$ is given by $\pm\partial_{N^\bullet}$. Thus

$$Q_1^t \cong \bigoplus_{t=i+j} A \otimes_{A_0} H^i(N^\bullet)_j = \bigoplus_{t=i+j} \mathcal{F}^j(H^i(N^\bullet)),$$

and the differential of $Q_1^\bullet$ is induced by that of $\mathcal{F}(N^i)$. Hence we can easily check that $Q_1^\bullet$, which can be identified with $\text{proj.lin}(\mathcal{F}(N^\bullet))$, is isomorphic to $\mathcal{F}(\mathcal{H}(N^\bullet)) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{F}(H^i(N^\bullet))[-i]$. We can prove the statement for $\text{inj.lin}(\mathcal{G}(M^\bullet))$ in the same way. □

4. Weakly Koszul modules

Let B be a noetherian Koszul algebra (e.g., $B = A$ or $A^!$) with the graded Jacobson radical $\mathfrak{m}$. For $M \in \text{gr}\, B$ and an integer i, $M_{\langle i\rangle}$ denotes the submodule of M generated by its degree i component M_i.

Proposition 4.1. *In the above situation, the following are equivalent.*

(1) *$M_{\langle i\rangle}$ has a linear projective resolution for all i.*
(2) *$H^i(\text{proj.lin}(M)) = 0$ for all $i \neq 0$.*
(3) *All indecomposable summands of $\text{gr}_\mathfrak{m} M$ have linear resolutions as $B\,(\cong \text{gr}_\mathfrak{m} B)$-modules.*

Proof. This result was proved in [20, Proposition 4.9] under the assumption that B is a polynomial ring. (Römer also proved this for a commutative Koszul algebra. See [18, Theorem 3.2.8].) In this proof, only the Koszul property of a polynomial ring is essential, and the proof also works in our case. But, to refer this, the reader should be careful with the following points.

(a) In [20], the grading of $\text{gr}_\mathfrak{m} M$ is given by a different way. There, $(\text{gr}_\mathfrak{m} M)_i = \mathfrak{m}^i M/\mathfrak{m}^{i+1} M$. It is easy to see that $\text{gr}_\mathfrak{m} M$ has a linear resolution in this grading if and only if the condition (3) of the proposition is satisfied in our grading.

(b) In the proof of [20, Proposition 4.9], the regularity $\text{reg}(N)$ of $N \in \text{gr}\, B$ is an important tool. Unless B is AS-regular, one cannot define $\text{reg}(N)$ using the local cohomologies of N. But if we set $\text{reg}(N) := \sup\{i + j \mid \beta^{i,j}(N) \neq 0\}$, then everything works well. It is not clear whether $\operatorname{reg}(N) < \infty$ for all $N \in \operatorname{gr} B$ (cf. [10]). But modules appearing in the argument similar to the proof of [20, Proposition 4.9] have finite regularities.

(c) In the proof of [20, Proposition 4.9], a few basic properties of the Castelnuovo–Mumford regularity (over a polynomial ring) are used. But $\operatorname{reg}(N)$ of $N \in \operatorname{gr} B$ also has these properties, if we define $\operatorname{reg}(N)$ as (b). For example, if $N \in \operatorname{gr} B$ satisfies $\dim_K N < \infty$, then $\operatorname{reg}(N) = \sigma(N)$. This can be proved by induction on $\dim_K N$. Using the short exact sequence $0 \to N_{\geq r} \to N \to N/N_{\geq r} \to 0$, we can also prove that $N_{\geq r}$ has an r linear resolution if and only if $r \geq \operatorname{reg}(N)$ (see also Proposition 2.11).

(d) For the implication (2) $\Rightarrow$ (3), [20] refers to another paper. But this implication can be proved by a spectral sequence argument, since $\operatorname{proj.lin}(M)$ can be constructed using a spectral sequence as we have seen in the previous section. □

Definition 4.2 (*[5,13]*). In the above situation, we say $M \in \operatorname{gr} B$ is *weakly Koszul*, if it satisfies the equivalent conditions of Proposition 4.1.

Remark 4.3. (1) If $M \in \operatorname{gr} B$ has a linear resolution, then it is weakly Koszul.

(2) The notion of weakly Koszul modules was first introduced by Green and Martinez-Villa [5]. But they used the name "strongly quasi Koszul modules". Weakly Koszul modules are also called "componentwise linear modules" by some commutative algebraists (see [7]).

Theorem 4.4. *Let $0 \neq N \in \operatorname{gr} A^!$ and set $N' := \mathbf{D}_{A^!}(N)$. Then the following are equivalent.*

(1) *N is weakly Koszul.*
(2) *$H^i(\mathcal{F}^{\mathrm{op}}(N'))$ has a $(-i)$-linear projective resolution for all i.*
(3) $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathrm{op}}(N')) = 0$.
(4) $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathrm{op}}(N')) \leq 0$.

Proof. Since $\iota(\mathcal{H} \circ \mathcal{F}^{\mathrm{op}}(N')) \geq 0$ (i.e., $\iota(H^i(\mathcal{F}^{\mathrm{op}}(N'))) \geq -i$ for all i), the equivalence among (2), (3) and (4) follows from Proposition 2.11. So it suffices to prove (1) $\Leftrightarrow$ (4). Since $\mathbf{D}_{(A^!)^{\mathrm{op}}}(\operatorname{inj.lin}(N')) \cong \operatorname{proj.lin}(N)$, N is weakly Koszul if and only if $H^i(\operatorname{inj.lin}(N')) = 0$ for all $i > 0$. By Proposition 3.4, we have

$$\operatorname{inj.lin}(N') = \operatorname{inj.lin}(\mathcal{G}^{\mathrm{op}} \circ \mathcal{F}^{\mathrm{op}}(N')) = \mathcal{G}^{\mathrm{op}} \circ \mathcal{H} \circ \mathcal{F}^{\mathrm{op}}(N').$$

Therefore, by Corollary 3.2, $H^i(\operatorname{inj.lin}(N')) = 0$ for all $i > 0$ if and only if the condition (4) holds. □

Remark 4.5. Martinez-Villa and Zacharia proved that if N is weakly Koszul then there is a filtration

$$U_0 \subset U_1 \subset \cdots \subset U_p = N$$

such that U_{i+1}/U_i has a linear resolution for each i (see [13, pp. 676–677]). We can interpret this fact using Theorem 4.4 in our case.

Let $N \in \operatorname{gr} A^!$ be a weakly Koszul module. Set $N' := \mathbf{D}_{A^!}(N)$ and $T^\bullet := \mathcal{F}^{\mathsf{op}}(N')$. Assume that N does not have a linear resolution. Then $H^i(T^\bullet) \neq 0$ for several i. Set $n = \min\{i \mid H^i(T^\bullet) \neq 0\}$. Consider the truncation

$$\sigma_{>n}T^\bullet : \cdots \longrightarrow 0 \longrightarrow \operatorname{im} d^n \longrightarrow T^{n+1} \longrightarrow T^{n+2} \longrightarrow \cdots$$

of $T^\bullet$. Then we have $H^i(T^\bullet) = H^i(\sigma_{>n}T^\bullet)$ for all $i > n$ and $H^i(\sigma_{>n}T^\bullet) = 0$ for all $i \leq n$. We have a triangle

$$H^n(T^\bullet)[-n] \to T^\bullet \to \sigma_{>n}T^\bullet \to H^n(T^\bullet)[-n+1]. \tag{4.1}$$

By Theorem 4.4, $H^n(T^\bullet)[-n]$ has a 0-linear resolution. On the other hand,

$$0 = \operatorname{reg}(\mathcal{H}(\sigma_{>n}T^\bullet)) \geq \operatorname{reg}(\sigma_{>n}T^\bullet) \geq \iota(\sigma_{>n}T^\bullet) \geq 0.$$

Hence $\sigma_{>n}T^\bullet$ also has a 0-linear resolution. Therefore, both $\mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(\sigma_{>n}T^\bullet)$ and $\mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(H^n(T^\bullet)[-n])$ are acyclic complexes (that is, the ith cohomology vanishes for all $i \neq 0$). Set

$$U := H^0(\mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(\sigma_{>n}T^\bullet)) \quad \text{and} \quad V := H^0(\mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(H^n(T^\bullet)[-n])).$$

Since $N = \mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(T^\bullet)$, the triangle (4.1) induces a short exact sequence $0 \to U \to N \to V \to 0$ in $\operatorname{gr} A^!$. It is easy to see that V has a linear resolution. Since $\mathcal{H} \circ \mathcal{F}^{\mathsf{op}} \circ \mathbf{D}_{A^!}(U) = \mathcal{H}(\sigma_{>n}T^\bullet)$, U is weakly Koszul by Theorem 4.4. Repeating this procedure, we can get the expected filtration.

Let $N \in \operatorname{gr} A^!$ and $\cdots \xrightarrow{f_2} P^{-1} \xrightarrow{f_1} P^0 \xrightarrow{f_0} N \to 0$ its minimal projective resolution. For $i \geq 1$, we call $\Omega_i(N) := \ker(f_{i-1})$ the ith *syzygy* of N. Note that $\Omega_i(N) = \operatorname{im}(f_i) = \operatorname{coker}(f_{i+1})$.

By the original definition of a weakly Koszul module given in [5,13], if $N \in \operatorname{gr} A^!$ is weakly Koszul then so is $\Omega_i(N)$ for all $i \geq 1$.

Definition 4.6 (*Herzog–Römer, [18]*). For $0 \neq N \in \operatorname{gr} A^!$, set

$$\operatorname{lpd}(N) := \inf\{i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul}\},$$

and call it the *linear part dominates* of N.

Since A is a noetherian ring of finite global dimension, $\operatorname{lpd}(N)$ is finite for all $N \in \operatorname{gr} A^!$ by [13, Theorem 4.5].

Theorem 4.7. *Let $N \in \operatorname{gr} A^!$ and set $N' := \mathbf{D}_{A^!}(N)$. Then we have*

$$\begin{aligned}\operatorname{lpd}(N) &= \operatorname{reg}(\mathcal{H} \circ \mathcal{F}^{\mathsf{op}}(N')) \\ &= \max\{\operatorname{reg}(H^i(\mathcal{F}^{\mathsf{op}}(N'))) + i \mid i \in \mathbb{Z}\}.\end{aligned}$$

Proof. Note that $P^\bullet := \mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}} \circ \mathcal{F}^{\mathsf{op}}(N')$ is a projective resolution of N, and $Q^\bullet := \mathbf{D}_{(A^!)^{\mathsf{op}}} \circ \mathcal{G}^{\mathsf{op}}(\mathcal{F}^{\mathsf{op}}(N')_{\geq i})$ is the truncation $\cdots \to P^{-i-1} \to P^{-i} \to 0 \to \cdots$ of $P^\bullet$ for each $i \geq 1$. Hence we have $H^j(Q^\bullet) = 0$ for all $j \neq -i$ and there is a projective module P such that $H^{-i}(Q^\bullet) \cong \Omega_i(N) \oplus P$. Since P is weakly Koszul, $\Omega_i(N)$ is weakly Koszul if and only if so is $Q := H^{-i}(Q^\bullet)$. We have

$$\text{proj.lin}(Q)[i] \cong \mathbf{D}_{(A^!)^{\text{op}}} \circ \mathcal{G}^{\text{op}} \circ \mathcal{H}(\mathcal{F}^{\text{op}}(N')_{\geq i}).$$

By Theorem 4.4, Q is weakly Koszul if and only if $\mathcal{H}(\mathcal{F}^{\text{op}}(N')_{\geq i})$ has an i-linear resolution, that is, $H^j(\mathcal{F}^{\text{op}}(N')_{\geq i})$ has an $(i-j)$-linear resolution for all j. But there is some $L \in \text{gr}\,(A^!)^{\text{op}}$ such that $L = L_{i-j}$ and $H^j(\mathcal{F}^{\text{op}}(N')_{\geq i}) \cong H^j(\mathcal{F}^{\text{op}}(N'))_{\geq i-j} \oplus L$. Note that L has an $(i-j)$-linear resolution. Therefore, $H^j(\mathcal{F}^{\text{op}}(N')_{\geq i})$ has an $(i-j)$-linear resolution if and only if so does $H^j(\mathcal{F}^{\text{op}}(N'))_{\geq i-j}$. Summing up the above facts, we have that $\Omega_i(N)$ is weakly Koszul if and only if $(\mathcal{H} \circ \mathcal{F}^{\text{op}}(N'))_{\geq i}$ has an i-linear resolution. By Proposition 2.11, the last condition is equivalent to the condition that $i \geq \text{reg}(\mathcal{H} \circ \mathcal{F}^{\text{op}}(N'))$. □

Remark 4.8. Assume that A is noetherian, Koszul, and has finite global dimension, but not necessarily AS-regular. Then $A^!$ is a finite-dimensional Koszul algebra, but not necessarily selfinjective. Even in this case, $\mathcal{G}(M^\bullet)$ for $M^\bullet \in D^b(\text{gr}\,A)$ is a complex of injective $A^!$-modules, and the results in Section 3 and Theorem 4.7 also hold. But now we should set $\text{reg}(M^\bullet) := \sup\{i + j \mid \beta^{i,j}(M^\bullet) \neq 0\}$ for $M^\bullet \in D^b(\text{gr}\,A)$ (local cohomology is not helpful to define the regularity). Since A is noetherian and has finite global dimension, we have $\text{reg}(M^\bullet) < \infty$ for all $M^\bullet$. In particular, we have $\text{lpd}(N) < \infty$ for all $N \in \text{gr}\,A^!$ (if A is *right* noetherian) as proved in [13, Theorem 4.5].

If $\text{lpd}(N) \geq 1$ for some $N \in \text{gr}\,A^!$, then $\sup\{\text{lpd}(T) \mid T \in \text{gr}\,A^!\} = \infty$. In fact, if $\Omega_{-i}(N)$ is the ith *cosyzygy* of N (since $A^!$ is selfinjective, we can consider cosyzygies), then $\text{lpd}(\Omega_{-i}(N)) > i$. But Herzog and Römer proved that if J is a monomial ideal of an exterior algebra $E = \bigwedge\langle y_1, \dots, y_d\rangle$ then $\text{lpd}(E/J) \leq d - 1$ (cf. [18, Section 3.3]). We will refine their results using Theorem 4.7.

In what follows, we regard the polynomial ring $S = K[x_1, \dots, x_d]$, $d \geq 1$, as an $\mathbb{N}^d$-graded ring with $\deg x_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is at the ith position. Similarly, the exterior algebra $E = S^! = \bigwedge\langle y_1, \dots, y_d\rangle$ is also an $\mathbb{N}^d$-graded ring. Let $^*\text{Gr}\,S$ be the category of $\mathbb{Z}^d$-graded S-modules and their degree preserving S-homomorphisms, and $^*\text{gr}\,S$ its full subcategory consisting of finitely generated modules. We have a similar category $^*\text{gr}\,E$ for E. For S-modules and graded E-modules, we do not have to distinguish left modules from right modules. Since $\mathbb{Z}^d$-graded modules can be regarded as $\mathbb{Z}$-graded modules in the natural way, we can discuss $\text{reg}(M^\bullet)$ for $M^\bullet \in D^b(^*\text{gr}\,S)$ and $\text{lpd}(N)$ for $N \in {}^*\text{gr}\,E$.

Note that $\mathbf{D}_E(-) = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} \text{Hom}_{^*\text{gr}\,E}(-, E(\mathbf{a}))$ gives an exact duality functor from $^*\text{gr}\,E$ to itself. Sometimes, we simply denote $\mathbf{D}_E(N)$ by N'. Set $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{Z}^d$. Then $\mathcal{D}_S^\bullet := S(-\mathbf{1})[d] \in D^b(^*\text{gr}\,S)$ is a $\mathbb{Z}^d$-graded normalized dualizing complex and $\mathbf{D}_S(-) := \mathbf{R}\underline{\text{Hom}}_S(-, \mathcal{D}_S^\bullet) = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} \mathbf{R}\text{Hom}_{^*\text{Gr}\,S}(-, \mathcal{D}_S^\bullet(\mathbf{a}))$ gives a duality functor from $D^b(^*\text{gr}\,S)$ to itself. As shown in [21, Theorem 4.1], we have the $\mathbb{Z}^d$-graded Koszul duality functors $\mathcal{F}^*$ and $\mathcal{G}^*$ giving an equivalence $D^b(^*\text{gr}\,S) \cong D^b(^*\text{gr}\,E)$. These functors are defined in the same way as in the $\mathbb{Z}$-graded case.

For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$, set $\text{supp}(\mathbf{a}) := \{i \mid a_i > 0\} \subset [d] := \{1, \dots, d\}$. We say $\mathbf{a} \in \mathbb{Z}^d$ is *squarefree* if $a_i = 0, 1$ for all $i \in [d]$. When $\mathbf{a} \in \mathbb{Z}^d$ is squarefree, we sometimes identify $\mathbf{a}$ with $\operatorname{supp}(\mathbf{a})$. For example, if $F \subset [d]$, then $S(-F)$ means the free module $S(-\mathbf{a})$, where $\mathbf{a} \in \mathbb{N}^d$ is the squarefree vector with $\operatorname{supp}(\mathbf{a}) = F$.

Definition 4.9 (*[20]*). We say $M \in {}^*\mathrm{gr}\ S$ is *squarefree*, if M has a presentation of the form

$$\bigoplus_{F\subset[d]} S(-F)^{m_F} \to \bigoplus_{F\subset[d]} S(-F)^{n_F} \to M \to 0$$

for some $m_F, n_F \in \mathbb{N}$.

The above definition seems different from the original one given in [20], but they coincide. Stanley–Reisner rings (that is, the quotient rings of S by squarefree monomial ideals) and many modules related to them are squarefree. Here we summarize the basic properties of squarefree modules. See [20,21] for further information. Let $\mathrm{Sq}(S)$ be the full subcategory of ${}^*\mathrm{gr}\ S$ consisting of squarefree modules. Then $\mathrm{Sq}(S)$ is closed under kernels, cokernels, and extensions in ${}^*\mathrm{gr}\ S$. Thus $\mathrm{Sq}(S)$ is an abelian category. Moreover, we have $D^b(\mathrm{Sq}(S)) \cong D^b_{\mathrm{Sq}(S)}({}^*\mathrm{Gr}\ S)$. If M is squarefree, then each term in a $\mathbb{Z}^d$-graded minimal free resolution of M is of the form $\bigoplus_{F\subset[d]} S(-F)^{n_F}$. Hence we have $\operatorname{reg}(M) \le d$. Moreover, $\operatorname{reg}(M) = d$ if and only if M has a summand which is isomorphic to $S(-\mathbf{1})$.

Definition 4.10 (*Römer [16]*). We say $N \in {}^*\mathrm{gr}\ E$ is *squarefree*, if $N = \bigoplus_{F\subset[d]} N_F$ (i.e., if $\mathbf{a} \in \mathbb{Z}^d$ is not squarefree, then $N_{\mathbf{a}} = 0$).

A monomial ideal of E is always a squarefree E-module. Let $\mathrm{Sq}(E)$ be the full subcategory of ${}^*\mathrm{gr}\ E$ consisting of squarefree modules. Then $\mathrm{Sq}(E)$ is an abelian category with $D^b(\mathrm{Sq}(E)) \cong D^b_{\mathrm{Sq}(E)}({}^*\mathrm{gr}\ E)$. If N is a squarefree E-module, then so is $\mathbf{D}_E(N)$. That is, $\mathbf{D}_E$ gives an exact duality functor from $\mathrm{Sq}(E)$ to itself. We have functors $\mathcal{S} : \mathrm{Sq}(E) \to \mathrm{Sq}(S)$ and $\mathcal{E} : \mathrm{Sq}(S) \to \mathrm{Sq}(E)$ giving an equivalence $\mathrm{Sq}(S) \cong \mathrm{Sq}(E)$. Here $\mathcal{S}(N)_F = N_F$ for $N \in \mathrm{Sq}(E)$ and $F \subset [d]$, and the multiplication map $\mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F\cup\{i\}}$ for $i \notin F$ is given by $\mathcal{S}(N)_F = N_F \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{F\cup\{i\}} = \mathcal{S}(N)_{F\cup\{i\}}$, where $\alpha(i, F) = \#\{j \in F \mid j < i\}$. See [16,21] for details. Since a free module $E(\mathbf{a})$ is *not* squarefree unless $\mathbf{a} = 0$, the syzygies of a squarefree E-module are *not* squarefree.

Proposition 4.11 (*Herzog–Römer, [18, Corollary 3.3.5]*). *If N is a squarefree E-module (e.g., $N = E/J$ for a monomial ideal J), then we have* $\operatorname{lpd}(N) \le d - 1$.

This result easily follows from Theorem 4.7 and the fact that $H^i(\mathcal{F}^*(N'))(-\mathbf{1})$ is a squarefree S-module for all i and $H^i(\mathcal{F}^*(N')) = 0$ unless $0 \le i \le d$. (Recall the remark on the regularity of squarefree modules given before Definition 4.10, and note that $M := H^d(\mathcal{F}^*(N'))(-\mathbf{1})$ is generated by M_0.)

We also remark that [18, Corollary 3.3.5] just states that $\operatorname{lpd}(N) \le d$. But their argument actually proves that $\operatorname{lpd}(N) \le d - 1$. In fact, they showed that

$$\operatorname{lpd}(N) \le \operatorname{proj.dim}_S \mathcal{S}(N).$$

But, if $\operatorname{proj.dim}_S \mathcal{S}(N) = d$ then $\mathcal{S}(N)$ has a summand which is isomorphic to $K = S/(x_1, \ldots, x_d)$ and hence N has a summand which is isomorphic to $K = E/(y_1, \ldots, y_d)$. But $K \in \mathrm{Sq}(E)$ has a linear resolution and irrelevant to $\operatorname{lpd}(N)$.

To refine Proposition 4.11, we need further properties of squarefree modules.

If $M^\bullet \in D^b(\mathrm{Sq}(S))$, then $\underline{\mathrm{Ext}}^i_S(M^\bullet, \mathcal{D}^\bullet_S)$ is squarefree for all i. Hence $\mathcal{D}^\bullet_S$ gives a duality functor on $D^b(\mathrm{Sq}(S))$. On the other hand, $\mathbf{A} := \mathcal{S} \circ \mathbf{D}_E \circ \mathcal{E}$ is an exact duality functor on $\mathrm{Sq}(S)$ and it induces a duality functor on $D^b(\mathrm{Sq}(S))$. Miller [14, Corollary 4.21] and Römer [17, Corollary 3.7] proved that $\mathrm{reg}(\mathbf{A}(M)) = \mathrm{proj.dim}_S M$ for all $M \in \mathrm{Sq}(S)$. I generalized this equation to a complex $M^\bullet \in D^b(\mathrm{Sq}(S))$ in [22, Corollary 2.10].

Lemma 4.12. *Let $N \in \mathrm{Sq}(E)$ and set $N' := \mathbf{D}_E(N)$. Then we have*

$$\mathrm{reg}(H^i(\mathcal{F}^*(N'))) = -\mathrm{depth}_S(\mathrm{Ext}^{d-i}_S(\mathcal{S}(N'), S)) \tag{4.2}$$

and

$$\mathrm{lpd}(N) = \max\{i - \mathrm{depth}_S(\mathrm{Ext}^{d-i}_S(\mathcal{S}(N'), S)) \mid 0 \le i \le d\}. \tag{4.3}$$

Here we set the depth of the 0 module to be $+\infty$.

If $M := \mathrm{Ext}^{d-i}_S(\mathcal{S}(N'), S) \neq 0$, then $\mathrm{depth}_S M \le \dim_S M \le i$. Therefore all members in the set of the right side of (4.3) are non-negative or $-\infty$.

Proof. By Theorem 4.7, (4.3) follows from (4.2). So it suffices to show (4.2). By [21, Proposition 4.3], we have $\mathcal{F}^*(N') \cong (\mathbf{A} \circ \mathbf{D}_S \circ \mathcal{S}(N'))(\mathbf{1})$. (The degree shifting "$(\mathbf{1})$" does not occur in [21, Proposition 4.3]. But E is a negatively graded ring there, and we need the degree shifting in the present convention.) Since $\mathbf{A}$ is exact, we have

$$\begin{aligned} H^i(\mathcal{F}^*(N')) \cong H^i(\mathbf{A} \circ \mathbf{D}_S \circ \mathcal{S}(N'))(\mathbf{1}) &\cong \mathbf{A}(H^{-i}(\mathbf{D}_S \circ \mathcal{S}(N')))(\mathbf{1}) \\ &= \mathbf{A}(\underline{\mathrm{Ext}}^{-i}_S(\mathcal{S}(N'), \mathcal{D}^\bullet_S))(\mathbf{1}). \end{aligned}$$

Recall that $\mathrm{reg}(\mathbf{A}(M)) = \mathrm{proj.dim}_S M$ for $M \in \mathrm{Sq}(S)$. On the other hand, since M is finitely generated, the underlying module of $\underline{\mathrm{Ext}}^{-i}_S(M, \mathcal{D}^\bullet_S)$ is isomorphic to $\mathrm{Ext}^{d-i}_S(M, S)$. So (4.2) follows from these facts and the Auslander–Buchsbaum formula. □

Corollary 4.13. *For $N \in \mathrm{Sq}(E)$, N is weakly Koszul (over E) if and only if $\mathcal{S}(N)$ is weakly Koszul (over S).*

In [17, Corollary 1.3], it was proved that N has a linear resolution if and only if so does $\mathcal{S}(N)$. Corollary 4.13 also follows from this fact and (the squarefree module version of) [7, Proposition 1.5].

Proof. We say $M \in \mathrm{gr}\, S$ is *sequentially Cohen–Macaulay*, if for each i $\mathrm{Ext}^i_S(M, S)$ is either the zero module or a Cohen–Macaulay module of dimension $d - i$ (cf. [19, III. Theorem 2.11]). By Lemma 4.12, N is weakly Koszul if and only if $\mathcal{S}(N')$ ($\cong \mathbf{A} \circ \mathcal{S}(N)$) is sequentially Cohen–Macaulay. By [17, Theorem 4.5], the latter condition holds if and only if $\mathcal{S}(N)$ is weakly Koszul. □

Many examples of squarefree monomial ideals of S which are weakly Koszul (dually, Stanley–Reisner rings which are sequentially Cohen–Macaulay) are known. So we can obtain many weakly Koszul monomial ideals of E using Corollary 4.13.

Proposition 4.14. *For an integer i with* $1 \leq i \leq d-1$*, there is a squarefree* E*-module* N *such that* $\mathrm{lpd} N = \mathrm{proj.dim}_S \mathcal{S}(N) = i$. *In particular, the inequality of Proposition* 4.11 *is optimal.*

Proof. Let M be the $\mathbb{Z}^d$-graded ith syzygy of $K = S/\mathfrak{m}$. Note that M is squarefree. We can easily check that $N := \mathbf{D}_E \circ \mathcal{E}(M) \in \mathrm{Sq}(E)$ satisfies the expected condition. In fact, $\mathrm{proj.dim}_S \mathcal{S}(N) = \mathrm{proj.dim}_S \mathbf{A}(M) = \mathrm{reg} M = i$. On the other hand, since $\mathrm{Ext}_S^{d-i}(\mathcal{S}(N'), S) = \mathrm{Ext}_S^{d-i}(M, S) = K$, $\mathrm{Ext}_S^{j}(\mathcal{S}(N'), S) = 0$ for all $j \neq d-i, 0$, and $\mathrm{depth}_S(\mathrm{Hom}_S(\mathcal{S}(N'), S)) = d - i + 1$, we have $\mathrm{lpd}\, N = i$. □

The above result also says that the inequality $\mathrm{lpd}(N) \leq \mathrm{proj.dim}_S \mathcal{S}(N)$ of [18, Corollary 3.3.5] is also optimal. But for a monomial ideal $J \subset E$, the situation is different.

Proposition 4.15. *If* $d \geq 3$*, then we have* $\mathrm{lpd}(E/J) \leq d-2$ *for a monomial ideal* J *of* E.

Proof. If $d = 3$, then easy computation shows that any squarefree monomial ideal $I \subset S$ is weakly Koszul. Hence J is weakly Koszul by Corollary 4.13. So we may assume that $d \geq 4$.

Note that $\mathbf{A} \circ \mathcal{S}(E/J)$ is isomorphic to a squarefree monomial ideal of S. We denote it by I. By Lemma 4.12, it suffices to show that $\mathrm{depth}_S(\mathrm{Hom}_S(I, S)) \geq 2$ and $\mathrm{depth}_S(\mathrm{Ext}_S^1(I, S)) \geq 1$. Recall that $\mathrm{Hom}_S(I, S)$ satisfies Serre's condition (S_2), hence its depth is at least 2. Since $\mathrm{Ext}_S^1(I, S) \cong \mathrm{Ext}_S^2(S/I, S)$, it suffices to prove that $\mathrm{depth}_S(\mathrm{Ext}_S^2(S/I, S)) \geq 1$.

If $\mathrm{ht}(I) > 2$, then we have $\mathrm{Ext}_S^2(S/I, S) = 0$. If $\mathrm{ht}(I) = 2$, then $\mathrm{Ext}_S^2(S/I, S)$ satisfies (S_2) as an S/I-module and $\mathrm{depth}_S \mathrm{Ext}_S^2(S/I, S) \geq \min\{2, \dim(S/I)\} \geq 2$. So we may assume that $\mathrm{ht}(I) = 1$. If the heights of all associated primes of I are 1, then I is a principal ideal and $\mathrm{Ext}_S^i(S/I, S) = 0$ for all $i \neq 1$. So we may assume that I has an prime of larger height. Then we have ideals I_1 and I_2 of S such that $I = I_1 \cap I_2$ and the heights of any associated prime of I_1 (resp. I_2) is 1 (at least 2). Since I is a radical ideal, we have $\mathrm{ht}(I_1 + I_2) \geq 3$. Hence $\mathrm{Ext}_S^2(S/(I_1 + I_2), S) = 0$ and $\mathrm{Ext}_S^3(S/(I_1 + I_2), S)$ is either the zero module or it satisfies (S_2) as an $S/(I_1 + I_2)$-module. In particular, if $\mathrm{Ext}_S^3(S/(I_1 + I_2), S) \neq 0$ (equivalently, if $\dim(S/(I_1 + I_2)) = d - 3$) then $\mathrm{depth}_S(\mathrm{Ext}_S^3(S/(I_1 + I_2), S)) \geq \min\{2, d-3\} \geq 1$. Note that $\mathrm{depth}_S(\mathrm{Ext}_S^2(S/I_2, S)) \geq 2$. From the short exact sequence

$$0 \to S/I \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0$$

and the above argument, we have the exact sequence

$$0 \to \mathrm{Ext}_S^2(S/I_2, S) \to \mathrm{Ext}_S^2(S/I, S) \to \mathrm{Ext}_S^3(S/(I_1 + I_2), S). \tag{4.4}$$

We have $\mathrm{depth}_S(\mathrm{Ext}_S^2(S/I, S)) \geq 1$ by (4.4), since the modules beside this module have positive depth. □

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