THE TOPOLOGY OF COMPLETE MINIMAL SURFACES OF FINITE TOTAL GAUSSIAN CURVATURE

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(Received 8 July 1980)

RECENTLY GREAT progress has been made in the classical theory of minimal surfaces in $\mathbb{R}^3$. For example, the proof of the embedding of the solution to Plateau’s problem for an extremal Jordan curve [10] and the embedding of a solution to the free boundary value problem [11], the proof of the bridge theorem [12], the regularity and finiteness of least area oriented surfaces bounding a smooth Jordan curve [5], the uniqueness and topological uniqueness of certain minimal surfaces [8, 9, 12], the proof that a complete stable minimal surface is a plane [1, 3], the existence of a collection of 4 Jordan curves which bound a continuous family of compact minimal surfaces [14], the existence of a complete minimal surface contained between two parallel planes [13] and a theorem [18] which states that if the Gauss map of a complete minimal surface misses 7 points on the sphere, then the minimal surface is a plane.

The aim of this work is to begin the exploration of the topological properties of complete minimal surfaces of finite total curvature. First we recall the fundamental classical result of Chern–Osserman [6] that states that such surfaces $M$ are conformally equivalent to compact Riemann surfaces with punctured in a finite number of points. Furthermore, they prove that the Gauss map on the surface $M$ extends conformally to $\hat{M}$. In particular, the theorem of Chern–Osserman implies that such surfaces have finite topological type with a finite number of topological ends and that the normal vectors at infinity on these ends are well defined. (Finite topological type means in this case that $M$ is diffeomorphic to the interior of a compact surface with boundary.)

Our first theorem gives a nice description of the topological placement in $\mathbb{R}^3$ of a complete surface of finite topological type which has well defined normal vectors at infinity. We prove that the ends of such surfaces behave like the ends of the catenoid at infinity. More precisely, Theorem 1 shows that if $f: M \to \mathbb{R}^3$ is an immersion of such a surface, then $f$ is proper and the image $f(M)$ viewed from infinity looks like a finite collection of flat planes (with multiplicity) that pass through the origin. For example, the catenoid viewed from infinity looks like two oppositely oriented copies of a plane passing through the origin. Theorem 1 has a natural generalization to submanifolds of arbitrary codimension in $\mathbb{R}^n$. This generalization which is Theorem 2 is discussed in §2.

In §3 we apply Theorems 1 and 2 to derive some interesting topological properties for complete minimal submanifolds of $\mathbb{R}^n$. For example, suppose that $M$ is a complete minimal surface with finite total curvature. If $M$ is embedded in $\mathbb{R}^3$, then we observe that $M$ viewed from infinity looks like a plane passing through the origin. Using this observation, we prove that $M$ is a flat plane or $M$ has at least two topological ends. In particular, if $M$ is not a flat plane, then $M$ is not simply connected. Of course, there are interesting immersed examples, such as Enneper’s surface, which are simply connected. Therefore, the embedding property is an essential part of this topological result.

In §4 we give a topological-geometric interpretation to a formula concerning the total curvature $C(M)$ for a complete minimal surface $M$ in $\mathbb{R}^3$. It follows from this
formula that if $M$ has $k$ ends and $C(M)$ is finite, then
\[ C(M) = 2\pi(\chi(M) - k) \]
if and only if the ends of $M$ are embedded. This formula gives rise to a characterization of the catenoid as being the unique embedded complete minimal annulus of finite total curvature.

In §5 we give for every $k > 1$ an example of a complete minimal planar domain $\Omega$ with $k$ ends and finite total curvature $C(\Omega) = 2\pi(\chi(\Omega) - k)$. Thus, by the earlier stated geometric interpretation for $C(M)$, the ends of these surfaces are embedded. In fact, these examples are also invariant under a large group of symmetries. The existence of these examples is interesting when compared with the nonexistence results in §6. In §6 it is shown that an embedded complete planar domain of finite total curvature in $\mathbb{R}^3$ cannot have three, four or five ends. The proofs of the existence and nonexistence of these surfaces are based on an analysis of the Weierstrass representation for these surfaces.

In §7, we show that a complete embedded minimal surface $M$ in $\mathbb{R}^3$ having $C(M)$ finite disconnects $\mathbb{R}^3$ into two regions $N_1, N_2$ such that
\[ \lim_{r \to \infty} \frac{\text{Vol}(B_r \cap N_1)}{\text{Vol}(B_r \cap N_2)} = 0, \infty, 1 \]
where $B_r$ denotes the ball of radius $r$ and where 1 occurs if and only if $M$ has an odd number of ends. We also give a formulation of the topology of $N_1$ and $N_2$ in terms of this limit.

In §8, we prove a theorem that deals with the topological uniqueness of complete embedded minimal surfaces $M$ with $C(M)$ finite. We prove that if $M_1$ and $M_2$ are two diffeomorphic examples with two ends, then there is a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(M_1) = M_2$. The existence of $f$ depends in an essential way on the topological uniqueness theorems in [9].

Having a good idea of what a complete minimal surface should look like and how it should behave at infinity is essential in finding embedded examples and in classifying these examples. It is the point of view of this work that a topological understanding of examples can be useful in acquiring a deeper understanding of the analytic properties of the examples. To date, the plane, the helicoid and the catenoid are the only known embedded examples with finite topological type. Hopefully, this work will provide a basis for the discovery of new surfaces (see [21]).

§1. SURFACES IN $\mathbb{R}^3$ WITH WELL DEFINED NORMAL VECTORS AT THEIR ENDS

Let $M$ be the complement of a finite number of points $p_1, \ldots, p_k$ on a compact oriented surface $\hat{M}$. We will consider those immersions of $M$ in $\mathbb{R}^3$ that are complete and which have well defined normal vectors at the points at infinity. In other words, the Gauss map has a continuous extension $\hat{g} : \hat{M} \to S^2$.

We will denote by $Y_r$, $r > 0$, the intersection of $M$ with the sphere of radius $r$ and center at the origin. Let $X_r$ be the radial projection of $Y_r$ onto $S^2$, or,

\[ X_r = \frac{1}{r} Y_r. \]

Suppose $n_r \in S^2$ is the normal vector $\hat{g}(p_r)$ at the point at infinity $p_r$.

**Theorem 1.** Let $M$ be an immersed complete surface in $\mathbb{R}^3$ that is diffeomorphic to the complement of a finite number of points $p_1, \ldots, p_k$ on a compact oriented surface $\hat{M}$. Suppose that the Gauss map on $M$ extends continuously to $\hat{M}$. Then (1) $M$ is proper.
(2) For large $r$, $X_r = \{y_1', y_2', \ldots, y_k'\}$ consists of $k$ immersed closed curves on $S^2$.

(3) $\gamma'$ converges $C^1$ to a geodesic with multiplicity on $S^2$ as $r$ goes to infinity.

(4) If $M$ is a minimal surface, then the convergence in (3) is $C^\infty$.

Thus $M$ viewed from infinity looks like a finite number of planes passing through the origin.

Since the proof of Theorem 1 is somewhat long and computational, we first give a brief idea behind the proof. First it is sufficient to understand the behavior of a closed annular end $A$ of $M$ corresponding to a punctured disk $D - \{p_i\}$ in $\bar{M}$ and where the normal vectors to $A$ form a small angle with the extended normal vector $v = \overrightarrow{G(p_i)}$ at infinity on $A$. If the dimensions involved were one less, in other words $M$ was a curve in $\mathbb{R}^2$ with well-defined unit tangent vector $T$ at an end $A$, then, after expressing $A$ as a graph over the $T$ axis, elementary trigonometry shows that $A$ is proper and that

$$\lim_{r \to \infty} \left(\frac{1}{r} A\right) \cap S^1 = T \in S^1$$

where $S^1$ is the unit circle in $\mathbb{R}^2$. The idea behind the proof of theorem 1 is to reduce the question one dimension by intersecting surface $M$ by planes $P$ perpendicular to $A$ at infinity and then analyzing the behavior of the curves in $P \cap A$ which we can understand with trigonometric formulae.

**Proof.** Let $W: A \to \mathbb{R}^3$ be a parametrization of a fixed closed annular neighbourhood of one of the points at infinity of $M$. Suppose that

$$v = \lim_{z \to \infty} \theta(W(z))$$

and that

$$\langle v, \theta(W(z)) \rangle = \cos \theta$$

$$\geq \sqrt{3}/2 \text{ for } 0 \leq \theta \leq \pi/6 \quad (1)$$

for all $z \in A$. Let $\pi$ be a plane containing the line generated by $v$ and let $\Gamma = W^{-1}(\pi)$. It follows by transversality of $W$ and $\pi$ that $\Gamma$ consists of points in $\partial A$ and connected curves. Let $\gamma$ be a connected component of $\Gamma$ that is a curve.

We will consider coordinates $(t, y)$ in $\pi$ such that the $y$-axis is the line generated by $v$. It follows from (1) that the tangent vector of $W(\gamma)$ is never colinear with $v$. Thus $W(\gamma)$ is the graph of a function $y(t)$. The angle between the normal vector $(-y', 1)$ of $W(\gamma)$ and the vector $v$ is less than or equal to $\theta$. Therefore,

$$\frac{1}{\sqrt{1 + (y')^2}} \geq \cos \theta$$

which implies that

$$|y'(t)| \leq \tan \theta, \text{ for all } t. \quad (2)$$

If $\gamma$ is compact it follows that the extremal points of $W(\gamma)$ are contained in $W(\partial A)$. Thus

$$\sup_{x \in W(\gamma)} \|x\| \leq d_0 + 2d_1 \quad (3)$$
where

\[ d_0 = \sup_{x \in \partial W} \|x\| \]
\[ d_1 = \text{diameter of } W(\partial A). \]

If \( W(\gamma) \) is unbounded in \( M \), then its length is infinite. It follows from (2) that \( y(t) \) is defined in the interval \((-\infty, a], [a, \infty) \) or \((-\infty, \infty)\).

Let \( C_r \) be the solid cylinder of radius \( r \) whose axis is the line generated by \( v \). Let \( \tilde{M} \) be the annulus \( A \) with the metric induced by \( W \) so that \( W: \tilde{M} \to \mathbb{R}^3 \) is an isometric immersion.

**Lemma 1.** \( W^{-1}(C_r) \) is a compact set of \( \tilde{M} \). In particular, the immersion \( M \) is proper.

*Proof of the lemma:* We will denote by \( \tilde{d} \) the distance on \( \tilde{M} \). Choose \( r > 0 \) such that \( W(\partial A) \) is contained in \( C_r \). Let \( x \in W(A) \cap C_r \) and let \( \pi' \) be a plane containing both the point \( x \) and the line generated by \( v \). Let \( \tilde{x} \in \tilde{M} \) such that \( W(\tilde{x}) \equiv x \). Consider a connected curve \( \gamma \) in \( W^{-1}(\pi') \) containing \( \tilde{x} \). We know that \( W(\gamma) \) is the graph of a function \( y(t) \) in \( \pi' \) with \( x = (t_0, y(t_0)) \). Observe that \( |t_0| \leq r \). If the domain of \( y(t) \) is the interval \((-\infty, a], [a, \infty) \) or \([a, b]\), then \((a, y(a)) \in W(\partial A) \) and

\[ 42r \leq 54r. \]

Assume now that \( t \) varies from \(-\infty\) to \(+\infty\). Let \( \pi_t \) be the plane passing through the point \((t, y(t))\) of \( W(\gamma) \), orthogonal to \( \pi' \) and parallel to the line generated by \( v \). Let \( \gamma_t \) be the connected curve in \( W^{-1}(\pi_t) \) that contains the point \( \tilde{x} \). If \( \gamma_t \) intersects \( \partial \tilde{M} \), then (4) holds. We assert that there exists \( t \in (-r, r) \) such that \( W^{-1}(\pi_t) \) contains some curve \( \gamma_t \) intersecting both \( \gamma \) and \( \partial \tilde{M} \). If not, then \( W(\gamma_t) \) is a graph in \( \pi_t \) over the \( t \)-axis of \( \pi_t \). As \( t_0 \) varies along the \( t \)-axis of \( \pi' \), \( W(\gamma_t) \) describes some surface \( \tilde{M}_0 \) that is a graph over the plane orthogonal to the vector \( v \). Then \( W^{-1}(\tilde{M}_0) \) is some connected component of \( \tilde{M} \) without border which contradicts the fact that \( \tilde{M} \) is connected and has border. This proves the assertion.

If \( \gamma_t \) is given by the assertion above, then in the same way as in (4) we have

\[ \tilde{d}(\gamma, \partial \tilde{M}) \leq 4r. \]

Let \( t_1 \) be a point on the \( t \)-axis of \( \pi' \) such that \( W(\gamma_t) \cap \pi' = (t_1, y(t_1)) \). It follows easily from the triangle inequality that

\[ \tilde{d}(\tilde{x}, \partial \tilde{M}) \leq 4r + \left| \int_{t_0}^{t_1} \sqrt{1 + \left( \frac{dy}{dt} \right)^2} \, dt \right| \leq 8r \]

which proves the lemma.

We now conclude the proof of Theorem 1. Let \( r_0 = d_0 + 2d_1 \) where \( d_0 \) and \( d_1 \) are defined in (3). Then \( W(\partial A) \) is contained inside the solid cylinder \( C_{r_0} \). By Lemma 1, the set \( W^{-1}(C_{r_0}) \) is compact. Set

\[ r_1 = \sup \{ \|W(z)\| \mid z \in W^{-1}(C_{r_0}) \} \]

and \( r_2 > \max \{r_0, r_1\} \) such that

\[ \frac{r_0 + r_1}{r_2} + \tan \frac{\pi}{6} < \frac{\sqrt{3}}{2}. \]
Then $W(\partial A)$ is contained inside the sphere $S^2(r_2)$ of radius $r_2$ and center at the origin. By Lemma 1 and by $\lim_{[z] \to \infty} \mathcal{Q}(W(z)) = \nu$, there exists a subannulus $A'$ contained in $A$ such that:

(i) (1) holds for $z \in A'$,

(ii) $W(z)$ is outside $C_{r_2}$ for $z \in A'$.

Let $\pi$ be the plane containing $W(z)$, $z \in A'$, and the axis of $C_{r_2}$. Let $\gamma$ be a connected component of $W^{-1}(\pi)$ containing $z$. Let $y(t)$ be such that $W(y(t))$ is the graph of $y(t)$ in $\pi$. By the transversality of $\pi$ and $W(A)$ and since $W(\partial A) \subset C_{r_2}$, $W(\gamma)$ intersects $C_{r_2}$.

Then $y$ is defined at the point $r_0$ or $-r_0$. We may assume that $y$ is defined at $r_0$. Then

$$|y(r_0)| \leq \|(x_0, y(r_0))\| \leq r_1.$$ 

Let $z \in A'$ and $W(z) = (r, y(r)), r > r_0$. By (2) it follows that

$$|y(t)| \leq |y(r_0)| + \left| \int_{r_0}^{r} y'(s) \, ds \right| \leq r_0 + r_1 + r \tan \theta.$$

Then, if $W(z) = (r, y(r))$, we have

$$\left( \frac{W(z)}{||W(z)||} \right) \leq \frac{r_0 + r_1}{r} + \tan \theta, z \in A'$$

$$< \sqrt{3}/2.$$ 

Set $r_2 > \sup \{ ||W(z)|| \mid z \in (A - A') \}$. We now prove that $W$ and $S^2(\gamma)$ are transverse for $r \geq r_3$. If $W$ and $S^2(\gamma)$ are not transverse, then there exists $z \in W^{-1}(S^2(\gamma))$ such that

$$\mathcal{Q}(W(z)) = \frac{W(z)}{||W(z)||}.$$ 

Since $W(A - A')$ lies inside $S^2(\gamma)$, we have that $z \in A'$ and (1) and (7) give a contradiction. Thus $W$ is transverse to $S^2(\gamma)$ for all $r \geq r_3$. We restrict $W$ to $A'$.

Then, by Lemma 1, the function $h: A' \to \mathbb{R}$ defined by

$$h(z) = ||W(z)||^2, z \in A',$$

is proper. By (1) and (7) it follows that $h$ does not have critical points. If $r > r_3$, then $h^{-1}(r)$ is a compact curve that not intersect $\partial A'$. Hence $h^{-1}(r)$ is a finite collection of Jordan curves. If $h^{-1}(r)$ has more than one Jordan curve, then there is a compact domain $\Omega \subset A'$ such that $\partial \Omega$ is the union of Jordan curves of $h^{-1}(r)$. Then $h$ has a maximum or minimum in the interior of $\Omega$ that is impossible. This shows that $h^{-1}(r)$ is a single Jordan curve, that says

$$\Gamma' = W(A) \cap S^2(\gamma)$$

is an immersion of $S^1$ and proves item (2) in Theorem 1.

We observe that $\theta$ of (7) goes to zero as $r$ goes to infinity. By (7) the curve $y' = 1/r\Gamma'$ is contained in a strip of $S^2$ that converges to a great circle $S$ as $r$ goes to infinity. Also, by (1), the angle between the tangent vector of $\Gamma'$ and $\nu$ goes to $\pi/2$ as $r$
goes to infinity. Hence, \( \Gamma' \) makes at least one loop around the direction \( v \) and \( \gamma' \) converges \( C^0 \) to \( S \) as \( r \) goes to infinity.

Let \( \alpha(\varphi), \varphi \in \mathbb{R} \), be a parametrization by arc length of the great circle \( S \). Let \( \beta(\varphi) \) be a parametrization of \( \gamma' \) such that \( \beta(\varphi) \) lies in the great circle of \( S^2 \) that contains \( v \) and \( \alpha(\varphi) \). We have that

\[
\left( \frac{\beta'}{\|\beta\|}, \alpha \right)^2 = 1 - \left( \frac{\beta'}{\|\beta\|}, \alpha \right)^2 - \left( \frac{\beta'}{\|\beta\|}, v \right)^2.
\]

Since \( \beta' \) is orthogonal to \( \alpha(\beta) \), we have that \( \langle \beta', \alpha \rangle \) goes to zero as \( r \) goes to infinity. Since \( \gamma' \) converges \( C^0 \) to \( \alpha \), it follows that \( \langle \beta', \alpha \rangle \) converges to \( 0 \) as \( r \) goes to infinity. Therefore, \( \gamma' \) converges \( C^1 \) to the great circle \( S \), with multiplicity, and item (3) is proved.

We now prove that if \( M \) is minimal, then \( \gamma' \) converges \( C^\alpha \) to \( S \). Let \( \pi \) be the plane orthogonal to \( v \) and containing the origin. Let \( \Omega \) be the annulus \( \{ p \in \pi : 1/2 \leq \| p \| \leq 2 \} \). Set

\[
M_\alpha = (1/r M) \cap (\Omega \times \mathbb{R}).
\]

The orthogonal projection of \( M_\alpha \) onto \( \Omega \) is a covering of \( \Omega \) and locally we may write \( M_\alpha \) as a graph of a function \( f \), defined over an angular sector of \( \Omega \). It follows from the \( C^0 \) convergence of \( M_\alpha \) and convergence properties for minimal surfaces (see, e.g. Corollary (15.7) in [4]) that all derivatives of \( f \), of order less than \( j + 1 \), \( j \) an integer, are uniformly bounded by a constant \( K_{j+1} \). Since \( f \) converges \( C^0 \) to \( f = 0 \) and the inclusion map of the space of \( C^{j+1} \) functions into the space of \( C^j \) functions is absolutely continuous, it follows that \( f \) converges \( C^j \) to \( f = 0 \). In particular, the intersection of \( M_\alpha \) with \( S^2 \) converges \( C^i \) to \( S \) with multiplicity for all \( j \). This completes the proof of the theorem.

\[ 82. \] **GENERALIZATIONS TO CODIMENSION K IMMERSIONS IN \( \mathbb{R}^n \)**

In this section \( M \) will denote the interior of an \( m \)-dimensional compact manifold \( \tilde{M} \) with boundary. The intersection of a closed regular neighborhood of a component of the boundary of \( \tilde{M} \) with \( M \) will be called an end of \( M \). We will consider those immersions \( f: M \rightarrow \mathbb{R}^n \) which are complete in the induced metric. Let \( G_{l,n} \) be the Grassmannian of \( l \)-planes in \( \mathbb{R}^n \). Let \( G: M \rightarrow G_{l,n}, l = n - m \), be the Gauss or normal map for the immersion \( f \) of \( M \).

**Theorem 2.** Let \( E \) be a closed end of a complete immersion \( f: M \rightarrow \mathbb{R}^n \) of dimension \( m \). Suppose that there exists an \( (n-m) \)-plane \( P \) such that the angle \( \theta \) between \( P \) and the tangent planes to \( E \) is greater than some positive number \( \epsilon \). Then

1. There is a closed subend \( E' \) of \( E \) which is diffeomorphic to \( S^{n-1} \times [0, \infty) \) and satisfies: (i) \( Y_r = f(E') \cap S^{n-1} \) is an immersed \((m - 1)\)-sphere for large \( r \) (or a point if \( m = 1 \)). (ii) \( f|E' \) is an embedding if \( m \neq 2 \).
2. If the normal planes converge to \( P \) at infinity, then

\[
X_r = \frac{1}{r} (f(E) \cap S^{n-1})
\]

converges \( C^1 \) to a totally geodesic subsphere \( S^{n-1} \) in \( S^{n-1} \). Furthermore, if \( m \neq 2 \) or if \( m = 2, n = 3 \) and \( f|E' \) is an embedding, then \( X_r \) converges to \( S^{n-1} \) with multiplicity one.
Proof. The proof of Theorem 2 follows closely the proof of Theorem 1. The proof when \( m = 1 \) follows immediately from the trigametric formulae given in Theorem 1. Suppose now that \( m \) is greater than 1. Let \( S^m \) be the sphere of radius \( r \) in the complementary orthogonal subspace \( P^1 \) of \( P \) in \( \mathbb{R}^n \) and let \( B^m \) be the ball of radius \( r \) in \( P^1 \). Let \( C = B^m \times P \) be the cylinder of radius \( r \) over \( B^m \).

For each \( x \in E \), consider the intersection of the \((n - m + 1)\) plane \( \Delta \) passing through \( P \) and \( f(x) \). The intersection of \( \Delta \) with \( f(E) \) consists of a collection of curves which are graphs over the "\( t \)" axis. With these notations, the proof of Theorem 1 shows that \( f \mid E \) is proper and transverse to the boundary of \( C \), and that \( \gamma' = \partial C \cap f(E) \) is a compact immersed manifold such that for large \( r \) \( \gamma' \) disconnects \( E \) into two components \( E' \) and \( E'' \) where \( E' \) is the unbounded component. Furthermore, the proof of Theorem 1 shows that if \( r: \mathbb{R}^n \to \mathbb{R}^1 \) is orthogonal projection, then \( r \mid E' \) is a covering space of \( Z = \mathbb{R}^1 - \mathbb{B}^m \).

When \( m \) is greater than two, \( Z \) is simply connected which implies that \( f \mid E' \) is an embedding and a graph over \( Z \). If \( m \) equals two, then it follows from the classification of two dimensional surfaces that \( E' \) is diffeomorphic to \( S^1 \times \{0, \infty\} \). As \( f \mid E \) is proper, \( \pi \mid E' \) is a finite-to-one covering mapping. In any case, \( \gamma' \) is an immersed sphere which proves part (1) of the theorem.

Part (2) follows from the proof of part (1) together with the arguments in the proof of Theorem 1. The only thing that is new would be the part of (2) concerned with the case when \( m = 2, n = 3 \) and \( f \mid E \) is an embedding. In this case the circles \( 1/r \gamma' \) defined above are embedded and converge \( C^1 \) to a closed geodesic with multiplicity. However, the embedding of \( 1/r \gamma' \) implies the convergence is one-to-one. This concludes the proof of the theorem.

§3. APPLICATIONS TO MINIMAL SUBMANIFOLDS OF \( \mathbb{R}^n \).

The following theorem is as easy corollary to Theorem 2.

**Theorem 3.** Let \( M^k \subset \mathbb{R}^{2k-1} \) be an embedded complete submanifold of finite topological type and with well defined limiting normal planes on each end. Then the components of

\[
X_r = \frac{1}{r} M_k \cap S^{2k-2}
\]

converge \( C^1 \) to the same subsphere \( S^{k-1} \) of \( S^{2k-2} \) as \( r \) goes to infinity. Thus, \( M^k \) viewed from infinity looks like a single \( k \)-plane \( P \) passing through the origin.

**Proof.** By part (2) of Theorem 2 each component \( C_i \) of \( X_r \) converges \( C^1 \) to a totally geodesic sphere \( S_i \) in \( S^{2k-1} \). If \( S_i \) and \( S_j \) are distinct, then they intersect transversely in \( S^{2k-1} \). The \( C^1 \)-convergence property in Theorem 2 implies \( C_i \) and \( C_j \) intersect transversely for large \( r \) which is impossible since \( M^k \) is embedded. This contradiction proves the theorem.

**Corollary.** If \( M \) is a complete embedded minimal surface in \( \mathbb{R}^3 \) with finite total curvature, then the normal vectors on the ends of \( M \) are parallel. Thus, after a rotation of \( M \), the Gauss map \( G: M \to S^2 = \mathbb{R} \cup \{-\infty, \infty\} \) has zeros or poles at the ends of \( M \).

**Theorem 4.** Suppose that \( f: M \to \mathbb{R}^n \) is a codimension-one complete minimal immersion and that \( M \) is diffeomorphic to the interior of a compact manifold with one boundary component. Then exactly one of the following cases occurs.
(1) \(M\) is a hyperplane.

(2) The closure of the image of the Gauss map on the end of \(M\) is not contained in the interior of an open hemisphere of \(S^{n-1}\).

(3) The end of \(M\) is not embedded, \(n = 3\), and the total curvature of \(M\) is finite.

Proof. If case 1 does not hold and \(n > 3\), then it follows from the proof of Theorem 2 that there is an open ball \(B\), of radius \(r\) contained in a hyperplane \(P'\) of \(\mathbb{R}^n\) so that an end \(E\) of \(M\) is graph over \(P' - B\). The proof of Theorem 1 in [7] applies to show that \(M\) is a graph over \(P'\). Theorem 2 in [14] states that if \(M\) is a graph but not a hyperplane, then part (2) must hold.

Suppose now that \(n = 3\) and that case 2 does not occur. Osserman [15] has shown that either \(C(M)\) is finite or for almost all points \(v\) in \(S^2\), \(G^{-1}(v)\) is an infinite set where \(G\) is the Gauss map for \(M\). In the case we are considering, there is a compact set \(X\) of \(M\) such that outside \(X\) the Gauss map is contained in an \(\epsilon\)-neighborhood \(N\) of a hemisphere. Therefore, for any \(v\) in \((S^2 - N)\), \(G^{-1}(v)\) is finite. Thus, \(C(M)\) must be finite. If the end of \(M\) is embedded, then Theorem 2 part (2) and Theorem 2 in [14] show that \(M\) is a graph and hence \(M\) is a plane. As the cases (1)–(3) are mutually exclusive, the theorem is proved.

Corollary. An embedded complete minimal surface of finite total curvature in \(\mathbb{R}^3\) is either a plane or has at least two ends.

§4. A GEOMETRIC FORMULA FOR THE TOTAL CURVATURE

In [6] it is shown that for a complete minimal surface \(M\) in \(\mathbb{R}^n\) of finite total Gaussian curvature \(C(M)\), \(C(M)\) is less than or equal to \(2\pi(\chi(M) - k)\) where \(k\) is the number of ends of \(M\). We shall now give an interpretation of this inequality in terms of the placement of the ends of \(M\) at infinity.

Recall that Theorem 1 and 2 imply that for each end \(E_i\) of \(M\), the immersed circles \(\Gamma_i^j = 1/r (E_i \cap S^{n-1})\) converge smoothly to closed geodesics \(\gamma^j\) on \(S^{n-1}\) with multiplicity \(I_j^r\). From the simple topology of the sphere \(S^2\) we note that this implies that when \(n\) equals three that \(I_j = 1\) if and only if \(\Gamma_i^j\) is an embedded circle for large \(r\).

Since the convergence of \(\Gamma_i^j\) to \(\gamma^j\) is smooth, the total curvatures of \(\Gamma_i^j\) converge to \(2\pi I_j\) as \(r\) goes to infinity. Let \(M_r = \frac{1}{r} (M \cap B_r)\) where \(B_r\) is the ball of radius \(r\). Then since the curvature vector of the geodesic \(\gamma^j\) lies in the plane containing the geodesic and this plane is the tangent plane to \(E_i\) at infinity, the total geodesic curvature of \(\Gamma_i^j\) as a boundary curve on \(M_r\), converges to \(2\pi I_j\) as \(r\) goes to infinity.

From the Gauss–Bonnet formula for \(M_r\) with \(k\) ends, we have

\[
\int_{M_r} K \ dA + \sum_{i=1}^{k} \int_{\Gamma_i^j} k_i \ dt = 2\pi \chi(M_r) = 2\pi\chi(M).
\]

Letting \(r\) go to infinity we have

\[
C(M) + \sum_{i=1}^{k} 2\pi I_i = 2\pi \chi(M).
\]

This formula together with the fact that \(I_i = 1\) in dimension three precisely when the end \(E_i\) is embedded proves the following theorem.

Theorem 4. Suppose \(M\) is a complete minimal surface in \(\mathbb{R}^n\) with \(k\) ends and finite total curvature \(C(M)\). Then

\[
C(M) = 2\pi \left(\chi(M) - \sum_{i=1}^{k} I_i\right) \leq 2\pi(\chi(M) - k).
\]
Furthermore, if \( n \) equals three, then
\[
C(M) = 2\pi(\chi(M) - k)
\]
if and only if the ends of \( M \) are embedded.

**Corollary 1.** The catenoid is the only embedded complete minimal annulus with finite total curvature.

**Proof.** Suppose that \( M \) is an embedded complete minimal annulus with finite total Gaussian curvature. From the above formula the total curvature of \( M \) must equal \(-4\pi\). Theorem 9.4 in [15] states that a complete minimal surface with total curvature \(-4\pi\) is either the catenoid or Enneper's surface. Thus, the surface \( M \) is the catenoid.

**Corollary 2.** Let \( M \) be a complete orientable non-flat minimal surface with one end and finite total curvature immersed in \( \mathbb{R}^n \). Then the index \( I_1 \) given above is greater than or equal to three for \( n \) equal to three and \( I_1 \) is greater than or equal to two for general \( n \).

**Proof.** First we consider the case \( n = 3 \). In this case Theorem 3 implies that the end of \( M \) is not embedded. Theorem 1 then implies \( I_1 \) is greater than one. Since \( M \) is orientable and \( C(M) \) is a multiple of \( 4\pi \), this implies that \( I_1 \) is odd. Hence, \( I_1 \) is greater than or equal to three.

Now suppose that \( n \) is greater than or equal to three. Suppose, after a translation of \( M \) in \( \mathbb{R}^n \), that the surface \( M \) is obtained from the Weierstrass representation (see, e.g. [20]). This means that the immersion \( h: M \to \mathbb{R}^n \) can be obtained as \( h = \text{Re} \int W \) where \( W \) is an \( n \) vector of holomorphic 1 forms of the form
\[
W = \psi \left( 1 - \sum_{i=1}^{n-2} g_i^2, \left( 1 + \sum_{i=1}^{n-2} g_i^2 \right), g_1, \ldots, g_{n-2} \right)
\]
where \( g_1, \ldots, g_{n-2} \) are meromorphic functions on \( M \) and \( \psi \) is a meromorphic 1-form on \( M \). Let \( z \) be a coordinate system around the point end \( p \) in \( M \) with \( z(0) = p \).

We may assume that the tangent plane to \( M \) at infinity is the \( X_1X_2 \) plane and that the Taylor expansion for \( g_1, \ldots, g_{n-2} \) about the point \( p \) on \( M \)
\[
g_i(z) = \sum_{k=0}^{\infty} a_{ik} z^k
\]
where the index of the first non-zero coefficient of \( g_i(z) \) is greater than or equal to the first non-zero coefficient of \( g_i(z) \) which is different from zero.

We now consider two cases. The first case is when \( a_{i1} \) is non-zero. If the winding number \( I_1 = 1 \), then \( f(z) \, dz = \psi = \sum_{k=2}^{\infty} b_k z^k \, dz \) where \( b_2 \neq 0 \). Then \( f g_1 \, dz = \sum_{k=2}^{\infty} c_k z^k \, dz \).

Since \( h_3(z) = \text{Re} \int f g_1 \, dz \) is well defined, \( c_{-1} \) is real. This implies \( h_3: M \to \mathbb{R} \) is a harmonic function which goes to \( \infty \) at \( p \). The maximum principle for harmonic functions implies \( h_3 \) is constant contrary to our situation.

Now suppose that \( a_{i1} \) is zero. In this case \( h_3 \) approaches a finite value and hence much be constant. These two cases prove the corollary.
Remark. In the proof of Theorem 4 we used Theorem 2. The special case of Theorem 2 for complete minimal surfaces of finite total curvature is easier to prove than is Theorem 2. The proof of Theorem 2 is easier in the case of these minimal surface because one can appeal directly to the coordinate function given in the Weierstrass representation. We refer the interested reader to the proof of Lemma 11.2 in [15] for the type of analytic reasoning involved in an analytic proof.

§5. NEW EXAMPLES OF COMPLETE MINIMAL SURFACES OF FINITE TOTAL CURVATURE

Now we construct a complete minimal surface in $\mathbb{R}^3$ conformal to the sphere minus $n$ points and with $C(M) = -4\pi(n - 1)$, or equivalently, $M$ has $n$ embedded ends.

Example. Let $n$ be a non-negative integer. Define $f(z) = (z^{n+1} - 1)^{-2}$ and $g(z) = z^n$. Using the Weierstrass representation, we have

$$\Phi_1 = \frac{1}{2} \int \frac{1 - z^{2n}}{(z^{n+1} - 1)^2} \, dz$$

$$\Phi_2 = i/2 \int \frac{1 + z^{2n}}{(z^{n+1} - 1)^2} \, dz$$

$$\Phi_3 = \frac{z^n}{(z^{n+1} - 1)^2} \, dz.$$  

If $\Phi_k = F_k \, dz$, $k = 1, 2, 3$, do not have real periods, then $\tau_k = \Re \int F_k$ gives an complete minimal surface in $\mathbb{R}^3$ with the characteristics above. Let $\theta$ be a $(n + 1)$-root of the unity. Denote by $\text{res}(F_k, \theta)$ the residue of $F_k$ at $z = 0$. Then, if $\gamma$ is a small circle around $\theta$ we have

the period of $\Phi_k$ at

$$\theta = \int_{\gamma} \Phi_k$$

$$= 2\pi i \text{res}(F_k, \theta).$$

If $\text{res}(F_k, \theta)$, $k = 1, 2, 3$ is real, then we obtain the example. Now we compute $\text{res}(F_k, \theta)$.

Calculation of $\text{res}(F_k, \theta)$: We consider two cases.

First case. $\theta = \pm 1$

Since $\theta$ is $(n + 1)$-root of unity, we have $\theta = \pm 1$ if and only if $\theta^{2n} - 1$.

Then

$$1 - z^{2n} = \theta^{2n} - z^{2n} = (\theta - z) \sum_{i=1}^{2n} \theta^{2n-1} z^{i-1}$$  \hspace{1cm} (1)$$

$$z^{n+1} - 1 = z^{n+1} - \theta^{n+1} - (z - \theta) \sum_{j=0}^{n} \theta^{n+1} z^{j}$$  \hspace{1cm} (2)$$

and

$$F_k = -\frac{1}{2(z - \theta)} \sum_{i=1}^{2n} \theta^{n-i} z^{i-1}.$$
If \( z = \theta \), we have

\[
\sum_{j=1}^{2n} \theta^{2n-j}\theta^{j-1} = 2n\theta^{2n-1}
\]  

(3)

and

\[
\sum_{j=0}^{n} \theta^{n-j}\theta^{j} = (n+1)\theta^n.
\]  

(4)

Thus, \( \theta \) is a pole of order one of \( F_1 \).

Therefore, the residue of \( F_1 \) at \( \theta = \pm 1 \) is

\[
\text{res}\{F_1, \theta\} = -\frac{1}{2} \frac{2n\theta^{2n-1}}{(n+1)^2\theta^{2n}} = \frac{n}{\theta(n+1)}, \text{ if } \theta = \pm 1.
\]

Second case. \( \theta \neq \pm 1 \).

In this case \( \theta \) is not zero of \( 1 - z^{2n} \). Using (2) we have

\[
F_1 = \frac{1}{2(z-\theta)^2} \frac{1-z^{2n}}{\left(\sum_{j=0}^{n} \theta^{n-j}\theta^{j}\right)}.
\]

and it follows that \( \theta \) is a pole of \( F_1 \) of order two. Then

\[
\text{res}\{F_1, \theta\} = \frac{d}{dz} \left[ \frac{1-z^n}{2\left(\sum_{j=0}^{n} \theta^{n-j}\theta^{j}\right)} \right] \text{ at } z = \theta.
\]

Now

\[
\frac{d}{dz} \left[ \frac{1-z^n}{2\left(\sum_{j=0}^{n} \theta^{n-j}\theta^{j}\right)^2} \right] = \frac{2(1-z^{2n}) \sum_{j=0}^{n} j\theta^{n-j}z^{j-1}}{2\left(\sum_{j=0}^{n} \theta^{n-j}\theta^{j}\right)^2}.
\]

It follows from (4) and

\[
\sum_{j=0}^{n} j\theta^{n-j}z^{j-1} = \frac{n(n+1)}{2} \theta^{n-1} \text{ at } z = \theta,
\]  

(5)

that at \( z = \theta \)

\[
\frac{d}{dz} \left[ \frac{1-z^n}{2\left(\sum_{j=0}^{n} \theta^{n-j}\theta^{j}\right)^2} \right] = -\frac{2n\theta^{2n-1}}{2(n+1)^2\theta^{2n}} - \frac{2(1-\theta^{2n})\theta^{n-1} \cdot 2^{n(n+1)}}{2(n+1)^2\theta^{3n}}
\]

\[
= -\frac{n}{2(n+1)^2} \left( 2\theta^{-1} + 1- \theta^{2n} \right).
\]
Since $\theta^{-1} = \bar{\theta}$ and $\theta^n = \bar{\theta}$, we have
\[
\frac{d}{dz} \frac{1 - z^{2n}}{2(\sum_{j=0}^{n} \theta^{n-j}z^j)} = -\frac{n}{2(n+1)^2} (\bar{\theta} + \theta), \text{ at } z = \theta.
\]
Then
\[
\text{res}\{F_1, \theta\} = \frac{n \Re(\theta)}{(n+1)^2}.
\]

**Calculation of res\{F_2, \theta\}.** We have also two cases.

**First case.** $\theta = \pm i$

Since $\theta$ is a $(n+1)$-root of unity, $\theta = \pm i$ is equivalent to $\theta^{2n} = -1$, or $\theta$ is zero of $1 + z^{2n}$. Therefore,
\[
1 + z^{2n} = z^{2n} - \theta^{2n} = (z - \theta) \sum_{j=1}^{2n} \theta^{2n-j}z^{j-1}
\]
and from (2) we have
\[
F_z = i/2 \frac{1 + z^{2n}}{(z^{2n} - 1)^2} = \frac{i}{2(z - \theta)} \frac{\sum_{j=1}^{2n} \theta^{2n-j}z^{j-1}}{\left(\sum_{j=0}^{n} \theta^{n-j}z^j\right)^2}.
\]
At $z = \theta$ we have
\[
\sum_{j=1}^{2n} \theta^{2n-j}z^{j-1} = 2n\theta^{2n-1}.
\]
It follows from this and from (4) that $\theta$ is a pole of $F_2$ of order one. Then
\[
\text{res}\{F_2, \theta\} = \frac{i}{2} \frac{\sum_{j=1}^{2n} \theta^{2n-j}z^{j-1}}{\left(\sum_{j=0}^{n} \theta^{n-j}z^j\right)^2}, \text{ at } z = \theta
\]
\[
= \frac{i}{2} \frac{2n\theta^{2n-1}}{(n+1)^2 \theta^{2n}}
\]
\[
= \frac{n}{(n+1)^2} \frac{i}{\theta}, \text{ if } \theta = \pm i.
\]

**Second case.** $\theta \neq \pm i$

Then $\theta$ is not zero of $1 + z^{2n}$ and
\[
F_z = \frac{i}{2(z - \theta)^2} \frac{1 + z^{2n}}{\left(\sum_{j=0}^{n} \theta^{n-j}z^j\right)^2}.
\]
It follows that $\theta$ is a pole of $F_2$ of order two, from where
\[
\text{res}(F_2, \theta) = \frac{d}{dz} \left( i \frac{1 + z^{2n}}{2 \left( \sum_{0}^{n} \theta^{n-i}z^i \right)} \right) \text{ at } z = \theta.
\]

Now
\[
\frac{d}{dz} \left( i \frac{1 + z^{2n}}{2 \left( \sum_{0}^{n} \theta^{n-i}z^i \right)} \right) = \frac{i}{2} \left[ \frac{2nz^{2n-1}}{\left( \sum_{0}^{n} \theta^{n-i}z^i \right)^2} \right. - \left. \frac{2(1 + z^{2n}) \sum_{0}^{n} j \theta^{n-i}z^i}{\left( \sum_{0}^{n} \theta^{n-i}z^i \right)^2} \right].
\]

It follows from (4) and (5) that at $z = \theta$
\[
\frac{d}{dz} \left( i \frac{1 + z^{2n}}{2 \left( \sum_{0}^{n} \theta^{n-i}z^i \right)} \right) = \frac{i}{2} \left[ \frac{2n\theta^{2n-1}}{(n+1)^2 \theta^{2n}} - \frac{2(1 + \theta^{2n}) \theta^{n-1} \theta^{(n+1)}}{(n+1)^2 \theta^{2n}} \right]
\]
\[
= \frac{\ln}{2(n+1)^2} \left( \frac{\theta^{-1} - 1 + \theta^{2n} \theta^{-n}}{\theta^{n}} \right)
\]
\[
= \frac{\ln}{2(n+1)^2} (\theta - \theta).
\]

Therefore
\[
\text{res}(F_2, \theta) = \frac{n}{(n+1)^2} \text{ Im}(\theta).
\]

\text{Calculation of res}(F_3, \theta). \text{ We observe that}
\[
F_3 = -\frac{1}{n+1} \frac{d}{dz} \left[ \frac{1}{z^{n+1} + 1} \right].
\]

Thus res$(F_3, \theta)$ are zero for all $\theta$.

\text{Remark.} \text{ The examples } X_n \text{ constructed above have a large group of linear symetries which consists of the natural } Z_2\text{-extension } D_n \subset C \mathbb{Z} \text{ of the dyhedral group } D_n. \text{ For a geometric construction (motivated in part by the existence of these examples) of these surfaces in terms of the conjugate surface we refer the reader to [21]. This geometric construction yields examples of higher genus that may be embedded. The following figure gives a rough sketch of a finite part of our example three ends. It should also be remarked that the other known examples given in [193 all arise as covering spaces of a fixed example of a planar domain of connectivity three. As such, only their example of a planar domain of connectivity three has the minimal possible total curvature with respect to its topology.}

\text{§6. THE NONEXISTENCE OF CERTAIN MINIMAL SURFACES}

\text{Theorem 5. The only complete finite total curvature minimal embeddings of } S^2 - \{a_1, \ldots, a_t\} \text{ in } R^3, \text{ for } 1 \leq k \leq 5, \text{ are the plane } (k = 1) \text{ and the catenoid } (k = 2). \text{ The cases } k = 3, 4 \text{ or } 5 \text{ do not occur.}

\text{Proof.} \text{ Let } M \text{ be minimally embedded in } R^3 \text{ and conformal to } S^2 \text{ minus } t \text{ points. We consider isothermal parameters globally defined in } M. \text{ Then, under composition}
with the stereographic projection, the Gauss map \( g \) is a meromorphic function \( g: \mathbb{C} \to \mathbb{C} \). By Theorem 4

\[
C(M) = -4\pi(t - 1).
\]

Since the Gauss map \( g \) is meromorphic function on \( S^2 \),

\[
g = \frac{Q}{P}
\]

where \( Q \) and \( P \) are polynomials and

\[t - 1 = \max\{\deg Q, \deg P\}\]

By the corollary Theorem 3, all the normals to \( M \) at the ends of \( M \) are parallel. After rotations of coordinates, and of \( M \), we may assume that the limit of the Gauss map \( g \) at the ends of \( M \) are \( \pm 1 \) and that \( \deg Q < \deg P \).

Let \( a_1, j = 1, 2, \ldots, t_0 \), be the ends of \( M \) such that \( g(a_j) = 1 \), and \( b_j, j = 1, 2, \ldots, t_1 \), be the ends such that \( g(b_j) = -1 \). Then

\[
\deg P = t - 1 \text{ and } t = t_0 + t_1.
\]

**Lemma 2.** We have \( t_0 = t_1 \) if \( t \) is even and \( t_0 + 1 = t_1 \) if \( t \) is odd.

**Proof of Lemma 2.** Let \( C_r \) be the solid cylinder of radius \( r \) and axis the straight line generated by \((1, 0, 0)\). By Lemma 1 in §1, if \( r \) is very large, \( M \) divides \( C_r \) into two connected open sets and \( M \cap \partial C_r \) is the union of \( t \) curves. Since \( M \) is oriented, the consecutive curves of \( \partial C_r \cap M \) have opposite orientations, and proves the lemma. (See also the proof of Theorem 6.)

By the Weierstrass representation, the embedding \( x = (x_1, x_2, x_3) \) of \( M \) into \( \mathbb{R}^3 \) is given by

\[
x_k = \text{Re} \int \Phi_k \, dz, \quad k = 1, 2, 3
\]

where

\[
\Phi_1 = \frac{1}{2} f(1 - g^2)
\]

\[
\Phi_2 = \frac{i}{2} f(1 + g^2)
\]

\[
\Phi_3 = fg.
\]

The zeros of \( f \) occur exactly at the poles of \( g \) in \( M \) and the order of the zeros of \( f \) is twice the order of the poles of \( g \) at the corresponding points. Since \( g(a_j) = 1 \) and \( g(b_j) = -1 \) and the ends are embedded, we have that \( f \) has poles at \( a_j \) and \( b_j \) of order exactly 2. (This follows immediately from Riemann's relation and the fact \( C(M) = -4\pi(t - 1) \).) Then

\[
f = c \frac{P^2}{\pi(z - a_j)^2 \pi(z - b_j)^2}, \quad z \in \mathbb{C},
\]

where \( c \) is constant. If we consider \( g = aQ/aP \) where \( a^{-2} = c \), then we may assume
that \( c = 1 \). This gives

\[
\Phi_1 = \frac{\pi^2}{\pi(z - a_j) \pi(z - b_j)}
\]

\[
\Phi_2 = \frac{i/2}{\pi(z - a_j) \pi(z - b_j)}
\]

\[
\Phi_3 = \frac{PQ}{\pi(z - a_j) \pi(z - b_j)}
\]

From (1) we have \( P(a_j) = Q(a_j) \) and \( P(b_j) = -Q(b_j) \). Then

\[
P - Q = G \prod_{j=1}^{t_0} (z - a_j)^{m_j},
\]

\[
P + Q = H \prod_{j=1}^{t_1} (z - b_j)^{n_j},
\]

where \( m_j > 1, n_j > 1 \) and

\[
G(a_j) \neq 0, G(b_j) \neq 0, \quad (4)
\]

\[
H(a_j) \neq 0, H(b_j) \neq 0,
\]

\( j = 1, 2, \ldots, t_0 \), and \( l = 1, 2, \ldots, t_1 \). Since the degree of \( g \) is \( t - 1 \), we have that \( P - Q \) and \( P + Q \) have \( t - 1 \) zeros with multiplicity. Then

\[
\deg G + \sum m_j = \deg H + \sum n_l = t - 1. \quad (5)
\]

Substituting (3) in the expression of \( \Phi_k \) we have

\[
\Phi_1 = \frac{1}{2} GH \pi(z - a_j)^{m_j - 2} \pi(z - b_j)^{n_l - 2},
\]

\[
\Phi_2 = i/4 \frac{G^2 \pi(z - a_j)^{2m_j - 2} \pi(z - b_j)^{2n_l - 2}}{\pi(z - a_j)^2 + \pi(z - b_j)^2}
\]

\[
\Phi_3 = 1/4 \frac{G^2 \pi(z - a_j)^{2m_j - 2} - H^2 \pi(z - b_j)^{2n_l - 2}}{\pi(z - a_j)^2}
\]

Since the \( \Phi_k \) \( dz, k = 1, 2, 3 \), do not have real periods, it follows that

\[
\frac{G^2 \pi(z - a_j)^{2m_j - 2}}{\pi(z - b_j)^2} \ dz
\]

and

\[
\frac{H^2 \pi(z - b_j)^{2n_l - 2}}{\pi(z - a_j)^3} \ dz
\]

do not have periods.

**Lemma 3.** If

\[
\prod_{j=1}^{t_0} (z - a_j)^{m_j} \prod_{l=1}^{t_1} (z - b_l)^2 \ dz, \quad \alpha_i = \beta_j, \quad i = 1, 2, \ldots, n
\]

\[
\prod_{j=1}^{t_0} (z - a_j)^{m_j} \prod_{l=1}^{t_1} (z - b_l)^2 \ dz, \quad \beta_i = \alpha_j, \quad j = 1, 2, \ldots, m
\]
does not have periods, then

$$\sum_{j=1}^{n} \frac{m_j}{\beta_k - \alpha_j} - \frac{2}{\sum_{j \neq k} \beta_k - \beta_j}, \quad k = 1, 2, \ldots, n.$$ 

**Proof of Lemma 3.** If the periods of the form given in the Lemma 3 are zero, then

$$0 = \frac{d}{dz} \left[ \frac{\pi(z - \alpha_j)^{m_j}}{\prod_{j \neq k} (z - \beta_j)^2} \right] = \frac{\prod_{j \neq k} (\beta_k - \alpha_j)^{m_j}}{\prod_{j \neq k} (\beta_k - \beta_j)^2} \left( \sum_{j \neq k} \frac{m_j}{\beta_k - \alpha_j} - \frac{2}{\sum_{j \neq k} \beta_k - \beta_j} \right)$$

that proves the lemma.

$$G = c_0 \pi (z - A_j)^{\delta}, \quad H = c_1 \pi (z - B_j)^{\epsilon}.$$ 

A straightforward computation of the periods of (6) using Lemma 3 gives

$$\sum_{j} \frac{\delta_j}{b_k - A_j} + \sum_{j=1}^{l_k} \frac{m_j - 1}{b_k - a_j} = \sum_{j=1}^{l_k} \frac{1}{b_k - b_j}, \quad k = 1, 2, \ldots, t_0. \quad (7)$$

and

$$\sum_{j} \frac{\epsilon_j}{a_k - B_j} + \sum_{j=1}^{l_k} \frac{n_j - 1}{a_k - a_j} = \sum_{j=1}^{l_k} \frac{1}{a_k - a_j}, \quad k = 1, 2, \ldots, t_0. \quad (8)$$

If \( t \in \{3, 4, 5\} \), we have, by Lemma 2 and (2), that \( t_0 \) or \( t_1 \) is equal to 2. If \( t_0 = 2 \), then from (5) we have

$$\sum \epsilon_j + \sum_{j=1}^{l_k} (n_j - 1) = t - 1 - t_1 = t_0 - 1 = 1.$$ 

Then one of the two assertions hold:

(i) \( \epsilon_j = 1 \) for some \( j \), \( \epsilon_k = 0 \), \( j \neq k \), and \( n_k = 0 \) for all \( k \).

(ii) \( n_j = 2 \) for some \( j \), \( n_k = 1 \), \( j \neq k \), and \( \epsilon_k = 0 \) for all \( k \).

Therefore, the equation (8) is of the type

$$\frac{1}{a_k - B_j} = \frac{1}{a_k - a_j},$$

or

$$\frac{1}{a_k - B_j} = \frac{1}{a_k - a_j}.$$ 

We know that \( b_j = a_k \), and by (4), \( B_j = a_k \). This gives a contradiction and then \( t_0 = 2 \). Analogously, we have \( t_1 = 2 \). Therefore, \( t_1 = 1 \) and \( t_0 = 0 \) or 1. For these cases, Theorem 9.4 in [15] implies that \( M \) is the plane or the catenoid.
§7. RATIOS OF VOLUME AND THE TOPOLOGY OF THE COMPLEMENTS

There is a well known relationship between critical points of oriented volume integrals and of surface area integrals. Analysis of this relationship leads one to the following definition and conjecture.

**Definition.** Suppose that $M$ is a complete properly embedded surface in $\mathbb{R}^3$. Let $N_1$ and $N_2$ be the two components of $\mathbb{R}^3 - M$. Define for the ball $B$,

$$V_M = \lim_{\text{vol}(B \cap N_1)} \frac{\text{vol}(B \cap N_2)}{\text{vol}(B)}$$

where $\text{vol}(X)$ is the volume of $X$.

**Conjecture.** If $M$ is minimal, then $V_M$ exists and equals $0, \infty, 1$. Furthermore, $V_M = 0, \infty$ if and only if $M$ has finite topological type and an even number of topological ends.

We now prove a generalized version of this conjecture in the case $M$ has finite total curvature.

**Theorem 6.** Suppose $M$ is an embedded complete minimal surface in $\mathbb{R}^3$ with $C(M)$ finite and with $k$ ends. Then:

1. If $M$ has an odd number of ends, then $V_M = 1$ and $M$ disconnects $\mathbb{R}^3$ into two regions diffeomorphic to the interior of a solid $(g + \frac{1}{2}(k - 1))$-holed torus where the number $g$ is equal to the genus of the associated compact surface $\tilde{M}$.
2. If $M$ has an even number of ends and $\tilde{M}$ has genus $g$, then $M$ disconnects $\mathbb{R}^3$ into $N_1$ diffeomorphic to the interior of a $(g + \frac{k}{2})$-holed solid torus and into $N_2$ diffeomorphic to the interior of a $(g + \frac{1}{2}(k - 2))$-holed solid torus. In this case, $V_M = 0$.

**Proof.** Suppose that $M$ has an odd number of ends as for example the plane. Consider a large ball $B$ such that the components of $M - B$ are graphs over $P$ where $P$ is the unique unoriented tangent plane at infinity. As the $(\mathbb{R}^3 - B) - M$ has a natural radial product structure, $\mathbb{R}^3 - M$ is diffeomorphic to $\tilde{B} - M$. By Proposition 2 in [9], $M \cap B$ is a Heegard surface in $B$. Hence $M$ disconnects $\tilde{B}$ into two solid tori. A simple Euler characteristic calculation shows that the boundary surfaces of these solid tori are diffeomorphic to $\tilde{M}$ with $\frac{1}{2}(k - 1)$ handles. This implies the topological property in 1. A similar calculation in the case that $M$ has an even number of ends shows that $M$ disconnects $B$ into a region $N_1$ which is the interior of a $(g + \frac{k}{2})$-holed solid torus and into $N_2$ which is the interior of a $(g + \frac{1}{2}(k - 2))$-holed solid torus. This implies the topological property in 2.

The calculations of $V_M$ in the theorem follows directly from Theorem 2. As $V_M$ is invariant under radial shrinking of $\mathbb{R}^3$, $V_M = \lim_{\text{vol}(B)} \frac{\text{vol}(B \cap N_2)}{\text{vol}(B)}$ where $M, \tilde{M}$ converge to a single plane $P$ with multiplicity. The surfaces $M, \tilde{M}$, disconnected the unit ball $B$ into two closed regions $N_1', N_2'$. By the separation properties of algebraic
topology in the case $M$ has an even number of ends, imply that $N_i'$ converges as a point set to $P \cap B$ and $N_i$ converges as a point set to $B$. If $M$ has an odd number of ends, then $N_i'$ converges to the closure of a component of $B-P$ in $B$. These calculations immediately imply the asserted values for $V_m$.

**Remark.** If $M$ is a codimension-one embedded submanifold of $\mathbb{R}^n$ that satisfies the hypothesis of Theorem 3, then the limit

$$V_M = \lim_{n \to \infty} \frac{\text{vol}(B, \cap N_i)}{\text{vol}(B, \cap N_2)}$$

has values 0, 1, $\infty$ with the value 1 occurring precisely when $M$ has an odd number of ends. This calculation is the same as the calculation in Theorem 6.

§8. A TOPOLOGICAL UNIQUENESS THEOREM.

**Theorem 7.** Suppose $M_1$ and $M_2$ are complete embedded minimal surfaces in $\mathbb{R}^3$ with $C(M)$ finite. If $M_1$ and $M_2$ are diffeomorphic with two topological ends, then there is an orientation preserving diffeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$ with $f(M_1) = M_2$. In particular, $M_1$ is ambiently isotopic to $M_2$.

The proof of this theorem will be based on the techniques developed in [9] to prove: If $\alpha$ is a Jordan curve on a plane $P_1$, $\beta$ is a Jordan curve on a parallel plane $P_2$, then any two embedded diffeomorphic compact minimal surfaces with borders $\alpha, \beta$ differ by an ambient isotopy in the region of $\mathbb{R}^3$ bounded by the parallel planes and the ambient isotopy is the identity on the parallel planes. The following lemma is a generalization of this uniqueness theorem.

**Lemma 4.** Let $\alpha, \beta$ be disjoint smooth Jordan curves on $S^2$ which are part of a smooth foliation $F$ of a closed annulus contained in $S^3$ such that every leaf of $F$ is a Jordan curve with total curvature less than $4\pi$. If $M$ is a compact embedded minimal surface in the ball $B^3$ with border $\alpha, \beta$, then $M$ is standardly embedded in $B^3$ up to isotopy.

**Proof of the lemma.** First we recall some definition in [8, 9, 17]. We also refer the reader to these papers for arguments and theorems. Let $\mathcal{C}$ denote the manifold of smooth Jordan curves in $\mathbb{R}^3$ considered without parametrizations. Let $\mathcal{M}$ be the manifold of immersed minimal disks in $\mathbb{R}^3$ with boundary curves in $\mathcal{C}$ and $\pi: \mathcal{M} \to \mathcal{C}$ be the natural projection.

The foliation $F$ can be considered to be a smooth embedding $\gamma_F: [0, 1] \to \mathcal{C}$. Extend the foliation $F$ to a smooth foliation $F'$ of the complement of two points on $S^2$ by Jordan curves. Let $\gamma_F': [-1, 2] \to \mathcal{C}$ be an extension of $\gamma_F$ corresponding to $F'$. As the curves in $F$ have total curvature less than $4\pi$, for each $t$ $\gamma_F(t)$ is a regular value for $\pi: \mathcal{M} \to \mathcal{C}$ because such minimal disks are stable [7]. The arguments in the proof of Theorem 3 in [8] show we may assume that $\gamma_F': [-1, 2]$ is transverse to the map $\pi$. The arguments at the end of the proof of Theorem 3 in [8] also imply that $M$ is contained in a solid cylinder $C$ foliated by minimal disks whose boundary curves are the leaves of the foliation $F$. To prove the lemma it is sufficient to prove that $M$ is standardly embedded in $C$.

The foliation of $C$ by disks corresponds to the level sets of a function $H: C \to \mathbb{R}$. By analyticity of $M$ and the foliation, $H$ has a finite number of critical points.

Suppose for the moment that $H$ only has non-degenerate critical points. In this case
the proof of Theorem 1 in [9] applies to show that $M$ is standardly embedded where one uses the maximum principle for minimal surfaces to show that $H: C \to \mathbb{R}$ has no maximum or minimum. In the case $H$ has degenerate critical points the foliation of $C$ can perturbed locally near each critical point of $H$ so that the new associated $H': C \to \mathbb{R}$ has only non-degenerate critical points of index 1. The existence of $H'$ together with Theorem 1 in [9] implies the lemma.

**Proof of the theorem.** Suppose now that $M_1$ and $M_2$ are as in Theorem 3. The proof of Theorem 6 implies that $M_1$ and $M_2$ have very simple topological placement in the complement of any large open ball. Thus to prove that $M_1$ and $M_2$ are isotopic it is sufficient to prove that the intersection of $M_i$ with a large ball $B$ is standardly embedded in this ball for $i = 1, 2$.

It is clear from Theorem 1 and the corollary to Theorem 3 that $M_1 \cap \partial B$ consists of two Jordan curves which are part of a foliation $F$ of $S^2 - \{p_1, p_2\}$ by Jordan curves with the total curvature of the leaves of $F$ being less than $3\pi$ when $B$ has sufficiently large radius. Lemma 4 implies that $M_i \cap B$ is standardly embedded in $B$ for $i = 1, 2$. This implies that $M_1$ and $M_2$ are isotopic which completes the proof of Theorem 7.

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