



Behavior of lacunary series at the natural boundary

O. Costin*, M. Huang

Mathematics Department, The Ohio State University, 231 W 18th Ave, Columbus, OH 43210, United States

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Abstract

We develop a local theory of lacunary Dirichlet series of the form $\sum_{k=1}^{\infty} c_k \exp(-zg(k))$, $\operatorname{Re}(z) > 0$ as z approaches the boundary $i\mathbb{R}$, under the assumption $g' \rightarrow \infty$ and further assumptions on c_k . These series occur in many applications in Fourier analysis, infinite order differential operators, number theory and holomorphic dynamics among others. For relatively general series with $c_k = 1$, the case we primarily focus on, we obtain blow up rates in measure along the imaginary line and asymptotic information at $z = 0$.

When sufficient analyticity information on g exists, we obtain Borel summable expansions at points on the boundary, giving exact local description. Borel summability of the expansions provides property-preserving extensions beyond the barrier.

The singular behavior has remarkable universality and self-similarity features. If $g(k) = k^b$, $c_k = 1$, $b = n$ or $b = (n + 1)/n$, $n \in \mathbb{N}$, behavior near the boundary is roughly of the standard form $\operatorname{Re}(z)^{-b'} Q(x)$ where $Q(x) = 1/q$ if $x = p/q \in \mathbb{Q}$ and zero otherwise.

The Bötcher map at infinity of polynomial iterations of the form $x_{n+1} = \lambda P(x_n)$, $|\lambda| < \lambda_0(P)$, turns out to have uniformly convergent Fourier expansions in terms of simple lacunary series. For the quadratic map $P(x) = x - x^2$, $\lambda_0 = 1$, and the Julia set is the graph of this Fourier expansion in the main cardioid of the Mandelbrot set.

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* Corresponding author.

E-mail address: costin@math.ohio-state.edu (O. Costin).

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1. Introduction

Natural boundaries (NBs) occur frequently in many applications of analysis, in the theory of Fourier series, in holomorphic dynamics (see [5–7] and references there), in analytic number theory, see [18], physics, see [12] and even in relatively simple ODEs such as the Chazy equation [10], an equation arising in conformal mapping theory, or the Jacobi equation.

The intimate structure of NBs turns out to be particularly rich, bridging analysis, number theory and complex dynamics.

Nonetheless (cf. also [12]), the study of NBs of concrete functions is yet to be completed from a pure analytic point of view. The aim of the present paper is a detailed study near the analyticity boundary of a prototypical functions exhibiting this singularity structure, classes of *lacunary series*. For such functions, we develop a theory of generalized local asymptotic expansions at

NBs, and explore their consequences and applications. The expansions are asymptotic in the sense that they become increasingly accurate as the singular curve is approach, and in many cases exact, in that the function can be recovered from these expansions.

Lacunary series, sums of the form $h(s) = \sum_{j \geq 1} c_j s^{g_j}$ (or written as Dirichlet series $f(z) = \sum_{j \geq 1} c_j e^{-zg(j)}$), where $j/g_j = o(1)$ for large j , often occur in applications and have deep connections with infinite order differential operators, as found and studied by Kawai [11,12]. Under the lacunarity assumption above, if the unit disk is the maximal disk of analyticity of h , then the unit circle is its NB [13]. For instance, the series

$$h(s) = \sum_{j=1}^{\infty} s^{2^j} \quad (|s| < 1) \tag{1}$$

studied by Jacobi [9] before the advent of modern Complex Analysis, clearly has the unit disk as a singular curve: $h(s) \rightarrow +\infty$ as $|s| \uparrow 1$ along any ray of angle $2^{-n}m\pi$ with $m, n \in \mathbb{N}$.

We show that if $c_j = 1$ and $g' \rightarrow \infty$, then, in a measure theoretic sense, $f(x + iy)$ blows up as $x \rightarrow 0^+$ uniformly in y at a calculable rate. We find interesting *universality properties in the blow-up profile*.

In special cases of interest, Borel summable power series, in powers of the distance to the boundary, and more generally convergent expansions as series of small exponentials multiplying Borel summed series power series¹ representations can be determined on a dense set on the singularity barrier. Examples are

- $\sum_{j \geq 1} e^{-zj^b}$ where $b > 1$, or its dual $d > 1$, where $b^{-1} + d^{-1} = 1$, is integer (relating to exponential sums and van der Corput dualities [15]; the special self-dual case $\sum_{j \geq 1} e^{-zj^2}$, is related to the Jacobi theta function);
- $\sum_{j \geq 1} e^{-za^j}$, $1 < a \in \mathbb{N}$;

More generally, if $c_j = c(j)$ and $g_j = g(j)$ have suitable analyticity properties in j , then the behavior at $z = 0^+$, and possibly at other points, is described in terms of Ecalle–Borel summed expansions (see footnote 1). Then the analysis leads to a natural, properties-preserving, continuation formulas across the boundary.

In general, the blow-up profile along the barrier is closely related to *exponential sums*, expressions of the form

$$S_N = \sum_{k=1}^N c_n \exp(2\pi i g(n)), \quad g(n) \in \mathbb{R} \tag{2}$$

where for us $g' \rightarrow \infty$ as $n \rightarrow \infty$. The corresponding lacunary series are in a sense the continuation of (2) in the complex domain, replacing $2\pi i$ by $-z$, $\text{Re}(z) > 0$, and letting $N \rightarrow \infty$. The asymptotic behavior of lacunary series as the imaginary line is approached in nearly-tangential directions is described by dual, van der Corput-like, expansions. The method we use extend to exponential sums, which will be the subject of a different paper.

¹ These are simple instances of Borel (or Ecalle–Borel) summed transseries. A brief summary of the definitions and properties of transseries and generalized Borel summability, and references to the literature are given in Section A.6.

2. Results

2.1. Results under general assumptions

2.1.1. Blow-up on a full measure set

We consider lacunary Dirichlet series of the form

$$f(z) = \sum_{k=0}^{\infty} e^{-zg(k)} \tag{3}$$

but, as it can be seen from the proofs, the analysis extends easily to series of the form

$$f(z) = \sum_{j=0}^{\infty} c_j e^{-zg(j)}$$

under suitable smoothness and growth conditions, see Section 3.8. The results in the paper apply under the further restriction,

Assumption 1. The function g is differentiable and $g'(j) \rightarrow \infty$ as $j \rightarrow \infty$.

In particular, g is eventually increasing. By subtracting a finite sum of terms from f (a finite sum is clearly entire), we arrange that g is **increasing**. If $g(0) = a$, we can multiply f by e^{-za} to arrange that $g(0) = 0$.

Normalization 2.

- (i) g is differentiable on $[0, \infty)$, $g' > 0$ and $g' \rightarrow \infty$ along \mathbb{R}^+ .
- (ii) $g(0) = 0$.

Notation. We write $|H(\cdot)| =^\mu 1 + o(1)$ if $|H(y)| dy$ converges to the Lebesgue measure $d\mu(y)$.

Under Assumption 1, after normalization, we have the following result, giving exact blow-up rates in measure, as well as sharp pointwise blow-up upper bounds.

Theorem 1.

- (i) We have the uniform blow-up rate in measure.²

$$|F(x + i\cdot)|^2 \stackrel{\mu}{=} \int_0^{\infty} e^{-2g(s)x} ds (1 + o(1)) \tag{4}$$

It can be checked that $\int_0^{\infty} e^{-2g(s)x} ds \geq g^{-1}(1/x) \rightarrow \infty$ as $x \rightarrow 0^+$; see also Note 3.

² It turns out that in general, L^1 or a.e. convergence of $|f|$ do not hold.

(ii) The following pointwise estimate holds:

$$\|f(x + i\cdot)\|_\infty \leq \int_0^\infty e^{-g(s)x} ds (1 + o(1)) \quad \text{as } x \rightarrow 0^+ \tag{5}$$

This is sharp at $z = 0$, cf. Proposition 2, and in many cases it is only reached at $z = 0$; see Proposition 5.

2.1.2. General behavior near $z = 0$

At $z = 0^+$ a more detailed asymptotic description is possible.

Theorem 2.

(i) As $z \rightarrow 0^+$ we have

$$f(z) = \int_0^\infty e^{-zg(s)} ds - \frac{1}{2} + o(1) \rightarrow \infty \quad \text{as } z \rightarrow 0^+ \tag{6}$$

In fact,

$$f(z) - \int_0^\infty e^{-zg(s)} ds = -z \int_0^\infty e^{-zu} \{g^{-1}(u)\} du \tag{7}$$

(where $\{\cdot\}$ denotes the fractional part, and we used $g(0) = 0$).³

(ii) If $g(s)$ has a differentiable asymptotic expansion as $s \rightarrow \infty$ in terms of (integer or noninteger) powers of s and $\log s$, and $g(s) \sim \text{const} \cdot s^b$, $b > 1$, then after subtracting the blowing up term, f has a Taylor series at $z = 0$ (generally divergent, even when g is analytic, which can be calculated explicitly),

$$f(z) = \int_0^\infty e^{-zg(s)} ds - \frac{1}{2} + zf_1(z), \quad f_1 \in C^\infty[0, \infty)$$

(as an example, see (61)).

Note 3. Often g has an asymptotic expansion starting with a combination of powers, exponentials and logs. Let $\phi = g^{-1}$. Then $\phi(\nu x)/\phi(\nu) \rightarrow \phi_1(x)$ as $\nu \rightarrow \infty$ and $x > 0$ is fixed and

$$\int_0^\infty e^{-xg(s)} ds = C_g g^{-1}(1/x)(1 + o(1)); \quad C_g = \int_0^\infty e^{-u} \phi_1(u) du$$

³ As mentioned, often this maximal growth is achieved at zero but in special cases it occurs, up to a bounded function, on a dense set of measure zero.

For instance, if $g(k) = k^b$, $b > 1$ we have, as $\rho \rightarrow 1$,

$$\begin{aligned} |f(z)| &\stackrel{\mu}{=} x^{-\frac{1}{2b}} \Gamma(1 + 1/b)^{-1/2} 2^{-\frac{1}{2b}} (1 + o(1)) \quad (x \rightarrow 0^+) \\ f(x) &= x^{-\frac{1}{b}} \Gamma(1 + 1/b) (1 + o(1)) \quad (x \rightarrow 0^+) \end{aligned} \tag{8}$$

2.1.3. Blow-up profile along barrier

Theorem 3.

(i) Assume that for some $y \in \mathbb{R}$ there is a smooth increasing function $\rho(N; y) =: \rho(N) \in (0, N]$ such that the following weighted exponential sum (see [15]) has a limit:

$$S_{\rho, N} := \rho(N)^{-1} \sum_{j=1}^N e^{iyg(j)} \rightarrow L(y) \quad \text{as } N \rightarrow \infty \tag{9}$$

where ρ'' is uniformly bounded and nonpositive for sufficiently large N . (Without loss of generality, we may assume $\rho''(N) \leq 0$ for all N .) Let

$$\Phi(x) = \int_0^\infty e^{-xg(u)} \rho'(u) du \tag{10}$$

Then, we have the asymptotic behavior

$$f(x + iy) = L(y)\Phi(x) + o(\Phi(x)) \quad (x \rightarrow 0^+) \tag{11}$$

(ii) As a pointwise upper bound we have: for given y ,

$$\text{If } \limsup_{N \geq 0} |S_{\rho, N}| < \infty \quad \text{then } f(x + iy) = O(\Phi(x)) \quad (x \rightarrow 0^+) \tag{12}$$

2.2. Results in specific cases

2.2.1. Settings leading to convergent expansions

The cases $g(j) = j^2$ and $g(j) = a^j$, $a > 1$ are distinguished, since the expansions at some points near the boundary converge.

Proposition 4. If $g(j) = j^2$, then we have the identity

$$f(z) = \frac{1}{2} \sqrt{\frac{\pi}{z}} - \frac{1}{2} + \sqrt{\frac{\pi}{z}} \sum_{k=1}^\infty e^{-\frac{k^2 \pi^2}{z}}$$

Clearly, this is most useful when $z \rightarrow 0$. It also shows the identity associated to the Jacobi theta function

$$f(z) = \frac{1}{2}\sqrt{\frac{\pi}{z}} - \frac{1}{2} + \sqrt{\frac{\pi}{z}} f\left(\frac{k^2\pi^2}{z}\right) \tag{13}$$

We have $f(\tau) = \frac{1}{2}[1 + \vartheta(0; \tau/\pi)]$, where ϑ is the Jacobi theta function defined, for $z \in \mathbb{C}$ and τ in the upper half plane by

$$\vartheta(z; \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

Proposition 5. *If $g(j) = a^j$, $a > 1$, then, as $z \rightarrow 0^+$, $f(z)$ is convergently given by*

$$\begin{aligned} f(z) &= -\frac{\log \zeta}{\log a} + \sum_{n=1}^{\infty} \frac{(-\zeta)^n}{n!(1-a^n)} + c_0 + \frac{1}{\log a} \sum_{k \neq 0} \Gamma\left(-\frac{2k\pi i}{\log a}\right) \zeta^{\frac{2k\pi i}{\log a}} \\ &= -\frac{\log \zeta}{\log a} + \sum_{n=1}^{\infty} \frac{(-\zeta)^n}{n!(1-a^n)} + c_0 - \frac{1}{2\pi i} \int_0^{\infty} \log_{\mathbb{R}}[-(s/\zeta)^{\frac{2\pi i}{\log a}}] e^{-s} ds \end{aligned} \tag{14}$$

where $\zeta = 1 - e^{-z}$. (Here $\log_{\mathbb{R}}$ is the usual branch of the log with a cut along \mathbb{R}^- and **not** the log on the universal covering of $\mathbb{C} \setminus \{0\}$.)

It is clear that for $a \in \mathbb{N}$ the transseries can be easily calculated for any $z = \rho \exp(2\pi i m/a^j)$, $(m, j) \in \mathbb{N}$, $0 \leq \rho < 1$ since

$$f(\rho \exp(2\pi i m/a^j)) = \sum_{n=1}^j \rho^{a^n} \exp(2\pi i m a^{n-j}) + f(\rho)$$

where the sum is a polynomial, thus analytic.

2.2.2. *Borel summable transseries representations; resurgence (cf. Section A.6)*

When $c_j \equiv 1$ it is clear that the growth rate as $z \rightarrow 0^+$ majorizes the rate at any point on $i\mathbb{R}$. There may be **no** other point with this growth, as is the case when $g(j) = j^b$, $b \in (1, 2)$ as seen in see Proposition 5 below, or densely many if, for instance, $g(j) = j^b$, $b \in \mathbb{N}$, or when $g(j) = a^j$, $a \in \mathbb{N}$. The behavior near points of maximal growth merits special attention.

For $g = j^b$, $1 < b \neq 2$, f has asymptotic expansions which do not, in general converge. They are however generalized Borel summable.

Define d by

$$\frac{1}{b} + \frac{1}{d} = 1 \tag{15}$$

Theorem 4. *Let $g = j^b$; $b > 1$. Then,*

(i) *the asymptotic series of $f(z)$ for small z ,*

$$\tilde{f}_0(z) = \Gamma\left(1 + \frac{1}{b}\right) z^{-\frac{1}{b}} - \frac{1}{2} + \frac{i}{2\pi} \sum_{j=1}^{\infty} (1 - (-1)^{jb}) \zeta(jb + 1) b_j z^j \tag{16}$$

is Borel summable in $X = z^{-1/(b-1)}$, along any ray $\arg(X) = c$ if $c \neq -\arg k^q s_{\pm}$, $k \in \mathbb{N}$, where $s_{\pm} = t_{\pm} \mp 2\pi i t_{\pm}^{1/b}$ and $t_{\pm} = (\pm 2\pi i/b)^{b/(b-1)}$. More precisely, for $b \in (1, 2)$ (a)

$$f(z) = \Gamma(1 + 1/b)z^{-1/b} - \frac{1}{2} + z^{-b/(b-1)} \int_0^{\infty} e^{-sz^{-1/(b-1)}} H(s) ds \tag{17}$$

where $H(s) =: H_b(s^{b-1})$, where H_b is analytic at zero and $H_b(0) = 0$; (b) H is analytic on the Riemann surface of the log, with square root branch points at all points of the form $k^q s_{\pm}$, $k \in \mathbb{N}$ and (c) making appropriate cuts (or working on Riemann surfaces), $u^{-(b-1)^2/b} H$ is bounded at infinity.

If $\arg(X) \in (\theta_-, \theta_+)$ where $\theta_{\pm} = \arg s_{\pm}$, then $\mathcal{LB} \tilde{f}_0 = f$.⁴ Here \mathcal{LB} is the Borel summation operator, see Section A.6, Definition 21. In a general complex direction, f has a nontrivial transseries, see (ii).

(ii) For a given direction φ , σ be ± 1 if $\pm\varphi > \theta_{\pm}$ and 0 otherwise. If $\sigma \arg z \in (\theta_{\sigma}, \pi/2)$, then the transseries of f is

$$\tilde{f}_0(z) + \sigma \frac{i}{2\pi} \sum_{k=1}^{\infty} e^{-s_{\sigma} k^q x^{-q+1}} \sum_{j=0}^{\infty} c_j(\sigma k)^{\frac{-2jb+2-b}{2(b-1)}} z^{\frac{2j-1}{2(b-1)}} \tag{18}$$

and it is Borel summable as well. For $b > 2$ the transseries is similar to (18) but σ is never zero.

Note 6. The duality $k^b \leftrightarrow k^d$ is the same as in van der Corput formulas; see [15].

2.2.3. Examples

(i) For $b = 3$, f the transseries is given by

$$\tilde{f}_0(z) + \sum_{k \in \mathbb{Z}} \sigma e^{-\kappa_3 k^{3/2} z^{-1/2}} \left[\left(\frac{\pi i k}{6} \right)^{\frac{1}{4}} z^{-\frac{1}{4}} + \frac{5}{32} \frac{(i k)^{\frac{7}{4}}}{6^{\frac{3}{4}} \pi^{\frac{5}{4}}} z^{\frac{1}{4}} + \dots \right] \tag{19}$$

where with $\kappa_3 = \pi^{3/2} \sqrt{\frac{32}{27}} (-1)^{\frac{1}{4}}$ and

$$\tilde{f}_0(z) = \frac{\Gamma(4/3)}{z^{1/3}} - \frac{1}{2} - \frac{z}{120} + \frac{z^3}{792} + \dots$$

(ii) For $b = 3/2$, with $\kappa = 32i\pi^3/27$, f has the Borel summable transseries

$$\tilde{f}_0(z) + \sigma \sum_{k=1}^{\infty} e^{\kappa \sigma k^3 z^{-2}} \left[\frac{4\sqrt{2}(\sigma i)^{-\frac{1}{2}} \pi k^{\frac{1}{2}}}{3} \frac{1}{z} + \frac{(\sigma i)^{\frac{1}{2}} z^2}{16\sqrt{2}\pi^{\frac{5}{4}} k^{\frac{5}{2}}} + \dots \right] \tag{20}$$

⁴ We note that the variable of Borel summation, or *critical time*, is not $1/z$ but $z^{-1/(b-1)}$.

where $\theta_{\pm} = \pi/4$ and

$$\tilde{f}_0(z) = \Gamma\left(\frac{5}{3}\right)z^{-\frac{2}{3}} - \frac{1}{2} - \frac{3\zeta\left(\frac{5}{2}\right)}{16\pi^2}z + \frac{1}{240}z^2 + \frac{315\zeta\left(\frac{11}{2}\right)}{2048\pi^5}z^3 + \dots$$

2.2.4. *Properties-preserving extensions beyond the barrier*

It is natural to require for an extension beyond the barrier that it has the following properties:

- (i) It reduces to usual analytic continuation when the latter exists.
- (ii) It commutes with all properties with which analytic continuation is compatible (principle of preservation of relations, a vaguely stated concept; this requirement is rather open-ended).

Borel summable series (more generally transseries) or suitable convergent representations allow for extension beyond the barrier, as follows. In the case $g(j) = j^b$ (and, in fact in others in which g has a convergent or summable expansion at infinity), $f(z)$ can be written, after Borel summation in the form (see Section 3.7)

$$z^{-d} \int_0^{\infty} e^{-z^{-1/(b-1)s}} H_1(s) ds = z^{-1} \int_0^{\infty} e^{-t} H_1(tz^{1/(b-1)}) dt =: \int_0^{\infty} e^{-s} F(s, z) ds \quad (21)$$

where H_1 is analytic near the origin and in \mathbb{C} except for arrays of isolated singularities along finitely many rays. Furthermore, H_1 is polynomially bounded at infinity. This means that the formal series is summable in all but finitely many directions in z .

Definition 7. We define the Borel sum continuation of f through point z on the natural boundary, in the direction d , to be the Borel sum of the formal series of f at z in the direction d , if the Borel sum exists.

This extension simply amounts to analytically continuing $F(s, z)$ in z , in (21), for small s and then analytically continuing $F(s, z)$ in s for fixed z along \mathbb{R}^+ .

2.2.5. *Notes (cf. Appendix A.5)*

- (i) The Borel sum provides a natural, properties-preserving, extension [3]. Borel summation commutes with all common operations such as addition, multiplication, differentiation. Thus, the function and its extensions will have the same properties.
- (ii) Also, when a series converges, the Borel sum coincides with the usual sum. Thus, when analytic continuation exists, it coincides with the extension.
- (iii) With or without a boundary, the Borel sum of a divergent series changes as the direction of summation crosses the *Stokes directions* in \mathbb{C} . Yet, the properties of the family of functions thus obtained are preserved. There may then exist extensions *along* the barrier as well. Of course, all this cannot mean that there is analytic continuation across/along the boundary.

See also Eq. (14), a convergent expansion, where z can also be replaced by $-z$. In this case, due to strong lacunarity, the extension *changes* along *every* direction, as though there existed densely many Stokes lines.

2.3. Universal behavior near boundary in specific cases

In many cases, $\phi(g^{-1}(u/x))$ has an asymptotic expansion, and then Φ in turn has an asymptotic expansion in x . Detailed behavior along the boundary can be obtained in special cases such as $g(j) = j^b$; $b \in \mathbb{N}$, or $g(j) = j^{(b+1)/b}$; $b \in \mathbb{N}$. Properly scaled sums converge to everywhere discontinuous functions.

Theorem 5.

(i) If $g(j) = j^b$, $b \in \mathbb{N}$, we have $\limsup x^{1/b} |f(x + iy)| < M < \infty$. We let $Q_{b,c}(s) = \lim_{x \rightarrow 0^+} x^{1/c} f(x + 2\pi is)$ (whenever it exists); then,

$$\frac{Q_{b,b}(s)}{\Gamma(1 + b^{-1})} = \begin{cases} \frac{1}{n} \sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^b}, & s = \frac{m}{n}, m, n \in \mathbb{N} \text{ relatively prime} \\ 0, & \text{a.e.} \end{cases} \tag{22}$$

(ii) For $b = 3/2$ and $z \rightarrow 0^+$, f grows like $z^{-2/3}$. For any other point on $i\mathbb{R}$ the growth is slower, at most $(\operatorname{Re} z)^{-1/2}$. Furthermore,

$$\frac{Q_{3/2,1/2}(\sqrt{s})}{\sqrt{6\pi} i s^{1/4}} = \begin{cases} \frac{1}{n} \sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^3}, & s = \frac{16}{27} \frac{n}{m}, n, m \text{ as in (i)} \\ 0, & \text{a.e.} \end{cases} \tag{23}$$

In particular, we have the profile duality relation

$$Q_{3/2,1/2}(s) = \frac{\sqrt{6\pi} i s^{1/2}}{\Gamma(4/3)} Q_{3,3}(2^8 3^{-6} s^{-2}) \tag{24}$$

for any $s \in \mathbb{R}$ for which $Q_{3/2,1/2}(s)$ and/or $Q_{3,3}(s)$ is well defined, for instance, in the cases given in (22) and (23).

For large n , the sum over l in Eq. (22) is, statistically, expected to be of order one. After $x^{1/b}$ rescaling, the template behavior “in the bulk” is given by the familiar function $Q(x) = n^{-1}$ if $x = m/n$, $(m, n) \in \mathbb{N}^2$ and zero otherwise, shown in Fig. 1. See also Figs. 2 and 3 for illustrations of relevant cases.

2.4. Fourier series of the Böttcher map and structure of Julia sets

We show how lacunary series are building blocks for fractal structures appearing in holomorphic dynamics (of the vast literature on holomorphic dynamics we refer here in particular to [2,6] and [14]). In Section 4 we mention a few known facts about polynomial maps. Consider for simplicity the quadratic map

$$x_{n+1} = \lambda x_n(1 - x_n) \tag{25}$$

It will be apparent from the proof that the results and method extend easily to polynomial iterations of the form

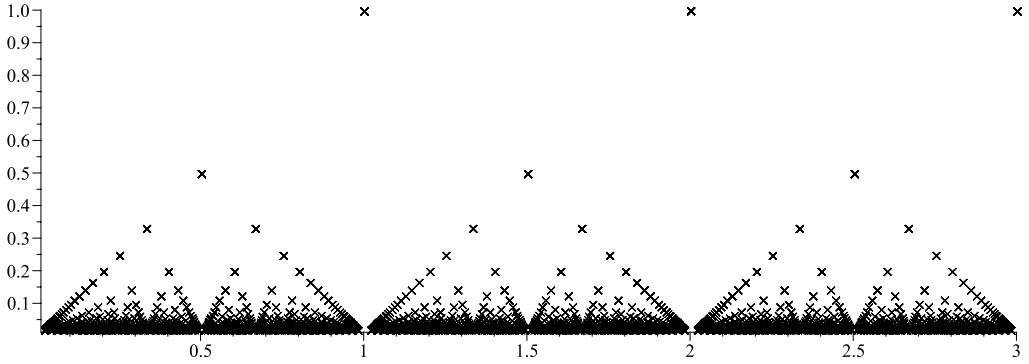


Fig. 1. The standard function $Q(x)$. The points above $Q = 0.05$ are the only ones present in the actual graph.

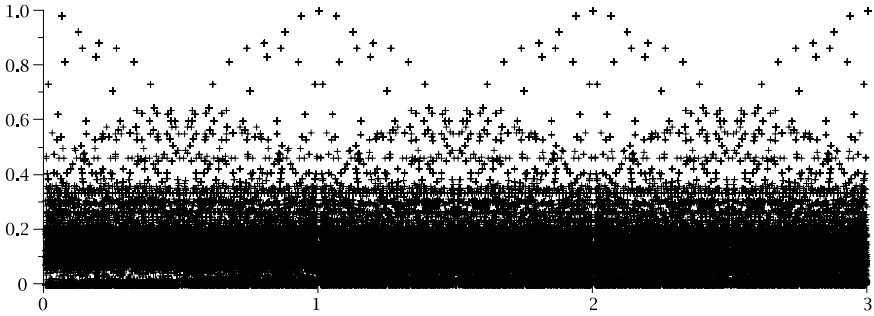


Fig. 2. Point-plot graph of $Q_{4,4}$, normalized to one.

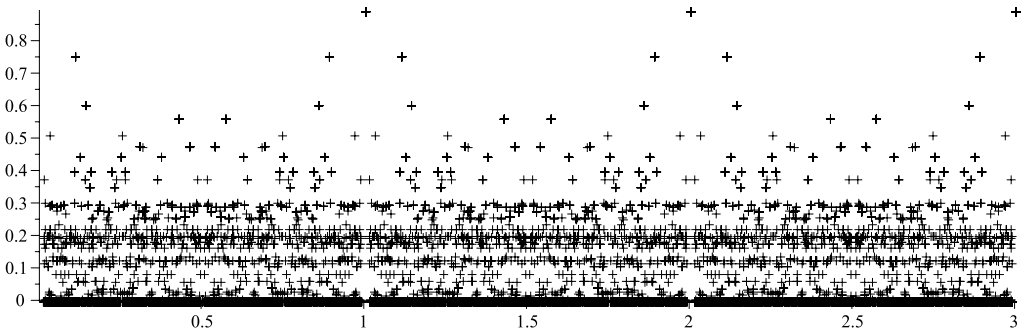


Fig. 3. Point-plot graph of $Q_{3,3}$; $Q_{3,2}$ follows from it through the transformation (24).

$$x_{n+1} = \lambda P_k(x_n) \tag{26}$$

with λ relatively small, where P_k is a polynomial of degree k . The substitution $x = -y^{-1}$ transforms (25) into

$$y_{n+1} = \frac{y_n^2}{\lambda(1 + y_n)} = f(y_n) \tag{27}$$

By Bötcher’s theorem (we give a self contained proof in Section 4, for (25), which extends in fact to the general case), there exists a map ϕ ,⁵ analytic near zero, with $\phi(0) = 0$, $\phi'(0) = \lambda^{-1}$ so that $(\phi f \phi^{-1})(z) = z^2$. Its inverse, ψ , conjugates (27) to the canonical map $z_{n+1} = z_n^2$, and it can be checked that

$$\psi(z)^2 = \lambda \psi(z^2)(1 + \psi(z)); \quad \psi(0) = 0, \quad \psi'(0) = \lambda \tag{28}$$

Let $\mathcal{A}(\mathbb{D})$ denote the Banach space of functions analytic in the unit disk \mathbb{D} and continuous in $\bar{\mathbb{D}}$, with the sup norm. We define the linear operator $\mathfrak{T} = \mathfrak{T}_2$, on $\mathcal{A}(\mathbb{D})$ by

$$(\mathfrak{T}f)(z) = \frac{1}{2} \sum_{k=0}^{\infty} 2^{-k} f(z^{2^k}) \tag{29}$$

This is the inverse of the operator $f \mapsto 2f - f^{\vee 2}$, where $f^{\vee p}(z) = f(z^p)$. Clearly, $\mathfrak{T}f$ is an isometry on $\mathcal{A}(\mathbb{D})$ and it maps simple functions, such as generic polynomials, to functions having $\partial\mathbb{D}$ as a natural boundary; it reproduces f across vanishingly small scales.

Theorem 6.

- (i) ψ and $H = 1/\psi$ are analytic in (λ, z) in $\mathbb{D} \times \mathbb{D}$ ($\lambda \in \mathbb{D}$ corresponds to the main cardioid in the Mandelbrot set, see Section 4), and continuous in $\mathbb{D} \times \bar{\mathbb{D}}$. The series

$$\psi(\lambda, z) = \lambda z + \sum_{k=1}^{\infty} z \lambda^{k+1} \psi_k(z) \tag{30}$$

converges in $\mathbb{D} \times \bar{\mathbb{D}}$ (and so does the series of H), but not in $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$. Here

$$\psi_1(z) = \mathfrak{T}z = \sum_{k=1}^{\infty} 2^{-k+1} z^{2^k} \tag{31}$$

(note that $\psi(z) = \frac{1}{2}z + \frac{1}{2} \int_0^z s^{-1} h(s) ds$ with h given in (1)), and in general

$$\psi_k = \mathfrak{T} \left(z \sum_{j=0}^{k-1} \psi_j^{\vee 2} \psi_{k-1-j} - \sum_{j=1}^{k-1} \psi_j \psi_{k-j} \right) \tag{32}$$

- (ii) All $\psi_k, k \geq 1$ have binarily lacunary series: In ψ_k , the coefficient of z^p is nonzero only if p has at most k binary-digits equal to 1, i.e.,

$$p = 2^{j_1} + \dots + 2^{j_k}, \quad j_i = 0, 1, 2, \dots \text{ (“}k\text{” is the same as in } \psi_k) \tag{33}$$

⁵ The notations ϕ, ψ, H , designate different objects than those in previous section.

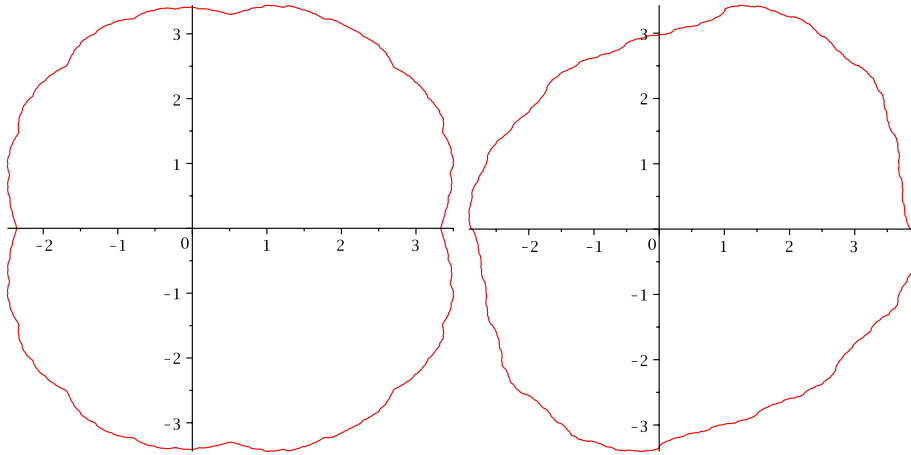


Fig. 4. The Julia set of $x_{n+1} = \lambda x_n (1 - x_n)$, for $\lambda = 0.3$ and $\lambda = 0.3i$, calculated from the Fourier series (30) discarding all $o(\lambda^2)$ terms. They coincide, within plot precision, with numerically calculated ones using standard iteration of maps algorithms.

(iii) For $|\lambda| < 1$, the Julia curve of (25) is given by a uniformly convergent Fourier series (see Fig. 4), by (i),

$$J = \{-(\operatorname{Re} H(e^{it}), \operatorname{Im} H(e^{it})) : t \in [0, 2\pi)\} \tag{34}$$

Remark 8. The effective lacunarity of the Fourier series makes calculations of the Julia set numerically effective if λ is not too large.

Note 9.

- (i) Lacunarity of ψ_k is a strong indication that ψ has a natural boundary (in fact, it does have one), but not a proof. Transseries at *singular* points and summation of convergent series (here in the parameter λ) **do not commute**. The transseries of H on the barrier can be calculated by transasymptotic matching, see [4], but in this case is more simply found directly from the functional relation; see Note 18.
- (ii) In assessing the fine structure of the fractal using the Fourier expansion truncated to $o(\lambda^n)$, the scale of analysis cannot evidently go below $O(\lambda^n)$.

Remark 10.

- (i) For $|\lambda|$ sufficiently small, Theorem 6 provides a convenient way to determine the Julia set as well as the discrete evolution on the boundary.
- (ii) For small λ , the self similar structure is seen in

$$\psi_1(\rho \exp(2\pi i m / 2^n)) = \sum_{k=1}^{n-1} \frac{\rho^{2^k} \exp(2\pi i m 2^{k-n})}{2^k} + \frac{\rho^{2^n-2}}{2^{n-1}} \psi_1(\rho)$$

where the sum is a polynomial, thus analytic. Up to a scale factor of 2^{-n+1} , if $|\lambda| \ll 2^{-n+1}$, the nontrivial structure of ψ at $\exp(2\pi im/2^n)$ and at 1 are the same, see Note 9; that is

$$\bar{\psi}(\exp(2\pi im/2^n)) = 2^{-n+1}\psi(1) + \text{regular} + o(\lambda)$$

Exact transseries can be obtained for ψ ; see also Note 18.

- (iii) For iterations of the form $x_{n+1} = \lambda^{k-1}P_k(x)$ where P_k is a polynomial of degree $k > 2$, the calculations and the results, for small λ , are essentially the same. The lacunary series would involve the powers z^{k^j} . For instance if the recurrence is $x_{n+1} = \lambda^2x_n(1 - 3x_n + x_n^2)$, then ψ is to be replaced by the solution of

$$\bar{\psi} = \lambda^2\psi^{\vee 3}(1 + 3\psi + \psi^2)$$

and the small λ series will now have $\psi_1 = \sum_{k=1}^{\infty} 3^{-k-1}z^{3^k}$ and so on.

3. Proofs

Lemma 11. *Under the assumptions of Theorem 1, (iii), $h(y) := g^{-1}(y)$ also has a differentiable asymptotic power series as $y \rightarrow \infty$.*

Proof. Straightforward inversion of power series asymptotics, cf. Appendix A. \square

3.1. Proof of Theorem 1

The proof of (4) (i) essentially amounts to showing that $|f|^2$ is diagonally dominant, in that terms containing $g(j)$ and $g(k)$ with $j \neq k$ are comparatively small, as shown in Section 3.8.

(ii) Since g is increasing on $(0, \infty)$, the result follows from the usual integral upper and lower bounds for a sum. Eq. (7) follows from simple calculations, cf. Appendix A.

3.2. Proof of Theorem 2

We prove part (ii); part (i) is similar, and simpler. By standard Fourier analysis we get

$$\{u\} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\sin 2k\pi u}{k\pi}, \quad u \notin \mathbb{Z} \tag{35}$$

where

$$\left\| \{u\} - \frac{1}{2} + \sum_{k=1}^M \frac{\sin 2k\pi u}{k\pi} \right\|_{\infty} \leq C_M \rightarrow \text{Si}(\pi) \quad \text{as } M \rightarrow \infty \tag{36}$$

where Si is the sine integral, and $\text{Si}(\pi)$ is the constant in the Gibbs phenomenon. Let g_m be an analytic function such that $g(s) - g_m(s) = o(s^{-m})$ for large s .

Lemma 12. *The analysis reduces to the case where h is a finite sum of powers. Indeed,*

$$f = e^{-xg(0)} + \int_0^\infty e^{-xu} g^{-1}(u) du + \int_{g(0)}^\infty e^{-xu} \{h_m(u)\} du + R_{m-1}(x) \tag{37}$$

where R_{m-1} is C^{m-1} and h_m is a truncation of the asymptotic expansion of h , such that $h(u) - h_m(u) = o(u^{-m})$.

Proof. We have

$$\begin{aligned} \int_0^\infty e^{-xu} \{h(u)\} du &= \sum_{N=0}^\infty \int_{g(N)}^{g(N+1)} e^{-xu} (h(u) - N) du \\ &= \sum_{N=0}^\infty \int_{g_m(N)}^{g_m(N+1)} e^{-xu} (h_m(u) - N) du + R_{m-1}(x) \end{aligned} \tag{38}$$

where

$$\begin{aligned} R_{m-1}(x) &= \sum_{N=0}^\infty \left(\int_{g(N)}^{g_m(N)} + \int_{g_m(N+1)}^{g(N+1)} \right) e^{-xu} \{h(u)\} du \\ &\quad + \sum_{N=0}^\infty \int_{g_m(N)}^{g_m(N+1)} e^{-xu} (h(u) - h_m(u)) du \end{aligned} \tag{39}$$

Using (35) and (36) the proof follows and the fact that $g(N) - g_m(N) = o(N^{-m})$ and $h(u) - h_m(u) = o(u^{-m})$, the sum is rapidly convergent, and the result follows. \square

Lemma 13. *If h is a finite sum of powers, then $f - e^{-xg(0)} - \int_0^\infty e^{-xu} g^{-1}(u) du \in C^\infty$.*

Proof. Using (35) and (36) we have

$$\begin{aligned} f_1(x) &= x \int_0^\infty e^{-xu} \{h_m(u)\} du = \frac{1}{2} - \sum_{k=1}^\infty \int_0^\infty e^{-xu} \frac{\sin 2k\pi h_m(u)}{k\pi} du \\ &= \frac{1}{2} - \sum_{k=1}^\infty \frac{1}{2k\pi i} \sum_{\sigma=\pm 1} \sigma \int_0^\infty e^{-xu + \sigma 2k\pi i h_m(u)} du \end{aligned} \tag{40}$$

We deform the contours of integration along the directions $\sigma\alpha$ respectively, say for $\alpha = \pi/2$. The integral

$$f_1 = \int_0^{i\infty} u^n e^{2k\pi i h_m(u)} du \tag{41}$$

exists for any n and it is estimated by

$$\left| \int_0^{i\infty} u^n e^{2k\pi i h_m(u)} du \right| < \text{const.} k^{-b(n+1)} \Gamma(n(b+1)) \tag{42}$$

The termwise n th derivatives at 0^+ of the series of $f_1(x)$ converge rapidly, and the result follows. \square

3.3. Proof of Theorem 3

Proof. Let $\epsilon > 0$ be arbitrary, let

$$S_N = \sum_{j=1}^N e^{iyg(j)}; \quad S_0 := 0$$

and let N_1 be large enough so that $|S_N/\rho(N) - L(y)| \leq \epsilon$ for $N > N_1$. Then, by looking at $e^{i\phi} f$ if necessary, we can assume that $L(y) \geq 0$. We have

$$\begin{aligned} f(x + iy) &= \sum_{k=1}^{\infty} e^{-xg(k)} (S_k - S_{k-1}) = \sum_{k=1}^{\infty} \frac{S_k}{\rho(k)} (e^{-xg(k)} - e^{-xg(k+1)}) \rho(k) \\ &= L(y) \sum_{k=1}^{\infty} (e^{-xg(k-1)} - e^{-xg(k)}) \rho(k) + \sum_{k=1}^{\infty} (e^{-xg(k)} - e^{-xg(k+1)}) d_k \rho(k) \end{aligned} \tag{43}$$

where $d_k = L(y) - S_k/\rho(k) \rightarrow 0$ as $k \rightarrow \infty$. Now,

$$\sum_{k=1}^{\infty} e^{-xg(k)} (\rho(k+1) - \rho(k)) \sim \int_0^{\infty} e^{-xg(k)} \rho'(k) dk =: \Phi_{\rho}(x) \tag{44}$$

and under the given assumptions $\Phi_{\rho} \rightarrow \infty$ as $x \rightarrow 0$. Note that $\Phi_{\rho} \rightarrow \infty$ as $x \rightarrow 0$ and the facts that $e^{-xg(k)} - e^{-xg(k+1)} > 0$, $\rho(k) > 0$ and $d_k \rightarrow 0$, readily imply that the last sum in (43) is $o(\Phi_{\rho}(x))$; (12) follows in a similar way. \square

3.4. Proof of Theorem 5

We rely on Theorem 3, and analyze the case $b = 3/2$; the case $b \in \mathbb{N}$ is simpler. We have $\beta = t/(2\pi)$. Let $\check{S}_j = \sum_{k=1}^j e^{-2\pi i \frac{m}{n} k^3}$. It is clear that for $m, n \in \mathbb{N}$ we have $j^{-1} \check{S}_j \rightarrow \sum_{k=1}^n e^{-2\pi i \frac{m}{n} k^3} = L_{mn}$. On the other hand, by summation by parts we get

$$\begin{aligned}
 S_N &= \frac{2\sqrt{2}e^{\pi i/4}}{3\beta} \sum_{k=1}^{k_N-1} k^{1/2} \exp\left(-\frac{8\pi i}{27\beta^2}k^3\right) + O(N^{1/4}) \\
 &= O(N^{1/4}) + \frac{2\sqrt{2}e^{\pi i/4}}{3\beta} (k_N - 1)^{3/2} \check{S}_{k_N-1} \\
 &\quad + \frac{2\sqrt{2}e^{\pi i/4}}{3\beta} \sum_{k=1}^{k_N-1} k^{-1} \check{S}_k (k^{1/2} - (k - 1)^{1/2})
 \end{aligned} \tag{45}$$

and it is easy to check that, as $N \rightarrow \infty$ we have

$$S_N \sim \left(\frac{2\sqrt{2}e^{\pi i/4}}{3\beta} (1 + 1/3)\right) k_N^{3/2} L_{mn} = \frac{8\sqrt{2}e^{\pi i/4}}{9\beta} \sqrt{\frac{3\beta}{2}} N^{3/4} \tag{46}$$

where we used the definition of k_N following Eq. (45), which implies $k_N \sim \frac{3}{2}N^{1/2}\beta$. The result follows by changes of variables, using (11) and noting that

$$\frac{3}{4} \int_0^\infty e^{-xk^{3/2}} k^{-1/4} dk = \frac{\sqrt{\pi}}{2\sqrt{x}} \tag{47}$$

It is also clear that $|j^{-1}\check{S}_j| \leq 1$ and a similar calculation provides an overall upper bound. Section A.4 provides an independent way to calculate the behavior along the boundary.

3.5. Proof of Proposition 4

For $b = 2$ using (40) and $\int_0^\infty e^{-px} \sin 2k\pi p^{1/2} dp = \pi^{3/2} k e^{-\frac{k^2\pi^2}{x}} x^{-3/2}$ we immediately obtain

$$f(x) = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2} + \frac{\sqrt{\pi}}{\sqrt{x}} \sum_{k=1}^\infty e^{-\frac{k^2\pi^2}{x}}$$

3.6. The case $g(j) = a^j$

It is easy to see that, for this choice of g , we have the functional relation

$$f(z) - f(az) = e^{-z}$$

Since $e^{-z} \rightarrow 1$ as $z \rightarrow 0+$, the leading behavior formally satisfies $f(z) - f(az) \sim 1$, i.e., $f(z) \sim -\frac{\log z}{\log a}$

In view of this we let $f(z) = -\frac{\log z}{\log a} + G(z)$ which gives $G(z) - G(az) = e^{-z} - 1$, i.e.,

$$G(z) = G(z/a) + (1 - e^{-z/a}) \tag{48}$$

We first obtain a solution \check{G} of the homogeneous equation and then write $G = \check{G} + h(a^z)$, where h now satisfies

$$h(y) = h(y + 1) \tag{49}$$

Iterating (48) we obtain

$$\check{G}(z) = \sum_{n=0}^{\infty} (1 - e^{-z/a^n}) = - \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-z)^k}{k! a^{nk}} = \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(1 - a^n)}$$

which is indeed an entire function and satisfies the functional relation. Now we return to (49). Since by its connection to $f(z)$, h is obviously smooth in y , it can be expressed in terms of its Fourier series

$$h(y) = \sum_{k=-\infty}^{\infty} c_k e^{2k\pi i y}$$

where the coefficients $c_k = \bar{c}_{-k}$ can be found by using the original function f . Recall that we have

$$f(z) = -\frac{\log z}{\log a} + \check{G}(z) + h\left(\frac{\log z}{\log a}\right)$$

which implies

$$h(y) = f(a^y) + y - \check{G}(a^y)$$

Lemma 14. *The Fourier coefficients of the periodic function h are given by*

$$c_k = \frac{1}{\log a} \Gamma\left(-\frac{2k\pi i}{\log a}\right) \quad (k \neq 0) \tag{50}$$

and

$$c_0 = \int_0^1 f(a^y) dy + \int_0^1 y dy - \int_0^1 \check{G}(a^y) dy$$

Since $c_k \sim \sqrt{\frac{1}{\pi \log a}} e^{\frac{-|k|\pi^2}{\log a}}$ (implying that the Fourier expansion for h is valid exactly for $\text{Re}(z) > 0$), we have

$$f(z) = -\frac{\log z}{\log a} + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(1 - a^n)} + c_0 + \frac{1}{\log a} \sum_{k \neq 0} \Gamma\left(-\frac{2k\pi i}{\log a}\right) z^{\frac{2k\pi i}{\log a}}$$

The series further resums to

$$f(z) = -\frac{\log z}{\log a} + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(1 - a^n)} + c_0 - \frac{z}{2\pi i} \int_0^{\infty} \log_{\mathbb{R}}\left(-s^{\frac{2\pi i}{\log a}}\right) e^{-sz} ds \tag{51}$$

valid for $z > 0$.

The proof is given in Section A.3.

Similar results hold for other rational angles if $b \in \mathbb{N}$ (by grouping terms with the same phase). For example,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} e^{-2^n(z + \frac{2}{5}\pi i)} \\
 &= e^{-\frac{2}{5}\pi i} \sum_{n=0}^{\infty} e^{-2^{4n}z} + e^{-\frac{4}{5}\pi i} \sum_{n=0}^{\infty} e^{-2^{4n+1}z} + e^{-\frac{8}{5}\pi i} \sum_{n=0}^{\infty} e^{-2^{4n+2}z} + e^{-\frac{6}{5}\pi i} \sum_{n=0}^{\infty} e^{-2^{4n+3}z} \\
 &= e^{-\frac{2}{5}\pi i} \sum_{n=0}^{\infty} e^{-16^n z} + e^{-\frac{4}{5}\pi i} \sum_{n=0}^{\infty} e^{-16^n(2z)} + e^{-\frac{8}{5}\pi i} \sum_{n=0}^{\infty} e^{-16^n(4z)} + e^{-\frac{6}{5}\pi i} \sum_{n=0}^{\infty} e^{-16^n(8z)}
 \end{aligned} \tag{52}$$

3.7. Proof of Theorem 4

(i) and (ii): By an argument similar to the one leading to (40) we have

$$f(z) = \int_0^{\infty} e^{-zg(s)} ds - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2k\pi i} \sum_{\sigma=\pm 1} \int_0^{\sigma i \infty} e^{-zu + \sigma 2k\pi i u^{1/b}} du \tag{53}$$

The exponential term ensures convergence in k . Taking $s^b = t$ we see that

$$\int_0^{\infty} e^{-zs^b} ds = \Gamma(1 + 1/b) z^{-1/b} \tag{54}$$

We now analyze the case $\sigma = -1$, the other case being similar. A term in the sum is

$$\begin{aligned}
 & \frac{1}{2k\pi i} \int_0^{-i\infty} e^{-zu - 2k\pi i u^{1/b}} du = \left(\frac{k}{z}\right)^d \int_0^{\infty} e^{-v_k(t + 2\pi i t^{1/b})} dt \\
 &= \left(\frac{k}{z}\right)^d \int_0^{e^{i\theta}\infty} e^{-v_k s} \check{H}_-(s) ds \\
 &= z^{-d} \int_0^{e^{i\theta}\infty} e^{-z^{-1/(b-1)}s} \check{H}_-(s/k^d) ds
 \end{aligned} \tag{55}$$

where θ is any angle between $(b - 2)/(2b - 2)\pi$ and $\pi/2$, $v_k = k^d z^{-1/(b-1)}$, $\check{H}_-(s) = \phi'(s)$ and

$$\Phi(\phi(u)) := \phi(u) + 2\pi i \phi(u)^{1/b} = u \tag{56}$$

(a) Near the origin, we write $\phi = u^b H_b$, and we get

$$H_b = \frac{1}{2\pi i} (1 - u^{b-1} H_b)^b$$

and analyticity of H_b in u^{b-1} follows, for instance, from the contractive mapping principle.

(b) The only singularities of ϕ , thus of \check{H}_- , are the points implicitly given by $\Phi' = 0$.

(c) It is also easy to check that for some $C > 0$ we have

$$|\check{H}_-(u)| \leq C \frac{|u^{b-1}|}{1 + |u^{b-1}|} \tag{57}$$

uniformly in \mathbb{C} , with a cut at the singularity, or on a corresponding Riemann surface. From (57) we see that the sum

$$H_- = \sum_{k=1}^{\infty} \check{H}_-(s/k^d) \tag{58}$$

converges, on compact sets in s , at least as fast as $\text{const.} \sum k^{-b}$, thus it is an analytic function wherever all $\check{H}_-(s/k^d)$ are analytic, that is, in \mathbb{C} except for the points $k^d s_0$. Using an integral estimate, we get the global bound

$$|H_-| \leq \text{const.} \sum_{k=1}^{\infty} \frac{|u|^{b-1}}{k^d + |u|^{b-1}} \leq \text{const.} |u|^{(b-1)^2/b} \tag{59}$$

as $|u| \rightarrow \infty$.

The function H in the lemma is simply $H_- + H_+$ where H_+ is obtained from H_- by replacing i by $-i$. The calculation of the explicit power series is straightforward, from (56), (58) and the similar formulae for H_+ , using dominated convergence based on (59). We provide the details for convenience.

We write the last term in (55) in the form

$$f_k(z) = \left(\frac{k}{z}\right)^{\frac{b}{b-1}} \int_C e^{-s\left(\frac{k^b}{z}\right)^{\frac{1}{b-1}}} \check{H}_- ds \tag{60}$$

where the contour C is a curve from the origin to ∞ in the first quadrant.

Watson’s Lemma implies

$$f_k(z) = \sum_{j=1}^N b_j k^{-j b} z^{j-1} + R_N(k, z)$$

where b_j can be calculated explicitly from Lagrange–Bürmann inversion formula used for the inverse function ϕ^{-1} and $R_N(k, z) \leq C k^{-(N+1)b} z^N$, for arbitrary $N \in \mathbb{N}$. It follows that

$$f(z) \sim \Gamma\left(1 + \frac{1}{b}\right)z^{-\frac{1}{b}} - \frac{1}{2} + \frac{i}{2\pi} \sum_{j=1}^{\infty} (1 - (-1)^{jb})\zeta(jb + 1)b_j z^j \tag{61}$$

which holds for $z \rightarrow 0$ in the right half plane in any direction not tangential to the imaginary axis.

(iii) We obtain the transseries (which gives us information near the imaginary axis) by using the global properties of \check{H}_- , and standard deformation of the Laplace contour.

As z goes around the complex plane, as usual in Laplace-like integrals, we rotate s in (60) simultaneously, to keep the exponent real and positive. In the process, as we cross singularities, we collect a contribution to the integral from the point s_- above; the contribution is an integral around a cut originating at s_- . The singularity is integrable, and collapsing the contour to the cut itself, we get a contribution again in the form of a Laplace transform. This is the Borel sum in the same variable, $z^{-1/(b-1)}$.

Generically s_- is a square root branch point and we have

$$\frac{ds_-}{ds} = \sum_{j=0}^{\infty} \tilde{c}_j \left[s - \left(\frac{bi}{2\pi} \right)^{\frac{b}{1-b}} \right]^{\frac{j-1}{2}}$$

The asymptotic expansion of the cut contribution, Borel summable as we mentioned, is

$$\exp\left(- (2\pi)^{\frac{b}{b-1}} \left(b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}} \right) i^{\frac{b}{1-b}} \left(\frac{k^b}{z} \right)^{\frac{1}{b-1}} \right) \sum_{j=0}^{\infty} c_j \left(\frac{z}{k^b} \right)^{\frac{2j+1}{2(b-1)}}$$

The exponential term ensures convergence in k of the Borel summed transseries. A similar result can be obtained for $-k$.

Thus, the transseries of f is of the form

$$\begin{aligned} \tilde{f}(z) = & \Gamma\left(1 + \frac{1}{b}\right)z^{-\frac{1}{b}} - \frac{1}{2} + \frac{i}{2\pi} \sum_{j=1}^{\infty} (1 - (-1)^{jb})\zeta(jb + 1)b_j z^j \\ & + \begin{cases} \frac{i}{2\pi} \sum_{k=1}^{\infty} e^{(b-1)(\frac{2\pi}{bi})^{\frac{b}{b-1}} (\frac{k^b}{z})^{\frac{1}{b-1}}} \sum_{j=0}^{\infty} c_j k^{\frac{-2jb+2-b}{2(b-1)}} z^{\frac{2j-1}{2(b-1)}}, & -\frac{\pi}{2} \leq \arg z \leq \theta_1 \\ -\frac{i}{2\pi} \sum_{k=1}^{\infty} e^{(b-1)(\frac{2\pi}{bi})^{\frac{b}{b-1}} (\frac{(-k)^b}{z})^{\frac{1}{b-1}}} \sum_{j=0}^{\infty} c_j (-k)^{\frac{-2jb+2-b}{2(b-1)}} z^{\frac{2j-1}{2(b-1)}}, & \theta_2 \leq \arg z \leq \frac{\pi}{2} \end{cases} \end{aligned}$$

where $\theta_{1,2}$ are the counterparts of θ_{\pm} in z .

Remark 15. The analysis can be extended to the case

$$f(z) = \sum_{k=1}^{\infty} F(k)e^{-k^b z} \quad (\lambda \neq 0)$$

by noticing that

$$f(z) = \int_0^\infty (zF(p) - F'(p))e^{-pz} [p^{\frac{1}{b}}] dp$$

If the expression of $F(k)$ is simple, for example $F(k)$ is a finite combination of terms of the form $\mu^{k^{\frac{1}{b}}} k^\lambda (\log k)^m$ ($m = 0, 1, 2, 3, \dots$), the method in this section applies with little change to calculate the transseries of $f(z)$.

For special values of b , asymptotic information as z approaches the imaginary line can be obtained in the following way. Let $z = \delta + 2\pi i\beta$ and

$$f(z) = \sum_{k=1}^\infty e^{-k^b(\delta+2\pi i\beta)} \quad (\text{Re}(\delta) > 0)$$

If $1 < b \in \mathbb{N}$, we may obtain the asymptotic behavior for all rational $\beta = \frac{m}{n}$, by noting that $e^{-k^b(2\pi i\beta)} = e^{-(k+n)^b(2\pi i\beta)}$ and splitting the sum into

$$f(z) = \sum_{j=0}^\infty \sum_{l=1}^n e^{-(nj+l)^b(\delta+2\pi i\frac{m}{n})} = \sum_{l=1}^n e^{-2\pi i\frac{m}{n}l^b} \sum_{j=0}^\infty e^{-(nj+l)^b\delta}$$

It follows that

$$\begin{aligned} \sum_{j=0}^\infty e^{-(nj+l)^b\delta} &= n^b\delta \int_{l^b/n^b}^\infty e^{-n^b\delta s} (s^{\frac{1}{b}} - l/n) ds \\ &= n^b\delta \int_{l^b/n^b}^\infty e^{-n^b\delta s} (s^{\frac{1}{b}} - l/n) ds - n^b\delta \int_{l^b/n^b}^\infty e^{-n^b\delta s} \{s^{\frac{1}{b}} - l/n\} ds \end{aligned}$$

by the argument above. Since the absolute value of the fractional part does not exceed one, we have the estimate

$$\sum_{j=0}^\infty e^{-(nj+l)^b\delta} = n^b\delta \int_{(\frac{l}{n})^b}^\infty e^{-n^b\delta s} s^{\frac{1}{b}} ds + O(1) = \frac{1}{n} \Gamma\left(1 + \frac{1}{b}\right) \delta^{-\frac{1}{b}} + O(1) \quad (z \rightarrow 0)$$

This implies

$$f(z) = \frac{1}{n} \sum_{l=1}^n e^{-2\pi i\frac{m}{n}l^b} \Gamma\left(1 + \frac{1}{b}\right) z^{-\frac{1}{b}} + O(1) \tag{62}$$

Therefore $f(z)$ either blows up like $z^{-\frac{1}{b}}$ (when $\sum_{l=1}^n e^{-2\pi i\frac{m}{n}l^b} \neq 0$) or it is bounded (when $\sum_{l=1}^n e^{-2\pi i\frac{m}{n}l^b} = 0$).

The Fourier expansion of the fractional part can be used to calculate the transseries as we did for $\beta = 0$, but we shall omit the calculation here.

For special values of b , asymptotic information is relatively easy to obtain on a dense set along the barrier. This is the case when $b = \frac{r+1}{r}$ where $r \in \mathbb{N}$; then, the transseries contains exponential sums in terms of integer powers, k^r , a consequence of the duality relation $\frac{1}{b} + \frac{1}{d} = 1$, which at the transseries level is of the form $\sum e^{-k^b z} \rightarrow g_1 + \sum e^{ck^d z^{-d/b}} g_2$ where g_1, g_2 are power series. We illustrate this for $b = \frac{3}{2}$.

Without loss of generality, we assume $\beta < 0$. The transseries of f is given in (87). To estimate the asymptotic behavior of $f(z)$ as z approaches the imaginary line, we rewrite (87) as

$$\begin{aligned}
 f(z) &= \Gamma\left(\frac{5}{3}\right)z^{-\frac{2}{3}} - \frac{1}{2} \\
 &+ \frac{i}{2\pi} \sum_{k=1}^{\infty} \left(\frac{k}{z}\right)^2 \int_0^{\infty} e^{-s\frac{k^3}{z^2}} \left(\frac{3}{4\sqrt{2}(\pi i)^{\frac{3}{2}}}s^{\frac{1}{2}} - \frac{3i}{8\pi^3}s - \frac{105i^{\frac{3}{2}}}{256\sqrt{2}\pi^{\frac{9}{2}}}s^{\frac{3}{2}} + \dots\right) ds \\
 &+ \sum_{k=1}^{\infty} e^{\frac{32i\pi^3 k^3}{27z^2}} \frac{4\sqrt{2}i^{-\frac{1}{2}}\pi}{3} k^{\frac{1}{2}} z^{-1} \\
 &+ \frac{i}{\pi} \sum_{k=1}^{\infty} \left(\frac{k}{z}\right)^2 e^{-\frac{32i\pi^3 k^3}{27z^2}} \int_0^{\infty} e^{-s\frac{k^3}{z^2}} \left(\frac{i^{-\frac{1}{2}}}{8\sqrt{2}\pi^{\frac{3}{4}}}(s - s_0)^{\frac{1}{2}} + \dots\right) ds
 \end{aligned} \tag{63}$$

Watson’s Lemma implies that

$$f(z) = \frac{4\sqrt{2}i^{-\frac{1}{2}}\pi}{3z} \sum_{k=1}^{\infty} k^{\frac{1}{2}} e^{\frac{32i\pi^3 k^3}{27z^2}} + O(1) \tag{64}$$

The sum in (64) is similar to the sum with $b = 3$, and can be estimated in a similar way:

$$\sum_{k=1}^{\infty} k^{\frac{1}{2}} e^{-(y+2\pi i \frac{m}{n})} = \left(\sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^3}\right) \frac{\sqrt{\pi}}{3n\sqrt{y}} + o\left(\frac{1}{\sqrt{y}}\right)$$

Setting $\frac{32i\pi^3 k^3}{27z^2} = y + 2\pi i \frac{m}{n}$, we have $z = -\frac{4\pi i}{3\sqrt{3}}\sqrt{\frac{n}{m}} + \left(\frac{n}{m}\right)^{\frac{2}{3}}y + o(y)$. The asymptotic behavior can be obtained for $\beta = -\frac{4\pi i}{3\sqrt{3}}\sqrt{\frac{n}{m}}$ (this includes all rationals) by substituting $y = \frac{32i\pi^3 k^3}{27z^2} - 2\pi i \frac{m}{n}$ in the above estimates. Setting $z = -\frac{4\pi i}{3\sqrt{3}}\sqrt{\frac{n}{m}} + \delta$ ($\delta > 0$), a direct calculation shows that

$$\sqrt{\operatorname{Re}(z)}f(z) = \frac{\sqrt{6\pi i}}{3^{\frac{7}{4}}}\left(\frac{n}{m}\right)^{\frac{1}{4}} \frac{1}{n} \sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^3} + o(1) \tag{65}$$

3.8. Details of the proof of Theorem 1

Consider, more generally,

$$f(\delta, \beta) = \sum_{k=0}^{\infty} a(k)e^{-g(k)(\delta+2\pi i\beta)}, \quad \delta > 0$$

where $g(k) > 0$ is a real function and $\int_0^{\infty} |a(t)|^2 dt = \infty$.

We find the behavior of

$$\begin{aligned} & \int_{\beta_0}^{\beta_1} |f(\delta, \beta)|^2 d\beta \\ &= \int_{\beta_0}^{\beta_1} \sum_{k=0}^{\infty} |a(k)|^2 e^{-2g(k)\delta} d\beta + \int_{\beta_0}^{\beta_1} \sum_{k \neq j} a(k)\bar{a}(j)e^{-(g(k)+g(j))\delta+(g(j)-g(k))2\pi i\beta} d\beta \\ &= (\beta_1 - \beta_0) \sum_{k=0}^{\infty} |a(k)|^2 e^{-2g(k)\delta} \\ & \quad + \frac{1}{2\pi i} \sum_{k \neq j} \frac{a(k)\bar{a}(j)}{g(j) - g(k)} e^{-(g(k)+g(j))\delta} e^{(g(j)-g(k))2\pi i\beta_0} (e^{(g(j)-g(k))2\pi i(\beta_1-\beta_0)} - 1) \end{aligned} \tag{66}$$

where $\beta_{0,1} \in \mathbb{R}$ are arbitrary, or after m integrations,

$$\begin{aligned} F(\delta) &= \int_{\beta_{m-1}}^{\beta_{m-1}+c_{m-1}} \cdots \int_{\beta_1}^{\beta_1+c_1} \int_{\beta_0}^{\beta_0+c_0} |f(\delta, \beta)|^2 d\beta d\beta_0 \cdots d\beta_{m-2} \\ &= c_0 c_1 \cdots c_m \sum_{k=0}^{\infty} |a(k)|^2 e^{-2g(k)\delta} + O\left(\sum_{k \neq j} \left| \frac{a(k)\bar{a}(j)}{(g(j) - g(k))^m} \right| e^{-(g(k)+g(j))\delta}\right) \end{aligned} \tag{67}$$

Note that

$$\begin{aligned} & \sum_{k \neq j} \left| \frac{a(k)\bar{a}(j)}{(g(j) - g(k))^m} \right| e^{-(g(k)+g(j))\delta} \\ &= 2 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \left| \frac{a(k)\bar{a}(k+n)}{(g(k+n) - g(k))^m} \right| e^{-(g(k)+g(k+n))\delta} \\ &= 2 \sum_{k=0}^{\infty} |a(k)| e^{-2g(k)\delta} \sum_{n=1}^{\infty} \left| \frac{a(k+n)}{(g(k+n) - g(k))^m} \right| e^{-(g(k+n)-g(k))\delta} \\ &= O\left(\sum_{k=0}^{\infty} |a(k)| e^{-2g(k)\delta} \sum_{n=1}^{\infty} \left| \frac{a(k+n)}{(g(k+n) - g(k))^m} \right|\right) \end{aligned} \tag{68}$$

under Assumption 1.

If furthermore we have

$$\sum_{n=1}^{\infty} \left| \frac{a(k+n)}{(g(k+n) - g(k))^m} \right| = o(a(k))$$

then we obtain

$$F(\delta) = c_0 c_1 \cdots c_m \sum_{k=0}^{\infty} |a(k)|^2 e^{-2g(k)\delta} + O\left(\sum_{k=0}^{\infty} \frac{|a(k)|^2}{G(k)} e^{-2g(k)\delta}\right) \tag{69}$$

where $G(k) > 1$, $G(k) \rightarrow \infty$ as $k \rightarrow \infty$, or

$$F(\delta) \left(\sum_{k=0}^{\infty} |a(k)|^2 e^{-2g(k)\delta}\right)^{-1} = c_0 \cdots c_m + o(1) \quad \text{as } \delta \rightarrow 0^+ \tag{70}$$

The result now follows from the following proposition.

Proposition 16. *Assume*

- (i) $h_n : \mathbb{R} \rightarrow [0, \infty)$ are locally L^1 , and
- (ii) $\lim_{n \rightarrow \infty} \int_B h_n(x_1 + \cdots + x_N) dx_1 \cdots dx_N = \text{meas}(B)$ for any box $B = \prod_{i=1}^N [a_i, b_i]$.

Then $h_n \rightarrow 1$ in the dual of $C[\alpha, \beta]$ for any $[\alpha, \beta]$.

Proof. We first take $N = 2$, the general case will follow by induction on N .

Consider the rectangle $B_c = (a, b - c) \times (0, c)$, $0 < c < b - a$. By changing coordinates to $x + y = s, y = y'$, we get that

$$c^{-1} \int_B h_n dy dx = \int_a^b h_n(s) T_{a,b;c}(s) ds \rightarrow \text{meas}(T_{a,b;c}) \quad \text{as } n \rightarrow \infty \tag{71}$$

where $T_{a,b;c}(\cdot)$ is the function having $T_{a,b;c}$ as a graph, $T_{a,b;c}$ being an isosceles trapezoid with lower base the interval (a, b) and upper base of length $b - a - 2c$ at height 1.

We also note that the indicator function of $[a, b]$, $\mathbf{1}_{ab}$, satisfies the inequalities $T_{a-c,b+c;c} \geq \mathbf{1}_{ab} \geq T_{a,b;c}$. Thus, since c is arbitrary and $h_n \geq 0$, and both $\text{meas}(T_{a-c,b+c;c})$, and $\text{meas}(T_{a,b;c})$ tend to $(b - a)$ as $c \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \int_a^b h_n(s) ds = (b - a) = \text{meas}([a, b]) \tag{72}$$

In particular, given $\alpha < \beta$, $\|h_n\|_{L^1[\alpha,\beta]}$ are uniformly bounded, that is, for some $C \geq 1$ we have

$$\sup_{n \geq 1} \|h_n\|_{L^1[\alpha,\beta]} \leq C(\beta - \alpha) \tag{73}$$

Since a continuous function on $[\alpha, \beta]$ is approximated arbitrarily well in sup norm by finite linear combinations of indicator functions of intervals, it follows from (72), (73)⁶ and the triangle inequality that

$$\int_{\alpha}^{\beta} h_n(s)f(s) ds \rightarrow \int_{\alpha}^{\beta} f(s) ds, \quad \forall f \in C[\alpha, \beta] \tag{74}$$

For general N we use $h_n^-(s) = \int_{B^r} h_n(s + x_1 + \dots + x_{N-1}) dx_1 \dots dx_{N-1}$ and (72) to reduce the problem to $N - 1$. \square

The condition

$$\sum_{n=1}^{\infty} \left| \frac{a(k+n)}{(g(k+n) - g(k))^m} \right| = o(a(k))$$

is satisfied, for instance, if

- (1) $\exists c > 0$ so that $c < |a(k)| < c^{-1}$, or $|a(k)|$ decreases to 0. (Note that $g(k+n) - g(k) = g'(k+tn)n$, where $0 \leq t \leq 1$, and $g'(k) \rightarrow \infty$ as $k \rightarrow \infty$.)
- (2) $\exists c, r$ so that $c < a(k) < k^r$. $g'(k) \geq k^\epsilon$ for some small $\epsilon > 0$.

4. Proof of Theorem 6

In Appendix A.5 we list some known facts about iterations of maps.

Proof of Bötcher’s theorem, for (28). (Note: this line of proof extends to general analytic maps.)

We write $\psi = \lambda z + \lambda^2 z g(z)$ and obtain

$$g(z) - \frac{1}{2}g(z^2) = \frac{1}{2}z + \frac{1}{2}\lambda[g(z)(z - g(z)) + g(z^2)] + \frac{\lambda^2 z}{2}g(z)g(z^2) = N(g) \tag{75}$$

Let \mathcal{A}_λ denote the functions analytic in the polydisk $\mathbb{P}_{1,\epsilon} = \mathbb{D} \times \{\lambda: |\lambda| < \epsilon\}$. We write (75) in the form (see (29))

$$g = 2\mathfrak{I}N(g) \tag{76}$$

This equation is manifestly contractive in the sup norm, in a ball of radius slightly larger than $1/2$ in \mathcal{A}_λ , if ϵ is small enough. For $\lambda \neq 0$, evidently $\psi = \phi^{-1}$ is also analytic at zero. \square

Lemma 17. ψ is analytic in \mathbb{D}_1 for all λ with $|\lambda| < 1$.

Proof. We have

$$\psi(z) = \frac{\lambda}{2}(X + \sqrt{X^2 + 4X/\lambda}) =: F(X); \quad X = \psi(z^2) \tag{77}$$

⁶ Alternatively, and somewhat more compactly, one can prove the result without the intermediate steps (72) and (73) by upper and lower bounding continuous functions by sums of trapezoids.

For small $z \neq 0$, $\psi(z) = O(z)$ and thus $F(\psi(z^2))$ is well defined and analytic. Note that (77) provides analytic continuation of ψ from \mathbb{D}_{ρ^2} to \mathbb{D}_ρ , provided nowhere in \mathbb{D}_{ρ^2} do we have $\psi = -4/\lambda$ (certainly the case if ρ is small). We assume, to get a contradiction, that there is a z_0 , $|z_0| = \lambda_0 < 1$ so that $\psi(z_0) = -4/\lambda$, and we choose the least λ_0 with this property. By the previous discussion, ψ is analytic in the open disk $\mathbb{D}_{\sqrt{\lambda_0}}$. Then we use the “backward” iteration $\psi(z^2) = \lambda^{-1}\psi(z)^2/(1 + \psi(z))$ to calculate $\psi(z^2)$ from $\psi(z)$, starting with $z = z_0$. This is in fact equivalent to (27); after the substitution $x = (-\lambda y)^{-1}$ we return to (25), with $x_0 = \lambda/4$. Using (vi) and (vii) of Appendix A.5, it follows that $1/x_n \not\rightarrow 0$, that is, $\psi(z_0^{2^n}) \not\rightarrow 0$. This impossible, since ψ is analytic and $\psi(0) = 0$. \square

Proof of Theorem 6, (i). We return to (28). Taking $a \in (0, 1)$ $m_n = \sup\{|\psi(z)| : |z| < a^{1/2^n}\}$ we note that

$$m_{n+1} \leq \frac{1}{2}|\lambda|(m_n + \sqrt{m_n^2 + 4m_n/|\lambda|}) \tag{78}$$

The sequence of m_n is bounded by the sequence of M_n , defined by replacing “ \leq ” with “ $=$ ” in (78). Since $\frac{1}{2}|\lambda|(x + \sqrt{x^2 + 4x/|\lambda|}) < x$ if $x > A := |\lambda|/(1 - |\lambda|)$, we have $\limsup_n M_n \leq A$. By the maximum principle, $|\psi(\lambda, z)| < A$ in $\mathbb{D} \times \mathbb{D}$. Thus, by Cauchy’s formula in λ we have $|\psi_n(z)| \leq A$ for all n and $z \in \mathbb{D}$. The radius of convergence of (30) in λ is at least one. By Appendix A.5, (vii), the radius of convergence is exactly one.

Indeed, note first that (a) if ψ is analytic in \mathbb{D} then $\psi' \neq 0$ in \mathbb{D} , otherwise $\psi'(z_1) = 0$ would imply $\psi'(z_1^{2^n}) = 0$ in contradiction with $\psi'(0) = \lambda$. This means that if there is a z_0 , $\psi(z_0) = -4/\lambda$, then $\sqrt{z_0}$ is a singular point of ψ .

Secondly, any λ of the form $1 + i\epsilon$ with small ϵ correspond to $c = 1/4 + 1/4\epsilon^2$, outside the Mandelbrot set. Thus, in the iteration (27), the initial condition $y_0 = -4/\lambda$ implies $y_n \rightarrow 0$. We can now use the implicit function theorem to suitably match y_n , once it is small enough, to some value of ψ near zero. Indeed, the equation $y_n = \lambda z_0^{2^n} + O(z_0^{2^{n+1}})$ has 2^n solutions. This means that for such a z_0 , using (77) to iterate backwards and to determine $\psi(z_0)$ (noting the parallel to (27)), we have $\psi(z_0) = -4/\lambda$, and by (a) above, ψ cannot be analytic in z in \mathbb{D} .

Formula (32) follows by straightforward expansion of (28) and identification of powers of λ . \square

Proof of Theorem 6, (ii). The stated type of lacunarity of ψ_k follows from (32) by induction, noting the discrete convolution structure in k . \square

Proof of Theorem 6, (iii). Continuity of ψ_k in $\overline{\mathbb{D}}$ also follows by induction from (32) and the properties of \mathfrak{F} . By dominated convergence (applied to the discrete measure $|\lambda|^n$), for $\lambda < 1$, ψ is continuous in $\overline{\mathbb{D}}$ and the Fourier series converges pointwise in $\partial\mathbb{D}$.

To show convergence of the Fourier series of H we only need to show $\inf_{\mathbb{D}} |\psi| > 0$. Now, ψ clearly cannot vanish for any $z_0 \in \mathbb{D}$, otherwise $\psi(z_0^{2^n}) = 0$, would imply by analyticity $\psi \equiv 0$. If $\min_{\mathbb{D}_\rho} |\psi(\rho)| = \epsilon$ would be small enough, then $\min_{\mathbb{D}_{\rho^2}} |\psi(\rho)| \leq O(\epsilon^2) \ll \epsilon$, contradicting the maximum principle for $z/\psi(z)$.

The rest of the proof is straightforward calculation, using the analyticity of ψ . \square

The extension of the small λ analysis to higher order polynomials is also straightforward.

Note 18. The transseries of the Bötcher map at binary rational numbers can be calculated rather explicitly. This is beyond the scope here, and will be the subject of a different paper. A less explicit expression has been obtained in [5]. We note that the constant $\log_2(2\pi)$ in (1.17) of [5] should be $(2\pi)/\log 2$.

Appendix A

A.1. Proof of Lemma 11

For every m , we write $g(s) = g_m + o(s^{-m})$ where g_m is a finite sum, an initial sum in the asymptotic series of g . It is straightforward to show that $g_m^{-1}(y)$ has an asymptotic power series as $y \rightarrow \infty$. Then, in the equation $g(s) = y$ we write $s = s_m + \epsilon$ where $g_m(s_m) = y$. Then, $y = g(s) = g(s_m + \epsilon) = g_m(s_m) + g'(\xi)\epsilon + (g(s_m) - g_m(s_m))$ implies $\epsilon = (g_m(s_m) - g(s_m))/g'(\xi) = o(s^{-m-\theta})$ where $g'(\xi) \sim ax^\theta$. Now θ is fixed and m is arbitrary, and then the result follows.

A.2. Proof of Theorem 2 part (i)

Letting $\lfloor \cdot \rfloor$ denote the floor function (greatest integer function), we have

$$\begin{aligned}
 f - e^{-xg(0)} &= - \sum_{k=1}^{\infty} k(e^{-xg(k+1)} - e^{-xg(k)}) \\
 &= x \int_0^{\infty} e^{-xg(s)} g'(s) \lfloor s \rfloor ds = x \int_{g(0)}^{\infty} e^{-xu} \lfloor g^{-1}(u) \rfloor du \\
 &= x \int_{g(0)}^{\infty} e^{-xu} g^{-1}(u) du - x \int_{g(0)}^{\infty} e^{-xu} \{g^{-1}(u)\} du
 \end{aligned} \tag{79}$$

and we also have

$$x \int_{g(0)}^{\infty} e^{-xu} g^{-1}(u) du = - \int_0^{\infty} (e^{-xg(u)})' u du = \int_0^{\infty} e^{-xg(u)} du \tag{80}$$

whereas

$$0 < \int_0^{\infty} e^{-xu} xg'(u) \{u\} du \leq \int_0^{\infty} e^{-xu} xg'(u) du = e^{xg(0)}$$

A.3. Proof of Lemma 14

By the Fourier coefficients formula we have

$$\begin{aligned}
 c_0 &= \int_0^1 f(a^y) dy + \int_0^1 y dy - \int_0^1 \check{G}(a^y) dy \\
 c_k &= \int_0^1 f(a^y) e^{-2k\pi iy} dy + \int_0^1 y e^{-2k\pi iy} dy - \int_0^1 \check{G}(a^y) e^{-2k\pi iy} dy \\
 &= \frac{i}{2k\pi} + \sum_{n=0}^{\infty} \int_0^1 e^{-a^{n+y} - 2k\pi iy} dy + \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^n a^{ny}}{n!(a^n - 1)} e^{-2k\pi iy} dy \\
 &= \frac{i}{2k\pi} + \frac{1}{\log a} \sum_{n=0}^{\infty} a^{\frac{2kn\pi i}{\log a}} \left(\Gamma\left(-\frac{2k\pi i}{\log a}, a^n\right) - \Gamma\left(-\frac{2k\pi i}{\log a}, a^{1+n}\right) \right) \\
 &\quad + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!(2k\pi i - n \log a)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!(2k\pi i - n \log a)} + \frac{1}{\log a} \Gamma\left(-\frac{2k\pi i}{\log a}, 1\right) \quad (k \neq 0)
 \end{aligned}
 \tag{81}$$

where $\Gamma(x, y)$ is the incomplete Gamma function.

Note that since $\mathcal{L}^{-1}\left(\frac{1}{2k\pi i - n \log a}\right) = \frac{1}{\log a} e^{\frac{2kn\pi ip}{\log a}}$ (the inverse Laplace transform is from the variable n to the variable p) we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!(2k\pi i - n \log a)} &= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} e^{-np}}{n!} \frac{1}{\log a} e^{\frac{2kn\pi ip}{\log a}} dp \\
 &= \frac{1}{\log a} \int_0^{\infty} e^{-e^{-p}} e^{\frac{2k\pi ip}{\log a}} dp = \frac{1}{\log a} \int_0^1 e^{-t} t^{\frac{-2k\pi i}{\log a} - 1} dt \\
 &= \frac{1}{\log a} \left(\Gamma\left(-\frac{2k\pi i}{\log a}\right) - \Gamma\left(-\frac{2k\pi i}{\log a}, 1\right) \right)
 \end{aligned}$$

The above procedure is justified for k in the upper half plane. By analytic continuation the expression holds for k real as well. Eq. (50) follows.

We can further resum the series in the above expression by noting that

$$\begin{aligned}
 \sum_{k \neq 0} \Gamma\left(-\frac{2k\pi i}{\log a}\right) z^{\frac{2k\pi i}{\log a}} &= \sum_{k=1}^{\infty} \frac{-\log a}{2k\pi i} \int_0^{\infty} t^{\frac{-2k\pi i}{\log a}} e^{-t} z^{\frac{2k\pi i}{\log a}} dt + \sum_{k=-\infty}^{-1} \frac{-\log a}{2k\pi i} \\
 &= \int_0^{\infty} \sum_{k=1}^{\infty} \frac{-\log a}{2k\pi i} \left(\frac{z}{t}\right)^{\frac{2k\pi i}{\log a}} e^{-t} dt + \int_0^{\infty} \sum_{k=-\infty}^{-1} \frac{-\log a}{2k\pi i} \left(\frac{z}{t}\right)^{\frac{2k\pi i}{\log a}} e^{-t} dt
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\log a}{2\pi i} \int_0^\infty \log_{\mathbb{R}} \left(1 - \left(\frac{z}{t} \right)^{\frac{2\pi i}{\log a}} \right) e^{-t} dt - \frac{\log a}{2\pi i} \int_0^\infty \log_{\mathbb{R}} \left(1 - \left(\frac{z}{t} \right)^{-\frac{2\pi i}{\log a}} \right) e^{-t} dt \\
 &= \frac{\log a}{2\pi i} \int_0^\infty \log_{\mathbb{R}} \left(- \left(\frac{z}{t} \right)^{\frac{2\pi i}{\log a}} \right) e^{-t} dt \\
 &= -\frac{\log a z}{2\pi i} \int_0^\infty \log_{\mathbb{R}} \left(-s^{\frac{2\pi i}{\log a}} \right) e^{-sz} ds \tag{82}
 \end{aligned}$$

which can be justified by analytic continuation, for the last expression and the sum are both analytic, and equal to each other on the real line. The logarithm $\log_{\mathbb{R}}$ is defined with a branch cut along \mathbb{R}^- .

We finally obtain the integral representation (51), valid for z in the right half plane.

Remark 19. Since $\log_{\mathbb{R}} \left(-s^{\frac{2\pi i}{\log a}} \right) = i \arg \left(\frac{2\pi \log s}{\log a} - \pi \right) = 2\pi i \left(\left\{ \frac{\log s}{\log a} \right\} - \frac{1}{2} \right)$, we actually recover the last term of (7).

A.4. Direct calculations for $b \in \mathbb{N}$ integer; the cases $b = 3, b = 3/2$

Proposition 20. *If b is an integer, the behavior of*

$$f(\delta + 2\pi i\beta) = \sum_{k=1}^\infty e^{-k^b(\delta + 2\pi i\beta)} \quad (\operatorname{Re}(\delta) > 0)$$

where $\beta = 2\pi im/n$, m and n being integers, as δ approaches 0 is

$$f(\delta + 2\pi i\beta) = \left[\frac{1}{n} \sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^b} \Gamma \left(1 + \frac{1}{b} \right) \right] \delta^{-\frac{1}{b}} + O(1)$$

Therefore $f(\delta)$ either blows up like $\delta^{-\frac{1}{b}}$ or is bounded.

The more general case $b = \frac{r+1}{r}$ where r is an integer can be treated similarly. In particular, if $b = \frac{3}{2}$, for $\beta = -\frac{4\pi i}{3\sqrt{3}}\sqrt{\frac{n}{m}}$ (this includes all rational numbers), we have

$$\sum_{k=1}^\infty e^{-k^{3/2}z} = \frac{\sqrt{6\pi i}}{3^{7/4} m^{1/4} n^{3/4}} \sum_{l=1}^n e^{-2\pi i \frac{m}{n} l^3} \frac{1}{\sqrt{\delta}} + o\left(\frac{1}{\sqrt{\delta}}\right)$$

with $z = -\frac{4\pi i}{3\sqrt{3}}\sqrt{\frac{n}{m}} + \delta$ ($\delta \rightarrow 0^+$).

Proof. In general, to find the asymptotic behavior of $f(z)$ we analyze the functions

$$f_k(z) = \int_0^\infty e^{-pz-2k\pi ip^{\frac{1}{b}}} dp, \quad k \in \mathbb{Z} \tag{83}$$

Letting $p = q(\frac{z}{k})^{\frac{b}{1-b}}$ we have

$$\int_0^\infty e^{-pz-2k\pi ip^{\frac{1}{b}}} dp = \left(\frac{k}{z}\right)^{\frac{b}{b-1}} \int_0^\infty e^{-(q+2\pi iq^{\frac{1}{b}})(\frac{k^b}{z})^{\frac{1}{b-1}}} dq$$

Next we let $s = h(q) = q + 2\pi iq^{\frac{1}{b}}$ and $f_k(z) = (\frac{k}{z})^{\frac{b}{b-1}} \int_{C_1} e^{-s(\frac{k^b}{z})^{\frac{1}{b-1}}} \frac{1}{h'(h^{-1}(s))} ds$ where the contour C_1 is a curve from the origin to ∞ in the first quadrant.

We can find the asymptotic behavior of $f_k(z)$ using Watson’s Lemma [1]. By iterating the contractive map $q \rightarrow (\frac{s-q}{2\pi i})^b$ near 0, we can easily see that $\frac{h^{-1}(s)}{s^b}$ is analytic in s^{b-1} , which implies $\frac{1}{h'(h^{-1}(s))}$ is analytic in s^{b-1} near 0 with no constant term.

Now let’s consider the examples $b = 3$ and $b = 3/2$; for $b = 3$ we have $s = h(q) = q + 2\pi iq^{\frac{1}{3}}$ and $\frac{ds}{dq} = \frac{3i}{8\pi^3}s^2 + \frac{15}{64\pi^6}s^4 - \frac{21i}{128\pi^9}s^6 \dots$. Thus, the asymptotic power series is

$$f(z) \sim \Gamma\left(\frac{4}{3}\right)z^{-\frac{1}{3}} - \frac{1}{2} - \frac{z}{120} + \frac{z^3}{792} + \dots$$

The branch point of h^{-1} is located at $s_0 = \pi^{\frac{3}{2}} \frac{4\sqrt{2}}{3\sqrt{3}}(-1)^{\frac{1}{4}}$, which is between the contour C_1 defined above and the x -axis. As we start rotating z from $z > 0$, we have, cf. (60),

$$\begin{aligned} f_k(z) &= \left(\frac{k}{z}\right)^{\frac{3}{2}} \int_{C_1} \frac{e^{-s(\frac{k^3}{z})^{\frac{1}{2}}}}{h'(h^{-1}(s))} ds \\ &= \left(\frac{k}{z}\right)^{\frac{3}{2}} \int_0^\infty \frac{e^{-s(\frac{k^3}{z})^{\frac{1}{2}}}}{h'(h^{-1}(s))} ds + \left(\frac{k}{z}\right)^{\frac{3}{2}} e^{-s_0(\frac{k^3}{z})^{\frac{1}{2}}} \int_{C_2} \frac{e^{-s(\frac{k^3}{z})^{\frac{1}{2}}}}{h'(h^{-1}(s+s_0))} ds \\ &= \left(\frac{k}{z}\right)^{\frac{3}{2}} \int_0^\infty \frac{e^{-s(\frac{k^3}{z})^{\frac{1}{2}}}}{h'(h^{-1}(s))} ds + 2\left(\frac{k}{z}\right)^{\frac{3}{2}} e^{-s_0(\frac{k^3}{z})^{\frac{1}{2}}} \int_0^\infty \frac{e^{-s(\frac{k^3}{z})^{\frac{1}{2}}}}{h'(h^{-1}(s+s_0))} ds \end{aligned} \tag{84}$$

where the contour C_2 starts at ∞ , goes clockwise around the origin, then ends at ∞ . Since now

$$\frac{ds}{dq} = \frac{(-\pi i)^{\frac{3}{4}}}{6^{\frac{1}{4}}}(s-s_0)^{-\frac{1}{2}} + \frac{5}{6} + \frac{5}{16}\left(\frac{i}{6\pi}\right)^{\frac{3}{4}}(s-s_0)^{\frac{1}{2}} + \dots$$

We have

$$\begin{aligned} & \left(\frac{k}{z}\right)^{\frac{3}{2}} e^{-s_0 z} \int_{C_2} \frac{e^{-s\left(\frac{k^3}{z}\right)^{\frac{1}{2}}}}{h'(h^{-1}(s+s_0))} ds \\ &= e^{-\pi^{\frac{3}{2}} \frac{4\sqrt{2}}{3\sqrt{3}} (-1)^{\frac{1}{4}} z} \left(\left(\frac{\pi i}{6}\right)^{\frac{1}{4}} k^{-\frac{1}{4}} z^{-\frac{1}{4}} + \frac{5}{32} \frac{i^{\frac{7}{4}}}{6^{\frac{3}{4}} \pi^{\frac{5}{4}}} k^{-\frac{7}{4}} z^{\frac{1}{4}} + \dots \right) \end{aligned} \tag{85}$$

Therefore the transseries is

$$\begin{aligned} \tilde{f}(z) &= \Gamma\left(\frac{4}{3}\right) z^{-\frac{1}{3}} - \frac{1}{2} - \frac{z}{120} + \frac{z^3}{792} + \dots \\ &+ \sum_{k=1}^{\infty} e^{-\pi^{\frac{3}{2}} \frac{4\sqrt{2}}{3\sqrt{3}} (-1)^{\frac{1}{4}} \left(\frac{k^3}{z}\right)^{\frac{1}{2}}} \left[\left(\frac{\pi i}{6}\right)^{\frac{1}{4}} k^{-\frac{1}{4}} z^{-\frac{1}{4}} + \frac{5}{32} \frac{i^{\frac{7}{4}}}{6^{\frac{3}{4}} \pi^{\frac{5}{4}}} k^{-\frac{7}{4}} z^{\frac{1}{4}} + \dots \right] \\ &- \sum_{k=1}^{\infty} e^{-\pi^{\frac{3}{2}} \frac{4\sqrt{2}}{3\sqrt{3}} (-1)^{\frac{1}{4}} \left(-\frac{k^3}{z}\right)^{\frac{1}{2}}} \left[\left(\frac{\pi i}{6}\right)^{\frac{1}{4}} (-k)^{-\frac{1}{4}} z^{-\frac{1}{4}} + \frac{5}{32} \frac{i^{\frac{7}{4}}}{6^{\frac{3}{4}} \pi^{\frac{5}{4}}} (-k)^{\frac{7}{4}} z^{\frac{1}{4}} + \dots \right] \end{aligned} \tag{86}$$

The calculation for $b = \frac{3}{2}$ is similar: in this case $\frac{ds}{dq} = \frac{3}{4\sqrt{2}(\pi i)^{\frac{3}{2}}} s^{\frac{1}{2}} - \frac{3i}{8\pi^3} s - \frac{105i^{\frac{3}{2}}}{256\sqrt{2}\pi^{\frac{9}{2}}} s^{\frac{3}{2}} + \dots$ and the asymptotic power series is

$$f(z) \sim \Gamma\left(\frac{5}{3}\right) z^{-\frac{2}{3}} - \frac{1}{2} - \frac{3\zeta\left(\frac{5}{2}\right)}{16\pi^2} z + \frac{1}{240} z^2 + \frac{315\zeta\left(\frac{11}{2}\right)}{2048\pi^5} z^3 + \dots$$

The exponential sum is slightly different than in the previous case, for now the branch point $s_0 = -\frac{32}{27}i\pi^3$ lies in the lower half plane, which means the contour C_2 can be deformed to $[0, +\infty)$ without passing through any singularity.

We collect the contribution from the branch point only when $\arg z$ decreases to $-\frac{\pi}{4}$ from 0. Since

$$\frac{ds}{dq} = \frac{-4\sqrt{2}i^{\frac{1}{2}}\pi^{\frac{3}{2}}}{3} (s-s_0)^{-\frac{1}{2}} + \frac{4}{3} + \frac{i^{-\frac{1}{2}}}{8\sqrt{2}\pi^{\frac{3}{4}}} (s-s_0)^{\frac{1}{2}} + \dots$$

we have for the exponential part of the sum

$$\left(\frac{4\sqrt{2}i^{-\frac{1}{2}}\pi^{\frac{1}{2}}}{3} k^{\frac{1}{2}} z^{-1} + \frac{i^{\frac{1}{2}}}{16\sqrt{2}\pi^{\frac{5}{4}}} k^{-\frac{5}{2}} z^2 + \dots \right) \exp\left(\frac{32i\pi^3 k^3}{27z^2}\right)$$

Therefore, for small z in the right half plane, the transseries is given by

$$\begin{aligned} \tilde{f}(z) = & \Gamma\left(\frac{5}{3}\right)z^{-\frac{2}{3}} - \frac{1}{2} - \frac{3\zeta\left(\frac{5}{2}\right)}{16\pi^2}z + \frac{1}{240}z^2 + \frac{315\zeta\left(\frac{11}{2}\right)}{2048\pi^5}z^3 + \dots \\ & + \begin{cases} \sum_{k=1}^{\infty} e^{\frac{32i\pi^3 k^3}{27z^2}} \left(\frac{4\sqrt{2}i^{-\frac{1}{2}}\pi}{3} k^{\frac{1}{2}} z^{-1} + \frac{i^{\frac{1}{2}}}{16\sqrt{2}\pi^{\frac{5}{4}}} k^{-\frac{5}{2}} z^2 + \dots \right); \\ -\frac{\pi}{2} \leq \arg z \leq -\frac{\pi}{4} \\ -\sum_{k=1}^{\infty} e^{-\frac{32i\pi^3 k^3}{27z^2}} \left(\frac{4\sqrt{2}i^{-\frac{1}{2}}\pi}{3} (-k)^{\frac{1}{2}} z^{-1} + \frac{i^{\frac{1}{2}}}{16\sqrt{2}\pi^{\frac{5}{4}}} (-k)^{-\frac{5}{2}} z^2 + \dots \right); \\ \frac{\pi}{4} \leq \arg z \leq \frac{\pi}{2} \end{cases} \end{aligned} \tag{87}$$

The effect of the exponential part of the transseries affects the leading order when $z \rightarrow 0$ nearly tangentially to the imaginary line.

For example, to see the effect of the first exponential term for $b = \frac{3}{2}$, we let $z^{-\frac{2}{3}} \sim e^{\frac{32i\pi^3}{27z^2}} z^{-1}$ and $z = -ire^{i\theta}$ near the negative imaginary line.

The critical curve along which the power term and the exponential term are of equal order is $\theta \sim \frac{9}{64\pi^3} r^2 \log \frac{1}{r}$. \square

A.5. *Notes about iterations of maps*

For the following, see e.g., [2,6,14].

- (i) For $|\lambda| < 1$, three types of behavior are possible for the solution of (25): if the initial condition $x_0 \in \mathcal{F}$, the connected component of the origin in the Fatou set, then $x_n \rightarrow 0$ as $n \rightarrow \infty$. Clearly, $x_0 \in \mathcal{F}$ if x_0 is small enough. If $x_0 \in \overline{\mathcal{F}}^c$, the connected component of infinity in the Fatou set, then $|x_n| \rightarrow \infty$. Clearly, $x_0 \in \overline{\mathcal{F}}^c$ if it is large enough. Finally, for $x_0 \in \partial F = J$, the Julia set, a (connected) curve of nontrivial Hausdorff dimension invariant under the map, $\{x_n\}_n$ are dense in J and the evolution is chaotic.
- (ii) J is the closure of the set of repelling periodic points.
- (iii) For polynomial maps, and more generally, for entire maps, J is the boundary of the set of points which converge to infinity under iteration.
- (iv) If the maximal disk of analyticity of ψ is the unit disk \mathbb{D}_1 , then ψ maps \mathbb{D}_1 biholomorphically onto the immediate basin \mathcal{A}_0 of zero. If on the contrary the maximal disk is \mathbb{D}_r , $r < 1$, then there is at least one other critical point in \mathcal{A}_0 , lying in $\psi(\partial\mathbb{D}_r) = J_y$, the Julia set of (27).
- (v) If $r = 1$, it follows that $\psi(\partial\mathbb{D}_1) = J_y$.
- (vi) By the change of variable $x_n = -(1/\lambda)t_n + 1/2$, (25) is brought to the “c form” $t_{n+1} = t_n^2 + c$, $c = \lambda/2 - \lambda^2/4$. The Mandelbrot set is defined as (see e.g. [6] and Fig. 5)

$$\mathcal{M} = \{c: t_n \text{ bounded if } t_0 = 0\} \tag{88}$$

If $c \in \mathcal{M}$, then clearly y_n in (27) are bounded away from zero. Note that $t_0 = 0$ corresponds to $x_0 = 1/2$ implying $x_1 = -\lambda/4$.

- (vii) \mathcal{M} is a compact set; it coincides with the set of c for which J is connected. The cardioid $\mathcal{H} = \{(2e^{it} - e^{2it})/4: t \in [0, 2\pi)\}$ is contained in \mathcal{M} ; see [6]. This means $\{\lambda: |\lambda| < 1\}$ corresponds to the interior of \mathcal{M} . We have $|\lambda| = 1 \Rightarrow c \in \partial\mathcal{M} \subset \mathcal{M}$.

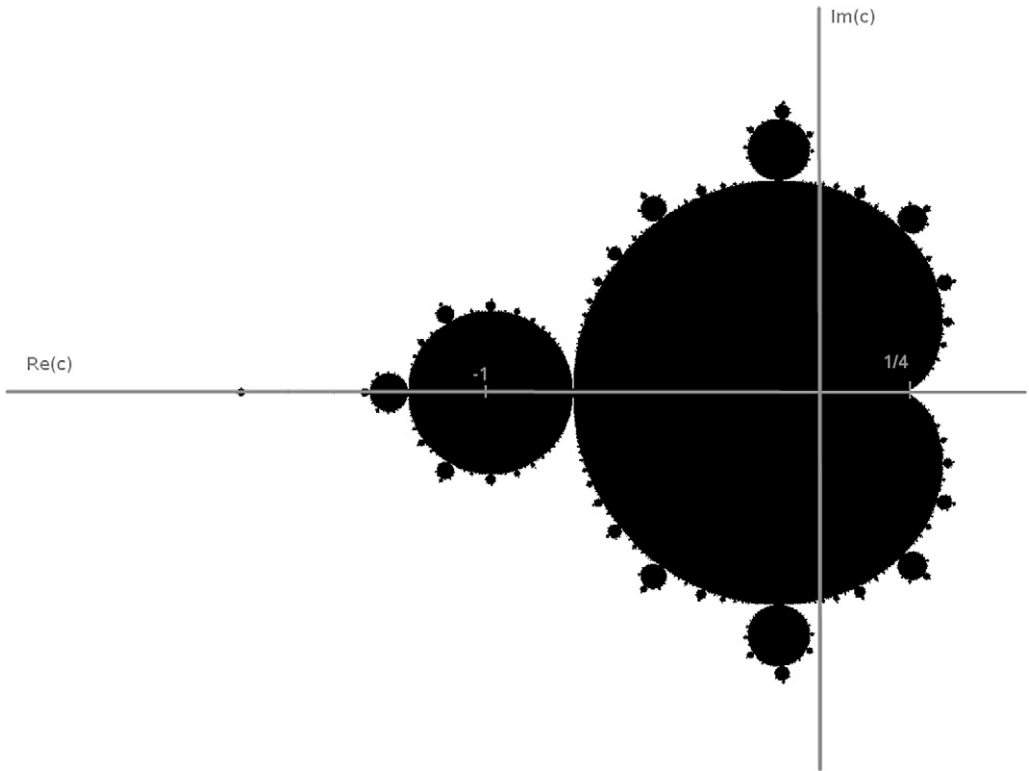


Fig. 5. The Madelbrot set (drawn with xaos 3.1 [16]).

A.6. Overview of Borel summability and transseries

There is a vast literature on transseries, Borel summability, and resurgence, see, for example [17]. Most of the modern theory originates in Ecalle’s work [8].

Definition 21. We say that f is given by a Borel summable transseries for $x > \nu$, if there exists a $\beta \in \mathbb{C}$, a sequence c_k , with $\text{Re } c_k \geq Ck$ for some $C > 0$, and a sequence of functions Y_k , analytic in a neighborhood of $(0, \infty)$, having convergent Puiseux series at zero, and $|Y_k(p)| \leq |B^k e^{\nu p}|$ (where B and ν are independent of k) such that

$$f(x) = \sum_{k=0}^{\infty} e^{-c_k x} x^{(k+1)\beta} \mathcal{L}Y_k \tag{89}$$

where \mathcal{L} is the usual Laplace transform:

$$(\mathcal{L}Y)(x) = \int_0^{\infty} e^{-px} Y(p) dp \tag{90}$$

The definition for other directions θ in the x complex domain is obtained by changing the variable to $x' = xe^{-i\theta}$.

The Borel–Laplace summation operator is denoted by \mathcal{LB} .

Definition 22. A formal power series in powers of $1/x$ is Borel summable as $x \rightarrow \infty$ if it is the asymptotic series of \mathcal{LY} , where Y is as in Definition 21. (We note that by Watson’s Lemma [1], $(\mathcal{LY})(x)$ has an asymptotic series as $x \rightarrow \infty$, which is the termwise Laplace transform of the Taylor series of Y at zero.)

Transseries representations contain therefore manifest asymptotic information.

Definition 23. A function $Y(p)$ is resurgent in p in the sense of Ecalle [8], if it is analytic on the Riemann surface of $\mathbb{C} \setminus J$, where J is a discrete set, and has uniform exponential bounds along any direction towards infinity cf. [17].⁷ By abuse of language, $f(x)$ is called resurgent if it satisfies the requirements in Definition 21 and **all** Y_k are resurgent.

This is especially useful when global information about f for $x \in \mathbb{C}$ is needed: deformation of contours in p , and collecting residues when/if singularities are crossed, provides a straightforward way to obtain this information.

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⁷ It is also required that certain relations, called bridge equations, hold; in our case it would be easy to derive them from the explicit form of H , but we omit this calculation.