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Large Time Behavior of Solutions to the Neumann Problem for a Quasilinear Second Order Degenerate Parabolic Equation in Domains with Noncompact Boundary

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We investigate the optimal rate of stabilization at large time of a solution to the Neumann problem

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(x, t, \nabla u)) - b(x, t, u), \qquad \text{in} \quad \Omega \times (0, T), \qquad T > 0$$

$$\sum_{i=1}^{N} a_i(x, t, \nabla u)\, n_i = 0, \qquad \text{on} \quad \partial\Omega \times (0, T)$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega, \qquad u_0(x) \geqslant 0 \qquad \text{in } \Omega,$$

where $\Omega \subset R^N$, $N \geqslant 2$, is an unbounded domain with sufficiently smooth noncompact boundary $\partial\Omega$ satisfying certain isoperimetrical inequality and $n = (n_i)$ is the outward normal to $\partial\Omega$. © 2001 Academic Press

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1. INTRODUCTION

In this paper we consider the following Neumann problem

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (a_i(x, t, \nabla u)) - b(x, t, u), \qquad \text{in} \quad D = \Omega \times (0, +\infty) \qquad (1)$$

$$\sum_{i=1}^{N} a_i(x, t, \nabla u)\, n_i = 0, \qquad \text{on} \quad \partial\Omega \times (0, +\infty) \qquad (2)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \qquad u_0(x) \geqslant 0 \quad \text{in } \Omega. \qquad (3)$$

Here $\Omega \subset R^N$, $N \geqslant 2$, is an unbounded domain with sufficiently smooth noncompact boundary $\partial\Omega$ and $n = (n_i)$ is the outward normal to $\partial\Omega$.

The coefficients $a_i(x, t, \xi)$, $i = 1, 2, ..., N$ and $b(x, t, u)$ are Carathéodory functions satisfying suitable growth conditions; moreover we assume that the following ellipticity condition holds

$$\sum_{i=1}^{N} a_i(x, t, \xi)\, \xi_i \geqslant \nu(x)\, \psi(t)\, |\xi|^{m+1} \quad \text{a.e. } x \in \Omega, \quad t \in\,]0, +\infty[, \quad \forall \xi \in R^N,$$

where $\nu(x)$ and $\psi(t)$ are nonnegative functions verifying additional conditions to be made precise later on. The function $b(x, t, u)$ is a lower order term playing the role of absorption.

A typical example of (1) is the following equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(|x|^{\theta}\, t^{\kappa}\, |\nabla u|^{m-1} \frac{\partial u}{\partial x_i} \right) - \lambda\, u^q,$$

where $0 \leqslant \theta < m$, $0 \leqslant \kappa < 1$, $m > 1$, $\lambda \geqslant 0$ and $q \geqslant 1$.

Our goal is to find the optimal bound of $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ for t large, where u is a nonnegative solution of the above problem with initial datum belonging to $L^1(\Omega)$ or slowly decaying at infinity. Moreover, when $b = 0$, $m > 1$, and u_0 is compactly supported we establish a sharp bound of the interface.

The results presented here obviously hold for the Cauchy problem corresponding to Eq. (1). Optimal bounds of maximum modulus of solution to the Cauchy problem for nonstationary p-laplacian in the nonweighted case can be found in [10, 14, 15]. Analogous results for a porous medium equation are contained in papers [1, 7, 17].

Nonweighted parabolic Neumann problems in domains with noncompact boundary and with L^1-initial datum have been studied in [13] in the linear case and in [5, 26] in the nonlinear one.

For further literature concerning qualitative properties of solutions of parabolic equations the reader can refer for instance to Kalashnikov's survey [16]. Concerning the existence of weak solutions in weighted spaces we quote, among others, the papers [9, 11, 19, 24].

Our approach relies on sharp energy estimates which are essentially based on a sharp form of a weighted Gagliardo–Nirenberg embedding result for a wide class of domains satisfying suitable isoperimetrical properties.

Let us briefly summarize the contents of the paper. After a section devoted to the notations and the statements of the main results, we state and prove an embedding theorem (see Section 3). Sections 4 and 5 contain the proofs of the optimal bounds of the maximum modulus of a non-negative solutions of the problem (1)–(3) with initial data respectively in L^1 and in $L^{p_o} \cap L^\infty$ with $p_o > 1$. In the last section we prove that a solution of the above problem with $m > 1$, $b \equiv 0$, and u_o compactly supported has the property of finite speed of propagation.

2. NOTATIONS, HYPOTHESES AND STATEMENTS OF THE MAIN RESULTS

Let $\Omega \subset R^N$, $N \geqslant 2$, be an unbounded domain, containing the origin, with sufficiently smooth and noncompact boundary $\partial\Omega$. We denote by $x \equiv (x_1, x_2, ..., x_N)$ a point in Ω.

As we have remarked in the Introduction, in order to prove the embedding theorem (see the next section) we need some assumptions on the geometry of Ω; to this aim we introduce the function

$$L(V) = \inf\{\text{meas}_{N-1}(\partial Q \cap \Omega),\ Q \subset \Omega \text{ open with lipschitz boundary},$$
$$meas_N Q = V\}$$

and we give the following

DEFINITION 2.1. Ω belongs to the class $\mathscr{B}_1(g)$ if there exists a positive, nondecreasing function $g \in C(0, +\infty)$ such that $V^{1-1/N}/g(V)$ is nondecreasing and

$$L(V) \geqslant g(V) \qquad \forall V > 0. \tag{1}$$

The above definition implies the existence of two positive constants γ_1, γ_2 such that

$$L(V) \geqslant \gamma_1 V^{(N-1)/N} \qquad \text{for } V \text{ small enough},$$

and

$$L(V) \geqslant \gamma_2 \qquad \text{for } V \text{ sufficiently large.}$$

The first estimate gives us back the classical isoperimetrical inequality in the case of bounded domains with lipschitz boundary (see [21, p. 301]), while the second one characterizes domains which do not contract at infinity.

Let us note, moreover, that examples of domain which do not satisfy condition (1) for small V can be found in [21, pp. 10, 164].

Given $R > 0$, let

$$\Omega_R = \Omega \cap \{x \in R^N : |x| < R\}$$

and

$$V(R) = \text{meas}_N\, \Omega_R.$$

Let us denote by $\mathscr{R}$ the inverse function of $V(R)$.

DEFINITION 2.2. Ω belongs to the class $\mathscr{B}_2(g)$ if $\Omega \in \mathscr{B}_1(g)$ and there exists a constant $c_0 > 0$ such that

$$\mathscr{R}(V) \geqslant c_0 \frac{V}{g(V)} \qquad \forall V > 0. \tag{2}$$

Domains belonging to classes similar to $\mathscr{B}_1(g)$, $\mathscr{B}_2(g)$ were considered by Gushchin [12, 13] and subsequent papers.

It is easy to prove that if $\Omega \in \mathscr{B}_1(g)$ then

$$\mathscr{R}(V) \leqslant N \frac{V}{g(V)} \qquad \forall V > 0. \tag{3}$$

Thus, assuming $\Omega \in \mathscr{B}_2(g)$ essentially amounts to requiring that the volume $V(\rho)$ is equivalent to $\rho g(V(\rho))$. As a matter of fact 5 and 6 are equivalent to

$$(1/N)\, \rho g(V(\rho)) \leqslant V(\rho) \leqslant (1/c_0)\, \rho g(V(\rho)) \qquad \forall \rho > 0. \tag{4}$$

Moreover, from (4) it follows that $|\Omega| = +\infty$, otherwise we would get a contradiction letting $\rho \to +\infty$ in (4).

An example of domain of class $\mathscr{B}_2(g)$ (and then of class $\mathscr{B}_1(g)$) is the paraboloid-like domain

$$\Omega^h = \{x \in R^N : |x'| < x_N^h\},$$

where $|x'| = (x_1^2 + \cdots + x_{N-1}^2)^{1/2}$, $x_N > 1$, $0 \leqslant h \leqslant 1$ (note that Ω^0 is a cylinder and Ω^1 is a cone). In this case (see [5])

$$g(V) = \gamma \min(V^{(N-1)/N}, V^{\eta}), \qquad \eta = \frac{h(N-1)}{h(N-1)+1} \leqslant \frac{N-1}{N}.$$

Let now $v(x)$ be a nonnegative function in Ω and $\tilde{v}(s)$ be the decreasing rearrangement of $\frac{1}{v(x)}$. We assume that

$$v(x) \in L^{\infty}_{loc}(\Omega), \tag{5}$$

$$\lim_{R \to +\infty} \frac{\sup_{\Omega_{2R} \setminus \Omega_R} v(x)}{R^m} = 0 \tag{6}$$

$$\frac{1}{v(x)} \in L^{\alpha}(\Omega), \qquad 1 + \frac{1}{\alpha} < m+1 < N, \qquad \alpha \geqslant \frac{N}{m+1} \tag{7}$$

$$\exists \kappa_1 \in \left]0, \frac{m+1}{N}\right[\qquad \text{such that} \quad h_{\kappa_1}(s) = s^{\kappa_1} \tilde{v}(s) \quad \text{is nondecreasing in}$$

$$]0, +\infty[. \tag{8}$$

Also, let ψ: $]0, +\infty[\to R$ be a monotone nondecreasing function such that

$$\psi \in L^1(0, T) \qquad \forall T > 0. \tag{9}$$

Set

$$\tilde{\psi}(t) = \int_0^t \psi(s)\, ds.$$

Assumptions (5), (7), and (9) are classical in the theory of weighted parabolic equations (see [22]); technical assumption (8) implies that $\tilde{v}(s)$ have power-like behavior and it is necessary, at least for power-like weight (see [8]); moreover hypothesis (6) will be used to obtain a mass estimate (see Corollary 4.1 later on).

Let $a_i(x, t, \xi)$, $i = 1, 2, ..., N$ be Carathéodory functions in $\Omega \times R^{N+1}$ such that the following structural assumptions are satisfied a.e. $(x, t) \in \Omega \times R$, $\forall \xi, \eta \in R^N$:

$$a_i(x, t, 0) = 0 \qquad i = 1, 2, ..., N, \tag{10}$$

$$\sum_{i=1}^{N} (a_i(x, t, \xi) - a_i(x, t, \eta))(\xi_i - \eta_i) \geqslant \nu(x)\, \psi(t)\, |\xi - \eta|^{m+1}, \tag{11}$$

$$\sum_{i=1}^{N} |a_i(x, t, \xi) - a_i(x, t, \eta)| \leqslant \nu(x)\, \psi(t)(|\xi| + |\eta|)^{m-1}\, |\xi - \eta|, \tag{12}$$

where $m > 1$.

Let $b(x, t, \chi)$ be a Carathéodory function in $\Omega \times R^2$ such that the following assumptions are satisfied a.e. $(x, t) \in \Omega \times R$, $\forall \chi, \tilde{\chi} \in R$:

$$b(x, t, 0) = 0, \tag{13}$$

$$(b(x, t, \chi) - b(x, t, \tilde{\chi}))(\chi - \tilde{\chi}) \geqslant \sigma\, |\chi - \tilde{\chi}|^{q+1}, \tag{14}$$

where $q > 1$ and σ is a positive constant.

The previous assumptions are classical in the theory of parabolic equations with general coefficients and are satisfied, for example, if we take

$$a_i(x, t, \xi) = \nu(x)\, \psi(t)\, |\xi|^{m-1}\, \xi_i \qquad i = 1, 2, ..., N,$$

$$b(x, t, \chi) = |\chi|^{q-1}\, \chi.$$

In order to give the definition of weak solution of the problem (1)–(3) we have to specify the functional setting we shall use.

Let $p, \gamma \geqslant 1$; then $W^{1,0}_{p,\gamma}(\nu, \Omega)$ is the space of functions u for which the norm

$$\|u\|_{W^{1,0}_{p,\gamma}(\nu, \Omega)} = \left(\int_\Omega |u|^\gamma\, dx\right)^{1/\gamma} + \left(\int_\Omega \nu(x)\, |\nabla u|^p\, dx\right)^{1/p}$$

is finite. $W^{1,0}_{p,\gamma}(\nu\psi, D_T)$ is the space of functions u for which the norm

$$\|u\|_{W^{1,0}_{p,\gamma}(\nu\psi, D_T)} = \left(\int_{D_T} |u|^\gamma\, dx\, dt\right)^{1/\gamma} + \left(\int_{D_T} \nu(x)\, \psi(t)\, |\nabla u|^p\, dx\, dt\right)^{1/p}$$

is finite. $W^{1,1}_{p,\gamma}(\nu\psi, D_T)$ is the space of functions u for which the norm

$$\|u\|_{W^{1,1}_{p,\gamma}(\nu\psi, D_T)} = \left(\int_{D_T} (|u|^\gamma + |u_t|^\gamma)\, dx\right)^{1/\gamma} + \left(\int_{D_T} \nu(x)\, \psi(t)\, |\nabla u|^p\, dx\right)^{1/p}$$

is finite. Due to assumptions (5), (7), and (9) the above weighted Sobolev spaces are Banach spaces (see [22]).

Now, we are in position to give the definition of a weak solution of problem (1)–(3).

Let $D_T = \Omega \times (0, T)$, $T > 0$, $u_0 \in L^{p_0}(\Omega) \cap L^\infty(\Omega)$, $u_0 \geqslant 0$ a.e. in Ω and $p_0 \geqslant 1$.

DEFINITION 2.3. Let $1 \leqslant p_0 \leqslant 2$. A weak solution of the problem (1)–(3) in D_T is a nonnegative function $u \in W^{1,0}_{m+1,2}(\nu\psi, D_T) \cap L^\infty(D_T)$ such that

$$\int_{D_T} \left[-uv_t + \sum_{i=1}^N a_i(x, t, \nabla u)\, v_{x_i} + b(x, t, u)\, v \right] dx\, dt = \int_\Omega u_0(x)\, v(x, 0)\, dx \tag{15}$$

holds for any $v \in W^{1,1}_{m+1,2}(\nu\psi, D_T)$ such that $v(x, T) = 0$.

A weak solution of the problem (1)–(3) in D is a weak solution of the problem (1)–(3) in D_T for any $T > 0$.

DEFINITION 2.4. Let $p_0 > 2$. A weak solution of the problem (1)–(3) is a nonnegative function $u \in W^{1,0}_{m+1,p_o}(\nu\psi, D_T) \cap L^\infty(D_T)$ such that the identity (15) is satisfied for any $v \in W^{1,1}_{m+1,p_0'}(\nu\psi, D_T)$ such that $v(x, T) = 0$.

A weak solution of the problem (1)–(3) in D is a weak solution of the problem (1)–(3) in D_T for any $T > 0$.

Under the hypotheses (5), (7), (9), (10)–(14) the existence of a solution u of the problem (1)–(3) follows from the results of [9, 11, 18–20, 23].

Now, let us denote by J^{-1} the inverse function of

$$J(V) = V^{m-1} \left[\frac{V}{g(V)} \right]^{m+1} \tilde{\nu}(V). \tag{16}$$

The following theorem concerns the large time behavior of a nonnegative solution of the problem (1)–(3) with initial datum in L^1.

THEOREM 2.1. *Let $\Omega \in \mathscr{B}_1(g)$. Assume that hypotheses (5)–(14) hold and let $u(x, t)$ be a solution of the problem (1)–(3) in D_T and $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Then there exist two positive constants C_1, Γ such that for any $t > 0$ the following estimate is true*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_1 \min\left(\frac{\|u_0\|_{L^1(\Omega)}}{J^{-1}(\Gamma \tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^1(\Omega)})},\ t^{-1/(q-1)} \right). \tag{17}$$

Remark 2.1. We point out that, using approximation arguments, the above theorem still holds under the assumption $u_0 \in L^1(\Omega)$.

Moreover, when $\Omega \in \mathscr{B}_2(g)$ the result of the previous theorem can be sharpened (see Theorem 4.1).

If we take $\Omega \equiv \Omega^h$, $v(x) = |x|^\theta$, $0 \leqslant \theta < m$, $\frac{m+1}{N} > \frac{\theta}{h(N-1)+1}$, $\psi(t) = t^\kappa$, $0 \leqslant \kappa \leqslant 1$, $u_0 \in L^1(\Omega^h) \cap L^\infty(\Omega^h)$, $u_0 \geqslant 0$ then Theorem 2.1 implies that for all $t > 1$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C(N, m)\ \|u_0\|_{L^1}^{(m+1-\theta)/\mathscr{K}}\ t^{-\lambda}, \qquad \text{if} \quad \lambda > \frac{1}{q-1} \tag{18}$$

while

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C(N, m)\ t^{-1/(q-1)} \qquad \text{if} \quad \lambda < \frac{1}{q-1} \tag{19}$$

where

$$\lambda = \frac{(1+\kappa)[1+(N-1)\,h]}{\mathscr{K}}$$

and

$$\mathscr{K} = \mathscr{K}(h, \theta) = (m-1)[1+(N-1)\,h] + m + 1 - \theta.$$

We notice that the number q^* defined by the relationship $\lambda = \frac{1}{q^*-1}$ plays the role of critical exponent for the problem (1)–(3). Moreover, we note that $\mathscr{K}(1, 0) = (m-1)\,N + m + 1$ is the well-known Barenblatt's exponent and the expression (18) when $\theta = 0$, $\kappa = 0$, $h = 1$ is the same as that obtained in [14].

In the case $\kappa = 0$ and $h = 1$ formula (18) was given in [25] for a solution to the Cauchy problem, while for $\kappa = 0$, $\theta = 0$ it was proven in [5].

Let us also remark that (18) provides sharp dependence of maximum modulus of a solution on the parameters of the problem, i.e. on $v(x)$, $\psi(t)$, m and the geometry of Ω^h when $b \equiv 0$ (see Corollary 6.1).

Now let us set

$$J_{p_0}(s) = s^{(m-1)/p_0} \left[\frac{s}{g(s)}\right]^{m+1} \tilde{v}(s), \qquad \forall s > 0. \tag{20}$$

If the initial datum belongs to $L^{p_0}(\Omega) \cap L^\infty(\Omega)$, $p_0 > 1$, we can prove the following

THEOREM 2.2. *Let $\Omega \in \mathscr{B}_1(g)$. Assume that hypotheses (5)–(14) hold and let $u(x, t)$ be a solution of the problem (1)–(3) in D_T and $u_0 \in L^{p_0}(\Omega) \cap L^\infty(\Omega)$, $p_0 > 1$. Then, there exist two positive constants C_2, Λ such that for any $t > 0$*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_2 \frac{\|u_0\|_{L^1(\Omega_{2\tilde{\mathscr{R}}(t)})}}{J^{-1}(\Gamma\tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^1(\Omega_{2\tilde{\mathscr{R}}(t)})})}, \tag{21}$$

where $\tilde{\mathscr{R}}(t)$ is defined by the relationship

$$\frac{\|u_0\|_{L^1(\Omega_{2\tilde{\mathscr{R}}(t)})}}{J^{-1}(\Gamma\tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^1(\Omega_{2\tilde{\mathscr{R}}(t)})})} = \frac{\|u_0\|_{L^{p_0}(\Omega \setminus \Omega_{\tilde{\mathscr{R}}(t)})}}{J^{-1}_{p_0}(\Lambda\tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^{p_0}(\Omega \setminus \Omega_{\tilde{\mathscr{R}}(t)})})^{1/p_0}}. \tag{22}$$

Remark 2.2. When Ω belongs to the class $\mathscr{B}_2(g)$ the result of the previous theorem can be sharpened (see Theorem 5.1). Moreover, in the particular case $\Omega \equiv \Omega^h$, $v(x) = |x|^\theta$ with $0 \leqslant \theta < m$, $\frac{m+1}{N} > \frac{\theta}{h(N-1)+1}$ $\psi(t) = t^\kappa$, $u_0(x) = (1 + |x|)^{-\beta}$, $0 < \beta < 1 + (N-1)\, h$ (see also Remark 5.2), for any $t > 1$ we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant \gamma_1 t^{-\lambda\beta}, \tag{23}$$

where

$$\lambda = \frac{1+\kappa}{m+1-\theta+\beta(m-1)}.$$

Let us note that if $\kappa = \theta = 0$, then (23) can be rewritten as

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant \gamma_1 t^{-\beta/(m+1+\beta(m-1))}$$

and the above estimate reduces to the results of [15, 26].

Let us denote

$$Z(t) = \inf\{\rho > 0 : \operatorname{supp} u(\cdot, t) \subset \Omega_\rho\}.$$

Obviously, $Z(t)$ gives a measure of the speed of propagation of the support of u. When $m > 1$ and the initial datum has compact support we shall prove the property of finite speed of propagation for a solution of problem (1)–(3) without an absorption term. As a matter of fact in Section 6 we prove the following

THEOREM 2.3. *Let the hypotheses of Theorem 2.1 be satisfied and* $b \equiv 0$. *Let* $\operatorname{supp} u_o \subset B_{R_0}$. *Then for all* $t > 0$

$$\tilde{Z}(t) \leqslant C_3(\tilde{Z}(0) + G_0(J^{-1}(\tilde{\psi}(t) \, \|u_0\|_{L^1(\Omega)}^{m-1}))), \tag{24}$$

where

$$\tilde{Z}(t) = \frac{Z(t)}{(\sup_{\Omega_{2Z(t)}} v(x))^{1/(m+1)}}$$

and

$$G_0(s) = \frac{s}{g(s)} \, (\tilde{v}(s))^{1/(m+1)}.$$

3. AN EMBEDDING RESULT

The description of the geometrical characteristics of the domain via isoperimetrical properties allows us to use naturally the symmetrization approach to prove an embedding result. This approach seems to be the most suitable for domains with noncompact boundary.

The following embedding lemma, which has interest in itself, will be crucial in the proofs of Theorems 2.1 and 2.2; the nondegenerate case has been considered in [27].

For the sake of simplicity, from now on we will always denote by c a positive constant, depending only on the data, which may vary from line to line.

LEMMA 3.1. *Let* $\Omega \in \mathscr{B}_1(g)$ *and* $v(x)$ *satisfy the condition* (5) *and*

$$\frac{1}{v(x)^{1/(p-1)}} \in L^1_{loc}(\Omega). \tag{1}$$

Assume, moreover, that there exist constants $\theta, \gamma_0 > 0$, *with* $1 < \theta < p < N$, *such that*

$$\int_0^s \tilde{v}(\tau)^{\theta/(p-\theta)} \, d\tau \leqslant \gamma_0 s \tilde{v}(s)^{\theta/(p-\theta)} \qquad \forall s > 0. \tag{2}$$

Then for any $u \in W^{1,0}_{p,p}(v, \Omega) \cap L^{\beta}(\Omega)$, $0 < \beta < q \leqslant \frac{N\theta}{N-\theta}$ *the inequality*

$$I \geqslant \bar{\gamma} \frac{E_q^{p/q}}{G_1(E_\beta^{q/(q-\beta)}/E_q^{\beta/(q-\beta)})} \tag{3}$$

holds, where

$$I = \int_\Omega v(x)\, |\nabla u|^p\, dx, \qquad E_\gamma = \int_\Omega |u|^\gamma\, dx,$$

$$G_1(s) = s^{p/q-1} \left[\frac{s}{g(s)} \right]^p \tilde{v}(s),$$

provided G_1 *is increasing.*

If $\theta \geqslant \beta$, $\bar{\gamma}$ *depends only on* θ, γ_o, N.

Proof. We first prove the theorem in the case $\theta < q \leqslant \frac{N\theta}{N-\theta}$.

Following the paper [3], we can construct for any $s \in (0, +\infty)$ a measurable set $D(s) \subset \Omega$ such that

(i) $\operatorname{meas}_N D(s) = s$,

(ii) $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$,

(iii) $D(s) = \{x \in \Omega : |u(x)| > r\}$ for $s = \mu(r)$,

where $\mu(r) = \operatorname{meas}_N\{|u(x)| > r\}$ is the distribution function.

Moreover, there exists $\underline{v}(s) > 0$, $s > 0$, such that

$$\int_{D(s)} \frac{1}{v(x)^{1/(p-1)}}\, dx = \int_0^s \frac{1}{\underline{v}(\tau)^{1/(p-1)}}\, d\tau. \tag{4}$$

Let us denote by $u^*(s)$ the decreasing rearrangement of $u(x)$; i.e.,

$$u^*(s) = \inf\{\tau > 0 : \mu(\tau) < s\},$$

and observe that if $\theta < q$ then

$$E_q = \int_0^{+\infty} [u^*(s)]^q\, ds \leqslant c \left(\int_0^{+\infty} [u^*(s)]^\theta\, s^{\theta/q-1}\, ds \right)^{q/\theta}. \tag{5}$$

For any fixed $k > 0$, to be chosen later on, we have

$$\int_0^{+\infty} [u^*(s)]^\theta s^{\theta/q-1}\, ds = \int_0^{\mu(k)} [u^*(s)]^\theta s^{\theta/q-1}\, ds + \int_{\mu(k)}^{+\infty} [u^*(s)]^\theta s^{\theta/q-1}\, ds$$

$$\equiv A_1 + A_2. \tag{6}$$

In order to estimate A_1 we integrate between 0 and $\mu(k)$ the identity

$$\frac{d}{d\sigma}\left(u^*(\sigma)^\theta \int_0^\sigma s^{\theta/q-1}\,ds\right)$$
$$= \theta(u^*(\sigma))^{\theta-1}\frac{du^*(\sigma)}{d\sigma}\int_0^\sigma s^{\theta/q-1}\,ds + (u^*(\sigma))^\theta\,\sigma^{\theta/q-1},$$

then we use Young and Chebychev inequalities and known properties of rearrangements so that

$$A_1 \leqslant c\left[\int_0^{\mu(k)}\left(-\frac{du^*(\sigma)}{d\sigma}\right)^\theta \sigma^{\theta/q+\theta-1}\,d\sigma + \mu(k)^{\theta/q-\theta/\beta}\,E_\beta^{\theta/\beta}\right]$$
$$\equiv c[A_3 + A_4]. \tag{7}$$

Discriminating the cases $\theta < \beta$ and $\theta \geqslant \beta$, one can easily prove that

$$A_2 \leqslant c\mu(k)^{\theta/q-\theta/\beta}\,E_\beta^{\theta/\beta} \tag{8}$$

(the constant c in (8) equals 1 if $\theta \geqslant \beta$).

Applying Hölder's inequality we can estimate A_3 getting

$$A_3 \leqslant \left(\int_0^{\mu(k)}\left(-\frac{du^*(\sigma)}{d\sigma}\right)^p (g(\sigma))^p\,\underline{v}(\sigma)\,d\sigma\right)^{\theta/p}$$
$$\times\left(\int_0^{\mu(k)}\frac{\sigma^{(\theta/q+\theta-1)\,p/(p-\theta)}}{(g(\sigma))^{p\theta/(p-\theta)}\,\underline{v}^{\theta/(p-\theta)}(\sigma)}\,d\sigma\right)^{1-\theta/p}. \tag{9}$$

Let us prove now that

$$\int_0^{+\infty}\left(-\frac{du^*(\sigma)}{d\sigma}\right)^p (g(\sigma))^p\,\underline{v}(\sigma)\,d\sigma \leqslant I. \tag{10}$$

Working as in [3] we get

$$\frac{1}{h}\int_{\{\tau<|u|\leqslant\tau+h\}}|\nabla u|\,dx \leqslant \left(\frac{1}{h}\int_{\{\tau<|u|\leqslant\tau+h\}} v(x)\,|\nabla u|^p\,dx\right)^{1/p}$$
$$\times\left(\frac{1}{h}\int_{\{\tau<|u|\leqslant\tau+h\}} v(x)^{-1/(p-1)}\,dx\right)^{(p-1)/p}$$
$$\forall h > 0.$$

Letting $h \to 0$ it follows that

$$-\frac{d}{d\tau}\int_{\{|u|>\tau\}} |\nabla u|\, dx \leqslant \left(-\frac{d}{d\tau}\int_{\{\tau<|u|\}} v(x)\,|\nabla u|^p\, dx\right)^{1/p} \times \left(-\frac{d}{d\tau}\int_{\{\tau<|u|\}} v(x)^{-1/(p-1)}\, dx\right)^{(p-1)/p}. \quad (11)$$

By the definition of $\underline{v}(s)$ we have

$$-\frac{d}{d\tau}\int_{\{\tau<|u|\}} v(x)^{-1/(p-1)}\, dx = -\mu'(\tau)\,\frac{1}{[\underline{v}(\mu(\tau))]^{1/(p-1)}}. \quad (12)$$

On the other hand by the isoperimetrical property of Ω and the Rishel–Fleming formula we deduce

$$g(\mu(\tau)) \leqslant L(\mu(\tau)) \leqslant -\frac{d}{d\tau}\int_{\{\tau<|u|\}} |\nabla u|\, dx,$$

so, from (11) and (12), it turns out that

$$(g(\mu(\tau)))^p\, \underline{v}(\mu(\tau))\left(-\frac{d\mu(\tau)}{d\tau}\right)^{-(p-1)} \leqslant -\frac{d}{d\tau}\int_{\{\tau<|u|\}} v(x)\,|\nabla u|^p\, dx.$$

Integrating in $]0, +\infty[$ the above inequality and using the identity

$$\frac{d\mu(\tau)}{d\tau} = \frac{1}{\dfrac{d}{d\tau}(u^*(\mu(\tau)))}$$

we obtain (10).

Now, by virtue of Lemma 2.2 in [3] there exists a sequence $\{v_n\}$ such that

$$\left(\frac{1}{v_n}\right)^* = \left(\frac{1}{v(x)}\right)^* \qquad \text{a.e.} \quad (13)$$

and

$$\int_0^{\mu(k)} \frac{ds}{(\underline{v}(s))^{\theta/(p-\theta)}}\, ds = \lim_{n\to+\infty}\int_0^{\mu(k)} \frac{ds}{(v_n(s))^{\theta/(p-\theta)}}. \quad (14)$$

Thus from (13) and (14) and our hypotheses on the weight v we infer

$$\int_0^{\mu(k)} \frac{ds}{(\underline{v}(s))^{\theta/(p-\theta)}}\, ds \leqslant \gamma_0 \mu(k)(\tilde{v}(\mu(k)))^{\theta/(p-\theta)}. \tag{15}$$

The monotonicity of the function

$$\frac{s^{(\theta/q+\theta-1)\,p/(p-\theta)}}{(g(s))^{p\theta/(p-\theta)}},$$

along with inequalities (9), (10), and (15) implies

$$A_3 \leqslant c I^{\theta/p} \frac{(\mu(k))^{\theta/q+\theta-\theta/p}}{(g(\mu(k)))^{\theta}} (\tilde{v}(\mu(k)))^{\theta/p}. \tag{16}$$

From (5), (7), (8), and (16) we deduce

$$\begin{aligned} E_q &\leqslant c\left[I^{\theta/p} \frac{(\mu(k))^{\theta/q+\theta-\theta/p}}{(g(\mu(k)))^{\theta}} ((\tilde{v}(\mu(k)))^{\theta/p} + \mu(k)^{\theta/q-\theta/\beta} E_\beta^{\theta/\beta}\right]^{q/\theta} \\ &\leqslant c\left[I^{1/p} \frac{(\mu(k))^{1/q+1-1/p}}{g(\mu(k))} ((\tilde{v}(\mu(k)))^{1/p} + \mu(k)^{1/q-1/\beta} E_\beta^{1/\beta}\right]^{q} \\ &= c[(IG_1(\mu(k)))^{1/p} + (\mu(k))^{1/q-1/\beta} E_\beta^{1/\beta}]^q. \end{aligned} \tag{17}$$

Now we choose k such that

$$I^{1/p}(G_1(\mu(k))^{1/p} = \mu(k)^{1/q-1/\beta} E_\beta^{1/\beta}$$

or, equivalently,

$$\mu(k) = \Phi^{-1}\left(\frac{E_\beta^{p/\beta}}{I}\right),$$

where $\Phi(s) = G_1(s)/s^{p/q-p/\beta}$ and $\Phi^{-1}(s)$ is its inverse function.

Substituting the above value in (17) we complete the proof in the case $q > \theta$.

To achieve the complete proof we argue in the following way: let q_1 be such that $\beta < q_1 \leqslant \theta$ and choose $q > \theta$; note

$$q_1 = q\frac{q_1-\beta}{q-\beta} + \beta\frac{q-q_1}{q-\beta}.$$

Applying Hölder's inequality we deduce

$$E_{q_1} \leqslant E_q^{(q_1-\beta)/(q-\beta)} E_\beta^{(q-q_1)/(q-\beta)}. \tag{18}$$

The thesis now follows immediately from the first part of the theorem and monotonicity of the right-hand side of (3) with respect to E_q.

Remark 3.1. The assumption (1) and that on the monotonicity of G_1 will be fulfilled in the instances of application of Lemma 3.1, as a consequence of (7), (8) and our choice of the parameters p, q, β, θ.

Remark 3.2. Assume that $v(x)$ satisfy conditions (8); then

$$\int_0^s \tilde{v}(\tau)^{N/(m+1)}\, d\tau \leqslant \gamma_0 s \tilde{v}(s)^{N/(m+1)} \qquad \forall s > 0 \tag{19}$$

with $\gamma_0 = 1/(1-\kappa_1 \frac{N}{m+1})$.

Remark 3.3. If $\Omega = \Omega^1$ and $v(x) \equiv 1$ then inequality (3) is the classical Nirenberg–Gagliardo inequality.

4. PROOF OF THEOREM 2.1

Before proving Theorem 2.1 let us premise some auxiliary results useful in the remainder of the paper.

PROPOSITION 4.1. *Let* u *be a weak solution of the problem* (1)–(3). *Assume that hypotheses* (5), (7), *and* (9) *hold. Let*

$$\operatorname{supp} u_0 \subset B_{R_o}.$$

Then for any $t > 0$ *there exists a positive constant* c, *depending on* $\|u(\cdot, t)\|_\infty$ *too, such that the estimate*

$$H_R(t) \leqslant e\, \|u_0\|^2_{2,\Omega} \exp\left\{-c\left[\left(\frac{(R-R_o)^{m+1}}{t\psi(t)\,\eta(R)}\right)^{1/m}\right]\right\} \qquad \forall R > R_o \tag{1}$$

holds, where

$$H_R(t) = \int_{\Omega\setminus\Omega_R} u^2\, dx + \int_0^t \int_{\Omega\setminus\Omega_R} v(x)\,\psi(\tau)\,|\nabla u|^{m+1}\, dx\, d\tau$$

and

$$\eta(R) = \sup_{\Omega_{2R}\setminus\Omega_R} v(x).$$

Proof. Let $R > R_o$, $\rho \in]0, R]$, and

$$\xi(x) = \begin{cases} 0 & \text{if} \quad |x| \leqslant R \\ \dfrac{|x| - R}{\rho} & \text{if} \quad R < |x| < R + \rho \\ 1 & \text{if} \quad |x| \geqslant R + \rho. \end{cases}$$

Choosing $\xi^{m+1}(|x|)\, u$ as a test function in the weak formulation of problem (1)–(3) we obtain:

$$\frac{1}{2} \int_{\Omega} u^2 \xi^{m+1}\, dx + \int_0^t \int_{\Omega} \sum_{i=1}^{N} a_i u_{x_i} \xi^{m+1}\, dx\, d\tau + \int_0^t \int_{\Omega} b \xi^{m+1} u\, dx\, d\tau$$

$$= -(m+1) \int_0^t \int_{\Omega} \sum_{i=1}^{N} a_i \xi_{x_i} \xi^m u\, dx\, d\tau.$$

Using Young's inequality and growth conditions in the right hand side of the previous inequality we have

$$\frac{1}{2} \int_{\Omega} u^2 \xi^{m+1}\, dx + \int_0^t \int_{\Omega} v(x)\, \psi(\tau)\, |\nabla u|^{m+1}\, \xi^{m+1}\, dx\, d\tau$$

$$\leqslant c\varepsilon^{(m+1)/m} \int_0^t \int_{\Omega} v(x)\, \psi(\tau)\, |\nabla u|^{m+1}\, \xi^{m+1}\, dx\, d\tau$$

$$+ c\varepsilon^{-(m+1)} \int_0^t \int_{\Omega} v(x)\, \psi(\tau)\, |u|^{m+1}\, |\nabla \xi|^{m+1}\, dx\, d\tau.$$

Hence, for $\varepsilon > 0$ sufficiently small, we obtain

$$H_{R+\rho}(t) \leqslant c\, \frac{\eta(R)}{\rho^{m+1}} \int_0^t \psi(\tau) (M(\tau))^{m-1}\, H_R(\tau)\, d\tau, \tag{2}$$

where

$$M(t) = \sup_{\Omega} u(x, t).$$

By the maximum principle (its formal proof can be readily carried out as in [28], Proposition 2) there exists a constant $M_o > 0$ such that

$$M(t) \leqslant M_o \qquad \forall t \in (0, T)$$

and, from (2), we obtain

$$H_{R+\rho}(t) \leqslant cM_o^{m-1} \frac{\eta(R)}{\rho^{m+1}} \int_0^t \psi(\tau)\, H_R(\tau)\, d\tau. \tag{3}$$

Let us absorb, for the sake of simplicity, M_o in the generic constant c.
Let us prove, by induction, the following inequality

$$H_{R_o+k\rho}(t) \leqslant c^k \, \|u_0\|_2^2 \frac{(t\psi(t))^k}{\rho^{(m+1)k} k!} [\eta(R)]^k. \tag{4}$$

Taking u as a test function in the weak formulation of problem (1)–(3),

$$H_{R_o}(t) \leqslant c \, \|u_0\|_2^2 \tag{5}$$

easily follows, which proves (4) for $k=0$.

Let us assume that inequality (4) holds for some integer $k>0$. By virtue of (3) and using (4), we can obtain

$$\begin{aligned} H_{R_o+(k+1)\rho}(t) &\leqslant c \frac{\eta(R)}{\rho^{m+1}} \int_0^t \psi(\tau)\, H_{R_o+k\rho}(\tau)\, d\tau \\ &\leqslant c^{k+1} \, \|u_0\|_2^2 \frac{(t\psi(t))^{k+1}}{\rho^{(m+1)(k+1)}(k+1)!} [\eta(R)]^{k+1}. \end{aligned}$$

Then, inequality (4) holds for any integer $k>0$.

Let $k \geqslant 1$. Choosing in (4) $\rho = (R - R_o)/k$ we obtain

$$H_R(t) \leqslant c^k \, \|u_0\|_2^2 \frac{k^{(m+1)k} [t\psi(t)]^k}{(R-R_o)^{(m+1)k} \, k!} [\eta(R)]^k.$$

From this inequality and also using Stirling's formula it follows that

$$H_R(t) \leqslant \|u_0\|_2^2 \exp\left\{ -k \log \frac{c(R-R_o)^{m+1}}{et\psi(t)\, \eta(R)\, k^m} \right\}. \tag{6}$$

Now, if

$$\frac{c(R-R_o)^{m+1}}{et\psi(t)\, \eta(R)} \leqslant \exp(1)$$

the estimate (1) easily follows from (5). Otherwise, we can obtain (1) from (6) taking as k the integer part of

$$\left\{\frac{c(R-R_o)^{m+1}}{e^2 t\psi(t)\,\eta(R)}\right\}^{1/m}.$$

COROLLARY 4.1 (Mass estimate). *Let u be a weak solution of the problem* (1)–(3). *Assume that hypotheses* (5)–(7) *and* (9) *hold and* supp $u_0 \subset B_{R_o}$. *Then,* $\forall t>0$

$$\int_\Omega u(x,t)\,dx \leqslant \int_\Omega u_0(x)\,dx. \tag{7}$$

Proof. Let $R>0$, $t>0$ and $u(x,\tau)$ be a solution of problem (1)–(3). Taking

$$\xi_R(x)=\begin{cases} 0 & \text{if } \; x\in\Omega_R \\ \dfrac{|x|-R}{R} & \text{if } \; x\in\Omega_{2R}\backslash\Omega_R \\ 1 & \text{if } \; x\in\Omega_{2R} \end{cases}$$

as a test function in the weak formulation of problem (1)–(3), we have

$$\int_0^t\int_\Omega u_\tau\xi_R\,dx\,d\tau+\int_0^t\int_\Omega\sum_{i=1}^N a_i(x,\tau,\nabla u)(\xi_R)_{x_i}\,dx\,d\tau$$
$$+\int_0^t\int_\Omega b(x,\tau,u)\,\xi_R\,dx\,d\tau=0. \tag{8}$$

Using the growth condition (12) and Holder's inequality (with exponents $m+1$ and $\frac{m+1}{m}$), it follows that

$$\left|\int_0^t\int_\Omega\sum_{i=1}^N a_i(x,\tau,\nabla u)(\xi_R)_{x_i}\,dx\,d\tau\right|$$
$$\leqslant\frac{c}{R}(H_R(t))^{m/(m+1)}\left(\int_0^t\int_{\Omega_{2R}\backslash\Omega_R} v(x)\,\psi(\tau)\,dx\,d\tau\right)^{1/(m+1)}$$
$$\leqslant c(H_R(t)\,R^{N/m})^{m/(m+1)}\left(\frac{t\psi(\tau)\,\eta(R)}{R^{m+1}}\right)^{1/(m+1)}. \tag{9}$$

By means of Proposition 4.1 we obtain the following estimate

$$H_R(t)\, R^{N/m} \leqslant c\, \|u_0\|^2_{2,\Omega} \exp\left\{ -\gamma \left[\frac{(R-R_o)^{m+1}}{t\psi(t)\,\eta(R)} \right]^{1/m} \right\} \times \left[\frac{R^{m+1}}{t\psi(t)\,\eta(R)} \right]^{N/m} \left[\frac{t\psi(t)\,\eta(R)}{R^m} \right]^{N/m}. \tag{10}$$

Therefore, from (9), (10) and by virtue of hypothesis (6) we obtain

$$\lim_{R\to+\infty} \int_0^t \int_\Omega \sum_{i=1}^N a_i(x,\tau,\nabla u)(\xi_R)_{x_i}\, dx\, d\tau = 0.$$

Letting $R \to +\infty$ in (8), (7) easily follows.

The next lemma will be crucial in the proof of Theorem 2.1.

Let us denote by $\bar{\gamma}$ the constant involved in (3) with $\theta = \frac{N(m+1)}{N+m+1}$ and $\gamma_0 = 1/(1-\kappa_1 N/(m+1))$. Moreover, for any $r \geqslant 1$ we set

$$\Gamma(r) = \bar{\gamma}(m-1)(r+1)\left(\frac{m+1}{r+m}\right)^{m+1}. \tag{11}$$

LEMMA 4.1. *Let $\Omega \in \mathscr{B}_1(g)$ and $u(x,t)$ be a weak solution of problem* (1)–(3) *in D_T. Suppose that hypotheses* (5)–(9) *are satisfied. Then, for any integer $r \geqslant 1$ the inequalities*

$$\left(\int_\Omega u^{r+1}\, dx\right)^{1/(r+1)} \leqslant \frac{\|u_o\|_{1,\Omega}}{(J^{-1}(\Gamma(r)\,\tilde{\psi}(t)\, \|u_o\|^{m-1}_{1,\Omega}))^{r/(r+1)}}, \tag{12}$$

$$\left(\int_\Omega u^{r+1}\, dx\right)^{1/(r+1)} \leqslant c^{r/(r+1)}\, \|u_o\|^{1/(r+1)}_{1,\Omega}\, t^{-(r/(r+1))(1/(q-1))} \tag{13}$$

hold, where c is a positive constant, independent of r, and $J(s)$ is the function defined in (16).

Proof. Let $r \geqslant 1$; multiplying by both sides of Eq. (1) by u^r and integrating on Ω, we get

$$\frac{d}{dt}\int_\Omega u^{r+1}\, dx = -r(r+1)\int_\Omega \sum_{i=1}^N a_i(x,t,\nabla u)\, u_{x_i} u^{r-1} dx - (r+1)\int_\Omega b(x,t,u)\, u^r\, dx. \tag{14}$$

Using the ellipticity and growth conditions, together with the identity

$$|\nabla u|^{m+1} u^{r-1} = \left(\frac{m+1}{m+r}\right)^{m+1} |\nabla u^{(r+m)/(m+1)}|^{m+1}$$

we obtain

$$\frac{d}{dt} \int_\Omega u^{r+1}\, dx \leqslant -r(r+1) \left(\frac{m+1}{r+m}\right)^{m+1} \psi(t) \int_\Omega \nu(x)\, |\nabla u^{(r+m)/(m+1)}|^{m+1}, \tag{15}$$

$$\frac{d}{dt} \int_\Omega u^{r+1}\, dx \leqslant -c(r+1) \int_\Omega u^{q+r}\, dx. \tag{16}$$

Now, let us estimate the right-hand side of (15) by applying Lemma 3.1 (with $p = m+1$, $q = \frac{m+1}{m+r}(r+1)$, $\theta = \frac{N(m+1)}{N+m+1}$, $\lambda = \frac{(m+1)(r+1)}{m+r}$, $\beta = \frac{m+1}{r+m}$) to the function $u^{(r+m)/(m+1)}$ and also by using the mass estimate (7). Thus it follows that

$$\frac{dE_{r+1}}{dt} \leqslant -r \frac{\Gamma(r)}{m-1} \frac{\psi(t)[E_{r+1}(t)]^{(m+r)/(r+1)}}{G_1(\|u_o\|_{1,\Omega}^{(r+1)/r} / E_{r+1}(t)^{1/r})}. \tag{17}$$

If we set

$$w(t) = \frac{\|u_o\|_1^{(r+1)/r}}{E_{r+1}^{1/r}(t)}$$

we can rewrite (17) as

$$w^{m-2} \tilde{G}_1(w)\, dw \geqslant \frac{\Gamma(r)}{m-1} \psi(t)\, \|u_o\|_{1,\Omega}^{m-1}\, dt, \tag{18}$$

where

$$\tilde{G}_1(s) = \left[\frac{s}{g(s)}\right]^{m+1} \tilde{\nu}(s).$$

Integrating the last inequality between $w(0)$ and $w(t)$ we obtain

$$J(w) \geqslant \Gamma(r)\, \tilde{\psi}(t)\, \|u_o\|_{1,\Omega}^{m-1}$$

which may be rewritten as

$$\frac{\|u_o\|_{1,\Omega}^{(r+1)/r}}{E_{r+1}(t)^{1/r}} \geqslant J^{-1}(\Gamma(r)\,\tilde{\psi}(t)\,\|u_o\|_{1,\Omega}^{m-1}). \tag{19}$$

From the last inequality immediately (12) follows.

Let us prove (13).

Applying the Holder's inequality and the mass estimate (7) we get

$$\int_\Omega u^{r+1}\,dx \leqslant \|u_o\|_{1,\Omega}^{(q-1)/(q+r-1)} \left(\int_\Omega u^{q+r}\,dx\right)^{r/(q+r-1)}.$$

Using this inequality we can estimate the right-hand side of (16) from above obtaining

$$\frac{d}{dt}\int_\Omega u^{r+1}\,dx \leqslant -c(r+1)\,\|u_o\|_{1,\Omega}^{-(q-1)/r}\left(\int_\Omega u^{r+1}\,dx\right)^{(q+r-1)/r}$$

and (13) easily follows integrating the above formula.

Proof of the Theorem 2.1. First of all we notice that letting $r \to +\infty$ in (13) we get

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant ct^{-1/(q-1)}, \qquad \forall t>0. \tag{20}$$

Let

$$p_k = (m+1)^k - \frac{1}{m}, \qquad k=1,2,....$$

Then

$$\frac{p_k+m}{m+1} = p_{k-1}+1.$$

Let us denote

$$E_k(t) = \int_\Omega u(x,t)^{p_k+1}\,dx,$$

$$\tilde{E}_k(t) = \|u_o\|_1^{p_k+1}\,A_k^{p_k+1}(J^{-1}(\Gamma\tilde{\psi}(t)\,\|u_o\|_1^{m-1}))^{-p_k},$$

where

$$\Gamma = \frac{(m-1)\,\bar{\gamma}}{2^{m+1}},$$

$$A_k = \prod_{i=1}^{k} (\theta_o p_i^{\theta_1})^{1/(p_i+1)},$$

with $\theta_0, \theta_1 > 1$ constants to be chosen later on.

Our goal will be to prove that for any $t > 0$ and $k \geqslant 1$ it holds that

$$\tilde{E}_k(t) \geqslant E_k(t). \tag{21}$$

In fact, once inequality (21) is achieved, since

$$\prod_{i=1}^{k} (\theta_o p_i^{\theta_1})^{1/(p_i+1)} \leqslant \exp\left(\sum_{i=1}^{+\infty} \frac{1}{p_i+1} \log(\theta_o p_i^{\theta_1})\right),$$

letting $k \to \infty$ we will obtain

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant c\, \|u_o\|_1\, (J^{-1}(\Gamma \tilde{\psi}(t)\, \|u_o\|_1^{m-1}))^{-1}. \tag{22}$$

Comparing estimates (20) and (22) we will get (17).

We now proceed by induction on k: the validity of (21), for $k = 1$, follows from Lemma 4.1, since $\Gamma(p_1) > \Gamma$ and $A_1 > 1$.

Rewriting (15) for $r = p_{k+1}$ and manipulating therein the constants we can deduce

$$\frac{dE_{k+1}}{dt} \leqslant -\frac{1}{p_{k+1}^{m-1}}\, \psi(t)\, J_k(t), \tag{23}$$

where

$$J_k(t) = \int_\Omega \nu(x)\, |\nabla u(x, t)^{p_k+1}|^{m+1}\, dx.$$

From the interpolation Lemma 3.1 it follows that

$$J_k(t) \geqslant \bar{\gamma}\, \frac{E_{k+1}^{(m-1)/(p_{k+1}-p_k)+1}(t)}{E_k^{(m-1)/(p_{k+1}-p_k)}(t)\, \tilde{G}_1(E_k^{(p_{k+1}+1)/(p_{k+1}-p_k)}(t)/E_{k+1}^{(p_k+1)/(p_{k+1}-p_k)}(t))}. \tag{24}$$

Now we set

$$w(z)=\left[\frac{z^{(N-1)/N}}{g(z)}\right]^{m+1}\tilde{v}(z),$$

$$\lambda_k=\frac{m-1}{p_{k+1}-p_k}+1+\frac{p_k+1}{p_{k+1}-p_k}\,\frac{m+1}{N},$$

$$\beta_k=\frac{m-1}{p_{k+1}-p_k}+\frac{p_{k+1}+1}{p_{k+1}-p_k}\,\frac{m+1}{N}.$$

Working as in [5], from (24) we deduce

$$J_k(t)\geqslant\bar{\gamma}\,\frac{(E_{k+1}(t))^{\lambda_k}}{(E_k(t))^{\beta_k}\,w(E_k^{(p_{k+1}+1)/(p_{k+1}-p_k)}(t)/E_{k+1}^{(p_k+1)/(p_{k+1}-p_k)}(t))}. \quad (25)$$

With the help of Hölder's inequality, mass estimate (7), and the inductive hypothesis we infer

$$J_k(t)\geqslant\bar{\gamma}\,\frac{(E_{k+1}(t))^{\lambda_k}}{(\tilde{E}_k(t))^{\beta_k}\,w(\|u_0\|_1^{(p_{k+1}+1)/(p_{k+1})}/E_{k+1}^{1/p_{k+1}}(t))}.$$

Therefore from (23) we get

$$\frac{dE_{k+1}}{dt}\leqslant-\frac{\bar{\gamma}}{p_{k+1}^{m-1}}\,\psi(t)\,\frac{E_{k+1}^{\lambda_k}}{(\tilde{E}_k(t))^{\beta_k}\,w(\|u_0\|_1^{(p_{k+1}+1)/p_{k+1}}/E_{k+1}(t)^{1/p_{k+1}})}. \quad (26)$$

If we set

$$f_k(t)=\frac{\|u_0\|_1^{(p_{k+1}+1)/p_{k+1}}}{E_{k+1}(t)^{1/p_{k+1}}},$$

from (26) it turns out that

$$w(f_k(t))(f_k(t))^{p_{k+1}\lambda_k-p_{k+1}-1}\,df_k\geqslant\frac{\bar{\gamma}}{p_{k+1}^m}\,\psi(t)\,\frac{\|u_0\|_1^{(p_{k+1}+1)(\lambda_k-1)}}{(\tilde{E}_k(t))^{\beta_k}}\,dt. \quad (27)$$

We notice that assumption (8) and the monotonicity property of $s^{1-1/N}/g(s)$ (recall the definition of the class $\mathscr{B}_1(g)$) imply the monotonicity of the function $w(s)\,s^{\kappa_1}$; moreover, there exists $\kappa_2>0$ such that the function $s^{\kappa_2}/J^{-1}(s)$ is nondecreasing (we can take $\kappa_2=m-1+(m+1)/N-\kappa_1$).

These two facts give

$$\int_{f_k(0)}^{f_k(t)} w(s)\, s^{p_{k+1}(\lambda_k-1)-1}\, ds \leqslant \frac{w(f_k(t))(f_k(t))^{(\lambda_k-1)\,p_{k+1}}}{(\lambda_k-1)\,p_{k+1}-\kappa_1},$$

$$\int_0^t \psi(\tau)(\tilde{E}_k(\tau))^{-\beta_k}\, d\tau \geqslant A_k^{-\beta_k(p_k+1)}\, \|u_0\|_1^{-(p_k+1)\,\beta_k} \frac{(J^{-1}(a(t)))^{p_k\beta_k}}{(\tilde{\psi}(t))^{p_k\beta_k\kappa_2}} \times \int_0^t \psi(\tau)(\tilde{\psi}(\tau))^{p_k\beta_k\kappa_2}\, d\tau, \tag{28}$$

where

$$a(t) = \Gamma\tilde{\psi}(t)\, \|u_0\|_1^{m-1}.$$

Moreover, we have

$$\int_0^t \psi(\tau)(\tilde{\psi}(\tau))^{p_k\beta_k\kappa_2}\, d\tau = \frac{(\tilde{\psi}(t))^{\kappa_2\beta_k p_k+1}}{\kappa_2\beta_k p_k+1}$$

and also

$$\int_0^t \psi(\tau)(\tilde{E}_k(\tau))^{-\beta_k}\, d\tau \geqslant \frac{A_k^{-\beta_k(p_k+1)}}{1+\kappa_2 p_k\beta_k}\, \|u_0\|_1^{-\beta_k(p_k+1)}(J^{-1}(a(t)))^{p_k\beta_k}\, \tilde{\psi}(t). \tag{29}$$

Integrating (27) and using inequalities (28) and (29) we have

$$w(f_k(t))(f_k(t))^{(\lambda_k-1)\,p_{k+1}} \geqslant \frac{\bar{\gamma}}{p_{k+1}^m}\, \frac{(\lambda_k-1)\,p_{k+1}-\kappa_1}{1+\kappa_2\beta_k p_k}\, \|u_0\|_1^{m-1} \times A_k^{-\beta_k(p_k+1)}(J^{-1}(a(t)))^{\beta_k p_k}\, \tilde{\psi}(t). \tag{30}$$

Let $\phi(z) = w(z)\, z^{p_{k+1}(\lambda_k-1)}$. Then from (30) it turns out that

$$f_k(t) \geqslant \varphi^{-1}[c(m, N)\, A_k^{-\beta_k(p_k+1)} p_{k+1}^{-m}(J^{-1}(a(t)))^{\beta_k p_k}\, a(t)], \tag{31}$$

where

$$c(m, N) = \frac{m-1}{2^{m+1}}\, \frac{m+1}{N+\kappa_2(m+1)} < 1.$$

Since the function

$$s^{1/(p_{k+1}(\lambda_k-1))}/\varphi^{-1}(s)$$

is nondecreasing, it follows that

$$\varphi^{-1}(\delta s) \geqslant \delta^{1/(p_{k+1}(\lambda_k - 1))} \varphi^{-1}(s), \qquad \forall 0 < \delta < 1.$$

Therefore, from (31), after calculations we obtain

$$E_{k+1}(t) \leqslant \frac{\|u_o\|_1^{p_{k+1}+1}}{[J_{-1}(a(t))]^{p_k+1}} \{A_k [(1/\sigma)^{1/N} p_{k+1}^{m/N}]^{1/(p_{k+1}+1)}\}^{p_{k+1}+1}$$

whence the estimate (21) follows by choosing

$$\theta_o = (1/\sigma)^{1/N}, \qquad \theta_1 > 1.$$

If $\Omega \in \mathscr{B}_2(g)$ then the result of Theorem 2.1 can be sharpened as the following theorem shows.

Theorem 4.1. *Let $\Omega \in \mathscr{B}_2(g)$ and the hypotheses of Theorem 2.1 be satisfied.*

Assume the function

$$\frac{\mathscr{R}(s)}{s^{1/N}}$$

is nondecreasing in $]0, +\infty[$.

Then, there exist two positive constants C_3, Γ such that for any $t > 0$ the estimate

$$\|u(\cdot, t)\|_{\infty, \Omega} \leqslant C_3 \min\left(\frac{\|u_o\|_{1,\Omega}}{\mathscr{P}^{-1}(\Gamma \tilde{\psi}(t)\, \|u_o\|_{1,\Omega}^{m-1})}, t^{-1/(q-1)}\right)$$

holds, where $\mathscr{P}^{-1}(s)$ is the inverse function of

$$\mathscr{P}(s) = s^{m-1} \mathscr{R}^{m+1}(s)\, \tilde{v}(s). \tag{32}$$

Proof. We can proceed as in the proof of Theorem 2.1, after observing that, due to (2), inequality (18) may be replaced by

$$w^{m-2}(t)\, \mathscr{R}^{m+1}(w(t))\, \tilde{v}(w(t))\, dw \geqslant \frac{c_0^{m+1} \Gamma(r)}{m-1}\, \psi(t)\, \|u_0\|_{1,\Omega}^{m-1}\, dt.$$

Since the function

$$\frac{\mathscr{R}(s)}{s^{1/N}}$$

is nondecreasing, then the function

$$\mathscr{R}^{m+1}(s)\,\tilde{v}(s)$$

is also nondecreasing and from the above inequality we obtain

$$\mathscr{R}^{m+1}(w(t))\,\tilde{v}(w(t))(w(t))^{m-1} \geqslant c_0^{m+1}\,\Gamma(r)\,\|u_0\|_{1,\Omega}^{m-1}\int_0^t \psi(\tau)\,d\tau.$$

Now the thesis readily follows.

5. PROOF OF THEOREM 2.2

Before going into the details of the proof the following technical results will be useful.

PROPOSITION 5.1. *Let* $u_1(x,t)$ *and* $u_2(x,t)$ *be solutions of problem* (1)–(3) *with initial data* $u_{o1}(x), u_{o2}(x) \in L^{p_0} \cap L^{\infty}(\Omega)$. *Assume that hypotheses* (10)–(14) *hold. If* $u_{01}(x) \leqslant u_{02}(x)$ *for a.e.* $x \in \Omega$, *then* $u_1(x,t) \leqslant u_2(x,t)$.

Proof. The proof follows immediately taking as a test function in the weak formulation of problem (1)–(3) $(w^+(x,t))^{p_o-1}$, where

$$w(x,t) = u_1(x,t) - u_2(x,t).$$

PROPOSITION 5.2. *Under the same assumptions of Proposition* 5.1, *for any* $p \geqslant p_o - 1$ *and* $t > 0$ *we have*

$$\int_\Omega |w|^{p+1}\,dx + \gamma\int_0^t\int_\Omega |w|^{p+q}\,dx\,dt \leqslant \int_\Omega |u_{01} - u_{02}|^{p+1}\,dx, \qquad (1)$$

where $w(x,t) = u_1(x,t) - u_2(x,t)$.

Proof. It is sufficient to take $|w|^{p-1}w$ as a test function in the weak formulation of problem (1)–(3).

LEMMA 5.1. *Let* $\Omega \in \mathscr{B}_1(g)$. *Assume hypotheses* (5)–(9), (11), (14) *be satisfied. Let* $u_i(x,t)$, $i = 1, 2$, *be solutions of the problem* (1)–(3) *in* D_T

respectively with initial data $u_{0i} \in L^{p_0}(\Omega) \cap L^{\infty}\Omega$, $p_0 > 1$, $i = 1, 2$. *Then there exist* C_4, C_5, $\Lambda > 0$ *such that for any* $t > 0$ *the inequalities*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_4 \frac{\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)}}{[J_{p_0}^{-1}(\Lambda\tilde{\psi}(t)(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)})^{m-1})]^{1/p_0}} \tag{2}$$

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_5 t^{-1/(q-1)} \tag{3}$$

hold, where J_{p_0} *is the function defined in* (20).

Proof. Set $w(x, t) = u_1(x, t) - u_2(x, t)$; let us multiply both sides of (1) by $v(x, t) = |w(x, t)|^r$, $r > 0$, and integrate.

Thus it turns out that

$$\frac{d}{dt}\int_\Omega |w|^{r+1}\,dx + (r+1)\,r\int_\Omega \left[\sum_{i=1}^N (a_i(x, t, u_1) - a_i(x, t, u_2))\, w_{x_i}\,|w|^{r-1}\right] dx$$
$$+ (r+1)\int_\Omega (b(x, t, u_1) - b(x, t, u_2))\,|w|^r\,dx = 0. \tag{4}$$

Due to hypotheses (11) and (14) we get

$$\frac{d}{dt}\int_\Omega |w|^{r+1}\,dx + r(r+1)\left(\frac{m+1}{m+r}\right)^{m+1}\psi(t)\int_\Omega v(x)\,|\nabla w^{(r+m)/(m+1)}|^{m+1}\,dx$$
$$+ \sigma(r+1)\int_\Omega |w|^{r+q}\,dx \leqslant 0. \tag{5}$$

Thus

$$\frac{d}{dt}\int_\Omega |w|^{r+1}\,dx \leqslant -r(r+1)\left(\frac{m+1}{m+r}\right)^{m+1}$$
$$\times \psi(t)\int_\Omega v(x)\,|\nabla w^{(r+m)/(m+1)}|^{m+1}\,dx, \tag{6}$$

$$\frac{d}{dt}\int_\Omega |w|^{r+1}\,dx \leqslant -(r+1)\,\sigma\int_\Omega |w|^{r+q}\,dx. \tag{7}$$

As in the proof of Theorem 2.1, inequality (7) implies the estimate (3). Now, setting

$$E_\gamma(t) = \int_\Omega |w|^\gamma\,dx$$

and estimating the right-hand side of (6) by applying Lemma 3.1 to the function $w^{(r+m)/(1+m)}$ with

$$p = m+1, \qquad q = \frac{(r+1)(m+1)}{m+r}, \qquad \beta = \frac{(m+1)\,p_0}{r+m}, \quad \theta = \frac{N(m+1)}{m+1+N}$$

we deduce

$$\frac{d}{dt}E_{r+1}(t) \leqslant -\frac{r\Gamma(r)}{m-1}\,\frac{\psi(t)(E_{r+1}(t))^{(r+m)/(r+1)}}{G_1((E_{p_0}(0))^{(r+1)/(r+1-p_0)}/(E_{r+1}(t))^{p_0/(r+1-p_0)})}, \tag{8}$$

where $\Gamma(r)$ is the constant defined by (11).

Assume $r > p_0 - 1$; arguing as in the proof of Lemma 4.1 the previous inequality can be rewritten as

$$(\omega(t))^{(m-1)/p_0 - 1}\,\tilde{G}_1(\omega(t)) \geqslant \frac{r\Gamma(r)}{m-1}\,\frac{p_0}{r+1-p_0}\,(E_{p_0}(0))^{(m-1)/p_0}\,\psi(t), \tag{9}$$

where

$$\omega(t) = \frac{E_p(0)^{(r+1)/(r+1-p_o)}}{E_{r+1}(t)^{p_o/(r+1-p_o)}}.$$

Integrating the last inequality between 0 and t we infer

$$E_{r+1}(t) \leqslant (E_{p_0}(0))^{(r+1)/p_0}\,[J_{p_0}^{-1}(\Lambda(r, p_0)(E_{p_0}(0))^{(m-1)/p_0}\,\tilde{\psi}(t))]^{(r-1-p_0)/p_0}, \quad \forall r > p_0 - 1,$$

where $\Lambda(r, p_0) = (r/(r+1-p_0))\,\Gamma(r)$.

Actually, starting from the above inequality, the proof of (2) can be rigorously performed as in Theorem 2.1 using a similar iterative process. In fact, choosing

$$p_k = p_0(m+1)^k - 1/m, \qquad k = 1, 2, ...,$$

by induction on k we can prove

$$E_{p_k+1}(t) \leqslant A_k^{p_k+1}\,\frac{\|w_0\|_{p_0}^{p_k+1}}{[J_{p_0}^{-1}(\Lambda\,\|w_0\|_{p_0}^{m-1}\,\tilde{\psi}(t))]^{(p_k+1-p_0)/p_0}},$$

where

$$\Lambda = \frac{m-1}{(p_0+1)^{m+1}}\,\bar{\gamma}$$

and

$$A_k = \prod_{i=1}^{k} (\theta_0 p_i^{\theta_1})^{1/(p_i+1)}$$

with θ_0, θ_1 suitably chosen positive constants.

Remark 5.1. If $\Omega \in \mathscr{B}_2(g)$, then in the thesis of Lemma 5.1 the function J_{p_0} may be replaced by

$$\mathscr{P}_{p_0}(s) = s^{(m-1)/p_0} \mathscr{R}^{m+1}(s)\, \tilde{\nu}(s).$$

In fact, we can start from (9) arguing as in the proof of Theorem 4.1.

Proof of the Theorem 2.2. Let $u_0 \in L^{p_0}(\Omega) \cap L^\infty(\Omega)$ and $\zeta_R(x)$ be the smooth cut-off function in Ω_{2R} introduced in the proof of Theorem 2.1 (that is $\zeta_R(x) = 1$ in Ω_R and $\zeta_R(x) = 0$ outside of Ω_{2R}).

Then $u_{0R}(x) = u_0(x)\, \zeta_R(x) \in L^1(\Omega) \cap L^\infty(\Omega)$.

Let $u_R(x, t)$ be a solution of (1)–(3) with initial datum u_{0R} and set

$$\omega_R(x, t) = u(x, t) - u_R(x, t).$$

From Theorem 2.1 and Lemma 5.1 we deduce

$$\begin{aligned} \|u(\cdot, t)\|_{L^\infty(\Omega)} &\leqslant \|u_R(\cdot, t)\|_{L^\infty(\Omega)} + \|\omega_R(\cdot, t)\|_{L^\infty(\Omega)} \\ &\leqslant c \left[\frac{\|u_0\|_{L^1(\Omega_{2R})}}{J^{-1}(\Gamma \tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^1(\Omega_{2R})})} + \frac{\|u_0\|_{L^{p_0}(\Omega \setminus \Omega_R)}}{J^{-1}_{p_0}(\Lambda \tilde{\psi}(t)\, \|u_0\|^{m-1}_{L^{p_0}(\Omega\setminus\Omega_R)})^{1/p_0}} \right]. \end{aligned} \tag{10}$$

If now $\tilde{\mathscr{R}}(t)$ is defined through formula (22), plugging in (10) $R = \tilde{\mathscr{R}}(t)$ the thesis immediately follows.

THEOREM 5.1. *Assume $\Omega \in \mathscr{B}_2(g)$ and the hypotheses of Theorem 2.2 hold.*

Let, in addition, the function

$$\frac{\mathscr{R}(w)}{w^{1/N}}$$

be nondecreasing in $]0, +\infty[$.

Then in the thesis of Theorem 2.2 the function J^{-1} may be replaced by $V(W(s))$ where $W(s)$ is the inverse function of

$$V(s)^{m-1}\, s^{m+1} \tilde{\nu}(V(s)).$$

Proof. Indeed, as we observed in Theorem 4.1, the function $J(s)$ may be replaced by $\mathscr{P}(s)$ and direct calculations show that $\mathscr{P}^{-1}(s)=V(W(S))$.

Remark 5.2. Suppose $\Omega \in \mathscr{B}_2(g)$ and $v(x)\equiv v(|x|)$ with $v(|x|)$ a non-decreasing function. Assume $u_0(x)=(V(|x|))^{-\sigma}$, $0<\sigma<1$, $p_0\sigma>1$, $|x|>1$. Then, as a consequence of the previous theorem, for any $t>1$ there exist positive constants $\gamma_1, \gamma_2, \gamma_3$ and a function $\mathscr{F}(t)$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant \frac{\gamma_1}{(V(\mathscr{F}(t)))^{\sigma}}, \tag{11}$$

where $\mathscr{F}(t)$ satisfies for any $t>1$

$$\gamma_2\tilde{\psi}(t) \leqslant \frac{\mathscr{F}(t)^{m+1}\,[V(\mathscr{F}(t))]^{\sigma(m-1)}}{v(\mathscr{F}(t))} \leqslant \gamma_3\tilde{\psi}(t).$$

6. PROOF OF THEOREM 2.3

We prove now the property of the finite speed of propagation for the problem (1)–(3) with $b\equiv 0$ following the approach of [5, 6].

Define a sequence of cut-off functions $\{\zeta_n\}$, $n\geqslant 1$, so that

$$\zeta_n \equiv 1, \qquad x\in \Omega_{\rho_n}\backslash\Omega_{\bar{\rho}_n},$$
$$\zeta_n = 0, \qquad x\notin \Omega_{\rho_{n-1}}\backslash\Omega_{\bar{\rho}_{n-1}},$$
$$|D\zeta_n| \leqslant 2^{n+1}/(\sigma\rho),$$

where $0<\sigma<1/2$ is given and

$$\rho_n=\rho+\sigma 2^{-n}\rho, \qquad \bar{\rho}_n=(\rho-\sigma 2^{-n}\rho)/2, \quad \rho\geqslant 4R_0.$$

Using in the weak formulation the test function $\zeta_n^{m+1}u^\theta$, $\theta>0$, after standard calculations as in [28, pp. 91–92], one gets

$$\sup_{0<\tau<t}\int_{\Omega(\tau)} u^{1+\theta}\zeta_n^{m+1}\,dx+\int_0^t\int_\Omega u^{\theta-1}v(x)\,\psi(\tau)\,|\nabla u|^{m+1}\,\zeta_n^{m+1}\,dx\,d\tau$$
$$\leqslant c\left(\frac{2^n}{\sigma}\right)^{m+1}\frac{v_\rho}{\rho^{m+1}}\int_0^t\psi(\tau)\int_{\Omega_{\rho_{n-1}}\backslash\Omega_{\bar{\rho}_{n-1}}} u^{m+\theta}\,dx\,d\tau \qquad \forall n\geqslant 1, \quad (1)$$

where $\Omega(\tau)=\{x\in R^N : (x,\tau)\in\Omega\times(0,T)\}$ and $v_\rho\equiv\sup_{\Omega_{2\rho}} v(x)$.

Setting $v_n = u^{(m+\theta)/(m+1)}\zeta_n^\lambda$, with $\lambda \geqslant \frac{m+\theta}{\theta+1}$, we have for $n \geqslant 1$

$$Y_n(t) \equiv \sup_{0<\tau<t} \int_{\Omega(\tau)} v_n^\varepsilon \, dx + \int_0^t \int_\Omega \nu(x)\, \psi(\tau)\, |\nabla v_n|^{m+1} \, dx \, d\tau$$

$$\leqslant c \left(\frac{2^n}{\sigma}\right)^{m+1} \frac{\nu_\rho}{\rho^{m+1}} \int_0^t \psi(\tau) \left(\int_\Omega v_{n-1}^{m+1} \, dx\right) d\tau, \tag{2}$$

where $\varepsilon = \frac{(m+1)(1+\theta)}{m+\theta}$.

Now we put

$$F_1(s) = \bar{\gamma} s^{(m-1)/(1+\theta)+m+1} (g(1/s))^{m+1} (\tilde{\nu}(1/s))^{-1},$$

$$F_2(s) = [F_1^{-1}(s)]^{(m-1)/(1+\theta)}$$

(as usual $\bar{\gamma}$ is the constant involved in (3) with $\theta = \frac{N(m+1)}{N+m+1}$ and $\gamma_0 = 1/(1 - \frac{\kappa_1 N}{m+1})$).

Thus the embedding result (Lemma 3.1, applied with $q = p = m+1$, $\beta = \varepsilon$, $u = v_{n-1}$) implies

$$\int_0^t \psi(\tau) \, d\tau \int_\Omega v_{n-1}^{m+1} \, dx$$

$$\leqslant \int_0^t \psi(\tau) \left(\int_\Omega v_{n-1}^\varepsilon \, dx\right)^{(m+1)/\varepsilon}$$

$$\cdot F_2 \left[\left(\int_\Omega \nu(x)\, |\nabla v_{n-1}|^{m+1} \, dx\right) \Big/ \left(\int_\Omega v_{n-1}^\varepsilon \, dx\right)^{(m+1)/\varepsilon} \right] d\tau.$$

Suppose by now F_2^{-1} is convex; then, by elementary reasoning, we infer that for any fixed constant $c > 0$ the function $sF_2(\frac{c}{s})$ is nondecreasing for $s > 0$.

Thus applying Jensen's inequality we obtain

$$\int_0^t \psi(\tau) \, d\tau \int_\Omega v_{n-1}^{m+1} \, dx \leqslant Y_{n-1}^{(m+1)/\varepsilon} \, \tilde{\psi}(t) \, F_2 \left(\frac{1}{\tilde{\psi}(t) \; Y_{n-1}^{(m+1)/\varepsilon - 1}} \right). \tag{3}$$

If F_2^{-1} were not convex then, reasoning as in [5], there would exist a convex function Φ such that

$$\gamma_0 \, F_2^{-1}(s) \leqslant \Phi(s) \leqslant F_2^{-1}(s);$$

therefore we might replace F_2^{-1} with Φ when invoking Jensen's inequality and then switch back to F_2^{-1} again, exploiting the above two-sided estimate.

Then from (2) it turns out

$$Y_n \leqslant c\left[\frac{2^n}{\sigma}\right]^{m+1} \frac{\nu_\rho}{\rho^{m+1}} \tilde{\psi}(t)\ Y_{n-1}^{(m+\theta)/(1+\theta)} F_2\left(\frac{1}{\tilde{\psi}(t)\ Y_{n-1}^{(m-1)/(1+\theta)}}\right). \qquad (4)$$

Due to hypothesis (8) and Definition 2.1, the function $F_2(1/s)\, s^{\alpha_1}$ with $\alpha_1 = N(m-1)/((m+1-\kappa_1 N)(1+\theta)+N(m-1))$ is nondecreasing and therefore from (4) we get

$$Y_n \leqslant c\,\frac{4^{n(m+1)}}{(2\sigma^2)^{m+1}}\,\frac{\nu_\rho}{\rho^{m+1}}\,\tilde{\psi}(t)\, f_t(I_0)\ Y_{n-1}^{1+\delta}, \qquad \forall n \geqslant 2, \qquad (5)$$

where

$$I_0 := \frac{\nu_\rho}{\rho^{m+1}} \int_0^t \int_{\Omega_{2\rho}} \psi(\tau)\, u^{m+\theta}\, dx\, d\tau,$$

$$f_t(s) = (\bar{\gamma}_1 s)^{((m-1)/(1+\theta))\,\alpha_1}\, F_2\left(\frac{1}{\tilde{\psi}(t)(\bar{\gamma}_1 s)^{(m-1)/(1+\theta)}}\right),$$

$$\delta = \frac{(m-1)(1-\alpha_1)}{1+\theta}.$$

Hence, using Lemma 5.6, Chapter II of [18] we have that $Y_n \to 0$ as $n \to +\infty$; i.e., $u(x, t) \equiv 0$, $x \in \Omega_\rho \backslash \Omega_{\rho/2}$, provided

$$Y_2^\delta\, \frac{\nu_\rho}{\rho^{m+1}}\, \tilde{\psi}(t)\, f_t(I_0) \leqslant \frac{(2\sigma^2)^{m+1}}{c4^{(m+1)/\delta}} \equiv \gamma_0.$$

The last inequality is true if

$$\left[c\left(\frac{2}{\sigma}\right)^{m+1}\right]^\delta \bar{\gamma}_1^{((m-1)(\alpha_1-1))/(1+\theta)}\, (\bar{\gamma}_1 I_0)^{(m-1)/(1+\theta)}\, \tilde{\psi}(t)$$
$$\times\left(F_1^{-1}\left(\frac{1}{\tilde{\psi}(t)(\bar{\gamma}_1 I_0)^{(m-1)/(1+\theta)}}\right)\right)^{(m-1)/(1+\theta)}$$
$$\leqslant \gamma_0\, \frac{\rho^{m+1}}{\nu_\rho}. \qquad (6)$$

Since (recalling the definition of G_0 in Theorem 2.3)

$$F_1(s) = \frac{s^{(m-1)/(1+\theta)}}{G_0^{m+1}(1/s)},$$

inequality (6) can be rewritten as

$$\bar{\gamma}_1 I_0 \leqslant \left[\tilde{\psi}(t)\, F_1 \left(1/G_0^{-1} \left(\tilde{\gamma}_0 \frac{\rho}{(v_\rho)^{1/(m+1)}} \right) \right) \right]^{-(1+\theta)/(m-1)}. \tag{7}$$

On the other hand, by the definition of I_0, Theorem 2.1, and Corollary 4.1 we have

$$I_0 \leqslant c_1 \frac{v_\rho}{\rho^{m+1}} \tilde{\psi}(t) \frac{\|u_0\|_{L^1(\Omega)}^{m+\theta}}{(J^{-1}(\Gamma \tilde{\psi}(t)\, \|u_0\|_{L^1(\Omega)}^{m-1}))^{m+\theta-1}}.$$

Therefore it is sufficient to estimate ρ from the following inequality

$$\frac{c_2}{\Gamma^{(m+\theta)/(m-1)}} \frac{v_\rho}{\rho^{m+1}} \frac{(a(t))^{(m+\theta)/(m-1)}}{(J^{-1}(a(t)))^{m+\theta-1}} \leqslant \left[F_1 \left(1/G_0^{-1} \left(\gamma_3 \frac{\rho}{(v_\rho)^{1/(m+1)}} \right) \right) \right]^{-(1+\theta)/(m-1)}, \tag{8}$$

where $a(t) = \Gamma \tilde{\psi}(t)\, \|u_0\|_{L^1(\Omega)}^{m-1}$.

Noticing now that

$$a(t) = (J^{-1}(a(t)))^{m-1}\, G_0(J^{-1}(a(t))),$$

formula (8) becomes

$$J^{-1}(a(t))\, G_0^{(m+1)((m+\theta)/(m-1))}(J^{-1}(a(t))) \leqslant c_3 G_0^{-1} \left(\gamma_3 \frac{\rho}{(v_\rho)^{1/(m+1)}} \right) \left(\frac{\rho^{m+1}}{v_\rho} \right)^{(m+\theta)/(m-1)}. \tag{9}$$

If we now put

$$\frac{\rho}{(v_\rho)^{1/(m+1)}} = \gamma^\star G_0^{-1}(J^{-1}(a(t))) \tag{10}$$

for a sufficiently large constant $\gamma^\star$, then (9) is satisfied and therefore (24) is proven.

COROLLARY 6.1. *Let* $\Omega \in \mathscr{B}_2(g)$ *and* $v(x) = |x|^\theta$, $0 < \theta < m$. *Suppose, moreover, that the assumptions of Theorem* 2.3 *be satisfied. Then*

$$Z(t) \sim W(a(t)) \tag{11}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \geqslant \frac{\|u_0\|_{L^1(\Omega)}}{V(W(a(t)))}, \tag{12}$$

where

$$a(t) = \Gamma\tilde{\psi}(t)\ \|u_0\|^{m-1}_{L^1(\Omega)}$$

and $W(s)$ is the inverse function of

$$V(s)^{m-1}\, s^{m+1}\tilde{v}(V(s)).$$

Proof. To prove (11) we observe first that, being

$$\tilde{v}(s) = \frac{1}{(\mathcal{R}(s))^\theta}, \qquad \eta(\rho) \sim \rho^\theta,$$

it follows that

$$G_0(s) \sim (\mathcal{R}(s))^{1-\theta/(m+1)}.$$

On the other hand, from Remark 5.1 and (10), it turns out that

$$\rho^{(m+1-\theta)/(m+1)} = \gamma^\star(W(a(t)))^{(m+1-\theta)/(m+1)} \tag{13}$$

which means

$$Z(t) \leqslant \gamma W(a(t)). \tag{14}$$

From the mass estimate (Corollary 4.1) and Theorem 2.1 we get

$$\|u_0\|_{L^1(\Omega)} \leqslant \|u(\cdot,t)\|_{L^\infty(\Omega)}\ V(Z(t)) \leqslant \gamma\,\frac{\|u_0\|_{L^1(\Omega)}}{V(W(a(t)))}\ V(Z(t)) \tag{15}$$

whence (11) follows by exploiting (14).

To conclude we observe that (12) easily follows from (11) and (15).

Remark 6.1. Let $\Omega \in \mathcal{B}_2(g)$, $b \equiv 0$ and $v(x) = |x|^\theta$, $0 < \theta < m$. Suppose that $u_0(x) = V(|x|)^{-1}$, $|x| \geqslant 1$.

Then, for sufficiently large t, the two-sided estimate

$$\gamma_1\,\frac{\ln \mathcal{P}(\tau)}{V(\mathcal{P}(\tau))} \leqslant \|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \gamma_2\,\frac{\ln \mathcal{P}(\tau)}{V(\mathcal{P}(\tau))}$$

holds, where $\mathscr{P}(\tau)$ is the inverse function of $V(R)^{m-1}\,R^{m+1-\theta}$ and τ is related to t through the relationship

$$\tilde{\psi}(t) = \frac{\tau}{(\ln \mathscr{P}(\tau))^{m-1}}.$$

In fact, following the outlines of the proof of Theorem 2.2 we can find the upper-bound putting in (10) $R = \mathscr{P}(\tau)$.

In order to obtain the lower–bound, we observe at first that from the mass conservation law we deduce

$$\int_\Omega u_{oR}\,dx = \int_\Omega u_R(x, t)\,dx \qquad \forall t > 0, \tag{16}$$

where $u_{oR} = u_o \zeta(|x|)$, ζ is the usual cut-off function in the ball B_R, and u_R is a solution of the problem with initial datum u_{oR}.

On the other hand by the comparison principle it follows that

$$u_R(x, t) \leqslant u(x, t) \qquad \text{a.e.} \quad x \in \Omega, \qquad \forall t > 0 \tag{17}$$

and easy calculations give

$$\gamma_1 \ln \mathscr{P}(\tau) \leqslant \int_{B_{Z(t)}} u(x, t)\,dx \leqslant \|u(\cdot, t)\|_{L^\infty(\Omega)}\, V(Z(t)). \tag{18}$$

Moreover, from formula (24) we deduce

$$Z(t) \leqslant \gamma_3 \mathscr{P}(\tau) \tag{19}$$

and combining together (18) and (19) we obtain

$$\gamma_1 \ln \mathscr{P}(\tau) \leqslant \|u(\cdot, t)\|_{L^\infty(\Omega)}\, V(\mathscr{P}(\tau)).$$

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