Positive Solutions of Superlinear Elliptic Equations

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In this paper, we study the existence of two positive solutions of superlinear elliptic equations without assuming the conditions which have been used in the literature to deduce either the P.S. condition or a priori bounds of positive solutions. The first solution is proved as the minimal positive solution, while the second one is obtained as the limit of a gradient flow whose starting point is properly chosen. The dependence of the minimal solution upon a parameter is also considered.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and $f: \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function. It will be always assumed that $f(0) \geq 0$. Consider the elliptic problems

\[
\begin{cases}
-\Delta u = f(u), & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u = 0, & x \in \partial \Omega
\end{cases}
\]  

(1)

and

\[
\begin{cases}
-\Delta u = \lambda f(u), & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u = 0, & x \in \partial \Omega
\end{cases}
\]  

(1)$_{\lambda}$

where $\lambda$ is a positive parameter. By a solution $u$ of (1) (or (1)$_{\lambda}$) we mean a classical solution which satisfies (1) (or (1)$_{\lambda}$) pointwise.

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The existence of solutions of semilinear elliptic problems has been extensively studied; see, for example, [1–7, 9–15, 17–22, 24–26]. The main tools that have been used for studying (1) and (1)_\lambda are topological degree theory, variational method, and the sub- and super-solutions method, and in many cases the results were obtained by comprehensive utilization of these three tools.

When a strict subsolution and a strict supersolution are known, we will deal with the general problem of proving the existence of two positive solutions for (1) and (1)_\lambda.

This type of problem has been frequently studied. Among the more original contributions we mention the work of Chang [13]. It is well known that if there are a strict subsolution and a strict supersolution, then a solution, lying in the “window” between them, can be chosen. This solution is a local minimum in the $C^1(\bar{\Omega})$ topology for the associated functional defined on $H^1_0(\Omega)$. The idea of Chang is then to try to benefit from the mountain pass geometry of the functional to find a second distinct solution. The main problem encountered is that the P.S. condition doesn’t hold for the functional in the $C^1(\bar{\Omega})$ topology. To overcome this difficulty Chang introduced the concept of retraction and proved that a gradient flow can be constructed (continuous in $C^1(\bar{\Omega})$) which permits recovery of the result that the standard mountain pass lemma would give.

An alternative approach was introduced by De Figueiredo and Solimini [19]. They showed that the sub–supersolution structure doesn’t only imply the presence of a local minimum in the $C^1(\bar{\Omega})$ topology but also one in the $H^1_0(\Omega)$ topology. Then they obtained a second solution directly using the mountain pass lemma. This result was rediscovered and popularized by Brezis and Nirenberg [10]. In [10] the critical exponent case was also treated and some applications of their result were given in [2] on a class of nonlinearities of the form or (1)_\lambda.

In this paper, we will consider only the case in which the nonlinear function $f$ is superlinear and subcritical at infinity, that is, $f$ satisfies

\[(H1) \quad \lim_{t \to +\infty} f(t) t^{-1} > \lambda_1,\]

and

\[(H2) \quad \text{for } l = (N + 2)/(N - 2) \text{ if } N \geqslant 3 \text{ and } l < \infty \text{ if } N = 1, 2, \]

\[\lim_{t \to +\infty} f(t) t^{-l} = 0,\]

where $\lambda_1$ is the first eigenvalue of $-A$ with 0-Dirichlet boundary condition. If we extend the definition of $f$ to the whole of $\mathbb{R}$ by letting $f(t) = f(0)$ for all $t < 0$, then solutions of (1) (or of (1)_\lambda) correspond to nontrivial critical points of the functional.
\[ J(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - F(u) \right) \, dx, \quad u \in H_0^1(\Omega), \]

(respectively,

\[ J_\lambda(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - \lambda F(u) \right) \, dx, \quad u \in H_0^1(\Omega), \]

where \( F(t) = \int_0^t f(s) \, ds \). We will assume (H1) and (H2) only but not the P.S. condition or an \textit{a priori} bound for positive solutions. We will restudy a conjecture of Lions which has already been studied by De Figueiredo and Lions in [18].

In addition to (H1) and (H2), assuming that \( \Omega \) is convex and \( f \) satisfies (H3) for some \( 0 < \theta < 2N/(N-2) \),

\[ \limsup_{t \to +\infty} \frac{tf(t) - \theta F(t)}{t^{2N/(N-2)}} \leq 0, \]

De Figueiredo \textit{et al.} [17] obtained an \textit{a priori} bound for the solutions of (1) (or of (1)\(^*\)) and studied the structure of all solutions of (1)\(^*\). They proved an existence result for (1) assuming (H1) and (H2) only, but not (H3). In [17, Theorem 2.2] the difficulties arising from the lack of (H3) were overcome by defining a sequence of modified functionals which satisfy the P.S. condition and whose critical points have an \textit{a priori} bound.

Using the results in [17], Lions [21] studied several classes of parameterized problems of the form (1)\(^*\). Under various conditions on \( f \) around the origin he showed that, for \( \lambda > 0 \) in a certain interval, there exist two positive solutions of (1)\(^*\), one being a minimal solution. These multiplicity results are obtained mainly using degree arguments, and \textit{a priori} bounds on the positive solutions are necessary in this approach. In [21] these bounds are assured by assuming that \( \Omega \) is convex and (H3) is present. Lions conjectured in [21] that these two conditions are just technical and that his results should hold without them. In the present paper, we manage to show that (H3) was indeed unnecessary. We also improve the conclusions of Lions on the behavior of the minimal solution with respect to \( \lambda \).

Some results in that direction have already been obtained by De Figueiredo and Lions in [18]. On (1)\(^*\), assuming that \( \Omega \) is convex but not (H3), they proved that there are two positive solutions. The first one is the minimal solution of the two and the second one is obtained by a mountain pass argument. For this they take advantage of the fact that the minimal solution is also a local minimum in the \( H_0^1(\Omega) \) topology, as proved in [19]. Since the condition (H3) is missing, the associated functional may have an unbounded P.S. sequence. Nevertheless, an approximation argument (previously developed in [17]) relying on Pohozaev’s identity permits us to overcome the difficulty.
In this paper, we will present a new approach to solving the Lions conjecture. In contrast to the works \[13, 18, 19,\text{ or } 2\], we do not make use of the fact that the first solution is a local minimum. The second solution is obtained as the limit (in $H^0_0(\Omega)$ and $C^1(\Omega)$) of a gradient flow whose starting point is properly chosen. From this starting point the flow is unlimited in time and the functional decreases only a finite quantity. This permits to construct along it a P.S. sequence which converges to a critical point distinct from the first solution. The flow lies completely out of the “window” defined by the sub- and supersolution which contains the first solution. The convergence of the P.S. sequence is insured by taking again the approximating procedure introduced in \[17\]. The tools developed to obtain the second solution then permit us to rederive the existence of the first solution and additional interesting properties on it. Related arguments are also used by Sun and Liu in \[28\], in which invariant sets of descending flow are studied and applied to problems of the form of (1).

We will mainly prove the following two theorems.

**Theorem 1.** Let us assume that $\Omega$ is convex, and that $f$ satisfies (H2) and the following conditions;

\begin{align*}
\text{(H4)} & \quad \lim_{t \to +\infty} f(t) t^{-1} = +\infty, \\
\text{(H5)} & \quad f(0) = 0 \text{ and } f'(0) = 1,
\end{align*}

and

\begin{align*}
\text{(H6)} & \quad f(t) > 0 \text{ for all } t > 0.
\end{align*}

Then there exists $\lambda^* > 0$ such that

\begin{enumerate}
  \item $+\infty > \lambda^* \geq \lambda_1$ and for $0 < \lambda < \lambda^*$, there exists at least one solution of (1)$_\lambda$;
  \item for $\lambda > \lambda^*$, there exists no solution of (1)$_\lambda$;
  \item if $\lambda^* > \lambda_1$, then for $\lambda_1 < \lambda \leq \lambda^*$ there exists a minimal solution $u_\lambda$ of (1)$_\lambda$, $u_\lambda$ is strictly increasing with respect to $\lambda_1 < \lambda \leq \lambda^*$ in $\Omega$, and $u_\lambda$ is continuous on the left from $(\lambda_1, \lambda^*)$ to $C^2(\Omega)$;
  \item if $\lambda^* > \lambda_1$, then for $\lambda_1 < \lambda < \lambda^*$ there exists a solution $u_\lambda$ of (1)$_\lambda$ distinct from $u_{\lambda_1}$ (therefore $u_\lambda < u_{\lambda_1}$ in $\Omega$), and for any $\lambda_1 < \lambda < \lambda^*$ and any solution $u_\gamma$ of (1)$_\gamma$, there exists a solution $u_\lambda$ of (1)$_\lambda$ such that $u_\lambda \leq u_\gamma$;
  \item $J_{\lambda_1}(u_{\lambda_1}) < 0$ and $J_{\lambda}(u_{\lambda})$ is strictly decreasing with respect to $\lambda_1 < \lambda \leq \lambda^*$.
\end{enumerate}

**Remark 1.** In Theorem 1, we can impose an additional condition on $f$ which implies $\lambda^* > \lambda_1$ (see \[21\]).
Theorem 2. Let us assume that $\Omega$ is convex and that $f$ satisfies (H1), (H2), and

(H7) there exists a $\beta > 0$ such that $f(\beta) = 0$ and $f(t) > 0$ for $t > \beta$.

Then there exists a solution $u$ of (1) satisfying $\max_{\Omega} u > \beta$. Moreover, the solution $u$ can be chosen such that $J(u) > 0$ if we assume in addition that $F(t) \leq \left((\lambda_1 - \alpha)/2\right) t^2$ for $0 \leq t \leq \beta$ for some $\alpha > 0$.

Remark 2. It should be pointed out that parts of the conclusions in these theorems are not new and can be found in [21] and [18]. However, the results in Theorems 1 and 2 contain more interesting properties on the solutions than those in [21] and [18], e.g., the properties concerning $u_2$. It should also be pointed out that the methods used here can also be used to deal with some other cases of the nonlinearity of $f$.

Remark 3. Just as in [17, 21], the assumption that $\Omega$ is convex in Theorem 1 and Theorem 2 can be replaced with either of the following two conditions.

(H8) $\partial(\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are closed, at every point of $\Gamma_1$, all sectional curvature of $\Gamma_1$ is bounded away from 0 by a positive constant $\kappa > 0$, and there exists $x_0 \in \mathbb{R}^N$ such that $(x - x_0, n(x)) \leq 0$ for all $x \in \Gamma_2$.

(H9) $f(t)^{t-(N+2)/(N-2)}$ is nonincreasing for $t \geq 0$ if $N \geq 3$ (if $N = 1, 2$, this condition is not necessary).

2. PRELIMINARIES

Let $X$ be a Banach space and $A: X \to X$ a Lipschitz continuous mapping. Consider the initial value problem in $X$

\[
\begin{aligned}
\frac{dx(t)}{dt} &= -x(t) + Ax(t), \\
x(0) &= x_0,
\end{aligned}
\]

for some fixed $x_0 \in X$. Let $x(t, x_0)$ denote the unique solution and $[0, \eta(x_0))$ the maximal interval of existence of $x(t, x_0)$.

Lemma 1. Assume that $D$ is a closed convex subset of $X$ and $A(D) \subseteq D$. If $x_0 \in D$, then

\[
\{x(t, x_0) \mid 0 \leq t < \eta(x_0)\} \subseteq D.
\]

(2)

This lemma is due to Sun [27]; we give the proof of it here only for reasons of convenience.
Proof of Lemma 1. Let
\[ t^* = \sup \{ t_1 \mid 0 \leq t_1 < \eta(x_0), x(t_1, x_0) \in D \text{ for all } t \in [0, t_1] \} . \]

If (2) were false, then \( t^* \) would satisfy \( 0 \leq t^* < \eta(x_0) \) and \( x(t^*, x_0) \in D \).

Consider the initial value problem
\[
\begin{aligned}
\frac{dx(t)}{dt} &= -x(t) + Ax(t), \\
x(0) &= x(t^*, x_0).
\end{aligned}
\]

(3)

For any \( x \in D \) and \( 0 < \lambda < 1 \), since \( A(D) \subset D \) and \( D \) is convex,
\[ x + \lambda(-x + Ax) = (1 - \lambda)x + \lambda Ax \in D, \]
therefore
\[
\lim_{\lambda \to 0+} \frac{1}{\lambda} d(x + \lambda(-x + Ax), D) = 0.
\]

Now using a theorem of Brezis [8] (see also [16, 23]), we assert that there exists \( \delta > 0 \) such that the unique solution \( x(t, x(t^*, x_0)) \) of (3) satisfies \( x(t, x(t^*, x_0)) \in D \) for \( 0 \leq t \leq \delta \), therefore \( x(t, x_0) \in D \) for all \( 0 \leq t \leq t^* + \delta \), which contradicts the definition of \( t^* \). The proof is finished.

The next lemma concerns a concrete mapping \( A = KG \) in which the following condition is assumed.

(H) \( f(t) \) is a Lipschitz continuous function from \( \mathbb{R}^1 \) to \( \mathbb{R}^1 \); there exist an increasing function \( M: \mathbb{R}_+ \to \mathbb{R}_+ \) and numbers \( C > 0 \) and \( \alpha \) with \( 1 < \alpha < (N + 2)/(N - 2) \) if \( N \geq 3 \) and \( \alpha > 1 \) if \( N = 1, 2 \), such that, for \( r \geq 0 \),
\[ M(r) \leq C(1 + r^{\alpha - 1}), \]
\[ |f(t_1) - f(t_2)| \leq M(r)|t_1 - t_2| \quad \text{for all} \quad t_1, t_2 \in [-r, r]. \]

Let \( G(u(x)) \triangleright f(u(x)), K = (-A + m)^{-1} \) denote the inverse operator of \(-A + m\) with 0-Dirichlet boundary condition, where \( m \) is a fixed positive number. Under condition (H), it is easy to see that \( A \triangleright KG: H^1_0(\Omega) \to H^1_0(\Omega) \) is Lipschitz continuous. Consider the initial value problem in \( H^1_0(\Omega) \),
\[
\begin{aligned}
\frac{du(t)}{dt} &= -u(t) + KG(t), \\
u(0) &= u_0.
\end{aligned}
\]

(4)
Lemma 2. Let \( u(t, u_0) \) be the unique solution of (4) with the maximal interval of existence \( [0, \eta(u_0)) \). We have the following conclusions:

(i) If \( u_0 \in C_0^1(\Omega) \), then \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \subset C_0^1(\Omega) \), and \( u(t, u_0) \) is continuous as a function of \( t \) from \( [0, \eta(u_0)) \) to \( C_0^1(\Omega) \).

(ii) If \( u_0 \), \( u^* \in C_0^1(\Omega) \), \( u^* = K \bar{u}^* \), and \( \| u(t, u_0) - u^* \|_{H_0^1(\Omega)} \to 0 \) as \( t \to \eta(u_0) \), then \( \| u(t, u_0) - u^* \|_{C_0^1(\Omega)} \to 0 \) as \( t \to \eta(u_0) \).

(iii) If \( u_0 \in C_0^{\mu}(|\Omega) \) for some \( \mu \in (0, 1) \) and \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \) is bounded in the \( H_0^1(\Omega) \) norm, then \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \subset C_0^{\mu}(|\Omega) \) and is bounded in the \( C_0^{\mu}(|\Omega) \) norm.

Lemma 2 is essentially due to Chang [13], but the manner of expression is somewhat different here and we give its proof also for reasons of convenience.

Proof of Lemma 2. We will only consider the case \( N \geq 3 \); the case \( N = 1, 2 \) can be handled similarly. Without loss of generality, the number \( \lambda \) in (H) may be assumed to satisfy \( 4/(N - 2) < \lambda < (N + 2)/(N - 2) \). Define \( q_i \) by

\[
q_0 = \frac{2N}{N - 2}, \quad \frac{1}{q_{i+1}} = \frac{\lambda}{q_i} = \frac{2}{N}, \quad i = 0, 1, 2, \ldots
\]

A direct computation shows that there exists a number \( n \geq 3 \) such that

\[
q_0 < q_1 < \cdots < q_{n-3} < \frac{N\lambda}{2} < q_{n-2}.
\]

Let

\[
q_i = q_i', \quad i = 0, 1, \ldots, n - 3,
\]

and choose \( q_{n-2} \) and \( q_{n-1} \) such that

\[
q_{n-3} < q_{n-2} < \frac{N\lambda}{2}, \quad \frac{\lambda}{q_{n-2}} = \frac{2}{N} < \frac{1}{2N\lambda}, \quad q_{n-1} = 2N\lambda.
\]

Let

\[
p_i = \frac{q_i}{\lambda}, \quad i = 0, 1, \ldots, n - 1,
\]

and define

\[
X_0 = L^{q_0}(\Omega), \quad X_{i+1} = W_0^{2, p_i}(\Omega), \quad Y_i = L^{q_i}(\Omega), \quad Z_i = L^{p_i}(\Omega), \quad i = 0, 1, \ldots, n - 1.
\]
Then we have the imbedding chains

\[ H^1_0(\Omega) \]

\[ \xymatrix{ X_n \ar[d] & X_{n-1} \ar[d] & X_{n-2} \ar[d] & \cdots & X_1 \ar[d] & X_0 \ar[d] \\ C^1(\Omega) \ar[d] & Z_{n-1} \ar[d] & Z_{n-2} \ar[d] & \cdots & Z_1 \ar[d] & Z_0 \ar[d] } \]

and

\[ Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0. \]

Moreover, we have the chains of bounded and continuous operators

\[ Z_i \overset{G}{\to} Y_i \overset{K}{\to} X_{i+1}, \quad i = 0, 1, 2, \ldots, n-1. \]

(i) Suppose that \( u_0 \in C^1_0(\bar{\Omega}) \). The solution \( u(t, u_0) \) of (4) satisfies

\[ u(t, u_0) = e^{-t}u_0 + \int_0^t e^{-t+s}KGu(s, u_0) \, ds, \quad 0 \leq t < \eta(u_0), \]

with respect to the \( H^1_0(\Omega) \) topology, that is, the integral in the right-hand side of (8) is taken in the \( H^1_0(\Omega) \) norm. For a Banach space \( X \), if \( KGu(t, u_0) \) is continuous with respect to \( t \) from \( [0, \eta(u_0)) \) to \( X \), by \( I_X(t, u_0) \) we denote the integral \( \int_0^t e^{-t+s}KGu(s, u_0) \, ds \) taken in the \( X \) norm. Since \( u(t, u_0) \) is continuous with respect to \( t \) from \( [0, \eta(u_0)) \) to \( H^1_0(\Omega) \), (5)-(7) imply that \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \subset Z_0 \); that \( u(t, u_0) \) and \( KGu(t, u_0) \) are continuous with respect to \( t \) from \( [0, \eta(u_0)) \) to \( Z_0 \); and that \( I_{H^1_0(\Omega)}(t, u_0) = I_{Z_0}(t, u_0) \), therefore (8) is true with respect to the \( Z_0 \) topology. Now (7) and the imbedding \( X_1 \to Z_1 \) imply that \( KGu(t, u_0) \) is continuous from \( [0, \eta(u_0)) \) to \( Z_1 \), hence the imbedding \( Z_1 \to Z_0 \) implies that \( I_{Z_0}(t, u_0) = I_{Z_1}(t, u_0) \), therefore, in view of the imbedding \( C^1_0(\bar{\Omega}) \to Z_1 \), (8) is true with respect to the \( Z_1 \) topology. \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \subset Z_1 \), and \( u(t, u_0) \) is continuous with respect to \( t \) from \( [0, \eta(u_0)) \) to \( Z_1 \). Repeating this discussion \( n \) times, we get that \( \{ u(t, u_0) | 0 \leq t < \eta(u_0) \} \subset C^1_0(\bar{\Omega}) \) and \( u(t, u_0) \) is continuous with respect to \( t \) from \( [0, \eta(u_0)) \) to \( C^1_0(\bar{\Omega}) \).

(ii) Since \( u^* = KGu^* \), we have

\[ u(t, u_0) - u^* = e^{-t}(u_0 - u^*) + \int_0^t e^{-t+s}(KGu(s, u_0) - KGu^*) \, ds, \quad 0 \leq t < \eta(u_0). \]
Without loss of generality, we assume that \( u_0 \neq u^* \). Since \( \|u(t, u_0) - u^*\|_{\mathcal{L}(\Omega)} \to 0 \) as \( t \to \eta(u_0) \) and \( \mathcal{A}: H^{1/2}_0(\Omega) \to H^{1/2}_0(\Omega) \) is Lipschitz continuous, it can be proved that \( \eta(u_0) = +\infty \). Indeed, there exists a constant \( C \) such that
\[
\frac{d}{dt}\|u(t, u_0) - u^*\|^2_{\mathcal{L}(\Omega)} \geq -C\|u(t, u_0) - u^*\|^2_{\mathcal{L}(\Omega)}, \quad 0 \leq t < \eta(u_0).
\]
Therefore,
\[
\|u(t, u_0) - u^*\|^2_{\mathcal{L}(\Omega)} \geq \|u_0 - u^*\|^2_{\mathcal{L}(\Omega)} e^{-Ct}, \quad 0 \leq t < \eta(u_0),
\]
which shows that \( \eta(u_0) = +\infty \). For any \( \varepsilon > 0 \), in view of the continuity of \( G: Z_0 \to Y_0 \), there exists a \( \delta > 0 \) such that \( \|Gu - Gu^*\|_{Y_0} \leq \varepsilon/(2 \|K\|_{L(\mathfrak{F}, x_1)}) \) for any \( u \) satisfying \( \|u - u^*\|_{x_0} \leq \delta \), where \( \|K\|_{L(\mathfrak{F}, x_1)} \) is the norm of the bounded linear operator \( K \) from \( Y_0 \) to \( X_1 \). Since \( \|u(t, u_0) - u^*\|_{\mathcal{L}(\Omega)} \to 0 \) as \( t \to \eta(u_0) \), in view of (5) there exists a \( T > 0 \) such that \( \|u(t, u_0) - u^*\|_{x_0} \leq \delta \) when \( t \geq T \), therefore \( \|Gu(t, u_0) - Gu^*\|_{Y_0} \leq \varepsilon/(2 \|K\|_{L(\mathfrak{F}, x_1)}) \) if \( t \geq T \).

Choose a number \( B_0 > 0 \) such that \( \|Gu(t, u_0) - Gu^*\|_{Y_0} \leq B_0 \) if \( 0 \leq t \leq T \). Such a \( B_0 \) exists because \( Gu(t, u_0) \) is continuous from \( [0, T] \) to \( Y_0 \). Choose a \( T_1 > T \) such that \( e^{-T_1} \|K\|_{L(\mathfrak{F}, x_1)} B_0 \leq \varepsilon/2 \). Then, if \( t \geq T_1 \),
\[
\left\| \int_0^t e^{-t+s} KGu(s, u_0) \, ds - \int_0^t e^{-t+s} KGu^* \, ds \right\|_{x_1} \leq \left\| \int_0^T e^{-t+s} K(Gu(s, u_0) - Gu^*) \right\|_{x_1} \, ds
\]
\[
\leq \left\| \int_0^T e^{-t+s} \|K\|_{L(\mathfrak{F}, x_1)} \|Gu(s, u_0) - Gu^*\|_{Y_0} \, ds
\]
\[
\leq \left\| \int_0^T e^{-t+s} \|K\|_{L(\mathfrak{F}, x_1)} \|Gu(s, u_0) - Gu^*\|_{Y_0} \, ds
\]
\[
+ \left\| \int_T^t e^{-t+s} \|K\|_{L(\mathfrak{F}, x_1)} \|Gu(s, u_0) - Gu^*\|_{Y_0} \, ds
\]
\[
\leq \|K\|_{L(\mathfrak{F}, x_1)} B_0 \left\| \int_0^T e^{-t+s} \, ds + \frac{e}{2} \right\| e^{-T_1} \, ds
\]
\[
\leq \|K\|_{L(\mathfrak{F}, x_1)} B_0 e^{-T_1} T + \frac{\varepsilon}{2}
\]
\[
\leq \varepsilon.
\]
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In view of the imbeddings $X_1 \rightarrow Z_1$ and $C_0^p(\Omega) \rightarrow Z_1$, we see that

$$\|u(t, u_0) - u^*\|_{Z_1} \rightarrow 0 \quad \text{as} \quad t \rightarrow \eta(u_0).$$

Repeating the above arguments $n$ times gives the result of (ii).

(iii) Obviously, $G: C_0^p(\Omega) \rightarrow C(\Omega)$ is bounded and continuous. The $L^p$ theory for elliptic equations and Sobolev imbedding theorems imply that $K: C(\Omega) \rightarrow C_0^p(\Omega)$ is a bounded linear operator. If $u_0 \in C_0^p(\Omega)$, using the result of (i) we see that $I_{c_3}\circ \eta(t, u_0) = I_{c_3}(t, u_0)$, therefore (8) is true with respect to the $C_0^p(\Omega)$ topology. \{u(t, u_0) \mid 0 \leq t < \eta(u_0)\} $\subset C_0^p(\Omega)$, and $u(t, u_0)$ is continuous from $[0, \eta(u_0))$ to $C_0^p(\Omega)$. When $\{u(t, u_0) \mid 0 \leq t < \eta(u_0)\}$ is bounded in $H_0^1(\Omega)$, in view of (5)–(8) and the fact that $G: C_0^p(\Omega) \rightarrow C(\Omega)$ and $K: C(\Omega) \rightarrow C_0^p(\Omega)$ are bounded operators, we get in turn that $\{u(t, u_0) \mid 0 \leq t < \eta(u_0)\}$ is bounded in $Z_1$, in $Z_2$, ..., in $C_0^p(\Omega)$, and in $C_0^p(\Omega)$.

3. PROOF OF THEOREM 1

We need to prove only (iii), (iv), and (v) since (i) and (ii) are well known results (see [17, 21, and 18]).

**Lemma 3.** If $\lambda^* > \lambda_1$, then for $\lambda_1 < \lambda < \lambda^*$ there exists a minimal solution $u_2$ of (1), and $u_2$ is strictly increasing with respect to $\lambda$. Let $u_2$ be any solution of (1), then $u_2$ is a strict supersolution of (1) by (H6); i.e.,

$$\begin{align*}
-\Delta u_2 > \lambda f(u_2), & \quad x \in \Omega, \\
|u_2| = 0, & \quad x \in \partial \Omega.
\end{align*}$$

Let $\phi(x)$ denote the eigenfunction of $-\Delta$ corresponding to $\lambda_1$, $\phi(x) > 0$ for all $x \in \Omega$ and $\max_{x \in \Omega} \phi = 1$. Take a $\delta_0 > 0$ such that, if $0 < \delta \leq \delta_0$, $f(t) > (\lambda_1/\lambda) t$ and $\partial_0 \phi(x) < u_2(x)$ for all $x \in \Omega$. For $0 < \delta \leq \delta_0$, $\delta \phi$ is a strict subsolution of (1), i.e.,

$$\begin{align*}
-\Delta (\delta \phi) < \lambda f(\delta \phi), & \quad x \in \Omega, \\
\delta \phi = 0, & \quad x \in \partial \Omega.
\end{align*}$$

It follows that (1) has a solution $u_4$ satisfying $\delta \phi < u_4 < u_2 \in \Omega$. We can choose $u_4$ to be the minimal solution of (1) in the order interval $(\delta \phi, u_2)$, but we do not know if $u_4$ is the minimal of all the solutions of (1). Next we prove that $u_4$ is the minimal solution of (1) if $\delta > 0$ is sufficiently small.
Let $M = \max_{\Omega} u$, and let $M_1 > M$ be such that $\inf_{t \leq M} f(t) > \sup_{0 < t < M} f(t)$, then choose a fixed number $m > 0$ such that $\lambda f(t) + mt$ is strictly increasing on the interval $[0, M_1]$. Such an $m$ exists because $f(t)$ satisfies the locally Lipschitz condition. Now, for $0 < \delta \leq \delta_0$, the iterative sequence $\{ u_n \}_{n=0}^{\infty}$ defined by

$$ u_0 = \delta \phi, \quad \begin{cases} -Au_n + mu_n = \lambda f(u_{n-1}) + mu_{n-1}, & x \in \Omega, \\ u_n = 0, & x \in \partial \Omega, \end{cases} \tag{9} $$

is increasing in $\Omega$ and has an upper bound $u^*_\Omega$; i.e.,

$$ u_0 < u_1 < \cdots < u_n < \cdots < u^*_\Omega. $$

This sequence has a limit, denoted by $u^*_\Omega$, that is,

$$ u^*_\Omega(x) = \lim_{n \to \infty} u_n(x) \quad \text{for} \quad x \in \Omega. $$

By the $L^p$ theory and the Schauder theory of elliptic equations, $\{ u_n \}_{n=0}^{\infty}$ is bounded in $C^{2,\alpha}_0(\bar{\Omega})$ for any $0 < \alpha < 1$. Then a compact argument (Arzela–Ascoli) shows that $u^*_\Omega \in C^{2,\alpha}_0(\bar{\Omega})$, that

$$ u^*_\Omega = \lim_{\delta \to 0^+} u_\delta \quad \text{in} \quad C^{2,\alpha}_0(\bar{\Omega}), $$

and that $u^*_\Omega$ is a solution of (1). It is easy to see that $u^*_\Omega$ is increasing with respect to $\delta$ and $u^*_\Omega < u^*_\Omega$, and therefore $\lim_{\delta \to 0^+} u^*_\Omega(x)$ exists for all $x \in \Omega$ and we denote this limit by $u^*$, i.e.,

$$ u^*(x) = \lim_{\delta \to 0^+} u^*_\delta(x) \quad \text{for} \quad x \in \Omega. $$

Repeating the discussion just made above, we see that $u^*_\Omega \in C^{2,\alpha}_0(\bar{\Omega})$ and

$$ u^*_\Omega \in C^{2,\alpha}_0(\bar{\Omega}), $$

therefore $u^*_\Omega$ satisfies

$$ \begin{cases} -Au^*_\Omega = \lambda f(u^*_\Omega), & x \in \Omega, \\ u^*_\Omega = 0, & x \in \partial \Omega. \end{cases} $$

We claim that $u^*_\Omega > 0$ in $\Omega$. If this were not true, then $u^*_\Omega \equiv 0$ and $\lim_{\delta \to 0^+} u^*_\delta = 0$ in $C^{2,\alpha}_0(\bar{\Omega})$, and there would be a $0 < \delta_1 \leq \delta_0$ such that

$$ 0 < u^*_\delta \leq \delta_0, \quad x \in \Omega. $$
Substituting $u^\phi_1$ for $u$ in (1)_1, multiplying (1)_1 by $\phi$, and integrating, we obtain

\[ \lambda_1 \int_{\Omega} u^\phi_1 \phi = \int_{\Omega} f(u^\phi_1) \phi > \lambda_1 \int_{\Omega} u^\phi_1 \phi, \]

which is a contradiction. Therefore $u_2$ is a solution of (1)_1. For any solution $u$ of (1)_1, there is a $0 < \delta \leq \delta_0$ such that $\delta \phi < u$ in $\Omega$. Then the sequence defined by (9) satisfies $u_\omega < u$ in $\Omega$, hence $u^\phi_2 < u$ in $\Omega$. We then have $u_2 < u^\phi_2 < u$ in $\Omega$, which means that $u_2$ is the minimal solution of (1)_1. If we choose $u_2$ to be $u_2$, the above discussion shows that $u_2$ is strictly increasing with respect to $\lambda$ in $\Omega$.

For any $\lambda_1 < \lambda_0 < \lambda_*$, the limit $\lim_{\lambda \to \lambda_0} u_\lambda(x)$ exists for any $x \in \Omega$. Denoting the limit by $u^\phi_0(x)$, by the $L^r$ theory and Schauder theory we get that

\[ u^\phi_0 \in C^2_0(\Omega) \] is a solution of (1)_0 and that

\[ \lim_{\lambda \to \lambda_0} u_\lambda = u^\phi_0 \text{ in } C^2_0(\Omega). \]

It is easy to see that $u^\phi_0 \leq u_\lambda$. Then the fact that $u_\lambda$ is the minimal solution of (1)_0 shows that $u^\phi_0 = u_\lambda$. Hence

\[ \lim_{\lambda \to \lambda_0} u_\lambda = u^\phi_0 \text{ in } C^2_0(\Omega). \]

The proof is complete.

Now we prove the existence of a second solution of (1)_1 in the case of $\lambda_1 < \lambda < \lambda_*$. In order to do this we use a variational argument. The inner product and the norm in the Hilbert space $H^1_0(\Omega)$ are taken to be

\[ (u, v) = \int_{\Omega} (Vu \cdot \nabla v + muw) \, dx, \quad u, v \in H^1_0(\Omega), \]

\[ \|u\|_{H^1_0(\Omega)} = \left( \int_{\Omega} (|\nabla u|^2 + mu^2) \, dx \right)^{\frac{1}{2}}, \quad u \in H^1_0(\Omega), \]

where $m$ is the number as in the proof of Lemma 3. Define

\[ \tilde{f}(t) = \begin{cases} f(0), & t < 0, \\ f(t), & t \geq 0, \end{cases} \]

\[ F(t) = \int_0^t \tilde{f}(s) \, ds, \quad t \in \mathbb{R}^1, \]

\[ J_\lambda(u) = \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + \lambda F(u) \right] \, dx, \quad u \in H^1_0(\Omega). \]
Then $J_\lambda$ is a $C^1$ functional defined on $H_0^1(\Omega)$ and $u$ is a nontrivial critical point of $J_\lambda$ if and only if $u$ is a solution of (1)*. Since $J_\lambda$ is not necessary to satisfy the P.S. condition, it is impossible for us to use directly the critical point theorems of linking type. In order to overcome this difficulty, just as in [17] we define a sequence of modified functionals. Note that condition (H4) implies that there exists a constant $t^* > 0$ such that

$$f(t) \frac{t-1}{t} \geq 2, \quad \forall t \geq t^*. \quad (10)$$

Select a sequence $\{s_n\}$ such that $s_1 > \max\{M_1, t^*\}$, $s_n < s_{n+1}$, and $s_n \to \infty$ as $n \to \infty$, where $M_1$ is as in the proof of Lemma 3. Choose a fixed number $\gamma$ such that $1 < \gamma < (N+2)/(N-2)$ and $\frac{1}{2} + (2\gamma)^{-\gamma(\gamma-1)}(1-\gamma) > 0$. Now define

$$f_n(t) = \begin{cases} f(0), & t < 0, \\ f(t), & 0 \leq t \leq s_n, \\ f(s_n) + s_n^{-\gamma}f(s_n)(t-s_n)^\gamma, & t > s_n, \end{cases}$$

$$F_n(t) = \int_0^t f_n(s) \, ds, \quad t \in \mathbb{R}^1,$$

$$J_{\lambda_n}(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \lambda F_n(u) \right) \, dx, \quad u \in H_0^1(\Omega).$$

**Lemma 4.** The functions $f_n$ satisfy

$$f_n(t) \frac{t-1}{t} \geq 1, \quad \forall t \geq t^*, \quad n = 1, 2, \ldots.$$  

Moreover, $f_n(t)$ satisfy (H2) uniformly in $n$.

**Proof.** For $x \geq 1$, let

$$h(x) = (x-1)^\gamma - \frac{1}{2}x + 1,$$

then

$$h'(x) = \gamma(x-1)^{\gamma-1} - \frac{1}{2}.$$

It is easy to see that $h(x)$ attains its minimum in $[1, \infty)$ at $(2\gamma)^{-\gamma(\gamma-1)} + 1$ and the minimum is $h_{\min} = \frac{1}{2} + (2\gamma)^{-\gamma(\gamma-1)}(1-\gamma)$. Hence $h(x) > 0$ for all $x \geq 1$ by the definition of $\gamma$. It follows that, for any $n$, if $t \geq s_n$,

$$f_n(t) \frac{t-1}{t} \geq 2 \left[ \frac{s_n}{t} + \frac{t}{s_n} \left( \frac{t}{s_n} - 1 \right)^\gamma \right] \geq 1.$$
For any \( \varepsilon > 0 \), choose a \( t^{**} \) such that \( f(t) t^{-1} \leq \varepsilon / 2 \) when \( t \geq t^{**} \). Then, for any \( n \), if \( t \geq s_n \),

\[
f_a(t) t^{-1} \leq \frac{\varepsilon}{2}\left(\frac{s_n}{t} + \left(1 - \frac{s_n}{t}\right)^{-\gamma}\right) \leq \varepsilon.
\]

The proof is complete.

It is well known that \( J_{\lambda,n}(u) \in C^{2-\theta} H^1_0(\Omega), R^1 \) satisfies the P.S. condition and

\[
dJ_{\lambda,n}(u) = u - KG_{\lambda,n}(u), \quad u \in H^1_0(\Omega),
\]

where \( K \triangleq (-A + m)^{-1} \) and \( G_{\lambda,n}(u) \triangleq f_a(u) + mu \), and \( m \) is the number specified in the proof of Lemma 3. It is known that \( K \) is a bounded linear operator from \( L^{2N(N+2)}(\Omega) \) to \( H^1_0(\Omega) \) as well as from \( C(\overline{\Omega}) \) to \( C^1(\overline{\Omega}) \), while \( G_{\lambda,n} \) is a bounded and continuous operator from \( H^1_0(\Omega) \) to \( L^{2N(N+2)}(\Omega) \) as well as from \( C^1(\overline{\Omega}) \) to \( C^1(\overline{\Omega}) \). Denote \( A_{\lambda,n} u = KG_{\lambda,n} u \), then \( A_{\lambda,n} : H^1_0(\Omega) \to H^1_0(\Omega) \) (as well as \( A_{\lambda,n} : C^1(\overline{\Omega}) \to C^1(\overline{\Omega}) \)) satisfies the Lipschitz condition on any bounded subset of \( H^1_0(\Omega) \) (respectively, of \( C^1(\overline{\Omega}) \)) uniformly. Indeed, it can be shown that there exist constants \( C_1, C_2 > 0 \) such that

\[
\|A_{\lambda,n} u - A_{\lambda,n} v\|_{H^1_0(\Omega)} \leq C_1 \|K\|_{L^{2N(N+2)}(\Omega), H^1_0(\Omega)} \|u\|_{H^1_0(\Omega)}^{-1} + \|v\|_{H^1_0(\Omega)}^{-1} + 1 \|u - v\|_{H^1_0(\Omega)},
\]

for \( u, v \in H^1_0(\Omega) \), and

\[
\|A_{\lambda,n} u - A_{\lambda,n} v\|_{C^1(\overline{\Omega})} \leq C_2 \|K\|_{L^{2N(N+2)}(\Omega), C^1(\overline{\Omega})} \|u\|_{C^1(\overline{\Omega})}^{-1} + \|v\|_{C^1(\overline{\Omega})}^{-1} + 1 \|u - v\|_{C^1(\overline{\Omega})},
\]

for \( u, v \in C^1(\overline{\Omega}) \).

For a fixed \( u_0 \in C^1(\overline{\Omega}) \), consider the initial value problem

\[
\begin{cases}
\frac{du(t)}{dt} = -u(t) + A_{\lambda,n} u(t), \\
u(0) = u_0,
\end{cases}
\tag{11}
\]

in both \( H^1_0(\Omega) \) and \( C^1(\overline{\Omega}) \).

**Lemma 5.** Let \( u(t, u_0) \) be the unique solution of \( (11) \) in \( H^1_0(\Omega) \) with maximal interval of existence \([0, \eta(u_0)]\) and let \( \bar{u}(t, u_0) \) be the unique solution of \( (11) \) in \( C^1(\overline{\Omega}) \) with maximal interval of existence \([0, \bar{\eta}(u_0)]\), then \( \eta(u_0) = \bar{\eta}(u_0) \) and \( u(t, u_0) = \bar{u}(t, u_0) \) for \( 0 \leq t < \eta(u_0) \).
Proof. The result is a direct consequence of the expression

\[ u(t, u_0) = e^{-t}u_0 + \int_0^t e^{-t+s} A_{\lambda, u} u(s, u_0) \, ds, \quad 0 \leq t < \eta(u_0), \]  

(12)

and Lemma 2.

Let \( \lambda_1 < \lambda < \lambda < \lambda \); let \( u_2 \) be any solution of (1) (the existence of solutions of (1) is proved in Lemma 13), and let \( \delta_0 \) be as in the proof of Lemma 3. Fix a \( \delta: 0 < \delta < \delta_0 \) such that \( \delta \phi < u_2 \) in \( \Omega \). Denote \( D = \{ u \in C^1_0(\Omega) \mid \delta \phi(x) \leq u(x) \leq u_2(x) \} \) for \( x \in \Omega \), then the interior part of \( D \) in \( C^1_0(\Omega) \) is \( D' = \{ u \in C^1_0(\Omega) \mid \delta \phi(x) < u(x) < u_2(x) \} \) for \( x \in \Omega \); \( \partial/\partial n)(u - u_2(x)) > 0 \) and \( \partial/\partial n)(\delta \phi - u)(x) > 0 \) for \( x \in \partial \Omega \).

**Lemma 6.** If \( u_0 \notin D \), then

\[ \{ u(t, u_0) \mid 0 < t < \eta(u_0) \} \subset D^c. \]  

(13)

**Proof.** First, we claim that \( A\lambda, u(D) \subset D^c \). In fact, for any \( u \in D \), set \( A\lambda, u = v \). Since \( f(t) + mt \) is strictly increasing in \([0, M]\), we have

\[
\begin{aligned}
&-A(u_2 - v) + m(u_2 - v) > 0, \quad x \in \Omega, \\
&u_2 - v = 0, \quad x \in \partial \Omega, \\
&-A(v - \delta \phi) + m(v - \delta \phi) > 0, \quad x \in \Omega, \\
&v - \delta \phi = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Then the Hopf strong maximum principle shows that

\[ \delta \phi(x) < v(x) < u_2(x) \quad \text{for} \quad x \in \Omega \]

and

\[
\frac{\partial}{\partial n}(v - u_2)(x) > 0 \quad \text{and} \quad \frac{\partial}{\partial n}(\delta \phi - v)(x) > 0 \quad \text{for} \quad x \in \partial \Omega;
\]

this means \( v \in D^c \). Hence \( A\lambda, u(D) \subset D^c \).

For \( u_0 \in D \), Lemma 1 asserts that

\[ \{ u(t, u_0) \mid 0 < t < \eta(u_0) \} \subset D. \]  

(14)

Let \( t \in (0, \eta(u_0)) \) be fixed and denote \( F_t = \{ A\lambda, u(s, u_0) \mid 0 \leq s \leq t \} \). We can use Lemma 2 to deduce that \( F_t \) is a compact set in \( C^1_0(\Omega) \). It follows from
A_{\lambda,n}(D) \subset D^\circ$ and (14) that $F_\lambda \subset D^\circ$, and hence $\overline{\mathcal{K}}F_\lambda \subset D^\circ$ since $D^\circ$ is convex, where $\overline{\mathcal{K}}F_\lambda$ is the closed convex hull of $F_\lambda$ in $C^1_0(\overline{\Omega})$. In the $C^1_0(\overline{\Omega})$ topology,

$$\frac{1}{e^{t-1}} \int_0^t e^s A_{\lambda,n} u(s, u_0) \, ds$$

$$= \frac{1}{e^{t-1}} \int_0^t A_{\lambda,n} u(t, u_0) \, ds$$

$$= \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^m A_{\lambda,n} u \left( \ln \left( 1 + \frac{i}{m} (e^t - 1) \right), u_0 \right),$$

therefore,

$$\frac{1}{e^{t-1}} \int_0^t e^s A_{\lambda,n} u(s, u_0) \, ds \in \overline{\mathcal{K}}F_\lambda \subset D^\circ.$$

It follows from (12) that

$$u(t, u_0) = e^{-t} u_0 + (1 - e^{-t}) \frac{1}{e^{t-1}} \int_0^t e^s A_{\lambda,n} u(s, u_0) \, ds \in D^\circ,$$

since $D^\circ$ is convex and $t > 0$. Hence (13) is valid.

According to Lemma 4, there exists a positive constant $C_3$ independent of $n$ such that

$$F_n(t) \geq \frac{1}{2} t^2 - C_3, \quad \forall t \geq 0.$$

It then follows that, for $t \geq 0$,

$$J_{\lambda,n}(t\phi) = \int_D \left[ \frac{1}{2} t^2 |\nabla \phi|^2 - \lambda F_n(t\phi) \right] \, dx$$

$$\leq \frac{1}{2} \lambda t^2 \int_D \phi^2 \, dx - \frac{1}{2} \lambda t^2 \int_D \phi^2 \, dx + \lambda C_3 \text{mes } \Omega.$$

Therefore, since $J_{\lambda,n}(u) = J_\lambda(u)$ is bounded from below on $D$, there exists a positive constant $T$ independent of $n$ such that

$$J_{\lambda,n}(T\phi) < \inf_{u \in D} J_{\lambda,n}(u) = \inf_{u \in D} J_\lambda(u). \quad (15)$$

For $\tau \in [0, T]$, let $v_\tau = \tau \phi$. Define $\tau^* = \sup \{ \tau_1 : \delta < \tau_1 < T \}$ and for any $\tau \in [\delta, \tau_1]$ there exists $\tau_\tau > 0$ such that $u(t, v_\tau) \in D^\circ$. 

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Lemma 7. We have $\delta < \tau^* < T$ and
\[ \{ u(t, v_r) \mid 0 \leq t < \eta(v_r) \} \cap D^\circ = \emptyset. \] (16)
\[ \inf_{u \in D} J_{\lambda_n}(u) \leq J_{\lambda_n}(u(t, v_r)) \leq \sup_{\delta < \tau < T} J_{\lambda_n}(v_r), \quad \forall 0 \leq t < \eta(v_r). \] (17)

Proof. If $\tau \geq \delta$ and $\tau$ is sufficiently close to $\delta$, $v_r \in D$. It can be deduced from Lemma 6 that
\[ \{ u(t, v_r) \mid 0 < t < \eta(v_r) \} \subset D^\circ. \]
If $\tau < T$ and $\tau$ is sufficiently close to $T$, we have
\[ \{ u(t, v_r) \mid 0 < t < \eta(v_r) \} \cap D = \emptyset \]
since $J_{\lambda_n}(u(t, u_0))$ is decreasing in $[0, \eta(u_0))$ and since $T$ satisfies (15). It follows that $\delta < \tau^* < T$. According to Lemma 5, we get (16) by the theory of ordinary differential equations. The second inequality in (17) is obvious since $J_{\lambda_n}(u(t, v_r))$ is decreasing in $[0, \eta(v_r))$ and $\delta < \tau^* < T$, while the reason for the first one is as follows. For any $0 \leq t < \eta(v_r)$ and any $\varepsilon > 0$, a number $\tau$ with $\delta < \tau < \tau^*$ exists such that
\[ J_{\lambda_n}(u(t, v_r)) \geq J_{\lambda_n}(u(t, v_r)) - \varepsilon. \]
Choose $t_1 > t$ such that $u(t_1, v_r) \in D^\circ$, then
\[ J_{\lambda_n}(u(t, v_r)) \geq J_{\lambda_n}(u(t_1, v_r)) - \varepsilon \]
\[ \geq \inf_{u \in D} J_{\lambda_n}(u) - \varepsilon \]
\[ = \inf_{u \in D} J_{\lambda}(u) - \varepsilon, \]
and we get the first inequality in (17) by letting $\varepsilon \to 0$. \hfill \Box

Lemma 8. $\eta(v_r) = +\infty$ and there are two constants $C_1$ and $C_2$ independent of $n$ such that
\[ C_1 \leq J_{\lambda_n}(u(t, v_r)) \leq C_2, \quad \forall 0 \leq t < +\infty. \] (18)

Proof. Since $T$ is independent of $n$, (17) implies that there are two constants $C_1$ and $C_2$ independent of $n$ such that
\[ C_1 \leq J_{\lambda_n}(u(t, v_r)) \leq C_2, \quad \forall 0 \leq t < \eta(v_r). \] (19)
This inequality can be used to assert that \( \eta(\nu,\nu^*) = +\infty \). In fact, if \( \eta(\nu,\nu^*) < +\infty \), we would have, for any \( 0 \leq t_1 < t_2 < \eta(\nu,\nu^*) \),

\[
\| u(t_2, \nu, \nu^*) - u(t_1, \nu, \nu^*) \|_{H_0^1(\Omega)} \\
\leq \int_{t_1}^{t_2} \| u'(t, \nu, \nu^*) \|_{H_0^1(\Omega)} \, dt \\
\leq \left( \int_{t_1}^{t_2} \| u'(t, \nu, \nu^*) \|_{H_0^1(\Omega)} \right)^{1/2} (t_2 - t_1)^{1/2} \\
\leq \left( - \int_{t_1}^{t_2} \frac{d}{dt} J_{\nu, \eta}(u(t, \nu, \nu^*)) \, dt \right)^{1/2} (t_2 - t_1)^{1/2},
\]

which, together with (19), induces

\[
\| u(t_2, \nu, \nu^*) - u(t_1, \nu, \nu^*) \|_{H_0^1(\Omega)} \leq (C_2 - C_1)^{1/2} (t_2 - t_1)^{1/2}.
\]

Then there exists \( u^* \in H_0^1(\Omega) \) such that

\[
\lim_{\nu \to \eta(\nu, \nu^*)} \| u(t, \nu, \nu^*) - u^* \|_{H_0^1(\Omega)} = 0,
\]

and \( u(t, \nu, \nu^*) \) can be extended to the interval \([0, \eta(\nu, \nu^*) + \eta(u^*)]\) for some \( \eta(u^*) > 0 \), which contradicts the maximality of the existence interval \([0, \eta(\nu, \nu^*)]\) of \( u(t, \nu, \nu^*) \). Hence \( \eta(\nu, \nu^*) = +\infty \).

**Lemma 9.** The boundary value problem

\[
\begin{cases}
- \Delta u = \lambda f_n(u), & x \in \Omega, \\
0, & x \in \partial \Omega,
\end{cases}
\]

has a solution \( u_{\lambda, n} \) distinct from \( u_\lambda \). Moreover, there exists an increasing sequence \( \{t_m\}_{m=1}^\infty \) with \( t_m \to \infty \) such that

\[
\lim_{m \to \infty} \| u(t_m, \nu, \nu^*) - u_{\lambda, n} \|_{H_0^1(\Omega)} = 0,
\]

\[
\lim_{m \to \infty} \| u(t_m, \nu, \nu^*) - u_{\lambda, n} \|_{C_0^1(\Omega)} = 0.
\]

**Proof.** In view of Lemma 8, there exists an increasing sequence \( \{t_m\}_{m=1}^\infty \) with \( t_m \to \infty \) such that

\[
\frac{d}{dt} J_{\nu, \eta}(u(t, \nu, \nu^*)) \bigg|_{t=t_m} = - \| dJ_{\nu, \eta}(u(t_m, \nu, \nu^*)) \|_{H_0^1(\Omega)}^2 \to 0, \quad \text{as} \quad m \to \infty.
\]
Since $J_{i,n}$ satisfies the P.S. condition, there is a subsequence of
\{u(t_m,v_\ast)\}_{m=1}^\infty \textit{ which has a limit in the } H^1_0(\Omega) \textit{ topology. Without loss of}
generality we assume that \{u(t_m,v_\ast)\}_{m=1}^\infty \textit{ itself has a limit } u_{i,n} \textit{ in } H^1_0(\Omega),
i.e., \textit{ } u_{i,n} \textit{ satisfies (22). Then } u_{i,n} \textit{ is both a critical point of } J_{i,n} \textit{ and a solution of (21)}.

Now we assert that \{u(t,v_\ast) | 0 \leq t < + \infty \} \textit{ is bounded in } H^1_0(\Omega). \textit{ By the P.S. condition, there exist positive constants } R \textit{ and } \mu \textit{ such that}
\begin{align*}
\|dJ_{i,n}(u(t,v_\ast))\|_{H^1_0(\Omega)} \geq \mu, \\
\|u\|_{H^1_0(\Omega)} \geq R.
\end{align*}
\textit{ If } \{u(t,v_\ast) | t \leq t < + \infty \} \subset \{u \|u-u_{i,n}\|_{H^1_0(\Omega)} \leq R\} \textit{ for some } R > 0, \textit{ the assertion is true. If such a } R > 0 \textit{ does not exist, there is a sequence of intervals } \{[S_i,T_i]\}_{i=1}^\infty \textit{, which are mutually disjoint, such that}
\begin{align*}
\|u(S_i,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} &= R, \\
\|u(T_i,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} &= R, \\
&i = 1, 2, 3, \ldots,
\end{align*}
and
\begin{align*}
\|u(t,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} &> R \textit{ if and only if } t \in (S_i,T_i) \textit{ for some } i.
\end{align*}

From Lemma 8, we have
\begin{align*}
C_2 - C_1 &\geq - \int_{[S_i,T_i]} \frac{d}{dt} J_{i,n}(u(t,v_\ast)) \, dt \\
&\geq \int_{[S_i,T_i]} \|dJ_{i,n}(u(t,v_\ast))\|^2_{H^1_0(\Omega)} dt \\
&\geq \mu^2 \sum_{i=1}^\infty (T_i-S_i).
\end{align*}
Therefore, for } t \in (S_i,T_i) \textit{, we have by using (20),}
\begin{align*}
\|u(t,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} &\leq \|u(S_i,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} + \|u(T_i,v_\ast) - u_{i,n}\|_{H^1_0(\Omega)} \\
&\leq (C_2 - C_1)^{1/2} (T_i - S_i)^{1/2} + R \\
&\leq \mu^{-1} (C_2 - C_1) + R.
\end{align*}
Hence } \{u(t,v_\ast) | 0 \leq t < + \infty \} \textit{ is bounded in } H^1_0(\Omega). \textit{ It follows that}
\{u(t,v_\ast) | 0 \leq t < + \infty \} \textit{ is bounded in } C^\alpha_0(\Omega) \textit{ for some } \alpha \in (0,1) \textit{ by the result of Lemma 2(iii) since } v_\ast \in C^\alpha_0(\Omega). \textit{ In view of (22), a compact}
argument is used to give (23).

Equations (16) and (23) imply that } u_{i,n} \notin D \textit{ and therefore that } u_{i,n} \notin D \textit{ since } u_{i,n} = A_{i,n}u_{i,n} \textit{ and } A_{i,n}(D) \subset D. \textit{ Hence } u_{i,n} \notin u_2. \]
Remark 4. It is clear that $u_{\lambda_n} \neq 0$ since the descending flow is invariant on the set $u \geq \delta \phi$.

In order to get a second solution of (1) in the case of $\lambda_1 < \lambda^*$, we should prove that $u_{\lambda_n} > u_{\lambda}$ and $\|u_{\lambda_n}\|_{C(\Omega)} < C$ for some constant $C$ independent of $n$, which is done with the following two lemmas. Indeed, $u_{\lambda_n} > u_{\lambda}$ is obvious since $u_{\lambda_n} \neq u_{\lambda}$ and since $u_{\lambda}$ is minimal among all positive solutions of (21). Nevertheless we will still prove it in Lemma 10 by a new argument which will be used in the proof of Lemma 12.

**Lemma 10.** For any $n$, $u_{\lambda_n} > u_{\lambda}$ in $\Omega$.

**Proof.** Note that $\{u(t, v_{\lambda}) | 0 < t < \eta(v_{\lambda})\} \subset D'$ by Lemma 6. Hence, for $0 \leq t < \eta(v_{\lambda})$,

$$\inf_{u \in D} J_{\lambda}(u) \leq J_{\lambda}(u(t, v_{\lambda})) = J_{\lambda}(u(t, v_{\lambda})) \leq J_{\lambda}(v_{\lambda}).$$

Then by an argument similar to the proof of Lemma 8 we have $\eta(v_{\lambda}) = +\infty$. Therefore, the proof of Lemma 9 implies that there exist an increasing sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \to +\infty$ and a solution $\tilde{u}_{\lambda_n}$ of (21) such that

$$\lim_{m \to +\infty} \|u(t_m, v_{\lambda}) - \tilde{u}_{\lambda_n} \|_{C^1(\Omega)} = 0.$$ (24)

It follows that $\tilde{u}_{\lambda_n} \in D$ and $\tilde{u}_{\lambda_n}$ is a solution of (1) in $\Omega$. Now we assert that, for $0 < t < +\infty$,

$$u(t, v_{\lambda}) < u_{\lambda} \text{ in } \Omega; \quad \frac{\partial}{\partial n} u(t, v_{\lambda}) > \frac{\partial}{\partial n} u_{\lambda} \text{ on } \partial \Omega. \quad \text{(25)}$$

Let

$$t^* = \sup \left\{ t_1 | u(t, v_{\lambda}) < u_{\lambda} \text{ in } \Omega \text{ and } \frac{\partial}{\partial n} u(t, v_{\lambda}) > \frac{\partial}{\partial n} u_{\lambda} \text{ on } \partial \Omega \right\}$$

for any $0 \leq t \leq t_1$.

then $t^* > 0$ since $v_{\lambda} < u_{\lambda}$ in $\Omega$, $(\partial/\partial n) v_{\lambda} > (\partial/\partial n) u_{\lambda}$ on $\partial \Omega$, and $u(t, v_{\lambda})$ is continuous in the $C^1(\partial \Omega)$ topology. In order to prove (25), it suffices to show that $t^* = +\infty$. Indeed, if $t^* < +\infty$, then either

$$u(t^*, v_{\lambda})(x_0) = u_{\lambda}(x_0) \quad \text{for some } x_0 \in \Omega \quad \text{(26)}$$
or
\[
\frac{\partial}{\partial n} u(t^*, v_d)(x_0) = \frac{\partial}{\partial n} y_j(x_0)
\]
for some \( x_0 \in \partial \Omega \). \( 27 \)

Since \( \max_{\partial \Omega} u_2 \leq M \) and \( \lambda f_d(t) + mt \) is strictly increasing on \([0, M]\), we see that
\[
G_{\lambda, u}^{\ast}(s, v_d) < G_{\lambda, u}^{\ast}(x_0) \quad \text{for some } x_0 \neq 0.
\]
for \( 0 < s < t^* \). Now the Hopf strong maximum principle shows that, for \( 0 < s < t^* \),
\[
A_{\lambda, u}^{\ast}(s, v_d) < A_{\lambda, u}^{\ast}(x_0) \quad \text{in } \Omega
\]
and
\[
\frac{\partial}{\partial n} A_{\lambda, u}^{\ast}(s, v_d) > \frac{\partial}{\partial n} A_{\lambda, u}^{\ast}(x_0) \quad \text{on } \partial \Omega.
\]
In the \( C^{1/2} \) topology,
\[
u(t^*, v_d) - u_2 = e^{-t^*}(v_d - u_2) + \int_0^{t^*} e^{-t^* + \tau} (A_{\lambda, u}^{\ast}(s, v_d) - A_{\lambda, u}^{\ast}(x_0)) \, ds.
\]
Therefore, we have
\[
u(t^*, v_d) < u_2 \quad \text{in } \Omega \quad \text{and } \quad \frac{\partial}{\partial n} \nu(t^*, v_d) > \frac{\partial}{\partial n} u_2 \quad \text{on } \partial \Omega,
\]
which contradicts either (26) or (27). Hence \( t^* = +\infty \). By (24) and (25) we get that \( \tilde{u}_{\lambda, u} \leq u_2 \). Then \( \tilde{u}_{\lambda, u} = u_2 \) since \( \tilde{u}_{\lambda, u} \) is a solution of \( (1)_2 \) and \( u_2 \) is the minimal solution of \( (1)_2 \). Now, (24) can be rewritten as
\[
\lim_{m \to +\infty} \|u(t^m, v_d) - u_2\|_{H^{1/2}(\Omega)} = 0.
\]
(28)

For \( 0 < t_1 < t_2 < +\infty \), since \( \{u(t, v_d) \mid 0 < t < \eta(v_d)\} \subset D^c \), we have
\[
u(t_2 - t_1, v_d) > v_d \quad \text{in } \Omega \quad \text{and } \quad \frac{\partial}{\partial n} \nu(t_2 - t_1, v_d) < \frac{\partial}{\partial n} v_d \quad \text{on } \partial \Omega.
\]
The argument for proving (25) shows that, for \( 0 < t < +\infty \),
\[
u(t, u(t_2 - t_1, v_d)) > u(t, v_d) \quad \text{in } \Omega;
\]
\[
\frac{\partial}{\partial n} \nu(t, u(t_2 - t_1, v_d)) < \frac{\partial}{\partial n} u(t, v_d) \quad \text{on } \partial \Omega.
\]
Choose $t = t_1$; then we arrive at

$$u(t_2, v_\delta) > u(t_1, v_\delta) \quad \text{in } \Omega;$$

(29)

$$\frac{\partial}{\partial n} u(t_2, v_\delta) < \frac{\partial}{\partial n} u(t_1, v_\delta) \quad \text{on } \partial \Omega.$$

Equations (25), (28), and (29) imply that

$$\lim_{t \to +\infty} u(t, v_\delta) = u_2, \quad \text{a.e. in } \Omega. \quad (30)$$

Using the argument for proving (25), we also have, for $0 < t < +\infty$,

$$u(t, v_\delta) < u(t, v_\star) \quad \text{in } \Omega; \quad \frac{\partial}{\partial n} u(t, v_\delta) > \frac{\partial}{\partial n} u(t, v_\star) \quad \text{on } \partial \Omega. \quad (31)$$

By (23), (30), and (31), we see that $u_2 \leq u_{j,n}$ a.e. in $\Omega$. Hence $u_2 \leq u_{j,n}$ in $\Omega$ since $u_2, u_{j,n} \in C_0^1(\Omega)$. By Lemma 9, $u_2 \neq u_{j,n}$. Then we have $u_{j,n} > u_j$ in $\Omega$ by the Hopf strong maximum principle.

Remark 5. The limit in (30) can be strengthened as

$$\lim_{t \to +\infty} \|u(t, v_\delta) - u_2\|_{C^1(\Omega)} = 0.$$

Indeed, since $u(t, v_\delta)$ satisfies

$$u(t, v_\delta) = e^{-t} v_\delta + \int_0^t e^{s-t} A_{j,n} u(s, v_\delta) \, ds, \quad 0 \leq t < +\infty,$$

Eq. (25), the $L^p$ theory for elliptic equations, and the Sobolev embedding theorem imply that $\{u(t, v_\delta) | 0 \leq t < +\infty\}$ is contained and bounded in $C_0^1(\Omega)$ for any $\delta \in (0, 1)$. Then Eq. (30) implies that $\lim_{t \to +\infty} \|u(t, v_\delta) - u_j\|_{C^1(\Omega)} = 0$.

Lemma 11. If $\bar{\lambda} > \bar{\lambda}_1$, then for $\bar{\lambda}_1 < \bar{\lambda} < \bar{\lambda}^*$ there exists a solution $u_3$ of (1) distinct from $y_j$ (therefore, $u_3 < u_j$ in $\Omega$), and for any $\lambda_1 < \lambda < \lambda^*$ and any solution $u_2$ of (1) the existence of solutions in the case of $\lambda = \lambda^*$ is proved in Lemma 13), there exists a solution $u_3$ of (1) such that $u_3 \neq u_2$.

Proof. If we can show $\|u_{j,n}\|_{C(\Omega)} \leq C$ for some constant $C$ independent of $n$ and if we denote $u_{j,n}$ by $u_j$ for a sufficiently large $n$, then the
arguments made above show that $u_1$ is a solution of (1) satisfying $u_1 > u_2$ in $\Omega$, and that for any $\lambda_1 < \lambda < \lambda^*$ and for any solution $u_2$ of (1) the solution $u_1$ of (1) thus obtained satisfies $u_1 \not\equiv u_2$.

It follows from (18) and (22) that

$$C_1 \leq J_{\lambda, n}(u_{\lambda, n}) = \int_{\Omega} \left[ \frac{1}{2} |\nabla u_{\lambda, n}|^2 - \lambda F_n(u_{\lambda, n}) \right] \, dx \leq C_2, \quad n = 1, 2, ... \quad (32)$$

Now an argument similar to Step 2 of the proof of Theorem 2.2 in [17] is used. Indeed, by Lemma 4 $\lambda f_\lambda(t)$ satisfies (H1) and (H2) uniformly with respect to $n$. Therefore, Steps 1 and 2 of the proof of Theorem 1.1 in [17] show that

$$\sum_{n \leq N} u_{\lambda, n}(x) \leq C_3$$

for all $x \in \partial \Omega$, $n = 1, 2, ...$, for some constant $C_3$ independent of $n$. By Pohozaev’s identity, we have

$$C_4 \leq \frac{1}{2} \int_{\Omega} |\nabla u_{\lambda, n}|^2 \, dx - \frac{\lambda N}{N-2} \int_{\Omega} F_n(u_{\lambda, n}) \, dx \leq C_5, \quad n = 1, 2, ... \quad (33)$$

where $C_4$ and $C_5$ are independent of $n$. Equations (32) and (33) imply that $\|u_{\lambda, n}\|_{H^1(\Omega)} \leq C_6$ for some $C_6$ independent of $n$. Now Step 4 of the proof of Theorem 1.1 in [17] implies that $\|u_{\lambda, n}\|_{C(\overline{\Omega})} \leq C$.  

**Lemma 12.** $J_\lambda(u_\lambda) < 0$ and $J_\lambda(u_\lambda)$ is strictly decreasing with respect to $\lambda_1 < \lambda < \lambda^*$.

**Proof.** First, we prove that $J_\lambda(u_\lambda)$ is strictly decreasing with respect to $\lambda_1 < \lambda < \lambda^*$. For $\lambda_1 < \lambda < \lambda^*$, choose $\lambda$ such that $\lambda_1 < \lambda < \lambda_2 < \lambda^*$. Let $u_{\lambda_2}$ be a solution of (1). Denote

$$D_1 = \{ u \in C^1_d(\overline{\Omega}) \mid u(x) \leq u_{\lambda_2}(x) \text{ for } x \in \overline{\Omega} \},$$

then $A_{\lambda, n}(D_1) \subset \bar{D}_1$ by Hopf strong maximum principle; therefore, by Lemma 6,

$$\{ u(t, y) \mid 0 \leq t < \eta(y) \} \subset D_1,$$

which implies that

$$C_1 \leq J_{\lambda, n}(u(t, y)) = J_\lambda(u(t, y)) \leq C_2, \quad \forall 0 \leq t < \eta(y)$$
for some constants $C_1$ and $C_2$ independent of $n$. The arguments made in Lemmas 8–10 show that
\[ \eta(u_2) = +\infty, \]
\[ \lim_{m \to +\infty} \|dJ_{\lambda_n}(u(t_m, u_2))\|_{H^1(\Omega)} = 0, \]
\[ \lim_{m \to +\infty} \|u(t_m, u_2) - u_2\|_{C^1(\bar{\Omega})} = 0, \]
and
\[ u(t, u_2) < u_2 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial}{\partial n} u(t, u_2) > \frac{\partial}{\partial n} u_2 \quad \text{in} \quad \partial \Omega \quad \text{for} \quad 0 \leq t < +\infty, \]
where \{t_m\} is some sequence with $t_m \to +\infty$ and $u^1_2 \in C^0_0(\bar{\Omega})$ is some function. It follows that $u^1_2 \in D_1$ is a solution of (1)$_2$ and $u^1_2 \leq u_2$ in $\bar{\Omega}$. Then $u^1_2 = u_2$ since $u_2$ is the minimal solution of (1)$_2$. Since $J_1(u(t, u_2))$ is strictly decreasing with respect to $0 \leq t < +\infty$, we get that
\[ J_1(u_2) = J_1(u^1_2) < J_1(u_2) < J_1(y_2); \]
this means that $J_1(u_2)$ is strictly decreasing with respect to $\lambda$.

Second, we prove that $J_1(u_2) < 0$ for $\lambda_1 < \lambda < \lambda^*$. Fix a $\delta > 0$ small enough such that $\delta \psi < u_2$ and
\[ J_1(\delta \phi) = \frac{1}{2} \lambda_1 \delta^2 \int_{\Omega} \phi^2 \, dx - \delta \int_{\Omega} F(\delta \phi) \, dx < 0. \]
Using $\delta \phi$ instead of $u_2$ in both $D_1$ and $u(t, u_2)$ and repeating the discussion just made, we can show that $\eta(\delta \phi) = +\infty$, that $\lim_{m \to +\infty} \|u(t_m, \delta \phi) - u_2\|_{C^0_0(\bar{\Omega})} = 0$ for some sequence $\{t_m\}$ with $t_m$ going to infinity, and that
\[ J_1(u_2) < J_1(\delta \phi) < 0. \]
The proof is complete.

**Lemma 13.** If $\lambda^* > \lambda_1$, there exists a minimal solution $u_2$ of (1)$_2$ with the properties that $u_2 > u_2$ in $\Omega$ for $\lambda_1 < \lambda < \lambda^*$, $\lim_{\lambda \to \lambda^*} \|u_2 - u_2\|_{C^0_0(\bar{\Omega})} = 0$, and $J_1(u_2) < J_1(y_2)$ for $\lambda_1 < \lambda < \lambda^*$.

**Proof.** Since
\[ \int_{\Omega} \left[ \frac{1}{2} |\nabla u_2|^2 - \lambda F(u_2) \right] \, dx < 0 \]
for \( \lambda_1 < \lambda < \lambda^* \) and \( f(t) \) satisfies (H2) and (H4) uniformly in \( \lambda_1 < \lambda < \lambda^* \), the same argument as that in the proof of Lemma 11 shows that \( \|u_1\|_{C^1(\Omega)} \leq C \) for some constant \( C \) independent of \( \lambda_1 < \lambda < \lambda^* \). Now the de Giorgi–Nash estimates and the Schauder estimates imply that \( \|u_1\|_{C^2(\Omega)} \leq C_1 \), for some constants \( C_1 \) independent of \( \lambda_1 < \lambda < \lambda^* \). When we can take a sequence \( \{u_{n}\} \) with \( \lambda_{n} \to \lambda^* \) such that \( \lim_{n \to +\infty} u_{n} \) exists in \( C^2(\Omega) \). Denote \( y_{n} = \lim_{n \to +\infty} u_{n} \), then \( u_{n} \) is a solution of (1). Note that \( u_{n} \) is strictly increasing with respect to \( \dot{\lambda} \) in \( \Omega \); it is easy to prove that \( \lim_{n \to +\infty} u_{n} = u_{\lambda} \) is the minimal solution of (1). The proof of Lemma 3 shows that \( u_{\lambda} > u_{\lambda} \) in \( \Omega \) for \( \lambda_1 < \lambda < \lambda^* \). The proof of Lemma 12 shows that \( J_{x}(u_{\lambda}) < J_{x}(u_{\lambda}) \) for \( \lambda_1 < \lambda < \lambda^* \).

**Proof of Theorem 1.** The proof is finished by combining Lemma 3, Lemma 11, Lemma 12, and Lemma 13.

**Remark 6.** We can say a little more about the result of (i) in Theorem 1. That is, with the methods used above it can be proved that, for any \( 0 < \lambda < \lambda^* \) and for any solution \( u_{\lambda} \) of (1), there exists a solution \( u_{\lambda} \) of (1) such that \( u_{\lambda} \in u_{\lambda} \).

### 4. PROOF OF THEOREM 2

**Proof of Theorem 2.** Since the proof of the first part of the conclusions is very similar to that of Theorem 1, we will only sketch it. Let \( M_1 > \beta \) be such that \( \inf_{\lambda > M_1} f(t) = \sup_{t \in [0, \beta]} f(t) \), then choose a fixed number \( m > 0 \) such that \( \dot{f}(t) + mt \) is strictly increasing on the interval \([0, M_1]\). In view of (H1), there exist \( \alpha > 0 \) and \( t^* > 0 \) such that

\[
\dot{f}(t) t^{-1} \geq \lambda_1 + \alpha, \quad \forall t \geq t^*. \tag{34}
\]

Choose a sequence \( \{s_n\} \) with \( s_n \to \infty \) and a number \( \gamma \) satisfying \( 1 < \gamma < (N + 2)/(N - 2) \) and \( 1 - \gamma^{-1}m^* \gamma^{-1}(1 - \gamma) > 0 \), where \( m^* = (\lambda_1 + \alpha/2)/(\lambda_1 + \alpha) \). Let the functions \( f_{\gamma}(t) \) and \( F_{\gamma}(t) \) be as in the proof of Theorem 1. Define

\[
J_{\gamma}(u) = \int_{D} \left[ \frac{1}{2} |\nabla u|^{2} - F_{\gamma}(u) \right] dx, \quad u \in H^1_{0}(\Omega),
\]

and

\[
D = \{ u \in C^1_{0}(\Omega) \mid 0 \leq u(x) \leq \beta \text{ for } x \in \Omega \}.
\]
For a fixed $u_0 \in C^1_0(\bar{\Omega})$, consider the initial value problem
\begin{equation}
\begin{aligned}
\frac{du(t)}{dt} &= -u(t) + A_n u(t), \\
u(0) &= u_0,
\end{aligned}
\tag{35}
\end{equation}
in both $H^1_0(\Omega)$ and $C^1_0(\bar{\Omega})$, where $A_n u = (-A + m)^{-1}(f_n(u) + mu)$. It can be deduced as in the proof of Theorem 1 that $A_n(D) \subset D^\circ$ and that
\begin{equation}
\{u(t, u_0) | 0 < t < \eta(u_0)\} \subset D^\circ \quad \text{for all } u_0 \in D,
\tag{36}
\end{equation}
where $u(t, u_0)$ is the unique solution of (35) with the maximal interval of existence $[0, \eta(u_0))$ and where $D^\circ = \{u \in C^1_0(\bar{\Omega}) | 0 \leq u(x) < \beta \text{ for } x \in \Omega\}$ is the interior part of $D$ in the cone $P = \{u \in C^1_0(\bar{\Omega}) | 0 \leq u(x) \text{ for } x \in \Omega\}$. From (34) and the definitions of $f_n$ and $J_n$ there is a number $T$ independent of $n$ such that
\begin{equation}
J_n(T) \leq \inf_{u \in D} J_n(u) = \inf_{u \in D} J(u).
\end{equation}

For $\tau \in [0, T]$, denote $v_\tau = \tau \theta$ and $\tau^* = \sup\{|\tau_1| | 0 < \tau_1 < T\}$, and for any $\tau \in [0, \tau_1]$ there exists a $t_\tau > 0$ such that $u(t_\tau, v_\tau) \in D^\circ$, then $0 < \tau^* < T$ and
\begin{equation}
\{u(t, v_\tau) | 0 \leq t < \eta(v_\tau)\} \cap D^\circ = \emptyset.
\tag{37}
\end{equation}

Just as in the proof of Theorem 1 we can show that $\eta(v_\tau^*) = +\infty$ and
\begin{equation}
\lim_{m \to +\infty} u(t_m, v_\tau^*) = u^* \quad \text{in both } H^1_0(\Omega) \text{ and } C^1_0(\bar{\Omega}),
\tag{38}
\end{equation}
for some $u^*$ and some sequence $\{t_m\}: t_m \to +\infty$, and $u^*$ is a solution of (1). From (37) and (38), we can use Hopf strong maximum principle to deduce that $\max_D u^* > \beta$.

Finally, we show that $J(u^*) > 0$ in the case $F(t) \leq ((2 - A) / 2) t^2$ for $0 \leq t < \beta$. According to the proof of Theorem 1, there are two constants $C_1$ and $C_2$ independent of $n$ such that
\begin{equation}
C_1 \leq J_n(u(t, v_\tau)) \leq C_2, \quad 0 \leq t < \eta(v_\tau), \quad \tau \in [0, \tau^*],
\end{equation}
from which we can show that $\eta(v_\tau) = +\infty$ for $\tau \in [0, \tau^*]$, and $\{u(t, v_\tau) | 0 \leq t < +\infty, 0 \leq \tau \leq \tau^*\}$ is bounded in $H^1_0(\Omega)$ and therefore bounded in
$C_0^k(\bar{\Omega})$ for any $x \in (0, 1)$ by Lemma 2. Take a constant $C_\lambda > 0$, which may be dependent on $n$, such that

$$\|u(t, v)\|_{C_0^k(\bar{\Omega})} \leq C_\lambda, \quad \forall 0 \leq t < + \infty, \quad \tau \in [0, \tau^*]. \quad (39)$$

We assert that there exists a constant $C_4 > 0$ such that

$$J_n(u(t_0, v_{x_0})) \geq C_4 \quad (40)$$

if $\max_{\Omega} u(t_0, v_{x_0}) = \beta$ for some $0 \leq t_0 < + \infty$ and $x_0 \in [0, \tau^*]$. Indeed, we have $u(t_0, v_{x_0}(x)) = \beta$ for some $x_0 \in \Omega$ and $0 \leq u(t_0, v_{x_0}(x)) \leq \beta$ for all $x \in \Omega$. Using (39) we can deduce that $\Omega_1 = \{ x \in \mathbb{R}^N \mid |x - x_0| \leq \beta/2C_3 \} \subset \Omega$ and $u(t_0, v_{x_0})(x) \geq \beta/2$ for all $x \in \Omega_1$. Hence

$$J_n(u(t_0, v_{x_0})) = \frac{1}{2} \int_{\Omega} |\nabla u(t_0, v_{x_0})|^2 \, dx - \int_{\Omega} F(u(t_0, v_{x_0})) \, dx$$

$$\geq \frac{\lambda_1}{2} \int_{\Omega} |u(t_0, v_{x_0})|^2 \, dx - \frac{\lambda_1 - \alpha}{2} \int_{\Omega} |u(t_0, v_{x_0})|^2 \, dx$$

$$\geq \frac{\lambda_1}{2} \int_{\Omega_1} |u(t_0, v_{x_0})|^2 \, dx$$

$$\geq \frac{2^N + 2\omega_N}{2N + 1} \frac{\lambda_1}{3} \geq C_4,$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. Using (38) we can take a $t_m$ such that

$$|J_n(u(t_m, v_{x_0})) - J_n(u^*)| < \frac{C_4}{4} \quad (41)$$

From (36) and the fact that $\max_{\Omega} u^* > \beta$ we see that

$$\min_{0 \leq t \leq t_m} \max_{t \geq t} u(t, v_{x_0}) > \beta.$$

Then we take a $0 < \tau < \tau^*$ such that

$$|J_n(u(t_m, v_{x_0})) - J_n(u(t_m, v_{x_0}))| < \frac{C_4}{4} \quad (42)$$

and

$$\min_{0 \leq t \leq t_m} \max_{t \geq t} u(t, v_{x_0}) > \beta. \quad (43)$$
In view of the definition of $\tau^*$, there exists $t_*>0$ such that $u(t_*, v_*) \in D^*$. It follows from (43) that $t_*>t_m$, therefore there exists $t'<t_m$ such that $\max_{\Omega} u(t', v_*) = \beta$. This, combined with (40), (41), and (42), leads us to

$$J(u^*) = J_u(u^*) > J_u(u(t_m, v_*)) - \frac{C_4}{4}$$

$$> J_u(u(t_m, v_*)) - \frac{C_4}{2}$$

$$> J_u(u(t_*, v_*)) - \frac{C_4}{2}$$

$$\geq \frac{C_4}{2} > 0,$$

where we have used the property that $J_u(u(t, v_*)$ is strictly decreasing in $t$.

The proof is complete. 

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