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Polynomial Closure

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Let D be a domain with quotient field K. The polynomial closure of a subset E of K is the largest subset F of K such that each polynomial (with coefficients in K),

polynomial closure of a subset is shown to contain its topological closure; the two closures are the same if D is a one-dimensional Notherian local domain, with finite residue field, which is analytically irreducible. A subset of D is said to be polynomially dense in D if its polynomial closure is D itself. The characterization of such subsets is applied to determine the ring R_α formed by the values $f(\alpha)$ of the integer-valued polynomials on a Dedekind domain R (at some element α of an extension of R). It is also applied to generalize a characterization of the Noetherian domains D such that the ring $\mathrm{Int}(D)$ of integer-valued polynomials on D is contained in the ring $\mathrm{Int}(D')$ of integer-valued polynomials on the integral closure D' of D.

INTRODUCTION

A few years ago R. Gilmer [13] characterized the subsets S of the ring of integers $\mathbb{Z}$ such that every polynomial taking integer values on S takes in fact integer values on every integer. This paper was immediately followed by comments by D. McQuillan [17] and ourselves [3]. Letting D be a domain, E a subset of its quotient field K, and

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$$

we discussed the subsets E of D such that

$$\mathrm{Int}(E, D) = \mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}.$$

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We thus recoverd easily Gilmer's result using topological arguments. More generally, McQuillan proposed to discuss when $\text{Int}(E_1, D) = \text{Int}(E_2, D)$ for two subsets E_1 and E_2. He observed that, for a given subset E, there is a largest subset, containing E, that defines the same ring of *integer-valued polynomials*. He thus defined and studied a closure operation, from which he recovered Gilmer's result (but with somewhat technical devices applying only to Dedekind domains with finite residue fields).

In this paper we study this *polynomial closure* in a general setting. Throughout, we let D be a domain (which is not a field), with quotient field K, and we denote by D' its integral closure.

We let the polynomial D-closure of a subset E of K be the largest subset F of K such that $\text{Int}(E, D) = \text{Int}(F, D)$. After some generalities (particularly on union and intersection), we discuss in a first section when the D-closure of E is K or equivalently $\text{Int}(E, D)$ is *trivial* (that is, does not contain any non constant polynomial). We show that, if D' is a fractional subset of D, then $\text{Int}(E, D)$ *is not* trivial if and only if E is a fractional subset of D (that is, is contained in a finitely generated D-module). In the next section we show that the closure of a fractional ideal is a fractional ideal and also that the divisorial ideals are always closed (generalizing McQuillan's result that each ideal of a Dedekind domain with finite residue fields is closed [17]). However we give examples of divisorial ideals which are not closed (even if D is Noetherian). We then look at localization properties. In particular, we show that integer-valued polynomials on a Krull domain have a somewhat good behavior under localization at the height-one primes (similarly to integer-valued polynomials on a Noetherian domain under localization at every prime). We conclude that the D-closure of a subset E of a Krull domain (resp., a Noetherian domain) is the intersection of its *local* $D_{\mathfrak{p}}$-closures, where $\mathfrak{p}$ runs among the height-one primes (resp., the maximal ideals) of D. It results that for a Krull domain, a fractional ideal is polynomally closed if and only if it is divisorial.

In the fourth and next section we then use the topological tools we had previously introduced. If D is a (Noetherian) Zariski domain (in particular if D is a local ring, with maximal ideal $\mathfrak{m}$ and if we consider the $\mathfrak{m}$-adic topology), we show that the polynomial closure of a fractional subset contains its topological closure. The two closures are the same if D is a one-dimensional Notherian local domain, with finite residue field, which is analytically irreducible (hence in particular if D is a discrete rank-one valuation domain). We thus recover and generalize McQuillan's characterization of the closure of a subset, when D is a Dedekind domain with finite residue fields.

A subset of D is said to be *polynomially dense* in D if its polynomial closure is D itself. Such subsets were studied in [3] (under the name of *parties pleines de D*) and polynomially dense subrings R of D had previously been characterized, originally in [8], then in [11] and [12] (as *sous-anneaux substituables à D*), in the cases where R or D (or both) are Dedekind domains. These characterizations allow us, in the next section, to interpret and generalize another result of D. McQuillan [18]. If R is a domain, we let R_α be the ring formed by the values $f(\alpha)$ of the integer-valued polynomials on R at some element α of an extension D of R. If R is a Dedekind domain, we show that R_α is the smallest subring of D, containing $R[\alpha]$, in which R is polynomially dense. If R is a Dedekind domain with finite residue fields and α is algebraic over R, then R_α contains the integral closure S of R in $K(\alpha)$ and we recover McQuillan's result according which R_α is the smallest overring of S, containing α, in which the primes of R split completely.

We lastly discuss integral closure. It is shown that $D'[X]$ is the inegral closure of $D[X]$. However, $\text{Int}(D)$ need not even be contained in $\text{Int}(D')$. R. Gilmer *et al.* have characterized the *one-dimensional* Noetherian domains for which this inclusion holds [14]. We generalize this characterization, whatever the dimension of D, showing that equivalently D is then a polynomally dense subset of D'. In fact, if D is Noetherian, the integral closure of $\text{Int}(D)$ is the ring $\text{Int}(D, D')$ and if D is a one-dimensional Noetherian domain with finite residue fields, then $\text{Int}(E, D')$ is the integral closure of $\text{Int}(E, D)$, for each fractional subset E of D. We conclude with examples showing that, if D is not Noetherian, then $\text{Int}(D, D')$ is not necessarily integral over $\text{Int}(D)$, and that $\text{Int}(E, D')$ is not necessarily integral over $\text{Int}(E, D)$, even if D is a one-dimensional Noetherian local domain, when its residue field if infinite.

Throughout we let the symbol "$\subset$" denot proper containment and the symbol "$\subseteq$" denote large containment.

1. EQUIVALENT SUBSETS AND POLYNOMIAL CLOSURE

DEFINITION 1.1. 1. Two subsets E and F of K are said to be *polynomially D-equivalent* (or simply *equivalent* if the context is clear) if $\text{Int}(E, D) = \text{Int}(F, D)$.

2. If E is a subset of K, we denote by $\bar{E}$ the subset

$$\bar{E} = \{x \in K \mid \forall f \in \text{Int}(E, D),\ f(x) \in D\}$$

and call it the *polynomial D-closure* of E (or simply the *polynomial closure* and even the *closure* of E).

3. If a subset E of K is equal to its polynomial closure, we say that E is *polynomially D-closed* (or simply (*polynomially*) *closed*).

4. If a subset E of D is D-equivalent to D, we say that E is a *polynomially dense* subset of D.

As noted by D. McQuillan [17], the closure $\bar{E}$ of E is clearly the largest subset of K equivalent to E. Right after introducing this polynomial closure (for Dedekind domains with finite residue fields), McQuillan noted a few easy facts [17, p. 246]. We expand here on them as follows.

LEMMA 1.2. 1. *For each subset E of K, $\bar{E}$ is polynomially closed.*

2. *If $E \subseteq F$, then $\bar{E} \subseteq \bar{F}$.*

3. *If (E_i) is a family of subsets of K, then $\overline{\bigcap_i E_i} \subseteq \bigcap_i \overline{E_i}$ and $\bigcup_i \overline{E_i} \subseteq \overline{\bigcup_i E_i}$.*

4. *For each $a \in K$, $a\bar{E} = \overline{aE}$ and $a + \bar{E} = \overline{a + E}$.*

5. *Each finite subset E of K is polynomially closed.*

Proof. The first two assertions are obvious and the third one can easily be derived. For the next one McQuillan argued that the maps $f(X) \to f(a+X)$ and $f(X) \to f(aX)$ define automorphisms of $K[X]$ for each $a \in K$ (the latter for $a \neq 0$, but if $a = 0$, then anyway $0\bar{E} = \overline{0E} = 0$). Lastly, let $E = x_1, ..., x_n$ be a finite subset of K, $x \notin E$, f the polynomial $f = \prod_{i=1}^{n} (X - x_i)$ and $\alpha = f(x)$. Since D is not a field, there is $\beta \in K$ such that $\alpha\beta \notin D$. Then $\beta f \in \text{Int}(E, D)$ but $\beta f(x) \notin D$, hence $x \notin \bar{E}$. ∎

Remark 1.3. 1. As stated by McQuillan [17, p. 246], it results in particular from assertion 3 that the intersection of closed sets is closed.

2. In general $\overline{\bigcap_i E_i}$ is strictly contained in $\bigcap_i \overline{E_i}$. For example both closures of $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geqslant 0\}$ and $\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \leqslant 0\}$ are equal to $\mathbb{Z}$ (from Gilmer's characterization of the polynomially dense subsets of $\mathbb{Z}$ [13, Theorem 2]). However there intersection is $\{0\}$ (a finite thus a closed subset).

3. In general also $\bigcup_i \overline{E_i}$ is strictly contained in $\overline{\bigcup_i E_i}$. In particular the finite union of closed sets need not be closed. For example, the ideals of $\mathbb{Z}$ are closed [17, Corollary 1] and [Proposition 2.1 below], whereas the union of two coprime ideals of $\mathbb{Z}$ is polynomially dense in $\mathbb{Z}$ [Corollary 3.14 below]. Hence, as noted by McQuillan [17, Remark 1], the polynomially closed sets do not define a topology.

Now we comment on subsets E of K such that $\text{Int}(E, D)$ is *trivial*, that is, equal to D, in other words E is D-equivalent to K (since one may check easily—or derive from the next proposition—that $\text{Int}(K, D) = D$). Recall that a subset E of K is said to be a *fractional* subset of D if there is a nonzero element d of D such that $dE \subseteq D$. In the case of a Dedekind domain with finite residue fields, D. McQuillan proved that E is D-equivalent to K

if and only if E is not fractional [17, Theorem 1]. With a very similar argument we prove here more generally the following.

PROPOSITION 1.4. 1. Int(E, D) *contains a polynomial of degree* 1 *if and only if* E *is a fractional subset of* D.

2. *If* Int(E, D) *contains a non-constant polynomial then* E *is a fractional subset of the integral closure* D' *of* D.

Proof. The first assertion is easily derived from the fact the dE is contained in D if and only if the polynomial dX belongs to Int(E, D). As for the second, let $f = \sum_{i=0}^{n} a_i X^i$ be a degree n polynomial in Int(E, D). We may as well assume f to be in $D[X]$ (multiplying by a common denominator of its coefficients). For each $x \in E$, $a_n^{n-1} f(x) \in D$, thus $(a_n x)^n + a_{n-1}(a_n x)^{n-1} + \cdots + a_o(a_n)^{n-1} = b$, where $b \in D$, hence $a_n x \in D'$. ▌

COROLLARY 1.5. *If* D' *is a fractional subset of* D, *then Int*(E, D) *contains a non-constant polynomial if and only if* E *is a fractional subset of* D.

This applies in particular if D is integrally closed. It does also apply a fortiori if the complete integral closure D'' of D is a fractional subset of D, a situation which is very common when D is a pullback.

Remark 1.6. If E is a fractional subset of D, then dE is a subset of D and $f(X) \in$ Int(dE, D) if and only if $f(dX) \in$ Int(E, D). Thus, for most of the results coming next, where E is supposed to be fractional, we may as well assume it is a subset of D.

It is worth giving examples such that (1) E is *not* a fractional subset of D but Int(E, D) contains some non-constant polynomials (however no polynomial of degree 1, according to the previous proposition) (2) E *is* a fractional subset of D' although Int(E, D) does not contain any non-constant polynomials. In both cases E is taken to be D' and D' is not a fractional subset of D, hence not a finitely generated D-module. The first example is quite classical, however we describe it completely for the sake of completeness.

EXAMPLE 1.7. 1. An example of a (dimension 1 and local) Noetherian domain D such that D' is not a finitely generated D-module and Int(D', D) contains a non-constant polynomial.

Let k be a field with characteristic p and $V = k[[x]]$ the power series ring with coefficients in k. Then V is a rank-one discrete valuation ring and we denote by v the corresponding valuation on its quotient field $k((x))$. It is known that $k((x))$ is a transcendental extension of $k(x)$, so let y be an element which is transcendental. We may assume that $y \in V$ (multiplying if necessary by a power of x). Let $K=k(x, y^p)$ and $L=k(x, y)$, then $K \subset L \subset k((x))$, L is a (purely inseparable) algebraic extension of degree p of K and $L^p \subseteq K$. Let $W = V \cap K$, then W is the ring of the restriction w of the valuation v to K, it is a rank-one discrete valuation domain. Let $D = W[y]$, then D is a Noetherian domain with field of quotient L. We claim its integral closure D' is the intersection $D' = V \cap L$ (that is, the ring of the restriction of v to L). First, it is clear that $V \cap L$ contains W and y, thus contains D. Since $L^p \subseteq K$, then $(V \cap L)^p \subseteq W$ and $V \cap L$ is integral over W. A fortiori $V \cap L$ is integral over D, hence it is contained in D'. Conversely, D' is contained in every valuation overring of D, thus in $V \cap L$. Moreover, the valuation w (with ring W) extends uniquely to L (the ring of the extension being D'). Since $x \in K$, it is easy to check that the ramification index and the residual degree of the extension $W \subset V$ are both 1. A fortiori the ramification index e and the residual degree f of the extension $W \subset D'$ are again 1. Thus $ef = 1$, although L is an extension of degree p of K, and therefore D' is not a finitely generated W-module (e.g. see [1, VI, §8, Theorem 2]). Hence D' is not a finitely generated D-module since $D = W[y]$ is a finitely generated W-module. But lastly, the polynomial X^p belongs to $\text{Int}(D', D)$, since $D'^p \subseteq W \subseteq D$.

2. An example of a domain D such that $\text{Int}(D', D)$ does not contain any non-constant polynomial.

Let k be a field, $B = k[x_1, x_2, \ldots]$ the ring in infinitely many indeterminates with coefficients in k (indexed by the positive integers) and D the subring generated by all the elements x_n^r, where $r \geqslant n$. Clearly D and B have the same quotient field K (since $x_n = x_n^{n+1}/x_n^n$), thus $B = D'$ is the integral closure of D. Now suppose, by way of contradiction, that a non-constant polynomial $f \in K[X]$ is such that $f(B) \subseteq D$. We may assume that the coefficients of f are in D (multiplying them, if need be, by their common denominator). Each of these coefficients is a polynomial in finitely many indeterminates. Let n be larger than the degree d of f and such that x_n is not any of these indeterminates. Write $f(X) = g_d X^d + \cdots + g_1 X + g_0$. In particular $f(x_n)$ should belong to D, whereas $f(x_n) = g_d x_n^d + \cdots + g_1 x_n + g_0$ is not in D.

2. POLYNOMIAL CLOSURE OF AN IDEAL

For a Dedekind domain with finite residue fields, McQuillan asserted that every ideal is closed [17, Corollary 1]. For every domain, we have in fact easily the following

PROPOSITION 2.1. *Each divisorial ideal of D is polynomially closed.*

Proof. Since the polynomial X is in $\mathrm{Int}(D)$, it is clear that D is polynomially closed. So is each fractional principle ideal, from assertion 4 of Lemma 1.2 and so is each divisorial from assertion 3 of this lemma, since a divisorial ideal is the intersection of fractional principal ideals. ▮

Remark 2.2. 1. A subset is a fractional subset of D if and only if it is contained in a (non-zero) principal fractional ideal. Hence it results from the previous proposition that the closure of a fractional subset is a fractional subset.

2. The converse of the previous proposition does not hold. Generalizing McQuillan's result in another direction, we shall see below that every ideal of D is closed when D is a one-dimensional, Noetherian, locally analytically irreducible domain with finite residue fields [Corollary 4.8]. However an ideal of D is not necessarily divisorial in this case. For example, if $D = k[[t^3, t^4, t^5]]$, the ring of series, with coefficients in a finite field k and no term in t nor t^2 (a local analytically irreducible domain with finite residue field), one may check that the ideal $\mathfrak{a} = (t^3, t^4)$ is not divisorial.

Although the converse of Proposition 2.1 does not hold in general, we shall prove in the next section that it does for a Krull domain [Proposition 3.8]. We also prove here that we have the following.

PROPOSITION 2.3. *Let D be a local Noetherian domain with maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is polynomially closed if and only if it is a divisorial ideal.*

Proof. Assume $\mathfrak{m}$ not to be divisorial, then it is not the conductor of a non-zero element of K in D, that is, it is not an associated prime of the D-module K/D. Necessarily the dimension of D is more than one and $\mathrm{Int}(D)$ is *trivial*, that is, equal to $D[X]$ [5, Corollary 1, p. 297]. We wish to prove that $\mathfrak{m}$ is not closed hence that $\mathrm{Int}(\mathfrak{m}, D) = D[X]$. Let $f \in \mathrm{Int}(\mathfrak{m}, D)$ and write $f = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n$. Suppose, by way of contradiction, that some coefficient α_k of f were not in D, its conductor $\mathfrak{a}_k = [D : \alpha_k]$ would then be a proper ideal of D. For all $x \in \mathfrak{m}$, $f(xX) \in D[X]$ (indeed, if $a \in D$, then $xa \in \mathfrak{m}$, thus $f(xa) \in D$ and $f(xX) \in \mathrm{Int}(D)$). Thus in particular $\alpha_k x^k \in D$. Hence $\mathfrak{m}$ would be minimal among the primes containing $\mathfrak{a}_k = [D : \alpha_k]$, which is the annihilator of a non-zero element of the D-module K/D. Since D is Noetherian, $\mathfrak{m}$ would then be an associated prime of K/D, contrary to our assumption. ▮

Although, in general, some ideals are not closed, we show now that the polynomial closure of an ideal is an ideal but first prove the following, where the sum $E + F$ (resp., the product EF), of two subsets E and F is the set of the elements of the form $x + y$ (resp., xy), where $x \in E$, $y \in F$.

LEMMA 2.4. *Let E and F be two subsets of K, then*

1. $\bar{E}+\bar{F}\subseteq\overline{E+F}$.
2. $\bar{E}\bar{F}\subseteq\overline{EF}$.

Proof. From assertion 4 of Lemma 1.2, if $\alpha\in E$, then $\alpha+\bar{F}=\overline{\alpha+F}$. Thus $E+\bar{F}\subseteq\overline{E+F}$. By symmetry, we then get

$$\bar{E}+\bar{F}\subseteq\overline{E+\bar{F}}\subseteq\overline{\overline{E+F}}=\overline{E+F}.$$

The second assertion is perfectly similar. ∎

Remark 2.5. These inclusions are strict in general: 1. The sum of two closed sets is not necessarily closed. For example, let $D=k[[x, y]]$ be the power series ring with coefficients in the field k, in two indeterminates x and y. The principal ideals (x) and (y) are closed [Proposition 2.1] whereas the maximal ideal $(x)+(y)$ is not [Proposition 2.3].

2. The product of two closed sets is not necessarily closed. For example, in $\mathbb{Z}$, consider the subsets $E=\{2, 3\}$ and $F=\mathbb{Z}$. Then E is closed since it is finite and so is obviously F [Proposition 2.1]. However their product is the union $2\mathbb{Z}\cup 3\mathbb{Z}$, which is polynomially dense in $\mathbb{Z}$ [Corollary 3.14 below].

PROPOSITION 2.6. *The polynomial closure of an additive subgroup* (*resp., a subring, resp., a fractional ideal*) *of D is an additive subgroup* (*resp., a subring, resp., a fractional ideal*) *of D.*

Proof. It results for example from the previous lemma that, if $\mathfrak{a}$ is an ideal of D, then $\bar{\mathfrak{a}}+\bar{\mathfrak{a}}\subseteq\overline{\mathfrak{a}+\mathfrak{a}}=\bar{\mathfrak{a}}$, and $D\bar{\mathfrak{a}}\subseteq\overline{D\mathfrak{a}}=\bar{\mathfrak{a}}$. ∎

Remark 2.7. In his paper D. McQuillan also considers *homogeneous sets* that he defines to be the (finite) union of cosets modulo an (integral) ideal of D. He then claims that the closure of an homogenous set (for a Dedekind domain with finite residue fields) is again homogeneous with respect to the same ideal [17, Lemma 3]. More generally we may note that an homogeneous set with respect to the ideal $\mathfrak{a}$ is a subset E such that $E+\mathfrak{a}\subseteq E$ (we do not insist that E be a finite union of cosets modulo $\mathfrak{a}$, a fact which is particular to Dedekind domains with *finite* residue fields). We may then take this inclusion as a definition and then allow $\mathfrak{a}$ to be a fractional ideal. It is then immediate to derive from Lemma 2.4 that the closure of E is homogeneous with respect to the closure $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ (in the case of a Dedekind domain, each ideal is closed and $\bar{E}$ is then homogeneous with respect to the same ideal $\mathfrak{a}$).

3. LOCALIZATION

Let us first recall a few elementary facts we had previously established [4, Proposition 1.5]: if E is a subset of K and S a multiplicative subset of D, then $S^{-1}\operatorname{Int}(E, D)$ is clearly a subring of $\operatorname{Int}(E, S^{-1}D)$. Under Noetherian hypotheses we showed there was equality. We restate the following for sake of completeness:

PROPOSITION 3.1. 1. *Let D be a Noetherian domain, E a fractional subset of D and S a multiplicative subset of D, then*

$$S^{-1}\operatorname{Int}(E, D) = \operatorname{Int}(E, S^{-1}D).$$

2. *Let D be a domain, R a Noetherian subring of D and S a multiplicative subset of R, then*

$$S^{-1}\operatorname{Int}(R, D) = \operatorname{Int}(R, S^{-1}D) = \operatorname{Int}(S^{-1}R, S^{-1}D).$$

Similarly we prove here that we have a good behavior under localization at the height-one primes of a Krull domain.

PROPOSITION 3.2. *Let D be a Krull domain, $\mathfrak{q}$ an height-one prime and E a fractional subset of D. Then*

$$\operatorname{Int}(E, D)_{\mathfrak{q}} = \operatorname{Int}(E, D_{\mathfrak{q}}).$$

Proof. We wish to prove that $\operatorname{Int}(E, D_{\mathfrak{q}}) \subseteq \operatorname{Int}(E, D)_{\mathfrak{q}}$. Let $f \in \operatorname{Int}(E, D_{\mathfrak{q}})$. There is $d \in D$, $d \neq 0$, such that $df \in D[X]$. Let $\mathscr{W}$ be the (finite) set of essential valuations w of D such that $w(d) > 0$ and w is not associated with $\mathfrak{q}$. For each height-one prime $\mathfrak{p}$ distinct from $\mathfrak{q}$, there is an element of $\mathfrak{p}$ which is not in $\mathfrak{q}$. Taking a product of such elements, we can thus get $a \in D$, $a \notin \mathfrak{q}$ such that, for each $w \in \mathscr{W}$, $w(a) \geqslant w(d)$. Hence, for each height-one prime $\mathfrak{p}$, but $\mathfrak{q}$, $af \in D_{\mathfrak{p}}[X]$. Assuming E to be a subset of D [Remark 1.6], then $af(E)$ is contained in $D_{\mathfrak{p}}$. But $af(E)$ is also contained in $D_{\mathfrak{q}}$, since $a \in D$ and $f \in \operatorname{Int}(E, D_{\mathfrak{q}})$ by hypothesis. Therefore $af \in \operatorname{Int}(E, D)$, hence $f \in \operatorname{Int}(E, D)_{\mathfrak{q}}$, since $a \notin \mathfrak{q}$. ∎

Remark 3.3. Let us note that in Propositions 3.1 and 3.2, it is essential to assume E to be a fractional subset. Indeed the ring of integers $\mathbb{Z}$ is obviously both a Noetherian and a Krull domain. But, for each nontrivial multiplicative subset S of $\mathbb{Z}$, $E = S^{-1}\mathbb{Z}$ is not a fractional subset of $\mathbb{Z}$. Therefore $S^{-1}\operatorname{Int}(E, \mathbb{Z}) = S^{-1}\mathbb{Z}$ [Corollary 1.5], whereas $\operatorname{Int}(E, S^{-1}\mathbb{Z}) = \operatorname{Int}(S^{-1}\mathbb{Z})$, a ring which contains non-constant polynomials.

If E is a subset of K and S a multiplicative subset (resp., $\mathfrak{p}$ a prime ideal) of D, we now consider the polynomial $S^{-1}D$-closure (resp., the polynomial $D_{\mathfrak{p}}$-closure) of E, that we denote by $\overline{S^{-1}E}$ (resp., $\overline{E_{\mathfrak{p}}}$). If $E=\mathfrak{a}$ is a fractional ideal of D, there could be some confusion with the $S^{-1}D$-closure of $S^{-1}\mathfrak{a}$, however next lemma shows there is no harm (there could still be some confusion with the closure of $S^{-1}\mathfrak{a}$ in D, but we do believe the context makes clear what we have in mind).

LEMMA 3.4. *Let $\mathfrak{a}$ be a fractional ideal and S be a multiplicative subset of D. Then $\mathfrak{a}$ and $S^{-1}\mathfrak{a}$ are $S^{-1}D$-equivalent.*

Proof. Let $f(X) \in \mathrm{Int}(\mathfrak{a}, S^{-1}D)$ and $(x/s) \in S^{-1}\mathfrak{a}$. Since $xD \subseteq \mathfrak{a}$, then $f(X) \in \mathrm{Int}(xD, S^{-1}D)$, thus $f(xX) \in \mathrm{Int}(D, S^{-1}D) = \mathrm{Int}(S^{-1}D)$ [Corollaire 5, p. 303]. Hence $f(X) \in \mathrm{Int}(xS^{-1}D, S^{-1}D)$ and $f(x/s) \in S^{-1}D$. ∎

We now relate the polynomial D-closure $\bar{E}$ of E to its *local* $S^{-1}D$-closures.

PROPOSITION 3.5. *Let E be a subset of K.*

1. *Let S be a multiplicative subset of D and assume that*

$$S^{-1}\mathrm{Int}(E, D) = \mathrm{Int}(E, S^{-1}D),$$

then $\bar{E} \subseteq \overline{S^{-1}E}$.

2. *Let $(S_i)_{i\in I}$ be a complete family of multiplicative subsets of D, that is, such that $D = \bigcap_{i\in I} S_i^{-1}D$, then $\bigcap_{i\in I} \overline{S_i^{-1}E} \subseteq \bar{E}$.*

Proof. 1. Let $x \in \bar{E}$, and $f \in \mathrm{Int}(E, S^{-1}D) = S^{-1}\,\mathrm{Int}(E, D)$. There is $s \in S$ such that $sf \in \mathrm{Int}(E, D)$, thus $sf(x) \in D$ and $f(x) \in S^{-1}D$. In conclusion $x \in \overline{S^{-1}E}$.

2. Let $x \bigcap_{i\in I} \overline{S_i^{-1}E}$ and $f \in \mathrm{Int}(E, D)$. Then, $\forall i \in I$, $f \in \mathrm{Int}(E, S_i^{-1}D)$, thus $f(x) \in S_i^{-1}D$. In conclusion $f(x) \in D = \bigcap_{i\in I} S_i^{-1}D$. ∎

Using Propositions 3.1 and 3.2, we then derive the following.

PROPOSITION 3.6. 1. *If D is Noetherian and $\mathscr{M}$ is the set of its maximal ideals, then $\bar{E} = \bigcap_{\mathfrak{m}\in\mathscr{M}} \overline{E_{\mathfrak{m}}}$.*

2. *If D is a Krull domain and $\mathscr{P}$ is the set of its height-one primes, then $\bar{E} = \bigcap_{\mathfrak{p}\in\mathscr{P}} \overline{E_{\mathfrak{p}}}$.*

In particular, we thus get the following (the Noetherian part of which we had already established [3, Proposition 1.3]).

COROLLARY 3.7. *Let E be a subset of D.*

1. *If D is Noetherian, then E is polynomially dense in D if and only if it is polynomially dense in $D_{\mathfrak{m}}$, for each maximal ideal $\mathfrak{m}$ of D.*

2. *If D is a Krull domain, then E is polynomially dense in D if and only if it is polynomially dense in $D_{\mathfrak{p}}$, for each height-one prime ideal $\mathfrak{p}$ of D.*

For a Krull domain D, we may note that, whatever the fractional ideal $\mathfrak{a}$ (whether it is closed or not closed), $\mathfrak{a}_{\mathfrak{p}}$ is a divisorial ideal of $D_{\mathfrak{p}}$ (a discrete rank one valuation domain), for each height-one prime ideal of D, hence is a $D_{\mathfrak{p}}$-closed ideal. From Proposition 3.6, the polynomial closure of $\mathfrak{a}$ is then the intersection $\bar{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in P} \mathfrak{a}_{\mathfrak{p}}$ and this intersection is known to be the v-closure of $\mathfrak{a}$ [1, VII, §1, Proposition 7] (that is, the smallest divisorial ideal containing $\mathfrak{a}$). We thus get the following (in particular the converse to Proposition 2.1 holds in this case).

COROLLARY 3.8. *Let D be a Krull domain and $\mathfrak{a}$ be a fractional ideal of D. The polynomial closure of $\mathfrak{a}$ is then the v-closure of $\mathfrak{a}$. In particular $\mathfrak{a}$ is polynomially D-closed if and only if it is a divisorial ideal of D.*

In general, a D-closed subset E is not $S^{-1}D$-closed, indeed the $S^{-1}D$-closure of an ideal $\mathfrak{a}$ contains $S^{-1}\mathfrak{a}$. It may even happen that an ideal $\mathfrak{a}$ is D-closed whereas $S^{-1}\mathfrak{a}$ is not $S^{-1}D$-closed, according to the next example (which stems from a classical pullback construction).

EXAMPLE 3.9. Let $B = \mathbb{Q}[[x, y]]$, the power series ring over the field of rationals. This is a Noetherian Krull domain, its maximal ideal $\mathfrak{m} = (x, y)$ is not divisorial hence it is not B-closed from the previous corollary (or Proposition 2.3). Let $D = \mathbb{Z} + \mathfrak{m}$. D is a subring of B and $\mathfrak{m}$ is a prime ideal of D. It is easy to check that $D_{\mathfrak{m}} = B$, indeed, on the one hand, $D_{\mathfrak{m}}$ is clearly contained in $B_{\mathfrak{m}} = B$, on the other hand, $\mathbb{Q}$ (and thus also $B = \mathbb{Q} + \mathfrak{m}$) is contained in $D_{\mathfrak{m}}$. In particular $\mathfrak{m}D_{\mathfrak{m}} = \mathfrak{m}$ and $\mathfrak{m}D_{\mathfrak{m}}$ is not $D_{\mathfrak{m}}$-closed. However, for each non-zero integer a, the ideal $\mathfrak{m}$ is contained in the principal ideal Da, in fact $\mathfrak{m} = \mathfrak{m}a$, indeed a is invertible in B and $\forall m \in \mathfrak{m}$, $m = (m/a)\, a$, where $(m/a) \in \mathfrak{m}$. The ideal $\mathfrak{m}$ is then the intersection of principal ideals in D (since in the quotient $D/\mathfrak{m} \simeq \mathbb{Z}$, the ideal (0) is the intersection of principal ideals). Therefore $\mathfrak{m}$ is a divisorial ideal of D and thus is D-closed.

However D is not Noetherian in the previous example and we ask the following.

QUESTION 3.10. *If D in Noetherian and $\mathfrak{a}$ is a D-closed fractional ideal, is $S^{-1}\mathfrak{a}$ a S^{-1} D-closed ideal?*

Conversely, we may derive immediately the following from Propostion 3.6.

Corollary 3.11. *Let* $\mathfrak{a}$ *be a fractional ideal of a Noetherian domain* D. *If* $\mathfrak{a}_{\mathfrak{m}}$ *is polynomially* $D_{\mathfrak{m}}$*-closed, for each maximal ideal* $\mathfrak{m}$ *of* D, *then* $\mathfrak{a}$ *is polynomially* D*-closed.*

For a Dedekind domain, with finite residue fields, McQuillan proved that the closure of the union $\mathfrak{a} \cup \mathfrak{b}$ of two fractional ideals is their *greatest common divisor* [17, Corollary 2]. More generally we get here the following.

Corollary 3.12. *Let* D *be a Krull or a Prüfer domain, and* $\mathfrak{a}, \mathfrak{b}$ *be two fractional ideals of* D. *Then* $\overline{\mathfrak{a} \cup \mathfrak{b}} = \overline{\mathfrak{a} + \mathfrak{b}}$. *In particular, if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are finitely generated and* D *is a Prüfer domain, then* $\overline{\mathfrak{a} \cup \mathfrak{b}} = \mathfrak{a} + \mathfrak{b}$.

Proof. The closure $\overline{\mathfrak{a} \cup \mathfrak{b}}$ of the union is clearly contained in the closure $\overline{\mathfrak{a} + \mathfrak{b}}$ of the sum. Conversely, form Proposition 3.5, $\overline{\mathfrak{a} \cup \mathfrak{b}}$ contains the intersection $\bigcap_{\mathfrak{p} \in \mathscr{P}} \overline{(\mathfrak{a} \cup \mathfrak{b})_{\mathfrak{p}}}$ of the local closures at each height-one prime (resp., maximal ideal) $\mathfrak{p}$ of D if D is a Krull (resp., a Prüfer) domain. Now, for each such prime, $D_{\mathfrak{p}}$ is a valuation domain, hence one of the ideals $\mathfrak{a}_{\mathfrak{p}}$ or $\mathfrak{b}_{\mathfrak{p}}$ is contained in the other and $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}} \cup \mathfrak{b}_{\mathfrak{p}}$. Thus $\overline{(\mathfrak{a} \cup \mathfrak{b})_{\mathfrak{p}}}$, which contains both closures of $\mathfrak{a}$ and $\mathfrak{b}$, contains both $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{b}_{\mathfrak{p}}$, hence their sum $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{b}_{\mathfrak{p}}$ and a fortiori contains $\mathfrak{a} + \mathfrak{b}$. If moreover $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated and D is a Prüfer domain, then $\mathfrak{a} + \mathfrak{b}$ is finitely generated, hence invertible, thus divisorial and it is closed. ∎

It is worth giving now an example of a (non-dimensional Noetherian) domain where the closure of the union of two ideals in not an ideal.

Example 3.13. Let k be a field of characteristic $p \neq 2$ and $D = k[t^2, t^3]$ the ring of polynomials with no terms in t. Let $f = x^2/t^4$. It is clear that f is integer-valued on t^2D and on t^3D, thus on the union of these two principal ideals. However $f(t^2 + t^3) = 1 + 2t + t^2$ is not in D.

About the closure of the union of two ideals we lastly have at least the following, whatever the domain D.

Corollary 3.14. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two coprime ideals of* D (*that is,* $\mathfrak{a} + \mathfrak{b} = D$). *Then* $\overline{\mathfrak{a} \cup \mathfrak{b}} = D$.

Proof. For each maximal ideal $\mathfrak{m}$, the $D_{\mathfrak{m}}$-closure of the union $\mathfrak{a} \cup \mathfrak{b}$ at $\mathfrak{m}$ clearly contains both $\mathfrak{a}_{\mathfrak{m}}$ and $\mathfrak{b}_{\mathfrak{m}}$. Since $D_{\mathfrak{m}}$ is a local ring and $\mathfrak{a}_{\mathfrak{m}} + \mathfrak{b}_{\mathfrak{m}} = D_{\mathfrak{m}}$, one of the ideals $\mathfrak{a}_{\mathfrak{m}}$ and $\mathfrak{b}_{\mathfrak{m}}$ must be equal to $D_{\mathfrak{m}}$, thus the local closure contains $D_{\mathfrak{m}}$. From Proposition 3.5, $\overline{\mathfrak{a} \cup \mathfrak{b}}$ contains the intersection of the local closures, thus it contains D. ∎

4. TOPOLOGICAL AND POLYNOMIAL CLOSURE

Recall a *Zariski ring* is a Noetherian ring R equipped with an $\mathfrak{a}$-adic topology, such that each ideal is topologically closed. Equivalently $\mathfrak{a}$ is contained in the Jacobson radical of R (e.g. see [19, Theorem 56]). For a Zariski domain, we now relate the polynomial closure to the topological closure.

THEOREM 4.1. *Let D be a Zariski domain. The polynomial D-closure of a fractional subset E of D contains the topologic closure of E.*

Proof. We may as well assume E to be a subset of D [Remark 1.6]. Let $f \in \text{Int}(E, D)$ and x in the topologic closure of E for the $\mathfrak{a}$-adic topology, we wish to show that $f(x) \in D$. Let $d \in D$, $d \neq 0$, such that $df \in D[X]$. For each integer n, there is $y \in E$ such that $(x-y) \in \mathfrak{a}^n$. Since $(x-y)$ divides $[df(x) - df(y)]$ in D, then $[df(x) - df(y)] \in \mathfrak{a}^n$. Since $df(y) \in dD$, then $df(x) \in dD + \mathfrak{a}^n$. Therefore $df(x)$ belongs to the topological closure of the ideal dD, which is dD itself, since D is a Zariski ring. Thus, $df(x) \in D$ and, dividing by d, $f(x) \in D$. ∎

COROLLARY 4.2. *Let D be a Zariski domain.*

1. *If E is a topologically dense subset of D, then E is a polynomially dense subset of D.*

2. *If E is a topologically closed fractional subset of D, then E is topologically closed.*

All this applies of course in particular to the $\mathfrak{m}$-adic topology, if D is a Noetherian local ring, with maximal ideal $\mathfrak{m}$.

Remark 4.3. If $\mathfrak{a}$ is not contained in the Jacobson radical of D, a dense subset of D for the $\mathfrak{a}$-adic topology need not be a polynomially dense subset of D and may even be polynomially closed. For example, let $D = k[t]$ be the polynomial ring with coefficients in a field k with q elements (a one-dimensional Noetherian domain). Let $\mathfrak{m}$ be the maximal ideal $\mathfrak{m} = (1+t)\,D$ and E be the complement of tD in D. Then E is easily seen to be dense in D for the $\mathfrak{m}$-adic topology. However E is polynomially closed, indeed $f = 1 - X^{q-1}/t$ belongs to $\text{Int}(E, D)$ but $\forall x \notin E$, $f(x) \notin D$.

If D is a Noetherian local domain, with maximal ideal $\mathfrak{m}$, integer-valued polynomials are continuous functions from D to D for the $\mathfrak{m}$-adic topology [9, Proposition 4.3]. If the dimension of D is 1 we show next that each polynomial with coefficients in K is a continuous functions from K to K.

LEMMA 4.4. *Let D be a one-dimensional Noetherian, local domain, with maximal ideal $\mathfrak{m}$. Each polynomial f with coefficients in K is uniformly continuous for the $\mathfrak{m}$-adic topology: there is an integer h such that, for each integer r and each $a, b \in K$, $(a-b) \in \mathfrak{m}^{r+h} \Rightarrow [f(a)-f(b)] \in \mathfrak{m}^r$.*

Proof. Let d be a non-zero element of D such that df is in $D[X]$. Then, for each a and b in D, $(a-b)$ divides $d[f(a)-f(b)]$ in D. Since D is one-dimensional, Noetherian and local, these is an integer h such that $\mathfrak{m}^h \subseteq dD$. Thus

$$(a-b) \in \mathfrak{m}^{r+h} \Rightarrow (a-b) \in d\mathfrak{m}^r \Rightarrow d[f(a)-f(b)] \in d\mathfrak{m}^r$$
$$\Rightarrow [f(a)-f(b)] \in \mathfrak{m}^r. \quad \blacksquare$$

In this case, we can use a topological argument to prove that the polynomial D-closure of a subset E contains the topological closure of E (for the $\mathfrak{m}$-adic topology): a polynomial f with coefficients in K is uniformly continuous, hence, if it takes E into D, it takes the (topological) closure of E into the closure of D, but D is obviously topologically closed in K.

Integer-valued polynomials being uniformly continuous functions, they can be considered as continuous functions from the completion $\hat{D}$ of D onto itself. Moreover, we have an analogous of the Stone–Weirstrass theorem if D is analytically irreducible (that is, $\hat{D}$ is a domain), with finite residue field: each continuous function from $\hat{D}$ to $\hat{D}$ can be arbitrarily and uniformly approximated by an integer-valued polynomial [6, p. 53] or [10, Theorem 3.3] (this holds in particular if D is a rank-one discrete valuation domain).

PROPOSITION 4.5. *Let D be a local, Noetherian, one-dimensional, analytically irreducible domain with finite residue field and $\mathfrak{m}$ be its maximal ideal. The polynomial and topologic closure (for the $\mathfrak{m}$-adic topology) of each fractional subset E of K are then equal.*

Proof. From Corollary 4.2, it remains to prove that if E is closed (for the $\mathfrak{m}$-adic topology), then it is polynomially closed. We may assume E to be a subset of D [Remark 1.6]. We know the polynomial closure $\bar{E}$ of E to be contained in D [Proposition 2.1] and wish to prove that, if $x \in D$ and $x \notin E$, then $x \notin \bar{E}$. Since E is topologically closed and the $\mathfrak{m}$-adic topology is ultrametric, there is an open and closed neigborhood U of x which does not meet E. Let T be a non-zero element of $\mathfrak{m}$ and n an integer such that $\mathfrak{m}^n \subseteq tD$ (such an integer exists, since D is one-dimensional). From the Stone–Weierstrass theorem, the characteristic function φ of U can be approximated modulo $\mathfrak{m}^n$ by a polynomial $f \in \text{Int}(D)$. Thus $f(x) = 1 + td$, where $d \in D$, whereas $\forall z \in E$, $f(z) \in tD$. Letting $g = f/t$, then $g \in \text{Int}(E, D)$ but $g(x) \notin D$. $\blacksquare$

In particular we recover the fact that the polynomially dense subsets are then topologically dense subsets of D [3, Proposition 2.1]. Similarly we get that the polynomially closed fractional subsets of D are then the topologically closed fractional subsets of D in this case.

Remark 4.6. In general, polynomially dense subsets are not necessarily topologically dense (and may even be closed for the $\mathfrak{m}$-adic topology), even if D is a local one-dimensional Noetherian domain. This may happen if D is not analytically irreducible [3, Example 5.3] or if the residue field of D is infinite [3, Remark 3.3].

From Proposition 3.6 we then derive the following characterization of the polynomial closure of a fractional subset in a domain D which is locally analytically irreducible, with finite residue fields. We thus generalize McQuillan's similar result for a Dedekind domain [17, Theorem 2] and recover in particular the characterization we had previously given of the polynomially dense subsets of D [3, p. 206] (as well as Gilmer's original characterization of the polynomially dense subsets of $\mathbb{Z}$ [13, Theorem 2]).

THEOREM 4.7. *Let D be a one-dimensional, Noetherian, locally analytically irreducible domain, with finite residue fields, and E a fractional subset of D. The polynomial closure of E is then the intersection of its $\mathfrak{m}$-adic topological closures in K, where $\mathfrak{m}$ runs over all the maximal ideals of D.*

Proof. We know, by localization properties, that the polynomial closure of E is the intersection of its $\mathfrak{m}D_{\mathfrak{m}}$-adic topological closures in K, where $\mathfrak{m}$ runs over all the maximal ideals of D. Assuming E to be a subset of D [Remark 1.6], its polynomial closure is contained in D and, for each $\mathfrak{m}$, we can rather consider the intersection of its $\mathfrak{m}D_{\mathfrak{m}}$-adic topological closure with D. But this is the topological closure of E for the topology induced by the $\mathfrak{m}D_{\mathfrak{m}}$-adic topology on D. And this topology is nothing else that the $\mathfrak{m}$-adic topology. ∎

Since each ideal of a one-dimensional Noetherian local domain D is clearly topologically closed, we derive in particular the following.

COROLLARY 4.8. *Let D be a one-dimensional, Noetherian, locally analytically irreducible domain with finite residue fields, then each fractional ideal of D is polynomially closed.*

5. RING OF VALUES

In this section we consider a domain D containing a domain R. We say that D is an extension of R and then set the following definition.

Definition 5.1. Let α be an element of an extension D of R. The set

$$R_\alpha = \{ f(\alpha) \mid f \in \mathrm{Int}(R) \}$$

is said to be the *ring of values* of $\mathrm{Int}(R)$ at α.

Clearly the quotient field L of D is a field extension of the quotient field K of R. With these notations, we first record a few obvious facts.

Propositon 5.2. 1. *The ring of values R_α is a domain.*

2. $R[\alpha] \subseteq R_\alpha \subseteq K[\alpha]$, *in particular the quotient field of R_α is $K(\alpha)$.*

3. $R_\alpha = R$ *if and only if* $\alpha \in R$.

Without any assumption on R nor α (α may be algebraic or transcendental) we have the following.

Lemma 5.3. *Let α be an element of an extension D of R.*

1. *If T is a domain such that $R \subseteq T \subseteq D$, if $\alpha \in T$ and if R is a polynomially dense subring of T, then $R_\alpha \subseteq T$.*

2. $\mathrm{Int}(R) \subseteq \mathrm{Int}(R_\alpha)$.

Proof. 1. Let $f \in \mathrm{Int}(R)$. A fortiori $f \in \mathrm{Int}(R, T) = \mathrm{Int}(T)$ (since R is polynomially dense in T). Therefore $f(\alpha) \in T$.

2. Let $f \in \mathrm{Int}(R)$ and $\beta = g(\alpha) \in R_\alpha$, where $g \in \mathrm{Int}(R)$. Then $f(g(X))$ is clearly also in $\mathrm{Int}(R)$, thus $f(\beta) = f(g(\alpha)) \in R_\alpha$. ∎

Remark 5.4. 1. If R is a subring of T then $\mathrm{Int}(R)$ need not be contained in $\mathrm{Int}(T)$ (we shall give examples below and in the next section). But clearly, if $\mathrm{Int}(R) \subseteq \mathrm{Int}(T)$, then $R_\alpha \subseteq T_\alpha$. However R_α may be contained in T_α (for example if $\alpha \in R$) even if $\mathrm{Int}(R)$ is not contained in $\mathrm{Int}(T)$.

2. It results from the previous lemma and the first remark above (letting $T = R[\alpha]$), that $R_\alpha = R[\alpha]$ if and only if $\mathrm{Int}(R) \subseteq \mathrm{Int}(R[\alpha])$.

Now, if R is a Dedekind domain, Gilbert Gerboud gave several characterizations of the domains D containing R which are such that $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(D)$ [12, Theorem 3]. For the sake of completeness we then record the following.

Lemma 5.5 (Gerboud). *Let D be an extension of the Dedekind domain R. The following assertions are equivalent.*

1. *R is a polynomially dense subring of D.*

2. $\mathrm{Int}(R)$ *is contained in* $\mathrm{Int}(D)$.

3. *For each prime ideal* $\mathfrak{m}$ *of* D *such that* $\mathfrak{p} = \mathfrak{m} \cap R$ *has a finite residue field in* R, $\mathfrak{m}D_{\mathfrak{m}} = \mathfrak{p}D_{\mathfrak{m}}$ *and* $D/\mathfrak{m} \simeq R/\mathfrak{p}$ (*in particular each such prime* $\mathfrak{m}$ *is maximal*).

The characterization in terms of the prime ideals results from localization properties [Corollary 3.7] and various characterizations of (discrete rank-one valuation) domains which are polynomially dense in a larger domain (already in [8] and to be found in [11, Proposition 6.1] or [3, Corollaire 2.5]). The restriction to the primes $\mathfrak{p}$ with finite residue fields results from the easy fact that a local ring R with infinite residue field is polynomially dense in every domain D containing it (clearly then $\mathrm{Int}(R, D) = \mathrm{Int}(D) = D[X]$ and this dates back, (with quite a different terminology) to [5, Proposition 3]). On the other hand it is clear that, whatever the domain R, $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(R, D)$, thus if R is polynomially dense in D (that is, $\mathrm{Int}(R, D) = \mathrm{Int}(D)$) then $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(D)$. The converse is more technical and requires R to be a Dedekind domain. Indeed we give below an example where, although R is an *almost Dedekind domain*, $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(D)$ whereas R is not polynomially dense in D.

EXAMPLE 5.6. Let R be an almost Dedekind domain with at least a maximal ideal $\mathfrak{m}_0$ with finite residue field but such that $R = \bigcap_{\mathfrak{m} \in \mathscr{M}} R_{\mathfrak{m}}$, where $\mathscr{M}$ denotes the set of maximal ideals with infinite residue fields. Then $\mathrm{Int}(R) = R[X]$ [5, Corollaire 3, p. 303], thus $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(D)$, whatever the domain D containing R. However, it results from the previous lemma that there exists a ring D, containing $R_{\mathfrak{m}_0}$ in which $R_{\mathfrak{m}_0}$ is not polynomially dense (for example a discrete rank-one valuation domain such that the ramification index or the residual degree of this extension is greater than 1). A fortiori R is not polynomially dense in D.

From the previous two lemmas we then get immediately the following characterization of R_α for a Dedekind domain R.

THEOREM 5.7. *Let* D *be an extension of the Dedekind domain* R *and* $\alpha \in D$. *The ring of values* R_α *is then the smallets subring of* D *containing* $R[\alpha]$ *in which* R *is polynomially dense.*

We gave above an example where $\mathrm{Int}(R)$ is contained in $\mathrm{Int}(D)$ but R is not polynomially dense in D. However D was not a ring of values R_α of $\mathrm{Int}(R)$ at each α. We thus raise the following question.

QUESTION 5.8. *Let* α *be an element of an extension of a domain* R. *Is* R *polynomially dense in* R_α?

From Theorem 5.7 we shall recover McQuillan's characterization of R_α [18, Theorem] in the case where R is a Dedekind domain with finite residue fields and α is algebraic over K. We first note (according to one of McQuillan's arguments) that in this case $\mathrm{Int}(R)$ is a Prüfer domain (for example see [7]) and so is its homomorphic image R_α. In particular R_α is then integrally closed in its quotient field $K(\alpha)$ and thus contains the integral closure of R in $K(\alpha)$. From the previous theorem we thus get the following.

Corollary 5.9. *Let R be a Dedekind domain with finite residue fields, α an element of an extension of its quotient field K and S the integral closure of R in $K(\alpha)$. Then R_α is the smallest overring of S containing α in which R is polynomially dense.*

Remark 5.10. In general R_α need not be integrally closed, even if R is integrally closed. For example if R is a Dedekind domain *with infinite residue fields*, then $\mathrm{Int}(R) = R[X]$, thus $R_\alpha = R[\alpha]$ and one may choose α in an algebraic extension of K such that $R[\alpha]$ is not integrally closed.

We did not restrict ourselves to the case where α is algebraic in the previous corollary. In fact, if α is transcendental then R_α is clearly isomorphic to $\mathrm{Int}(R)$ and without any assumption on R, we indeed have the following.

Proposition 5.11. *Let R be an infinite domain then* $\mathrm{Int}(R)$ *is the smallest overring of $R[X]$ in which R is polynomially dense.*

Proof. From Lemma 5.3, $\mathrm{Int}(R)$ is contained in every overring of $R[X]$ in which R is polynomially dense. We want to prove that R is polynomially dense in $\mathrm{Int}(R)$ itself. Let $f \in \mathrm{Int}(R, \mathrm{Int}(R))$, and write $f = g_0 + g_1 T + \cdots + g_n T^n$ as a polynomial T with coefficients in the quotient field $K(X)$ of $\mathrm{Int}(R)$. Consider $n+1$ distinct elements $r_0, r_1, \ldots, r_n$ of R, the values $f(r_i)$ of f on these elements are in $\mathrm{Int}(R)$ and a fortiori in $K[X]$. We thus get a system, whose Vandermonde determinant $d = \prod_{i<j} (r_j - r_i)$ is invertible in K. Hence the coefficients of f are in fact in $K[X]$. Now, let $g \in \mathrm{Int}(R)$, then $f(g)$ is an element of $K[X]$ and, $\forall r \in R$, $f(g)(r) = f(g(r)) \in R$ (since $g(r) \in R$), thus $f \in \mathrm{Int}(\mathrm{Int}(R))$. ∎

If R is a Dedekind domain and α is algebraic over its quotient field K, it results from the Krull–Akizuki theorem that S is a Dedekind domain (whether α is separable or not). Thus each overring T of S is determined by a set χ_T of primes of S (namely the primes that lift in T). If the residue fields of R are finite, it results from Lemma 5.5 that R is polynomially dense in T if and only each prime $\mathfrak{m}$ in χ_T is such that, letting $\mathfrak{p} = \mathfrak{m} \cap R$, the ramification index $e(\mathfrak{m}/\mathfrak{p})$ and the residual degree $f(\mathfrak{m}/\mathfrak{p})$ are both equal to 1. The primes of S satisfying this condition form a set Π_1 (and are called the *split primes* of S). On the other hand an overring T of S contains α, if and only if each prime $\mathfrak{m}$ in χ_T is such that $v_{\mathfrak{m}}(\alpha) \geqslant 0$ (where $v_{\mathfrak{m}}$ is the valuation associated to $\mathfrak{m}$). Let Π_α be the subset of Π_1 formed by all the split primes of S which satisfy this second condition. The larger is the set χ_T that determines T, the smaller is the overring T, hence the smallest overring T of S, containing α, in which R is polynomially dense is determined by Π_α. From Corollary 5.9 we thus recover McQuillan's result.

COROLLARY 5.12 (D. McQuillan). *Let R be a Dedekind domain with quotient field K and finite residue fields, $L = K(\alpha)$ an algebraic extension of K and S the integral closure of R in L. The ring of value R_α is then the overring of S determined by Π_α. If in particular $\alpha \in S$, then R_α is determined by Π_1 and so is independent of the choice of α in S.*

6. INCLUSION OF INT(D) IN INT(D')

If D is a domain and D' its integral closure, Int(D) need not be contained in Int(D'). It is however obvious that it is contained in Int(D, D') and R. Gilmer *et al.* [14, Proposition 2.2] have show, that, if D is Noetherian, then Int(D, D') is the integral closure of Int(D). They characterized also the *one-dimensional* Noetherian domains such that Int(D) is contained in Int(D') [14, Proposition 6.1]. We generalize here this result, whatever the dimension of D, proving in particular that, equivalently, D is polynomially dense in D'. Now, from the *Mori-Nagota integral closure theorem* [19, p. 264], D' is a Krull domain (although not necessarily Noetherian). We thus first record the following characterization of the polynomially dense subrings of a Krull domain, which is easily derived from the local properties of the polynomially dense subsets of a Krull domain [Corollary 3.7], and the characterization of the dense subrings of a discrete rank-one valuation domain (e.g. [3, Corollaire 2.5]).

LEMMA 6.1. *Let D be a Krull domain and R a subring of D. The following assertions are equivalent*

1. *R is a polynomially dense subring of D.*

2. *For each height-one prime $\mathfrak{q}$ of D, letting $\mathfrak{p} = \mathfrak{q} \cap R$, if $D/\mathfrak{q}$ is finite, then $R/\mathfrak{p} = D/\mathfrak{q}$ and $\mathfrak{p}D_{\mathfrak{q}} = \mathfrak{q}D_{\mathfrak{q}}$ and if $D/\mathfrak{q}$ is infinite, then $R/\mathfrak{p}$ is infinite.*

Now comes the main theorem of this section.

THEOREM 6.2. *Let D be a Noetherian domain. The following assertions are equivalent*

1. *$\mathrm{Int}(D)$ is contained in $\mathrm{Int}(D')$.*

2. *D is a polynomially dense subset of D'.*

3. *For each height-one prime $\mathfrak{q}$ of D' with finite residue field, letting $\mathfrak{p} = \mathfrak{q} \cap D$, then $D/\mathfrak{p} \simeq D'/\mathfrak{q}$ and $\mathfrak{p}D'_{\mathfrak{q}} = \mathfrak{q}D'_{\mathfrak{q}}$.*

Proof. We first prove the first two assertions to be equivalent: if D is a polynomially dense subset of D', then $\mathrm{Int}(D') = \mathrm{Int}(D, D')$ contains $\mathrm{Int}(D)$. Conversely, if $\mathrm{Int}(D) \subseteq \mathrm{Int}(D')$, then $\mathrm{Int}(D') = \mathrm{Int}(D, D')$, since $\mathrm{Int}(D')$ is integrally closed [4, Proposition 2.1] and contained in $\mathrm{Int}(D, D')$, which is the integral closure of $\mathrm{Int}(D)$.

In the previous lemma we characterized the polynomially dense subrings of a Krull domain. The condition on an height-one prime $\mathfrak{q}$ with finite residue field is exactly the last assertion above and the condition on an height-one prime $\mathfrak{q}$ with infinite residue field is always satisfied since, from the *Mari-Nagata integral closure theorem*, if $D'/\mathfrak{q}$ is infinite, then $D/\mathfrak{p}$ is infinite.

If D is not Noetherian, then $\mathrm{Int}(D, D')$ need not be the integral closure of $\mathrm{Int}(D)$, according to the next example, which stems from a classical pullback construction.

EXAMPLE 6.3. Let $\mathbb{F}_q$ be the finite field with q elements, $\overline{\mathbb{F}}_q$ its algebraic closure, $V = \overline{\mathbb{F}_q}[[t]]$ and $D = \mathbb{F}_q + t\overline{\mathbb{F}}_q[[T]]$. Then D is a pseudo-valuation one-dimensional domain [15] sharing its maximal ideal $\mathfrak{m} = t\overline{\mathbb{F}}_q[[t]]$ with its integral closure $D' = V$. We prove now that $\mathrm{Int}(D, D')$ is not integral over $\mathrm{Int}(D)$. It is easy to check that the polynomial $f = (X^q - X)/t$ is in $\mathrm{Int}(D, D')$. Now, let $a \in \overline{\mathbb{F}}_q$, then $f(at) \equiv -a \pmod{\mathfrak{m}}$ in D'. If f were a root of a monic polynomial of degree n over $\mathrm{Int}(D)$, every element of $\overline{\mathbb{F}}_q$ would then be of degree n over $\mathbb{F}_q$.

Considering more generally a subset E of the quotient field K of D, we had shown that $\mathrm{Int}(E, D')$ is the integral closure of $\mathrm{Int}(E, D)$, if D is a *local* one-dimensional Noetherian domain with finite residue field [4, Proposition 2.3]. From Proposition 3.1, this local result may easily be globalized, hence the same holds if D is a one-dimensional Noetherian domain with finite residue fields. It can even more generally be seen that, if D is a one-dimensional Noetherian domain and E is a subset of K such that, for each maximal ideal $\mathfrak{m}'$ of D' with infinite residue field, E contains infinitely many classes modulo $\mathfrak{m}'$, then $\mathrm{Int}(E, D')$ is the integral closure of $\mathrm{Int}(E, D)$ [4, Remark 2.4]. We lastly give an example of a (pseudo-valuation) one-dimensional Noetherian domain D (with infinite residue field) and a subset E of D, such that $\mathrm{Int}(E, D')$ *is not* the integral closure of $\mathrm{Int}(E, D)$.

EXAMPLE 6.4. Let $V = \mathbb{C}[[t]]$, the ring of power series with coefficients in the field of complex numbers $\mathbb{C}$, and $D = \mathbb{R} + t\mathbb{C}[[t]]$. Then D is a pseudo-valuation one-dimensional Noetherian domain sharing its maximal ideal $\mathfrak{m} = t\mathbb{C}[[t]]$ with its integral closure $D' = V$. Since $D/\mathfrak{m} \simeq \mathbb{C}$ is infinite, then $\mathrm{Int}(D') = D'[X]$ and $\mathrm{Int}(\mathfrak{m}, D') = \mathrm{Int}(tD', D') = D'[X/t]$.

Now we determine $\mathrm{Int}(\mathfrak{m}, D)$. Let $f \in \mathrm{Int}(\mathfrak{m}, D)$, a fortiori $f(tX) \in \mathrm{Int}(D)$, thus $f \in D[X/t]$. Write $f \in \sum_{i=1}^{n} a_i (X/t)^i$, where $a_i \in D$. For each $z \in \mathbb{C}$, $f(zt) \in D$. If some coefficient a_i were not in $\mathfrak{m}$, for $i > 0$, there would be a non-constant real polynomial taking real values on every complex number. Thus $\mathrm{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[X/t]$.

We finally determine the integral closure of $\mathrm{Int}(\mathfrak{m}, D)$. Let $Y = X/t$, then $\mathrm{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[Y]$. The ring $R = D' + \mathfrak{m}[Y]$ shares the ideal $\mathfrak{m}D'[Y]$ with the ring of polynomials $D'[Y]$, which is integrally closed. The quotient $R/\mathfrak{m}D'[Y] \simeq D'/\mathfrak{m}$ is integrally closed in $D'/\mathfrak{m}D'[Y] \simeq (D'/\mathfrak{m})[Y]$, thus R is integrally closed (in its quotient field) [2, Proposition 2]. On the other hand, R is clearly integral over $\mathrm{Int}(\mathfrak{m}, D) = D + \mathfrak{m}D[Y]$. Hence the integral closure of $\mathrm{Int}(\mathfrak{m}, D)$ is the ring $R = D' + \mathfrak{m}D'[X/t]$, which is strictly contained in $\mathrm{Int}(\mathfrak{m}, D') = D'[X/t]$.

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