Linear structure of sets of divergent sequences and series

A. Aizpuru a, C. Pérez-Eslava a, J.B. Seoane-Sepúlveda b,*

a Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain
b Department of Mathematics, Kent State University, Kent, OH 44242, USA

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Abstract

We show that there exist infinite dimensional spaces of series, every non-zero element of which, enjoys certain pathological property. Some of these properties consist on being (i) conditional convergent, (ii) divergent, or (iii) being a subspace of $l_\infty$ of divergent series. We also show that the space $l^\omega_1(X)$ of all weakly unconditionally Cauchy series in $X$ has an infinite dimensional vector space of non-weakly convergent series, and that the set of unconditionally convergent series on $X$ contains a vector space $E$, of infinite dimension, so that if $x \in E \setminus \{0\}$ then $\sum_j \|x_j\| = \infty$.

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1. Introduction

This paper contributes to the search for large vector spaces of functions on $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) which have special properties. Given such a property, we say that the subset $M$ of functions on $\mathbb{K}$ which satisfy it is spaceable if $M \cup \{0\}$ contains a closed infinite dimensional subspace. The set $M$ will be called lineable if $M \cup \{0\}$ contains an infinite dimensional vector space. We will say that the set

* Corresponding author.

E-mail addresses: antonio.aizpuru@uca.es (A. Aizpuru), sonriencanto@hotmail.com (C. Pérez-Eslava), jseoane@math.kent.edu (J.B. Seoane-Sepúlveda).

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$M$ is $\mu$-lineable if it contains a vector space of dimension $\mu$. Also, we denote $\lambda(M)$ the maximum dimension of such a vector space [1].

One of the first results in this area was proved by Fonf et al. [5], who showed that the set of nowhere differentiable functions on $[0, 1]$ is spaceable in $C[0, 1]$. Moreover, Rodríguez-Piazza proved [7] that the $X$ in [5] can be chosen to be isometrically isomorphic to any separable Banach space. In this work, we continue the search for pathological vector spaces. Particularly, we study large vector spaces of sequences and series enjoying several special properties. There are many examples of scalar series in $\mathbb{K}$ which are conditionally convergent, unconditionally convergent, and examples of unconditionally convergent vector series $\sum_{i} x_i$ in a Banach space $X$ so that $\sum_i \|x_i\| = \infty$ [4]. Here we study the lineability of sets of scalar and vector series enjoying, amongst others, these special properties.

2. Sequences, scalar series, and vector series

A family $\{A_\alpha : \alpha \in I\}$ of infinite subsets of $\mathbb{N}$ is called almost disjoint if $A_\alpha \cap A_\beta$ is finite whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$. The usual procedure to generate such a family is the following. Let us denote $I$ the irrationals in $[0, 1]$ and $\{q_n : n \in \mathbb{N}\}$ denotes $[0, 1] \cap \mathbb{Q}$. For every $\alpha \in I$ we choose a subsequence $(q_{n_k})_k$ of $(q_n : n \in \mathbb{N})$ so that $\lim_{k \to \infty} q_{n_k} = \alpha$ and we define $A_\alpha = \{n_k : k \in \mathbb{N}\}$. By construction we obtain that $\{A_\alpha : \alpha \in I\}$ is an almost disjoint family of subsets of $\mathbb{N}$. In this paper, we use this notion on several occasions.

If $V$ denotes the set of conditionally convergent series then, clearly, $V \cup \{0\}$ is not a vector space in $CS(\mathbb{K})$, the set of convergent series. The following theorem shows that the set of conditionally convergent series is $c$-lineable.

**Theorem 2.1.** $CS(\mathbb{K})$ contains a vector space $E$ verifying the following properties:

(i) Every $x \in E \setminus \{0\}$ is a conditionally convergent series.

(ii) $\dim(E) = c$.

(iii) span($E \cup c_{00}$) is an algebra and its elements are either elements of $c_{00}$ or conditionally convergent series.

**Proof.** Let us fix any conditionally convergent series $\sum_{i} a_i$ such that $a_i \neq 0$ for every $i \in \mathbb{N}$. Consider the family $(A_\alpha)_{\alpha \in I}$ of almost disjoint subsets of $\mathbb{N}$. We have that card$(I) = c$. For every $\alpha \in I$ we define $x_\alpha$ given by $x_\alpha^n = a_n$ if $n$th element of $A_\alpha$ and $x_\alpha^n = 0$ otherwise. For every $\alpha \in I$ we have that the series $\sum_{i} x_\alpha^i$ has a conditionally convergent subseries and, therefore, it is conditionally convergent. Let $E = \text{span}(x_\alpha : \alpha \in I)$. We have that $\{x_\alpha : \alpha \in I\}$ is a linearly independent family, and so dim($E$) = $c$.

Suppose that $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{K} \setminus \{0\}$ and $\{\alpha_1, \ldots, \alpha_n\} \subset I$. We now see that $z = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}$ is a conditionally convergent series. Indeed, it is easy to check that there exists $A \subset A_{\alpha_1}$, infinite, so that $A_{\alpha_1} \setminus A$ is finite, and verifying that $A \cap (A_{\alpha_2} \cup \cdots \cup A_{\alpha_n}) = \emptyset$. Then $\sum_{i \in A} z_i = \sum_{i \in A} \lambda_i x_i^\alpha$ is conditionally convergent and, therefore, so is $z$.

To check that span($E \cup c_{00}$) is an algebra it suffices to notice that

$$(\lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n})(\mu_1 x_{\beta_1} + \cdots + \mu_m x_{\beta_m}) \in c_{00}$$

if $\{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m\} \subset \mathbb{K} \setminus \{0\}$ and $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\} \subset I$. 

\hspace{1cm} □
This same technique can be also used to prove the following results in a similar way:

**Theorem 2.2.** There exists a vector space \( E \subset BS(\mathbb{K}) \) (the set of all series with bounded partial sums) such that:

(i) Every \( x \in E \setminus \{0\} \) is a divergent series.
(ii) \( \dim(E) = c \) and \( E \) is non-separable.
(iii) \( \text{span}\{E \cup c_{00}\} \) is an algebra and every element of it is either a divergent series or is an element of \( c_{00} \).

**Theorem 2.3.** There exists a vector space \( E \subset l_\infty \) such that:

(i) \( \dim(E) = c \).
(ii) Every \( x \in E \setminus \{0\} \) is a divergent sequence.
(iii) \( E \oplus c_0 \) is an algebra.
(iv) Every element in \( \text{cl}(E) + c_0 \) is either a divergent sequence or a sequence in \( c_0 \).

If \( X \) is a Banach space and \( \sum_i x_i \) is a series in \( X \), we say that \( \sum_i x_i \) is unconditionally convergent (UC) if, for every permutation \( \pi \) of \( \mathbb{N} \), we have that \( \sum_{i=1}^{\infty} x_{\pi(i)} \) converges. We say that \( \sum_i x_i \) is weakly unconditionally Cauchy (WUC) if \( \sum_{i=1}^{\infty} |f(x_i)| < \infty \) for every \( f \in X^* \), the dual space of \( X \). It is also known that [2,3,6] if \( X \) is a Banach space, then there exists a WUC series in \( X \) which is convergent but which is not unconditionally convergent if and only if \( X \) has a copy of \( c_0 \). It is a well known fact that every infinite dimensional Banach space has a series \( \sum_i x_i \) which is unconditionally convergent and so that \( \sum_i \|x_i\| = \infty \) [4]. The technique from the proof of Theorem 2.1 lead to the following final results:

**Theorem 2.4.** There exists a vector space \( E \subset l_1^w(c_0) \) (the space of all weakly unconditionally Cauchy series in \( c_0 \)) verifying:

(i) \( \dim(E) = c \).
(ii) If \( x \in E \setminus \{0\} \) then \( \sum_i x_i \) is not weakly convergent.

**Theorem 2.5.** Let \( X \) be an infinite dimensional Banach space. Then there exists a vector subspace \( E \) of \( \text{UC}(X) \) such that \( \dim(E) = c \), and if \( x \in E \setminus \{0\} \) then \( \sum_i \|x_i\| = \infty \).

To conclude, notice that from Theorem 2.4 it follows that \( X \) has a copy of \( c_0 \) if and only if there exists a vector subspace \( E \) of \( l_1^w(X) \) with \( \dim(E) = c \), so that every non-zero element of \( E \) is a non-weakly convergent series.

**References**

